

410 (VI.21) Surfaces

A. The Notion of a Surface

The notion of a surface may be roughly expressed by saying that by moving a curve we get a surface or that the boundary of a solid body is a surface. But these propositions cannot be considered mathematical definitions of a surface. We also make a distinction between surfaces and planes in ordinary language, where we mean by surfaces only those that are not planes. In mathematical language, however, planes are usually included among the surfaces.

A surface can be defined as a 2-dimensional \dagger continuum, in accordance with the definition of a curve as a 1-dimensional continuum.

However, while we have a theory of curves based on this definition, we do not have a similar theory of surfaces thus defined (\rightarrow 93 Curves).

What is called a surface or a curved surface is usually a 2-dimensional \dagger topological manifold, that is, a topological space that satisfies the \dagger second countability axiom and of which every point has a neighborhood \dagger homeomorphic to the interior of a circular disk in a 2-dimensional Euclidean space. In the following sections, we mean by a surface such a 2-dimensional topological manifold.

B. Examples and Classification

The simplest examples of surfaces are the 2-dimensional \dagger simplex and the 2-dimensional \dagger sphere. Surfaces are generally \dagger simplicially decomposable (or triangulable) and hence homeomorphic to 2-dimensional polyhedra (T. Radó, *Acta Sci. Math. Szeged.* (1925)). A \dagger compact surface is called a **closed surface**, and a noncompact surface is called an **open surface**. A closed surface is decomposable into a finite number of 2-simplexes and so can be interpreted as a \dagger combinatorial manifold. A 2-dimensional topological manifold having a boundary is called a **surface with boundary**. A 2-simplex is an example of a surface with boundary, and a sphere is an example of a closed surface without boundary.

Surfaces are classified as \dagger orientable and \dagger nonorientable. In the special case when a surface is \dagger embedded in a 3-dimensional Euclidean space E^3 , whether the surface is orientable or not depends on its having two sides (the "surface" and "back") or only one side. Therefore, in this special case, an orientable surface is called **two-sided**, and a nonorientable

surface, **one-sided**. A nonorientable closed surface without boundary cannot be embedded in the Euclidean space E^3 (\rightarrow 56 Characteristic Classes, 114 Differential Topology).

The first example of a nonorientable surface (with boundary) is the so-called **Möbius strip** or **Möbius band**, constructed as an \dagger identification space from a rectangle by twisting through 180° and identifying the opposite edges with one another (Fig. 1).

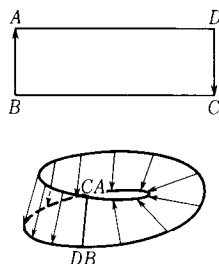


Fig. 1

As illustrated in Fig. 2, from a rectangle $ABCD$ we can obtain a closed surface homeomorphic to the product space $S^1 \times S^1$ by identifying the opposite edges AB with DC and BC with AD . This surface is the so-called 2-dimensional **torus** (or **anchor ring**). In this case, the four vertices A, B, C, D of the rectangle correspond to one point p on the surface, and the pairs of edges AB, DC and BC, AD correspond to closed curves a' and b' on the surface. We use the notation $aba^{-1}b^{-1}$ to represent a torus. This refers to the fact that the torus is obtained from an oriented four-sided polygon by identifying the first side and the third (with reversed orientation), the second side and the fourth (with reversed orientation). Similarly, aa^{-1} represents a sphere (Fig. 3), and $a_1b_1a_1^{-1}b_1^{-1}a_2b_2a_2^{-1}b_2^{-1}$ represents the closed surface shown in Fig. 4.

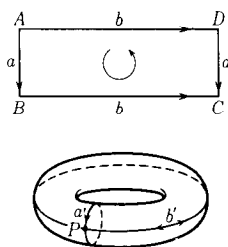


Fig. 2

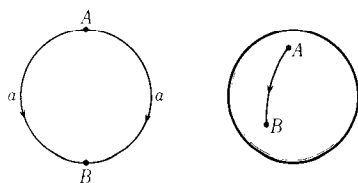


Fig. 3

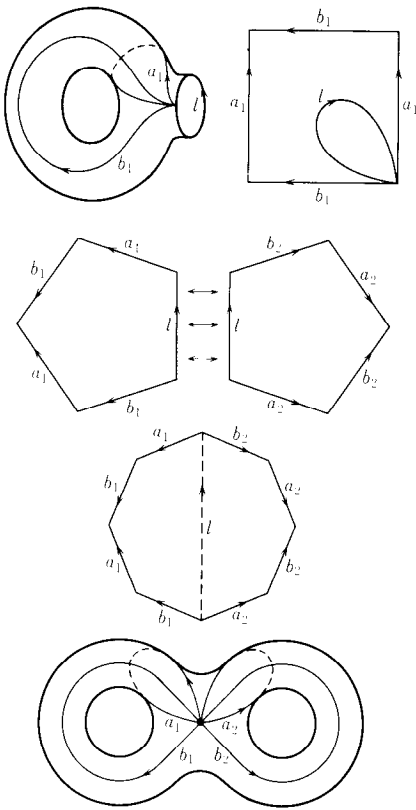


Fig. 4

All closed surfaces without boundary are constructed by identifying suitable pairs of sides of a $2n$ -sided polygon in a Euclidean plane E^2 . Furthermore, a closed orientable surface without boundary is homeomorphic to the surface represented by aa^{-1} or

$$a_1 b_1 a_1^{-1} b_1^{-1} \dots a_p b_p a_p^{-1} b_p^{-1}. \tag{1}$$

The 1-dimensional \ast Betti number of this surface is $2p$, the 0-dimensional and 2-dimensional \ast Betti numbers are 1, the \ast torsion coefficients are all 0, and p is called the **genus** of the surface. Also, a closed orientable surface of genus p with boundaries c_1, \dots, c_k is represented by

$$w_1 c_1 w_1^{-1} \dots w_k c_k w_k^{-1} a_1 b_1 a_1^{-1} b_1^{-1} \dots a_p b_p a_p^{-1} b_p^{-1} \tag{2}$$

(Fig. 5). A closed nonorientable surface without boundary is represented by

$$a_1 a_1 a_2 a_2 \dots a_q a_q. \tag{3}$$

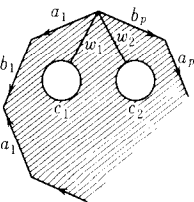


Fig. 5

The 1-dimensional Betti number of this surface is $q - 1$, the 0-dimensional and 2-dimensional Betti numbers are 1 and 0, respectively, the 1-dimensional torsion coefficient is 2, the 0-dimensional and 2-dimensional torsion coefficients are 0, and q is called the **genus** of the surface. A closed nonorientable surface of genus q with boundaries c_1, \dots, c_k is represented by

$$w_1 c_1 w_1^{-1} \dots w_k c_k w_k^{-1} a_1 a_1 \dots a_q a_q. \tag{4}$$

Each of forms (1)–(4) is called the **normal form** of the respective surface, and the curves a_i, b_j, w_k are called the **normal sections** of the surface. To explain the notation in (3), we first take the simplest case, aa . In this case, the surface is obtained from a disk by identifying each pair of points on the circumference that are end-points of a diameter (Fig. 6). The surface aa is then homeomorphic to a \ast projective plane of which a decomposition into a complex of triangles is illustrated in Fig. 7. On the other hand, $aabb$ represents a surface like that shown in Fig. 8, called the **Klein bottle**. Fig. 9 shows a **handle**, and Fig. 10 shows a **cross cap**.

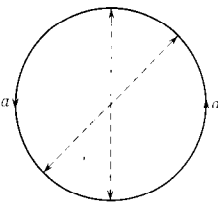


Fig. 6

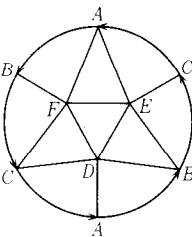


Fig. 7

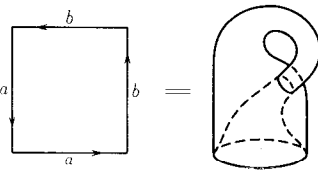


Fig. 8

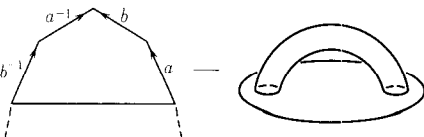


Fig. 9

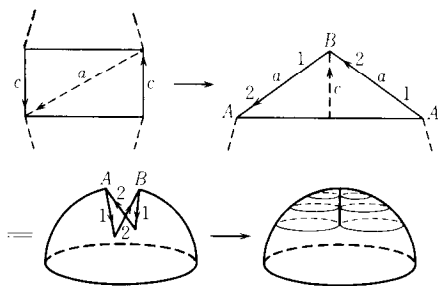


Fig. 10

The last two surfaces have boundaries; a handle is orientable, while a cross cap is non-orientable and homeomorphic to the Möbius strip. If we delete p disks from a sphere and replace them with an equal number of handles, then we obtain a surface homeomorphic to the surface represented in (1), while if we replace the disks by cross caps instead of by handles, then the surface thus obtained is homeomorphic to that represented in (3). Now we decompose the surfaces (1) and (3) into triangles and denote the number of i -dimensional simplexes by α_i ($i=0, 1, 2$). Then in view of the \dagger Euler-Poincaré formula, the surfaces (1) and (3) satisfy the respective formulas

$$\alpha_0 - \alpha_1 + \alpha_2 = 2(1 - p),$$

$$\alpha_0 - \alpha_1 + \alpha_2 = 2 - q.$$

The \dagger Riemann surfaces of \dagger algebraic functions of one complex variable are always surfaces of type (1), and their genera p coincide with those of algebraic functions.

All closed surfaces are homeomorphic to surfaces of types (1), (2), (3), or (4). A necessary and sufficient condition for two surfaces to be homeomorphic to each other is coincidence of the numbers of their boundaries, their orientability or nonorientability, and their genera (or \dagger Euler characteristic $\alpha^0 - \alpha^1 + \alpha^2$). This proposition is called the **fundamental theorem of the topology of surfaces**. The \dagger homeomorphism problem of closed surfaces is completely solved by this theorem. The same problem for n ($n \geq 3$) manifolds, even if they are compact, remains open. (For surface area \rightarrow 246 Length and Area. For the differential geometry of surfaces \rightarrow 111 Differential Geometry of Curves and Surfaces.)

References

- [1] B. Kerékjártó, *Vorlesungen über Topologie*, Springer, 1923.
- [2] H. Seifert and W. Threlfall, *Lehrbuch der Topologie*, Teubner, 1934 (Chelsea, 1945).
- [3] S. Lefschetz, *Introduction to topology*, Princeton Univ. Press, 1949.

[4] D. Hilbert and S. Cohn-Vossen, *Anschauliche Geometrie*, Springer, 1932; English translation, *Geometry and the imagination*, Chelsea, 1952.

[5] W. S. Massey, *Algebraic topology: An introduction*, Springer, 1967.

[6] E. E. Moise, *Geometric topology in dimensions 2 and 3*, Springer, 1977.

411 (I.4) Symbolic Logic

A. General Remarks

Symbolic logic (or **mathematical logic**) is a field of logic in which logical inferences commonly used in mathematics are investigated by use of mathematical symbols.

The **algebra of logic** originally set forth by G. Boole [1] and A. de Morgan [2] is actually an algebra of sets or relations; it did not reach the same level as the symbolic logic of today. G. Frege, who dealt not only with the logic of propositions but also with the first-order predicate logic using quantifiers (\rightarrow Sections C and K), should be regarded as the real originator of symbolic logic. Frege's work, however, was not recognized for some time. Logical studies by C. S. Peirce, E. Schröder, and G. Peano appeared soon after Frege, but they were limited mostly to propositions and did not develop Frege's work. An essential development of Frege's method was brought about by B. Russell, who, with the collaboration of A. N. Whitehead, summarized his results in *Principia mathematica* [4], which seemed to have completed the theory of symbolic logic at the time of its appearance.

B. Logical Symbols

If A and B are propositions, the propositions (A and B), (A or B), (A implies B), and (not A) are denoted by

$$A \wedge B, \quad A \vee B, \quad A \rightarrow B, \quad \neg A,$$

respectively. We call $\neg A$ the **negation** of A , $A \wedge B$ the **conjunction** (or **logical product**), $A \vee B$ the **disjunction** (or **logical sum**), and $A \rightarrow B$ the **implication** (or B by A). The proposition $(A \rightarrow B) \wedge (B \rightarrow A)$ is denoted by $A \leftrightarrow B$ and is read " A and B are **equivalent**." $A \vee B$ means that at least one of A and B holds. The propositions (For all x , the proposition $F(x)$ holds) and (There exists an x such that $F(x)$ holds) are denoted by $\forall x F(x)$ and $\exists x F(x)$, respectively. A proposition of the form $\forall x F(x)$

is called a **universal proposition**, and one of the form $\exists xF(x)$, an **existential proposition**. The symbols $\wedge, \vee, \rightarrow, \leftrightarrow, \neg, \forall, \exists$ are called **logical symbols**.

There are various other ways to denote logical symbols, including:

$$A \wedge B: A \& B, A \cdot B,$$

$$A \vee B: A + B,$$

$$A \rightarrow B: A \supset B, A \Rightarrow B,$$

$$A \leftrightarrow B: A \rightleftharpoons B, A \equiv B, A \sim B, A \supset \subset B, A \Leftrightarrow B,$$

$$\neg A: \sim A, \bar{A},$$

$$\forall xF(x): (x)F(x), \prod xF(x), \bigwedge xF(x),$$

$$\exists xF(x): (Ex)F(x), \sum xF(x), \bigvee xF(x).$$

C. Free and Bound Variables

Any function whose values are propositions is called a **propositional function**. $\forall x$ and $\exists x$ can be regarded as operators that transform any propositional function $F(x)$ into the propositions $\forall xF(x)$ and $\exists xF(x)$, respectively. $\forall x$ and $\exists x$ are called **quantifiers**; the former is called the **universal quantifier** and the latter the **existential quantifier**. $F(x)$ is transformed into $\forall xF(x)$ or $\exists xF(x)$ just as a function $f(x)$ is transformed into the definite integral $\int_0^1 f(x)dx$; the resultant propositions $\forall xF(x)$ and $\exists xF(x)$ are no longer functions of x . The variable x in $\forall xF(x)$ and in $\exists xF(x)$ is called a **bound variable**, and the variable x in $F(x)$, when it is not bound by $\forall x$ or $\exists x$, is called a **free variable**. Some people employ different kinds of symbols for free variables and bound variables to avoid confusion.

D. Formal Expressions of Propositions

A formal expression of a proposition in terms of logical symbols is called a **formula**. More precisely, formulas are constructed by the following **formation rules**: (1) If \mathfrak{A} is a formula, $\neg \mathfrak{A}$ is also a formula. If \mathfrak{A} and \mathfrak{B} are formulas, $\mathfrak{A} \wedge \mathfrak{B}$, $\mathfrak{A} \vee \mathfrak{B}$, $\mathfrak{A} \rightarrow \mathfrak{B}$ are all formulas. (2) If $\mathfrak{F}(a)$ is a formula and a is a free variable, then $\forall x\mathfrak{F}(x)$ and $\exists x\mathfrak{F}(x)$ are formulas, where x is an arbitrary bound variable not contained in $\mathfrak{F}(a)$ and $\mathfrak{F}(x)$ is the result of substituting x for a throughout $\mathfrak{F}(a)$.

We use formulas of various scope according to different purposes. To indicate the scope of formulas, we fix a set of formulas, each element of which is called a **prime formula** (or **atomic formula**). The scope of formulas is the set of formulas obtained from the prime formulas by formation rules (1) and (2).

E. Propositional Logic

Propositional logic is the field in symbolic logic in which we study relations between propositions exclusively in connection with the four logical symbols $\wedge, \vee, \rightarrow$, and \neg , called **propositional connectives**.

In propositional logic, we deal only with operations of **logical operators** denoted by propositional connectives, regarding the variables for denoting propositions, called **proposition variables**, only as prime formulas. We examine problems such as: What kinds of formulas are identically true when their proposition variables are replaced by any propositions, and what kinds of formulas can sometimes be true?

Consider the two symbols \vee and \wedge , read **true** and **false**, respectively, and let $\mathbf{A} = \{\vee, \wedge\}$. A univalent function from \mathbf{A} , or more generally from a Cartesian product $\mathbf{A} \times \dots \times \mathbf{A}$, into \mathbf{A} is called a **truth function**. We can regard $\wedge, \vee, \rightarrow, \neg$ as the following truth functions: (1) $A \wedge B = \vee$ for $A = B = \vee$, and $A \wedge B = \wedge$ otherwise; (2) $A \vee B = \wedge$ for $A = B = \wedge$, and $A \vee B = \vee$ otherwise; (3) $A \rightarrow B = \wedge$ for $A = \vee$ and $B = \wedge$, and $A \rightarrow B = \vee$ otherwise; (4) $\neg A = \wedge$ for $A = \vee$, and $\neg A = \vee$ for $A = \wedge$.

If we regard proposition variables as variables whose domain is \mathbf{A} , then each formula represents a truth function. Conversely, any truth function (of a finite number of independent variables) can be expressed by an appropriate formula, although such a formula is not uniquely determined. If a formula is regarded as a truth function, the value of the function determined by a combination of values of the independent variables involved in the formula is called the **truth value** of the formula.

A formula corresponding to a truth function that takes only \vee as its value is called a **tautology**. For example, $\mathfrak{A} \vee \neg \mathfrak{A}$ and $((\mathfrak{A} \rightarrow \mathfrak{B}) \rightarrow \mathfrak{A}) \rightarrow \mathfrak{A}$ are tautologies. Since a truth function with n independent variables takes values corresponding to 2^n combinations of truth values of its variables, we can determine in a finite number of steps whether a given formula is a tautology. If $\mathfrak{A} \leftrightarrow \mathfrak{B}$ is a tautology (that is, \mathfrak{A} and \mathfrak{B} correspond to the same truth function), then the formulas \mathfrak{A} and \mathfrak{B} are said to be **equivalent**.

F. Propositional Calculus

It is possible to choose some specific tautologies, designate them as axioms, and derive all tautologies from them by appropriately given rules of inference. Such a system is called a **propositional calculus**. There are many ways

to stipulate axioms and rules of inference for a propositional calculus.

The abovementioned propositional calculus corresponds to the so-called classical propositional logic (\rightarrow Section L). By choosing appropriate axioms and rules of inference we can also formally construct intuitionistic or other propositional logics. In intuitionistic logic the law of the \dagger excluded middle is not accepted, and hence it is impossible to formalize intuitionistic propositional logic by the notion of tautology. We therefore usually adopt the method of propositional calculus, instead of using the notion of tautology, to formalize intuitionistic propositional logic. For example, V. I. Glivenko's theorem [5], that if a formula \mathfrak{A} can be proved in classical logic, then $\neg \neg \mathfrak{A}$ can be proved in intuitionistic logic, was obtained by such formalistic considerations. A method of extending the classical concepts of truth value and tautology to intuitionistic and other logics has been obtained by S. A. Kripke. There are also studies of logics intermediate between intuitionistic and classical logic (T. Umezawa).

G. Predicate Logic

Predicate logic is the area of symbolic logic in which we take quantifiers in account. Mainly propositional functions are discussed in predicate logic. In the strict sense only single-variable propositional functions are called **predicates**, but the phrase **predicate of n arguments** (or **n -ary predicate**) denoting an n -variable propositional function is also employed. Single-variable (or unary) predicates are also called **properties**. We say that a has the property F if the proposition $F(a)$ formed by the property F is true. Predicates of two arguments are called **binary relations**. The proposition $R(a, b)$ formed by the binary relation R is occasionally expressed in the form aRb . Generally, predicates of n arguments are called **n -ary relations**. The domain of definition of a unary predicate is called the **object domain**, elements of the object domain are called **objects**, and any variable running over the object domain is called an **object variable**. We assume here that the object domain is not empty. When we deal with a number of predicates simultaneously (with different numbers of variables), it is usual to arrange things so that all the independent variables have the same object domain by suitably extending their object domains.

Predicate logic in its purest sense deals exclusively with the general properties of quantifiers in connection with propositional connectives. The only objects dealt with in this

field are **predicate variables** defined over a certain common domain and object variables running over the domain. Propositional variables are regarded as predicates of no variables. Each expression $F(a_1, \dots, a_n)$ for any predicate variable F of n variables a_1, \dots, a_n (object variables designated as free) is regarded as a prime formula ($n = 0, 1, 2, \dots$), and we deal exclusively with formulas generated by these prime formulas, where bound variables are also restricted to object variables that have a common domain. We give no specification for the range of objects except that it be the common domain of the object variables.

By designating an object domain and substituting a predicate defined over the domain for each predicate variable in a formula, we obtain a proposition. By substituting further an object (object constant) belonging to the object domain for each object variable in a proposition, we obtain a proposition having a definite truth value. When we designate an object domain and further associate with each predicate variable as well as with each object variable a predicate or an object to be substituted for it, we call the pair consisting of the object domain and the association a **model**. Any formula that is true for every model is called an **identically true formula** or **valid formula**. The study of identically true formulas is one of the most important problems in predicate logic.

H. Formal Representations of Mathematical Propositions

To obtain a formal representation of a mathematical theory by predicate logic, we must first specify its object domain, which is a non-empty set whose elements are called **individuals**; accordingly the object domain is called the **individual domain**, and object variables are called **individual variables**. Secondly we must specify **individual symbols**, **function symbols**, and **predicate symbols**, signifying specific individuals, functions, and \dagger predicates, respectively. Here a function of n arguments is a univalent mapping from the Cartesian product $D \times \dots \times D$ of n copies of the given set to D . Then we define the notion of **term** as in the next paragraph to represent each individual formally. Finally we express propositions formally by formulas.

Definition of terms (formation rule for terms): (1) Each individual symbol is a term. (2) Each free variable is a term. (3) $f(t_1, \dots, t_n)$ is a term if t_1, \dots, t_n are terms and f is a function symbol of n arguments. (4) The only terms are those given by (1)–(3).

As a prime formula in this case we use any

formula of the form $F(t_1, \dots, t_n)$, where F is a predicate symbol of n arguments and t_1, \dots, t_n are arbitrary terms. To define the notions of term and formula, we need logical symbols, free and bound individual variables, and also a list of individual symbols, function symbols, and predicate symbols.

In pure predicate logic, the individual domain is not concrete, and we study only general forms of propositions. Hence, in this case, predicate or function symbols are not representations of concrete predicates or functions but are **predicate variables** and **function variables**. We also use free individual variables instead of individual symbols. In fact, it is now most common that function variables are dispensed with, and only free individual variables are used as terms.

I. Formulation of Mathematical Theories

To formalize a theory we need **axioms** and **rules of inference**. Axioms constitute a certain specific set of formulas, and a rule of inference is a rule for deducing a formula from other formulas. A formula is said to be **provable** if it can be deduced from the axioms by repeated application of rules of inference. Axioms are divided into two types: **logical axioms**, which are common to all theories, and **mathematical axioms**, which are peculiar to each individual theory. The set of mathematical axioms is called the **axiom system** of the theory.

(I) Logical axioms: (1) A formula that is the result of substituting arbitrary formulas for the proposition variables in a tautology is an axiom. (2) Any formula of the form

$$\forall x \mathfrak{F}(x) \rightarrow \mathfrak{F}(t) \quad \text{or} \quad \mathfrak{F}(t) \rightarrow \exists x \mathfrak{F}(x)$$

is an axiom, where $\mathfrak{F}(t)$ is the result of substituting an arbitrary term t for x in $\mathfrak{F}(x)$.

(II) Rules of inference: (1) We can deduce a formula \mathfrak{B} from two formulas \mathfrak{A} and $\mathfrak{A} \rightarrow \mathfrak{B}$ (**modus ponens**). (2) We can deduce $\mathfrak{A} \rightarrow \forall x \mathfrak{F}(x)$ from a formula $\mathfrak{A} \rightarrow \mathfrak{F}(a)$ and $\exists x \mathfrak{F}(x) \rightarrow \mathfrak{A}$ from $\mathfrak{F}(a) \rightarrow \mathfrak{A}$, where a is a free individual variable contained in neither \mathfrak{A} nor $\mathfrak{F}(x)$ and $\mathfrak{F}(a)$ is the result of substituting a for x in $\mathfrak{F}(x)$.

If an axiom system is added to these logical axioms and rules of inference, we say that a **formal system** is given.

A formal system S or its axiom system is said to be **contradictory** or to contain a **contradiction** if a formula \mathfrak{A} and its negation $\neg \mathfrak{A}$ are provable; otherwise it is said to be **consistent**. Since

$$(\mathfrak{A} \wedge \neg \mathfrak{A}) \rightarrow \mathfrak{B}$$

is a tautology, we can show that any formula is provable in a formal system containing a

contradiction. The validity of a proof by **reductio ad absurdum** lies in the fact that

$$(\mathfrak{A} \rightarrow (\mathfrak{B} \wedge \neg \mathfrak{B})) \rightarrow \neg \mathfrak{A}$$

is a tautology. An affirmative proposition (formula) may be obtained by reductio ad absurdum since the formula (of propositional logic) representing the **discharge of double negation**

$$\neg \neg \mathfrak{A} \rightarrow \mathfrak{A}$$

is a tautology.

J. Predicate Calculus

If a formula has no free individual variable, we call it a **closed formula**. Now we consider a formal system S whose mathematical axioms are closed. A formula \mathfrak{A} is provable in S if and only if there exist suitable mathematical axioms $\mathfrak{C}_1, \dots, \mathfrak{C}_n$ such that the formula

$$(\mathfrak{C}_1 \wedge \dots \wedge \mathfrak{C}_n) \rightarrow \mathfrak{A}$$

is provable without the use of mathematical axioms. Since any axiom system can be replaced by an equivalent axiom system containing only closed formulas, the study of a formal system can be reduced to the study of pure logic.

In the following we take no individual symbols or function symbols into consideration and we use predicate variables as predicate symbols in accordance with the commonly accepted method of stating properties of the pure predicate logic; but only in the case of **predicate logic with equality** will we use predicate variables and the equality predicate $=$ as a predicate symbol. However, we can safely state that we use function variables as function symbols.

The formal system with no mathematical axioms is called the **predicate calculus**. The formal system whose mathematical axioms are the equality axioms

$$a = a, \quad a = b \rightarrow (\mathfrak{F}(a) \rightarrow \mathfrak{F}(b))$$

is called the **predicate calculus with equality**.

In the following, by being provable we mean being provable in the predicate calculus.

(1) Every provable formula is valid.

(2) Conversely, any valid formula is provable (K. Gödel [6]). This fact is called the **completeness** of the predicate calculus. In fact, by Gödel's proof, a formula \mathfrak{A} is provable if \mathfrak{A} is always true in every interpretation whose individual domain is of countable cardinality. In another formulation, if $\neg \mathfrak{A}$ is not provable, the formula \mathfrak{A} is a true proposition in some interpretation (and the individual domain in this case is of countable cardinality). We can

extend this result as follows: If an axiom system generated by countably many closed formulas is consistent, then its mathematical axioms can be considered true propositions by a common interpretation. In this sense, **Gödel's completeness theorem** gives another proof of the †Skolem-Löwenheim theorem.

(3) The predicate calculus is consistent. Although this result is obtained from (1) in this section, it is not difficult to show it directly (D. Hilbert and W. Ackermann [7]).

(4) There are many different ways of giving logical axioms and rules of inference for the predicate calculus. G. Gentzen gave two types of systems in [8]; one is a natural deduction system in which it is easy to reproduce formal proofs directly from practical ones in mathematics, and the other has a logically simpler structure. Concerning the latter, Gentzen proved **Gentzen's fundamental theorem**, which shows that a formal proof of a formula may be translated into a "direct" proof. The theorem itself and its idea were powerful tools for obtaining consistency proofs.

(5) If the proposition $\exists x\mathfrak{A}(x)$ is true, we choose one of the individuals x satisfying the condition $\mathfrak{A}(x)$, and denote it by $\varepsilon x\mathfrak{A}(x)$. When $\exists x\mathfrak{A}(x)$ is false, we let $\varepsilon x\mathfrak{A}(x)$ represent an arbitrary individual. Then

$$\exists x\mathfrak{A}(x) \rightarrow \mathfrak{A}(\varepsilon x\mathfrak{A}(x)) \quad (1)$$

is true. We consider εx to be an operator associating an individual $\varepsilon x\mathfrak{A}(x)$ with a proposition $\mathfrak{A}(x)$ containing the variable x . Hilbert called it the **transfinite logical choice function**; today we call it **Hilbert's ε -operator** (or **ε -quantifier**), and the logical symbol ε used in this sense **Hilbert's ε -symbol**. Using the ε -symbol, $\exists x\mathfrak{A}(x)$ and $\forall x\mathfrak{A}(x)$ are represented by

$$\mathfrak{A}(\varepsilon x\mathfrak{A}(x)), \quad \mathfrak{A}(\varepsilon x \neg \mathfrak{A}(x)),$$

respectively, for any $\mathfrak{A}(x)$. The system of predicate calculus adding formulas of the form (1) as axioms is essentially equivalent to the usual predicate calculus. This result, called the **ε -theorem**, reads as follows: When a formula \mathfrak{C} is provable under the assumption that every formula of the form (1) is an axiom, we can prove \mathfrak{C} using no axioms of the form (1) if \mathfrak{C} contains no logical symbol ε (D. Hilbert and P. Bernays [9]). Moreover, a similar theorem holds when axioms of the form

$$\forall x(\mathfrak{A}(x) \leftrightarrow \mathfrak{B}(x)) \rightarrow \varepsilon x\mathfrak{A}(x) = \varepsilon x\mathfrak{B}(x) \quad (2)$$

are added (S. Maehara [10]).

(6) For a given formula \mathfrak{A} , call \mathfrak{A}' a normal form of \mathfrak{A} when the formula

$$\mathfrak{A} \leftrightarrow \mathfrak{A}'$$

is provable and \mathfrak{A}' satisfies a particular condition. For example, for any formula \mathfrak{A} there is

a normal form \mathfrak{A}' satisfying the condition: \mathfrak{A}' has the form

$$Q_1 x_1 \dots Q_n x_n \mathfrak{B}(x_1, \dots, x_n),$$

where Qx means a quantifier $\forall x$ or $\exists x$, and $\mathfrak{B}(x_1, \dots, x_n)$ contains no quantifier and has no predicate variables or free individual variables not contained in \mathfrak{A} . A normal form of this kind is called a **prenex normal form**.

(7) We have dealt with the classical first-order predicate logic until now. For other predicate logics (\rightarrow Sections K and L) also, we can consider a predicate calculus or a formal system by first defining suitable axioms or rules of inference. Gentzen's fundamental theorem applies to the intuitionistic predicate calculus formulated by V. I. Glivenko, A. Heyting, and others. Since Gentzen's fundamental theorem holds not only in classical logic and intuitionistic logic but also in several systems of first-order predicate logic or propositional logic, it is useful for getting results in modal and other logics (M. Ohnishi, K. Matsumoto). Moreover, Glivenko's theorem in propositional logic [5] is also extended to predicate calculus by using a rather weak representation (S. Kuroda [12]). G. Takeuti expected that a theorem similar to Gentzen's fundamental theorem would hold in higher-order predicate logic also, and showed that the consistency of analysis would follow if that conjecture could be verified [13]. Moreover, in many important cases, he showed constructively that the conjecture holds partially. The conjecture was finally proved by M. Takahashi [14] by a nonconstructive method. Concerning this, there are also contributions by S. Maehara, T. Simauti, M. Yasuhara, and W. Tait.

K. Predicate Logics of Higher Order

In ordinary predicate logic, the bound variables are restricted to individual variables. In this sense, ordinary predicate logic is called **first-order predicate logic**, while predicate logic dealing with quantifiers $\forall P$ or $\exists P$ for a predicate variable P is called **second-order predicate logic**.

Generalizing further, we can introduce the so-called **third-order predicate logic**. First we fix the individual domain D_0 . Then, by introducing the whole class D_1^n of predicates of n variables, each running over the object domain D_0 , we can introduce predicates that have D_1^n as their object domain. This kind of predicate is called a **second-order predicate** with respect to the individual domain D_0 . Even when we restrict second-order predicates to one-variable predicates, they are divided into vari-

ous types, and the domains of independent variables do not coincide in the case of more than two variables. In contrast, predicates having D_0 as their object domain are called **first-order predicates**. The logic having quantifiers that admit first-order predicate variables is second-order predicate logic, and the logic having quantifiers that admit up to second-order predicate variables is third-order predicate logic. Similarly, we can define further **higher-order predicate logics**.

Higher-order predicate logic is occasionally called **type theory**, because variables arise that are classified into various types. Type theory is divided into **simple type theory** and **ramified type theory**.

We confine ourselves to variables for single-variable predicates, and denote by P such a bound predicate variable. Then for any formula $\mathfrak{F}(a)$ (with a a free individual variable), the formula

$$\exists P \forall x (P(x) \leftrightarrow \mathfrak{F}(x))$$

is considered identically true. This is the point of view in simple type theory.

Russell asserted first that this formula cannot be used reasonably if quantifiers with respect to predicate variables occur in $\mathfrak{F}(x)$. This assertion is based on the point of view that the formula in the previous paragraph asserts that $\mathfrak{F}(x)$ is a first-order predicate, whereas any quantifier with respect to first-order predicate variables, whose definition assumes the totality of the first-order predicates, should not be used to introduce the first-order predicate $\mathfrak{F}(x)$. For this purpose, Russell further classified the class of first-order predicates by their **rank** and adopted the axiom

$$\exists P^k \forall x (P^k(x) \leftrightarrow \mathfrak{F}(x))$$

for the predicate variable P^k of rank k , where the rank i of any free predicate variable occurring in $\mathfrak{F}(x)$ is $\leq k$, and the rank j of any bound predicate variable occurring in $\mathfrak{F}(x)$ is $< k$. This is the point of view in ramified type theory, and we still must subdivide the types if we deal with higher-order propositions or propositions of many variables. Even Russell, having started from his ramified type theory, had to introduce the **axiom of reducibility** afterwards and reduce his theory to simple type theory.

L. Systems of Logic

Logic in the ordinary sense, which is based on the **law of the excluded middle** asserting that every proposition is in principle either true or false, is called **classical logic**. Usually, propo-

sitional logic, predicate logic, and type theory are developed from the standpoint of classical logic. Occasionally the reasoning of intuitionistic mathematics is investigated using symbolic logic, in which the law of the excluded middle is not admitted (\rightarrow 156 Foundations of Mathematics). Such logic is called **intuitionistic logic**. Logic is also subdivided into propositional logic, predicate logic, etc., according to the extent of the propositions (formulas) dealt with.

To express **modal propositions** stating **possibility**, **necessity**, etc., in symbolic logic, J. Łukasiewicz proposed a propositional logic called **three-valued logic**, having a third truth value, neither true nor false. More generally, **many-valued logics** with any number of truth values have been introduced; classical logic is one of its special cases, **two-valued logic** with two truth values, true and false. Actually, however, many-valued logics with more than three truth values have not been studied much, while various studies in **modal logic** based on classical logic have been successfully carried out. For example, studies of **strict implication** belong to this field.

References

- [1] G. Boole, An investigation of the laws of thought, Walton and Maberly, 1854.
- [2] A. de Morgan, Formal logic, or the calculus of inference, Taylor and Walton, 1847.
- [3] G. Frege, Begriffsschrift, eine der arithmetischen nachgebildete Formalsprache des reinen Denkens, Halle, 1879.
- [4] A. N. Whitehead and B. Russell, Principia mathematica I, II, III, Cambridge Univ. Press, 1910–1913; second edition, 1925–1927.
- [5] V. Glivenko, Sur quelques points de la logique de M. Brouwer, Acad. Roy. de Belgique, Bulletin de la classe des sciences, (5) 15 (1929), 183–188.
- [6] K. Gödel, Die Vollständigkeit der Axiome des logischen Funktionenkalküls, Monatsh. Math. Phys., 37 (1930), 349–360.
- [7] D. Hilbert and W. Ackermann, Grundzüge der theoretischen Logik, Springer, 1928, sixth edition, 1972; English translation, Principles of mathematical logic, Chelsea, 1950.
- [8] G. Gentzen, Untersuchungen über das logische Schliessen, Math. Z., 39 (1935), 176–210, 405–431.
- [9] D. Hilbert and P. Bernays, Grundlagen der Mathematik II, Springer, 1939; second edition, 1970.
- [10] S. Machara, Equality axiom on Hilbert's ε -symbol, J. Fac. Sci. Univ. Tokyo, (I), 7 (1957), 419–435.

- [11] A. Heyting, Die formalen Regeln der intuitionistischen Logik I, S.-B. Preuss. Akad. Wiss., 1930, 42–56.
- [12] S. Kuroda, Intuitionistische Untersuchungen der formalistischen Logik, Nagoya Math. J., 2 (1951), 35–47.
- [13] G. Takeuti, On a generalized logic calculus, Japan. J. Math., 23 (1953), 39–96.
- [14] M. Takahashi, A proof of the cut-elimination theorem in simple type-theory, J. Math. Soc. Japan, 19 (1967), 399–410.
- [15] S. C. Kleene, Mathematical logic, Wiley, 1967.
- [16] J. R. Shoenfield, Mathematical logic, Addison-Wesley, 1967.
- [17] R. M. Smullyan, First-order logic, Springer, 1968.

412 (IV.13)

Symmetric Riemannian Spaces and Real Forms

A. Symmetric Riemannian Spaces

Let M be a * Riemannian space. For each point p of M we can define a mapping σ_p of a suitable neighborhood U_p of p onto U_p itself so that $\sigma_p(x_t) = x_{-t}$, where x_t ($|t| < \varepsilon$, $x_0 = p$) is any * geodesic passing through the point p . We call M a **locally symmetric Riemannian space** if for any point p of M we can choose a neighborhood U_p so that σ_p is an * isometry of U_p . In order that a Riemannian space M be locally symmetric it is necessary and sufficient that the * covariant differential (with respect to the * Riemannian connection) of the * curvature tensor of M be 0. A locally symmetric Riemannian space is a * real analytic manifold. We say that a Riemannian space M is a **globally symmetric Riemannian space** (or simply **symmetric Riemannian space**) if M is connected and if for each point p of M there exists an isometry σ_p of M onto M itself that has p as an isolated fixed point (i.e., has no fixed point except p in a certain neighborhood of p) and such that σ_p^2 is the identity transformation on M . In this case σ_p is called the **symmetry** at p . A (globally) symmetric Riemannian space is locally symmetric and is a * complete Riemannian space. Conversely, a * simply connected complete locally symmetric Riemannian space is a (globally) symmetric Riemannian space.

B. Symmetric Riemannian Homogeneous Spaces

A * homogeneous space G/K of a connected * Lie group G is a **symmetric homogeneous**

space (with respect to θ) if there exists an **involution automorphism** (i.e., automorphism of order 2) θ of G satisfying the condition $K_\theta^0 \subset K \subset K_\theta$, where K_θ is the closed subgroup consisting of all elements of G left fixed by θ and K_θ^0 is the connected component of the identity element of K_θ . In this case, the mapping $aK \rightarrow \theta(a)K$ ($a \in G$) is a transformation of G/K having the point K as an isolated fixed point; more generally, the mapping $\theta_{a_0}: aK \rightarrow a_0\theta(a_0)^{-1}\theta(a)K$ is a transformation of G/K that has an arbitrary given point a_0K of G/K as an isolated fixed point. If there exists a G -invariant Riemannian metric on G/K , then G/K is a symmetric Riemannian space with symmetries $\{\theta_{a_0} | a_0 \in G\}$ and is called a **symmetric Riemannian homogeneous space**. A sufficient condition for a symmetric homogeneous space G/K to be a symmetric Riemannian homogeneous space is that K be a compact subgroup. Conversely, given a symmetric Riemannian space M , let G be the connected component of the identity element of the Lie group formed by all the isometries of M ; then M is represented as the symmetric Riemannian homogeneous space $M = G/K$ and K is a compact group. In particular, a symmetric Riemannian space can be regarded as a Riemannian space that is realizable as a symmetric Riemannian homogeneous space.

The Riemannian connection of a symmetric Riemannian homogeneous space G/K is uniquely determined (independent of the choice of G -invariant Riemannian metric), and a geodesic x_t ($|t| < \infty$, $x_0 = a_0K$) passing through a point a_0K of G/K is of the form $x_t = (\exp tX)a_0K$. Here X is any element of the Lie algebra \mathfrak{g} of G such that $\theta(X) = -X$, where θ also denotes the automorphism of \mathfrak{g} induced by the automorphism θ of G and $\exp tX$ is the * one-parameter subgroup of G defined by the element X . The covariant differential of any G -invariant tensor field on G/K is 0, and any G -invariant * differential form on G/K is a closed differential form.

C. Classification of Symmetric Riemannian Spaces

The * simply connected * covering Riemannian space of a symmetric Riemannian space is also a symmetric Riemannian space. Therefore the problem of classifying symmetric Riemannian spaces is reduced to classifying simply connected symmetric Riemannian spaces M and determining * discontinuous groups of isometries of M . When we take the * de Rham decomposition of such a space M and represent M as the product of a real Euclidean space and a number of simply connected irre-

ducible Riemannian spaces, all the factors are symmetric Riemannian spaces. We say that M is an **irreducible symmetric Riemannian space** if it is a symmetric Riemannian space and is irreducible as a Riemannian space.

A simply connected irreducible symmetric Riemannian space is isomorphic to one of the following four types of symmetric Riemannian homogeneous spaces (here Lie groups are always assumed to be connected):

(1) The symmetric Riemannian homogeneous space $(G \times G)/\{(a, a) | a \in G\}$ of the direct product $G \times G$, where G is a simply connected compact \dagger simple Lie group and the involutive automorphism of $G \times G$ is given by $(a, b) \rightarrow (b, a)$ ($(a, b) \in G \times G$). This space is isomorphic, as a Riemannian space, to the space G obtained by introducing a two-sided invariant Riemannian metric on the group G ; the isomorphism is induced from the mapping $G \times G \ni (a, b) \rightarrow ab^{-1} \in G$.

(2) A symmetric homogeneous space G/K_θ of a simply connected compact simple Lie group G with respect to an involutive automorphism θ of G . In this case, the closed subgroup $K_\theta = \{a \in G | \theta(a) = a\}$ of G is connected. We assume here that θ is a member of the given complete system of representatives of the \dagger conjugate classes formed by the elements of order 2 in the automorphism group of the group G .

(3) The homogeneous space G^C/G , where G^C is a complex simple Lie group whose \dagger center reduces to the identity element and G is an arbitrary but fixed maximal compact subgroup of G^C .

(4) The homogeneous space G_0/K , where G_0 is a noncompact simple Lie group whose center reduces to the identity element and which has no complex Lie group structure, and K is a maximal compact subgroup of G . In Section D we shall see that (3) and (4) are actually symmetric homogeneous spaces. All four types of symmetric Riemannian spaces are actually irreducible symmetric Riemannian spaces, and G -invariant Riemannian metrics on each of them are uniquely determined up to multiplication by a positive number. On the other hand, (1) and (2) are compact, while (3) and (4) are homeomorphic to Euclidean spaces and not compact. For spaces of types (1) and (3) the problem of classifying simply connected irreducible symmetric Riemannian spaces is reduced to classifying \dagger compact real simple Lie algebras and \dagger complex simple Lie algebras, respectively, while for types (2) and (4) it is reduced to the classification of noncompact real simple Lie algebras (\rightarrow Section D) (for the result of classification of these types \rightarrow Appendix A, Table 5.II). On the other hand, any (not necessarily simply connected) irreducible

symmetric Riemannian space defines one of (1)–(4) as its \dagger universal covering manifold; if the covering manifold is of type (3) or (4), the original symmetric Riemannian space is necessarily simply connected.

D. Symmetric Riemannian Homogeneous Spaces of Semisimple Lie Groups

In Section C we saw that any irreducible symmetric Riemannian space is representable as a symmetric Riemannian homogeneous space G/K on which a connected semisimple Lie group G acts \dagger almost effectively (\rightarrow 249 Lie Groups). Among symmetric Riemannian spaces, such a space $M = G/K$ is characterized as one admitting no nonzero vector field that is \dagger parallel with respect to the Riemannian connection. Furthermore, if G acts effectively on M , G coincides with the connected component $I(M)^0$ of the identity element of the Lie group formed by all the isometries of M .

We let $M = G/K$ be a symmetric Riemannian homogeneous space on which a connected semisimple Lie group G acts almost effectively. Then G is a Lie group that is \dagger locally isomorphic to the group $I(M)^0$, and therefore the Lie algebra of G is determined by M . Let \mathfrak{g} be the Lie algebra of G , \mathfrak{k} be the subalgebra of \mathfrak{g} corresponding to K , and θ be the involutive automorphism of G defining the symmetric homogeneous space G/K . The automorphism of \mathfrak{g} defined by θ is also denoted by θ . Then $\mathfrak{k} = \{X \in \mathfrak{g} | \theta(X) = X\}$. Putting $\mathfrak{m} = \{X \in \mathfrak{g} | \theta(X) = -X\}$, we have $\mathfrak{g} = \mathfrak{m} + \mathfrak{k}$ (direct sum of linear spaces), and \mathfrak{m} can be identified in a natural way with the tangent space at the point K of G/K . The \dagger adjoint representation of G gives rise to a representation of K in \mathfrak{g} , which induces a linear representation $\text{Ad}_\mathfrak{m}(k)$ of K in \mathfrak{m} . Then $\{\text{Ad}_\mathfrak{m}(k) | k \in K\}$ coincides with the \dagger restricted homogeneous holonomy group at the point K of the Riemannian space G/K .

Now let φ be the \dagger Killing form of \mathfrak{g} . Then \mathfrak{k} and \mathfrak{m} are mutually orthogonal with respect to φ , and denoting by $\varphi_\mathfrak{k}$ and $\varphi_\mathfrak{m}$ the restrictions of φ to \mathfrak{k} and \mathfrak{m} , respectively, $\varphi_\mathfrak{k}$ is a negative definite quadratic form on \mathfrak{k} . If $\varphi_\mathfrak{m}$ is also a negative definite quadratic form on \mathfrak{m} , \mathfrak{g} is a compact real semisimple Lie algebra and G/K is a compact symmetric Riemannian space; in this case we say that G/K is of **compact type**. In the opposite case, where $\varphi_\mathfrak{m}$ is a \dagger positive definite quadratic form, G/K is said to be of **noncompact type**. In this latter case, G/K is homeomorphic to a Euclidean space, and if the center of G is finite, K is a maximal compact subgroup of G . Furthermore, the group of isometries $I(G/K)$ of G/K is canonically

isomorphic to the automorphism group of the Lie algebra \mathfrak{g} . When G/K is of compact type (noncompact type), there exists one and only one G -invariant Riemannian metric on G/K , which induces in the tangent space \mathfrak{m} at the point K the positive definite inner product $-\varphi_{\mathfrak{m}}(\varphi_{\mathfrak{m}})$.

A symmetric Riemannian homogeneous space G/K_{θ} of compact type defined by a simply connected compact semisimple Lie group G with respect to an involutive automorphism θ is simply connected. Let $\mathfrak{g} = \mathfrak{m} + \mathfrak{k}_{\theta}$ be the decomposition of the Lie algebra \mathfrak{g} of G with respect to the automorphism θ of \mathfrak{g} , and let $\mathfrak{g}^{\mathbb{C}}$ be the \ast complex form of \mathfrak{g} . Then the real subspace $\mathfrak{g}_{\theta} = \sqrt{-1} \mathfrak{m} + \mathfrak{k}_{\theta}$ in $\mathfrak{g}^{\mathbb{C}}$ is a real semisimple Lie algebra and a \ast real form of $\mathfrak{g}^{\mathbb{C}}$. Let G_{θ} be the Lie group corresponding to the Lie algebra \mathfrak{g}_{θ} with center reduced to the identity element, and let K be the subgroup of G_{θ} corresponding to \mathfrak{k}_{θ} . Then we get a (simply connected) symmetric Riemannian homogeneous space of noncompact type G_{θ}/K .

When we start from a symmetric Riemannian space of noncompact type G/K instead of the symmetric Riemannian space of compact type G/K_{θ} and apply the same process as in the previous paragraphs, taking a simply connected G_{θ} as the Lie group corresponding to \mathfrak{g}_{θ} , we obtain a simply connected symmetric Riemannian homogeneous space of compact type. Indeed, each of these two processes is the reverse of the other, and in this way we get a one-to-one correspondence between simply connected symmetric Riemannian homogeneous spaces of compact type and those of noncompact type. This relationship is called **duality** for symmetric Riemannian spaces; when two symmetric Riemannian spaces are related by duality, each is said to be the **dual** of the other.

If one of the two symmetric Riemannian spaces related by duality is irreducible, the other is also irreducible. The duality holds between spaces of types (1) and (3) and between those of types (2) and (4) described in Section C. This fact is based on the following theorem in the theory of Lie algebras, where we identify isomorphic Lie algebras. (i) Complex simple Lie algebras $\mathfrak{g}^{\mathbb{C}}$ and compact real simple Lie algebras \mathfrak{g} are in one-to-one correspondence by the relation that $\mathfrak{g}^{\mathbb{C}}$ is the complex form of \mathfrak{g} . (ii) Form the Lie algebra \mathfrak{g}_{θ} in the above way from a compact real simple Lie algebra \mathfrak{g} and an involutive automorphism θ of \mathfrak{g} . We assume that θ is a member of the given complete system of representatives of conjugate classes of involutive automorphisms in the automorphism group of \mathfrak{g} . Then we get from the pair (\mathfrak{g}, θ) a noncompact real simple Lie algebra \mathfrak{g}_{θ} , and any noncompact real

simple Lie algebra is obtained by this process in one and only one way.

Consider a Riemannian space given as a symmetric Riemannian homogeneous space $M = G/K$ with a semisimple Lie group G , and let K be the \ast sectional curvature of M . Then if M is of compact type the value of K is ≥ 0 , and if M is of noncompact type it is ≤ 0 . On the other hand, the **rank** of M is the (unique) dimension of a commutative subalgebra of \mathfrak{g} that is contained in and maximal in \mathfrak{m} . (For results concerning the group of isometries of M , distribution of geodesics on M , etc. — [3].)

E. Symmetric Hermitian Spaces

A connected \ast complex manifold M with a \ast Hermitian metric is called a **symmetric Hermitian space** if for each point p of M there exists an isometric and \ast biholomorphic transformation of M onto M that is of order 2 and has p as an isolated fixed point. As a real analytic manifold, such a space M is a symmetric Riemannian space of even dimension, and the Hermitian metric of M is a \ast Kähler metric. Let $I(M)$ be the (not necessarily connected) Lie group formed by all isometries of M , and let $A(M)$ be the subgroup consisting of all holomorphic transformations in $I(M)$. Then $A(M)$ is a closed Lie subgroup of $I(M)$. Let G be the connected component $A(M)^0$ of the identity element of $A(M)$. Then G acts transitively on M , and M is expressed as a symmetric Riemannian homogeneous space G/K .

Under the de Rham decomposition of a simply connected symmetric Hermitian space (regarded as a Riemannian space), all the factors are symmetric Hermitian spaces. The factor that is isomorphic to a real Euclidean spaces as a Riemannian space is a symmetric Hermitian space that is isomorphic to the complex Euclidean space \mathbb{C}^n . A symmetric Hermitian space defining an irreducible symmetric Riemannian space is called an **irreducible symmetric Hermitian space**. The problem of classifying symmetric Hermitian spaces is thus reduced to classifying irreducible symmetric Hermitian spaces.

In general, if the symmetric Riemannian space defined by a symmetric Hermitian space M is represented as a symmetric Riemannian homogeneous space G/K by a connected semisimple Lie group G acting effectively on M , then M is simply connected, G coincides with the group $A(M)^0$ introduced in the previous paragraph, and the center of K is not a \ast discrete set. In particular, an irreducible symmetric Hermitian space is simply connected. Moreover, in order for an irreducible symmetric Riemannian homogeneous space G/K to be defined by an irreducible symmetric Hermitian

space M , it is necessary and sufficient that the center of K not be a discrete set. If G acts effectively on M , then G is a simple Lie group whose center is reduced to the identity element, and the center of K is of dimension 1. For a space G/K satisfying these conditions, there are two kinds of structures of symmetric Hermitian spaces defining the Riemannian structure of G/K .

As follows from the classification of irreducible symmetric Riemannian spaces, an irreducible Hermitian space defines one of the following symmetric Riemannian homogeneous spaces, and conversely, each of these homogeneous spaces is defined by one of the two kinds of symmetric Hermitian spaces.

(I) The symmetric homogeneous space G/K of a compact simple Lie group G with respect to an involutive automorphism θ such that the center of G reduces to the identity element and the center of K is not a discrete set. Here θ may be assumed to be a representative of a conjugate class of involutive automorphisms in the automorphism group of G .

(II) The homogeneous space G_0/K of a noncompact simple Lie group G_0 by a maximal compact subgroup K such that the center of G_0 reduces to the identity element and the center of K is not a discrete set.

An irreducible symmetric Hermitian space of type (I) is compact and is isomorphic to a \dagger rational algebraic variety. An irreducible symmetric Hermitian space of type (II) is homeomorphic to a Euclidean space and is isomorphic (as a complex manifold) to a bounded domain in \mathbb{C}^n (Section F).

By the same principle as for irreducible symmetric Riemannian spaces, a duality holds for irreducible symmetric Hermitian spaces which establishes a one-to-one correspondence between the spaces of types (I) and (II). Furthermore, an irreducible symmetric Hermitian space M_b of type (II) that is dual to a given irreducible symmetric Hermitian space $M_a = G/K$ of type (I) can be realized as an open complex submanifold of M_a in the following way. Let $G^{\mathbb{C}}$ be the connected component of the identity element in the Lie group formed by all the holomorphic transformations of M_a . Then $G^{\mathbb{C}}$ is a complex simple Lie group containing G as a maximal compact subgroup, and the complex Lie algebra $\mathfrak{g}^{\mathbb{C}}$ of $G^{\mathbb{C}}$ contains the Lie algebra \mathfrak{g} of G as a real form. Let θ be the involutive automorphism of G defining the symmetric homogeneous space G/K , and let $\mathfrak{g} = \mathfrak{m} + \mathfrak{k}$ be the decomposition of \mathfrak{g} determined by θ . We denote by G_0 the real subgroup of $G^{\mathbb{C}}$ corresponding to the real form $\mathfrak{g}_0 = \sqrt{-1}\mathfrak{m} + \mathfrak{k}$ of $\mathfrak{g}^{\mathbb{C}}$. Then G_0 (i) is a closed subgroup of $G^{\mathbb{C}}$ whose center reduces to the identity element and (ii) contains K as a maximal com-

pact subgroup. By definition the space M_b is then given by G_0/K . Now the group G_0 acts on M_a as a subgroup of $G^{\mathbb{C}}$, and the orbit of G_0 containing the point K of M_a is an open complex submanifold that is isomorphic to M_b (as a complex manifold). M_a regarded as a complex manifold can be represented as the homogeneous space $G^{\mathbb{C}}/U$ of the complex simple Lie group $G^{\mathbb{C}}$.

F. Symmetric Bounded Domains

We denote by D a bounded domain in the complex Euclidean space \mathbb{C}^n of dimension n . We call D a **symmetric bounded domain** if for each point of D there exists a holomorphic transformation of order 2 of D onto D having the point as an isolated fixed point. On the other hand, the group of all holomorphic transformations of D is a Lie group, and D is called a **homogeneous bounded domain** if this group acts transitively on D . A symmetric bounded domain is a homogeneous bounded domain. The following theorem gives more precise results: On a bounded domain D , \dagger Bergman's kernel function defines a Kähler metric that is invariant under all holomorphic transformations of D . If D is a symmetric bounded domain, D is a symmetric Hermitian space with respect to this metric, and its defining Riemannian space is a symmetric Riemannian homogeneous space of noncompact type G/K with semisimple Lie group G . Conversely, any symmetric Hermitian space of noncompact type is isomorphic (as a complex manifold) to a symmetric bounded domain. When D is isomorphic to an irreducible symmetric Hermitian space, we call D an **irreducible symmetric bounded domain**. A symmetric bounded domain is simply connected and can be decomposed into the direct product of irreducible symmetric bounded domains.

The connected component of the identity element of the group of all holomorphic transformations of a symmetric bounded domain D is a semisimple Lie group that acts transitively on D . Conversely, D is a symmetric bounded domain if a connected semisimple Lie group, or more generally, a connected Lie group admitting a two-sided invariant \dagger Haar measure, acts transitively on D . Homogeneous bounded domains in \mathbb{C}^n are symmetric bounded domains if $n \leq 3$ but not necessarily when $n \geq 4$.

G. Examples of Irreducible Symmetric Riemannian Spaces

Here we list irreducible symmetric Riemannian spaces of types (2) and (4) (\rightarrow Section C) that

can be represented as homogeneous spaces of classical groups, using the notation introduced by E. Cartan. We denote by $M_u = G/K$ a simply connected irreducible symmetric Riemannian space of type (2), where G is a group that acts almost effectively on M_u and K is the subgroup given by $K = K_\theta^0$ for an involutive automorphism θ of G . For such an M_u , the space of type (4) that is dual to M_u is denoted by $M_\theta = G_\theta/K$. Clearly $\dim M_u = \dim M_\theta$. (For the dimension and rank of M_u and for those M_u that are represented as homogeneous spaces of simply connected * exceptional compact simple Lie groups \rightarrow Appendix A, Table 5.III.) In this section (and also in Appendix A, Table 5.III), $O(n)$, $U(n)$, $Sp(n)$, $SL(n, \mathbf{R})$, and $SL(n, \mathbf{C})$ are the * orthogonal group of degree n , the * unitary group of degree n , the * symplectic group of degree $2n$, and the real and complex * special linear groups of degree n , respectively. Let $SO(n) = SL(n, \mathbf{R}) \cap O(n)$ and $SU(n) = SL(n, \mathbf{C}) \cap U(n)$. We put

$$I_{p,q} = \begin{pmatrix} -I_p & 0 \\ 0 & I_q \end{pmatrix}, \quad J_n = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix},$$

$$K_{p,q} = \begin{pmatrix} -I_p & 0 & 0 & 0 \\ 0 & I_q & 0 & 0 \\ 0 & 0 & -I_p & 0 \\ 0 & 0 & 0 & I_q \end{pmatrix},$$

where I_p is the $p \times p$ unit matrix.

Type AI. $M_u = SU(n)/SO(n)$ ($n > 1$), where $\theta(s) = \bar{s}$ (with \bar{s} the complex conjugate matrix of s). $M_\theta = SL(n, \mathbf{R})/SO(n)$.

Type AII. $M_u = SU(2n)/Sp(n)$ ($n > 1$), where $\theta(s) = J_n s J_n^{-1}$. $M_\theta = SU^*(2n)/Sp(n)$. Here $SU^*(2n)$ is the subgroup of $SL(2n, \mathbf{C})$ formed by the matrices that commute with the transformation $(z_1, \dots, z_n, z_{n+1}, \dots, z_{2n}) \rightarrow (\bar{z}_{n+1}, \dots, \bar{z}_{2n}, -\bar{z}_1, \dots, -\bar{z}_n)$ in \mathbf{C}^{2n} ; $SU^*(2n)$ is called the **quaternion unimodular group** and is isomorphic to the commutator group of the group of all regular transformations in an n -dimensional vector space over the quaternion field \mathbf{H} .

Type AIII. $M_u = SU(p+q)/S(U_p \times U_q)$ ($p \geq q \geq 1$), where $S(U_p \times U_q) = SU(p+q) \cap (U(p) \times U(q))$, with $U(p) \times U(q)$ being canonically identified with a subgroup of $U(p+q)$, and $\theta(s) = I_{p,q} s I_{p,q}$. This space M_u is a * complex Grassmann manifold. $M_\theta = SU(p, q)/S(U_p \times U_q)$, where $SU(p, q)$ is the subgroup of $SL(p+q, \mathbf{C})$ consisting of matrices that leave invariant the Hermitian form $z_1 \bar{z}_1 + \dots + z_p \bar{z}_p - z_{p+1} \bar{z}_{p+1} - \dots - z_{p+q} \bar{z}_{p+q}$.

Type AIV. This is the case $q = 1$ of type AIII. M_u is the $(n-1)$ -dimensional complex projective space, and M_θ is called a **Hermitian hyperbolic space**.

Type BDI. $M_u = SO(p+q)/SO(p) \times SO(q)$ ($p \geq q \geq 1, p > 1, p+q \neq 4$), where $\theta(s) = I_{p,q} s I_{p,q}$. M_u is the * real Grassmann manifold formed by

the oriented p -dimensional subspaces in \mathbf{R}^{p+q} . $M_\theta = SO_0(p, q)/SO(p) \times SO(q)$, where $SO(p, q)$ is the subgroup of $SL(n, \mathbf{R})$ consisting of matrices that leave invariant the quadratic form $x_1^2 + \dots + x_p^2 - x_{p+1}^2 - \dots - x_{p+q}^2$, and $SO_0(p, q)$ is the connected component of the identity element.

Type BDII. This is the case $q = 1$ of type BDI. M_u is the $(n-1)$ -dimensional sphere, and M_θ is called a **real hyperbolic space**.

Type DIII. $M_u = SO(2n)/U(n)$ ($n > 2$), where $U(n)$ is regarded as a subgroup of $SO(2n)$ by identifying $s \in U(n)$ with

$$\begin{pmatrix} \operatorname{Re} s & \operatorname{Im} s \\ -\operatorname{Im} s & \operatorname{Re} s \end{pmatrix} \in SO(2n),$$

and $\theta(s) = J_n s J_n^{-1}$. $M_\theta = SO^*(2n)/U(n)$. Here $SO^*(2n)$ denotes the group of all complex orthogonal matrices of determinant 1 leaving invariant the skew-Hermitian form $z_1 \bar{z}_{n+1} - z_{n+1} \bar{z}_1 + z_2 \bar{z}_{n+2} - z_{n+2} \bar{z}_2 + \dots + z_n \bar{z}_{2n} - z_{2n} \bar{z}_n$; this group is isomorphic to the group of all linear transformations leaving invariant a nondegenerate skew-Hermitian form in an n -dimensional vector space over the quaternion field \mathbf{H} .

Type CI. $M_u = Sp(n)/U(n)$ ($n \geq 1$), where $U(n)$ is considered as a subgroup of $Sp(n)$ by the identification $U(n) \subset SO(2n)$ explained in type DIII and $\theta(s) = \bar{s} (= J_n s J_n^{-1})$. $M_\theta = Sp(n, \mathbf{R})/U(n)$, where $Sp(n, \mathbf{R})$ is the real symplectic group of degree $2n$.

Type CII. $M_u = Sp(p+q)/Sp(p) \times Sp(q)$ ($p \geq q \geq 1$), where $Sp(p) \times Sp(q)$ is identified with a subgroup of $Sp(p+q)$ by the mapping

$$\left(\begin{pmatrix} A_1 & B_1 \\ C_1 & D_1 \end{pmatrix}, \begin{pmatrix} A_2 & B_2 \\ C_2 & D_2 \end{pmatrix} \right) \rightarrow \begin{pmatrix} A_1 & 0 & B_1 & 0 \\ 0 & A_2 & 0 & B_2 \\ C_1 & 0 & D_1 & 0 \\ 0 & C_2 & 0 & D_2 \end{pmatrix}$$

and $\theta(s) = K_{p,q} s K_{p,q}$. $M_\theta = Sp(p, q)/Sp(p) \times Sp(q)$. Here $Sp(p, q)$ is the group of complex symplectic matrices of degree $2(p+q)$ leaving invariant the Hermitian form $(z_1, \dots, z_{p+q}) K_{p,q} (\bar{z}_1, \dots, \bar{z}_{p+q})$; this group is interpreted as the group of all linear transformations leaving invariant a nondegenerate Hermitian form of index p in a $(p+q)$ -dimensional vector space over the quaternion field \mathbf{H} . For $q = 1$, M_u is the quaternion projective space, and M_θ is called the **quaternion hyperbolic space**.

Among the spaces introduced here, there are some with lower p, q, n that coincide (as Riemannian spaces) (\rightarrow Appendix A, Table 5.III).

H. Space Forms

A Riemannian manifold of * constant curvature is called a **space form**; it is said to be **spherical**,

Euclidean, or **hyperbolic** according to the constant curvature K is positive, zero, or negative. A space form is a locally symmetric Riemannian space; a simply connected complete space form is a sphere if $K > 0$, a real Euclidean space if $K = 0$, and a real hyperbolic space if $K < 0$. More generally, a complete spherical space form of even dimension is a sphere or a projective space, and one of odd dimension is an orientable manifold. A complete 2-dimensional Euclidean space form is one of the following spaces: Euclidean plane, cylinder, torus, \dagger Möbius strip, \dagger Klein bottle. Except for these five spaces and the 2-dimensional sphere, any \dagger closed surface is a 2-dimensional hyperbolic space form (for details about space forms \rightarrow [6]).

I. Examples of Irreducible Symmetric Bounded Domains

Among the irreducible symmetric Riemannian spaces described in Section H, those defined by irreducible symmetric Hermitian spaces are of types AIII, DIII, BDI ($q = 2$), and CI. We list the irreducible symmetric bounded domains that are isomorphic to the irreducible Hermitian spaces defining these spaces. Positive definiteness of a matrix will be written $\gg 0$.

Type I_{m,m'} ($m' \geq m \geq 1$). The set of all $m \times m'$ complex matrices Z satisfying the condition $I_m - {}^t\bar{Z}Z \gg 0$ is a symmetric bounded domain in $\mathbb{C}^{mm'}$, which is isomorphic (as a complex manifold) to the irreducible symmetric Hermitian space defined by M_θ of type AIII ($p = m$, $q = m'$).

Type II_m ($m \geq 2$). The set of all $m \times m$ complex \dagger skew-symmetric matrices Z satisfying the condition $I_m - {}^t\bar{Z}Z \gg 0$ is a symmetric bounded domain in $\mathbb{C}^{m(m-1)/2}$ corresponding to the type DIII ($n = m$).

Type III_m ($m \geq 1$). The set of all $m \times m$ complex symmetric matrices satisfying the condition $I_m - {}^t\bar{Z}Z \gg 0$ is a symmetric bounded domain in $\mathbb{C}^{m(m+1)/2}$ corresponding to the type CI ($n = m$). This bounded domain is holomorphically isomorphic to the \dagger Siegel upper half-space of degree m .

Type IV_m ($m \geq 1, m \neq 2$). This bounded domain in \mathbb{C}^m is formed by the elements (z_1, \dots, z_m) satisfying the condition $|z_1|^2 + \dots + |z_m|^2 < (1 + |z_1^2 + \dots + z_m^2|)/2 < 1$, and corresponds to the type BDI ($p = m, q = 2$).

Among these four types of bounded domains, the following complex analytic isomorphisms hold: $I_{1,1} \cong II_2 \cong III_1 \cong IV_1$, $II_3 \cong I_{1,3}$, $IV_3 \cong III_2$, $IV_4 \cong I_{2,2}$, $IV_6 \cong II_4$. (For details about these symmetric bounded domains \rightarrow [2].) There are two more kinds of irreducible symmetric bounded domains,

which are represented as homogeneous spaces of exceptional Lie groups.

J. Weakly Symmetric Riemannian Spaces

A generalization of symmetric Riemannian space is the notion of weakly symmetric Riemannian space introduced by Selberg. Let M be a Riemannian space. M is called a **weakly symmetric Riemannian space** if a Lie subgroup G of the group of isometries $I(M)$ acts transitively on M and there exists an element $\mu \in I(M)$ satisfying the relations (i) $\mu G \mu^{-1} = G$; (ii) $\mu^2 \in G$; and (iii) for any two points x, y of M , there exists an element m of G such that $\mu x = my$, $\mu y = mx$. A symmetric Riemannian space M becomes a weakly symmetric Riemannian space if we put $G = I(M)$ and $\mu =$ the identity transformation; as the element m in condition (iii) we can take the symmetry σ_p at the midpoint p on the geodesic arc joining x and y . There are, however, weakly symmetric Riemannian spaces that do not have the structure of a symmetric Riemannian space. An example of such a space is given by $M = G = SL(2, \mathbb{R})$ with a suitable Riemannian metric, where μ is the inner automorphism defined by

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

(Selberg [4]). On a weakly symmetric Riemannian space, the ring of all G -invariant differential-integral operators is commutative; this fact is useful in the theory of spherical functions (\rightarrow 437 Unitary Representations).

References

- [1] S. Helgason, Differential geometry, Lie groups, and symmetric spaces, Academic Press, 1978.
- [2] C. L. Siegel, Analytic functions of several complex variables, Princeton Univ. Press, 1950.
- [3] J.-L. Koszul, Exposés sur les espaces homogènes symétriques, São Paulo, 1959.
- [4] A. Selberg, Harmonic analysis and discontinuous groups in weakly symmetric Riemannian spaces with application to Dirichlet series, J. Indian Math. Soc., 20 (1956), 47–87.
- [5] E. Cartan, Sur certaines formes riemanniennes remarquables des géométries à groupe fondamental simple, Ann. Sci. Ecole Norm. Sup., 44 (1927), 345–467 (Oeuvres complètes, Gauthier-Villars, 1952, pt. I, vol. 2, 867–989).
- [6] E. Cartan, Sur les domaines bornés homogènes de l'espace de n -variables complexes, Abh. Math. Sem. Univ. Hamburg, 11 (1936), 116–162 (Oeuvres complètes, Gauthier-Villars, 1952, pt. I, vol. 2, 1259–1305).

[7] J. A. Wolf, Spaces of constant curvature, McGraw-Hill, 1967.

[8] S. Kobayashi and K. Nomizu, Foundations of differential geometry II, Interscience, 1969.

[9] O. Loos, Symmetric spaces. I, General theory; II, Compact spaces and classification, Benjamin, 1969.

413 (VII.7) Symmetric Spaces

A \dagger Riemannian manifold M is called a **symmetric Riemannian space** if M is connected and if for each $p \in M$ there exists an involutive \dagger isometry σ_p of M that has p as an isolated fixed point. For the classification and the group-theoretic properties of symmetric Riemannian spaces \rightarrow 412 Symmetric Riemannian Spaces and Real Forms. We state here the geometrical properties of a symmetric Riemannian space M . Let M be represented by G/K , a \dagger symmetric Riemannian homogeneous space. The \dagger Lie algebras of G and K are denoted by \mathfrak{g} and \mathfrak{k} respectively. Let us denote by τ_a the \dagger left translation of M defined by $a \in G$, and by X^* the vector field on M generated by $X \in \mathfrak{g}$. We denote by θ the differential of the involutive automorphism of G defining G/K and identify the subspace $\mathfrak{m} = \{X \in \mathfrak{g} \mid \theta(X) = -X\}$ of \mathfrak{g} with the tangent space $T_o(M)$ of M at the origin $o = K$ of M . The \dagger representation of \mathfrak{k} on \mathfrak{m} induced from the \dagger adjoint representation of \mathfrak{g} is denoted by $\text{ad}_{\mathfrak{m}}$.

A. Riemannian Connections

M is a complete real analytic \dagger homogeneous Riemannian manifold. If M is a \dagger symmetric Hermitian space, it is a \dagger homogeneous Kählerian manifold. The \dagger Riemannian connection ∇ of M is the \dagger canonical connection of the homogeneous space G/K and satisfies $\nabla_Y X^* = [X, Y]$ ($Y \in \mathfrak{m}$) for each $X \in \mathfrak{k}$ and $\nabla_Y X^* = 0$ ($Y \in \mathfrak{m}$) for each $X \in \mathfrak{m}$. For each $X \in \mathfrak{m}$, the curve γ_X of M defined by $\gamma_X(t) = (\exp tX)o$ ($t \in \mathbf{R}$) is a \dagger geodesic of M such that $\gamma_X(0) = o$ and $\dot{\gamma}_X(0) = X$. In particular, the \dagger exponential mapping Exp_o at o is given by $\text{Exp}_o X = (\exp X)o$ ($X \in \mathfrak{m}$). For each $X \in \mathfrak{m}$, the \dagger parallel translation along the geodesic arc $\gamma_X(t)$ ($0 \leq t \leq t_0$) coincides with the differential of $\tau_{\exp t_0 X}$. If M is compact, for each $p \in M$ there exists a smooth simply closed geodesic passing through p . Any G -invariant tensor field on M

is \dagger parallel with respect to ∇ . Any G -invariant \dagger differential form on M is closed. The Lie algebra \mathfrak{h} of the \dagger restricted homogeneous holonomy group of M at o coincides with $\text{ad}_{\mathfrak{m}}[\mathfrak{m}, \mathfrak{m}]$. If the group $I(M)$ of all isometries of M is \dagger semisimple, one has $\mathfrak{h} = \{A \in \text{gl}(\mathfrak{m}) \mid A \cdot g_o = 0, A \cdot R_o = 0\} = \text{ad}_{\mathfrak{m}}\mathfrak{k}$. Here, g_o and R_o denote the values at o of the Riemannian metric g and the \dagger Riemannian curvature R of M , respectively, and $A \cdot$ is the natural action of A on the tensors over \mathfrak{m} . If, moreover, M is a symmetric Hermitian space, the value J_o at o of the \dagger almost complex structure J of M belongs to the center of \mathfrak{h} . In general, $\mathfrak{h} = \{0\}$ if and only if M is \dagger flat, and \mathfrak{h} has no nonzero invariant on \mathfrak{m} if and only if $I(M)$ is semisimple.

B. Riemannian Curvature Tensors

The Riemannian curvature tensor R of M is parallel and satisfies $R_o(X, Y) = -\text{ad}_{\mathfrak{m}}[X, Y]$ ($X, Y \in \mathfrak{m}$). Assume that $\dim M \geq 2$ in the following. Let P be a 2-dimensional subspace of \mathfrak{m} , and $\{X, Y\}$ an orthonormal basis of P with respect to g_o . Then the \dagger sectional curvature $K(P)$ of P is given by $K(P) = g_o([X, Y], X)$. $K = 0$ everywhere if and only if M is flat. If M is of \dagger compact type (resp. of \dagger noncompact type), then $K \geq 0$ (resp. $K \leq 0$) everywhere. $K > 0$ (resp. $K < 0$) everywhere if and only if the \dagger rank of M is 1 and M is of compact type (resp. of noncompact type). For any four points p, q, p', q' of a manifold M of any of these types satisfying $d(p, q) = d(p', q')$, d being the \dagger Riemannian distance of M , there exists a $\phi \in I(M)$ such that $\phi(p) = p'$ and $\phi(q) = q'$. Other than the aforementioned M 's, the only Riemannian manifolds having this property are circles and Euclidean spaces. If $K > 0$ everywhere, any geodesic of M is a smooth simply closed curve and all geodesics are of the same length. For a symmetric Hermitian space M , the \dagger holomorphic sectional curvature H satisfies $H = 0$ (resp. $H > 0$, $H < 0$) everywhere if and only if M is flat (resp. of compact type, of noncompact type).

C. Ricci Tensors

The \dagger Ricci tensor S of M is parallel. If $\varphi_{\mathfrak{m}}$ denotes the restriction to $\mathfrak{m} \times \mathfrak{m}$ of the \dagger Killing form φ of \mathfrak{g} , the value S_o of S at o satisfies $S_o = -\frac{1}{2}\varphi_{\mathfrak{m}}$. If M is \dagger irreducible, it is an \dagger Einstein space. $S = 0$ (resp. positive definite, negative definite, nondegenerate) everywhere if and only if M is flat (resp. M is of compact type, M is of noncompact type, $I(M)$ is semisimple). If M is a \dagger symmetric bounded domain and g is the \dagger Bergman metric of M , one has $S = -g$.

D. Symmetric Riemannian Spaces of Noncompact Type

Let M be of noncompact type. For each $p \in M$, p is the only fixed point of the \dagger symmetry σ_p , and the exponential mapping at p is a diffeomorphism from $T_p(M)$ to M . In particular, M is diffeomorphic to a Euclidean space. For each pair $p, q \in M$, a geodesic arc joining p and q is unique up to parametrization. For each $p \in M$ there exists neither a \dagger conjugate point nor a \dagger cut point of p . If M is a symmetric Hermitian space, that is, if it is a symmetric bounded domain, then it is a \dagger Stein manifold and holomorphically homeomorphic to a \dagger Siegel domain.

E. Groups of Isometries

The isotropy subgroup at o in $I(M)$ is denoted by $I_o(M)$. Then the smooth mapping $I_o(M) \times \mathfrak{m} \rightarrow I(M)$ defined by the correspondence $\phi \times X \mapsto \phi \tau_{\exp X}$ is surjective, and it is a diffeomorphism if M is of noncompact type. If M is of noncompact type, $I(M)$ is isomorphic to the group $A(\mathfrak{g})$ of all automorphisms of \mathfrak{g} in a natural way, and $I_o(M)$ is isomorphic to the subgroup $A(\mathfrak{g}, \mathfrak{f}) = \{\phi \in A(\mathfrak{g}) \mid \phi(\mathfrak{f}) = \mathfrak{f}\}$ of $A(\mathfrak{g})$, provided that G acts almost effectively on M . Moreover, in this case the center of the identity component $I(M)^0$ of $I(M)$ reduces to the identity, and the isotropy subgroup at a point in $I(M)^0$ is a maximal compact subgroup of $I(M)^0$. If $I(M)$ is semisimple, any element of $I(M)^0$ may be represented as a product of an even number of symmetries of M . In the following, let M be a symmetric Hermitian space, and denote by $A(M)$ (resp. $H(M)$) the group of all holomorphic isometries (resp. all holomorphic homeomorphisms) of M , and by $A(M)^0$ and $H(M)^0$ their identity components. All these groups act transitively on M . If M is compact or if $I(M)$ is semisimple, one has $A(M)^0 = I(M)^0$. If $I(M)$ is semisimple, M is simply connected and the center of $I(M)^0$ reduces to the identity. If M is of compact type, M is a \dagger rational \dagger projective algebraic manifold, and $H(M)^0$ is a complex semisimple Lie group whose center reduces to the identity, and it is the \dagger complexification of $I(M)^0$. In this case, the isotropy subgroup at a point in $H(M)^0$ is a \dagger parabolic subgroup of $H(M)^0$. If M is of noncompact type, one has $H(M)^0 = I(M)^0$. In the following we assume that G is compact.

F. Cartan Subalgebras

A maximal Abelian \dagger Lie subalgebra in \mathfrak{m} is called a **Cartan subalgebra** for M . Cartan sub-

algebras are conjugate to each other under the \dagger adjoint action of K . Fix a Cartan subalgebra \mathfrak{a} and introduce an inner product $(\ , \)$ on \mathfrak{a} by the restriction to $\mathfrak{a} \times \mathfrak{a}$ of g_θ . For an element α of the dual space \mathfrak{a}^* of \mathfrak{a} , we put $m_\alpha = \{X \in \mathfrak{m} \mid [H, [H, X]] = -\alpha(H)^2 X \text{ for any } H \in \mathfrak{a}\}$. The subset $\Sigma = \{\alpha \in \mathfrak{a}^* - \{0\} \mid m_\alpha \neq \{0\}\}$ of \mathfrak{a}^* is called the **root system** of M (relative to \mathfrak{a}). We write $m_\alpha = \dim m_\alpha$ for $\alpha \in \Sigma$. The subset $D = \{H \in \mathfrak{a} \mid \alpha(H) \in \pi\mathbb{Z} \text{ for some } \alpha \in \Sigma\}$ of \mathfrak{a} is called the **diagram** of M . A connected component of $\mathfrak{a} - D$ is called a **fundamental cell** of M . The quotient group W of the normalizer of \mathfrak{a} in K modulo the centralizer of \mathfrak{a} in K is called the **Weyl group** of M . W is identified with a finite group of orthogonal transformations of \mathfrak{a} .

G. Conjugate Points

For a geodesic arc γ with the initial point o , any \dagger Jacobi field along γ that vanishes at o and the end point of γ is obtained as the restriction to γ of the vector field X^* generated by an element $X \in \mathfrak{f}$. For $H \in \mathfrak{a} - \{0\}$, $\text{Exp}_o H$ is a conjugate point to o along the geodesic γ_H if and only if $\alpha(H) \in \pi\mathbb{Z} - \{0\}$ for some $\alpha \in \Sigma$. In this case, the multiplicity of the conjugate point $\text{Exp}_o H$ is equal to $\frac{1}{2} \sum_{\alpha \in \Sigma, \alpha(H) \in \pi\mathbb{Z} - \{0\}} m_\alpha$. From this fact and Morse theory (\rightarrow 279 Morse Theory), we get a \dagger cellular decomposition of the \dagger loop space of M . The set of all points conjugate to o coincides with $K \text{Exp}_o D$ and is stratified to a disjoint union of a finite number of connected regular submanifolds with dimension $\leq \dim M - 2$.

H. Cut Points

We define a \dagger lattice group Γ of \mathfrak{a} by $\Gamma = \{A \in \mathfrak{a} \mid \text{Exp}_o A = o\}$, and put $C_a = \{H \in \mathfrak{a} \mid \text{Max}_{A \in \Gamma - \{0\}} 2(H, A)/(A, A) = 1\}$. Then, for $H \in \mathfrak{a} - \{0\}$, $\text{Exp}_o H$ is a cut point of o along the geodesic γ_H if and only if $H \in C_a$. The set C_o of all cut points of o coincides with $K \text{Exp}_o C_a$ and is stratified to a disjoint union of a finite number of connected regular submanifolds with dimension $\leq \dim M - 1$. The set of all points \dagger first conjugate to o coincides with C_o if and only if M is simply connected.

I. Fundamental Groups

Let Γ_o denote the subgroup of Γ generated by $\{(2\pi/(\alpha, \alpha))\alpha \mid \alpha \in \Sigma\}$, identifying \mathfrak{a}^* with \mathfrak{a} by means of the inner product $(\ , \)$ of \mathfrak{a} . This is a subgroup of Γ . We regard Γ as a subgroup of the group $I(\mathfrak{a})$ of all motions of \mathfrak{a} by parallel

translations. The subgroup $\tilde{W} = W\Gamma$ of $I(\alpha)$ generated by Γ and the Weyl group W is called the **affine Weyl group** of M . \tilde{W} leaves the diagram D invariant and acts transitively on the set of all fundamental cells of M . Take a fundamental cell σ such that its closure $\bar{\sigma}$ contains 0, and put $\tilde{W}_\sigma = \{w \in \tilde{W} \mid w(\sigma) = \sigma\}$. Then the fundamental group $\pi_1(M)$ of M is an \dagger Abelian group isomorphic to the groups \tilde{W}_σ and Γ/Γ_0 . $\pi_1(M)$ is a finite group if and only if M is of compact type. In this case, the order of $\pi_1(M)$ is equal to the cardinality of the set $\Gamma \cap \bar{\sigma}$ as well as to the index $[\Gamma : \Gamma_0]$. Moreover, if we denote by \tilde{W}_σ^* the group \tilde{W}_σ for the symmetric Riemannian space $M^* = G^*/K^*$ defined by the \dagger adjoint group G^* of G and $K^* = \{a \in G^* \mid a\theta = \theta a\}$, then \tilde{W}_σ is isomorphic to a subgroup of \tilde{W}_σ^* . If M is irreducible, \tilde{W}_σ^* is isomorphic to a subgroup of the group of all automorphisms of the \dagger extended Dynkin diagram of the root system Σ .

J. Cohomology Rings

Let $P(\mathfrak{g})$ (resp. $P(\mathfrak{f})$) be the \dagger graded linear space of all \dagger primitive elements in the \dagger cohomology algebra $H(\mathfrak{g})$ of \mathfrak{g} (resp. $H(\mathfrak{f})$ of \mathfrak{f}), and $P(\mathfrak{g}, \mathfrak{f})$ the intersection of $P(\mathfrak{g})$ with the image of the natural homomorphism $H(\mathfrak{g}, \mathfrak{f}) \rightarrow H(\mathfrak{g})$, where $H(\mathfrak{g}, \mathfrak{f})$ denotes the relative cohomology algebra for the pair $(\mathfrak{g}, \mathfrak{f})$. Then one has $\dim P(\mathfrak{g}, \mathfrak{f}) + \dim P(\mathfrak{f}) = \dim P(\mathfrak{g})$. Denote by $\Lambda P(\mathfrak{g}, \mathfrak{f})$ the exterior algebra over $P(\mathfrak{g}, \mathfrak{f})$. The \dagger graded algebra of all G -invariant polynomials on \mathfrak{g} (resp. all K -invariant polynomials on \mathfrak{f}) is denoted by $I(G)$ (resp. $I(K)$), where the degree of a homogeneous polynomial with degree p is defined to be $2p$. We denote by $I^+(G)$ the ideal of $I(G)$ consisting of all $f \in I(G)$ such that $f(0) = 0$, and regard $I(K)$ as an $I^+(G)$ -module through the restriction homomorphism. Then the \dagger real cohomology ring $H(M)$ of M is isomorphic to the tensor product $\Lambda P(\mathfrak{g}, \mathfrak{f}) \otimes (I(K)/I^+(G)I(K))$. If K is connected and the \dagger Poincaré polynomials of $P(\mathfrak{g})$, $P(\mathfrak{f})$, and $P(\mathfrak{g}, \mathfrak{f})$ are $\sum_{i=1}^r t^{2m_i-1}$, $\sum_{i=1}^s t^{2n_i-1}$, and $\sum_{i=s+1}^r t^{2m_i-1}$, respectively, then the Poincaré polynomial of $H(M)$ is given by $\prod_{i=s+1}^r (1 + t^{2m_i-1}) \prod_{i=1}^s (1 - t^{2m_i}) \prod_{i=1}^s (1 - t^{2n_i})^{-1}$.

References

- [1] E. Cartan, Sur certaines formes riemanniennes remarquables des géométries à groupe fondamental simple, Ann. Sci. Ecole Norm. Sup., 44 (1927), 345–467.
- [2] S. Helgason, Differential geometry, Lie groups, and symmetric spaces, Academic Press, 1978.

- [3] S. Kobayashi and K. Nomizu, Foundations of differential geometry II, Interscience, 1969.
- [4] H. C. Wang, Two point homogeneous spaces, Ann. Math., (2) 55 (1952), 177–191.
- [5] A. Korányi and J. A. Wolf, Realization of Hermitian symmetric spaces as generalized half-planes, Ann. Math., (2) 81 (1965), 265–288.
- [6] R. Bott and H. Samelson, Applications of the theory of Morse to symmetric spaces, Amer. J. Math., 80 (1958), 964–1029.
- [7] T. Sakai, On cut loci of compact symmetric spaces, Hokkaido Math. J., 6 (1977), 136–161.
- [8] M. Takeuchi, On conjugate loci and cut loci of compact symmetric spaces I, Tsukuba J. Math., 2 (1978), 35–68.
- [9] R. Crittenden, Minimum and conjugate points in symmetric spaces, Canad. J. Math., 14 (1962), 320–328.
- [10] J. L. Koszul, Sur un type d'algèbre différentielles avec la transgression, Colloque de Topologie (Espaces fibrés), Brussels, 1950, 73–81.

414 (XX.1) Systems of Units

A. International System of Units

Units representing various physical quantities can be derived from a certain number of **fundamental (base) units**. By a **system of units** we mean a system of fundamental units. Various systems of units have been used in the course of the development of physics. Today, the standard is set by the **international system of units** (système international d'unités; abbreviated SI) [1], which has been developed in the spirit of the meter-kilogram system. This system consists of the seven fundamental units listed in Table 1, units induced from them, and unit designations with prefixes representing the powers of 10 where necessary. It also contains two **auxiliary units** for plane and solid angles, and a large number of derived units [1].

B. Systems of Units in Mechanics

Units in mechanics are usually derived from length, mass, and time, and SI uses the meter, kilogram, and second as base units. Neither the CGS system, derived from centimeter, gram, and second, nor the **system of gravitational units**, derived from length, force, and time, are recommended for general use by

Table 1

Quantity	SI unit	Symbol	Description
Length	meter	m	The meter is the length equal to 1,650,763.73 wavelengths in vacuum of the radiation corresponding to the transmission between the levels $2p^{10}$ and $5d^5$ of the krypton-86 atom.
Mass	kilogram	kg	The kilogram is equal to the mass of the international prototype of the kilogram.
Time	second	s	The second is the duration of 9,192,631.770 periods of the radiation corresponding to the transmission between the two hyperfine levels of the ground state of the cesium-133 atom.
Intensity of electric current	ampere	A	The ampere is the intensity of the constant current maintained in two parallel, rectilinear conductors of infinite length and of negligible circular section, placed 1 m apart in vacuum, and producing a force between them equal to 2×10^{-7} newton ($\text{m} \cdot \text{kg} \cdot \text{s}^{-2}$) per meter of length.
Temperature	kelvin	K	The kelvin, the unit of thermodynamical temperature, is $1/273.16$ of the thermodynamical temperature of the triple point of water.
Amount of substance	mole	mol	The mole is the amount of substance of a system containing as many elementary entities as there are atoms in 0.012 kg of carbon-12.
Luminous intensity	candela	cd	The candela is the luminous intensity in a given direction of a source emitting monochromatic radiation of frequency 540×10^{12} hertz ($= \text{s}^{-1}$), the radiant intensity of which in that direction is $1/683$ watt per steradian. (This revised definition of candela was adopted in 1980.)

Table 2

Quantity	SI unit	Symbol	Unit in terms of SI base or derived units
Frequency	hertz	Hz	$1 \text{ Hz} = 1 \text{ s}^{-1}$
Force	newton	N	$1 \text{ N} = 1 \text{ kg} \cdot \text{m}/\text{s}^2$
Pressure and stress	pascal	Pa	$1 \text{ Pa} = 1 \text{ N}/\text{m}^2$
Work, energy, quantity of heat	joule	J	$1 \text{ J} = 1 \text{ N} \cdot \text{m}$
Power	watt	W	$1 \text{ W} = 1 \text{ J}/\text{s}$
Quantity of electricity	coulomb	C	$1 \text{ C} = 1 \text{ A} \cdot \text{s}$
Electromotive force, potential difference	volt	V	$1 \text{ V} = 1 \text{ W}/\text{A}$
Electric capacitance	farad	F	$1 \text{ F} = 1 \text{ C}/\text{V}$
Electric resistance	ohm	Ω	$1 \Omega = 1 \text{ V}/\text{A}$
Electric conductance	siemens	S	$1 \text{ S} = 1 \Omega^{-1}$
Flux of magnetic induction magnetic flux	weber	Wb	$1 \text{ Wb} = 1 \text{ V} \cdot \text{s}$
Magnetic induction, magnetic flux density	tesla	T	$1 \text{ T} = 1 \text{ Wb}/\text{m}^2$
Inductance	henry	H	$1 \text{ H} = 1 \text{ Wb}/\text{A}$
Luminous flux	lumen	lm	$1 \text{ lm} = 1 \text{ cd} \cdot \text{sr}$
Illuminance	lux	lx	$1 \text{ lx} = 1 \text{ lm}/\text{m}^2$
Activity	becquerel	Bq	$1 \text{ Bq} = 1 \text{ s}^{-1}$
Adsorbed dose	gray	Gy	$1 \text{ Gy} = 1 \text{ J}/\text{kg}$
Radiation dose	sievert	Sv	$1 \text{ Sv} = 1 \text{ J}/\text{kg}$

the SI Committee. Besides the base units, minute, hour, and day, degree, minute, and second (angle), liter, and ton have been approved by the SI Committee. Units such as the electron volt, atomic mass unit, astronomical unit, and parsec (not SI) are empirically defined and have been approved. Several other units, such as nautical mile, knot, are (area), and bar, have been provisionally approved.

C. System of Units in Thermodynamics

The base unit for temperature is the degree Kelvin ($^{\circ}\text{K}$; formerly called the absolute temperature). Degree Celsius ($^{\circ}\text{C}$), defined by $t = T - 273.15$, where T is in $^{\circ}\text{K}$, is also used.

The unit of heat is the joule J, the same as the unit for other forms of energy. Formerly, one calorie was defined as the quantity of heat that must be supplied to one gram of water to raise its temperature from 14.5°C to 15.5°C ; now one calorie is defined by $1 \text{ cal} = 4.1855 \text{ J}$.

D. Systems of Units in Electricity and Magnetism

Three distinct systems of units have been developed in the field of electricity and magnetism: the electrostatic system, which originates from Coulomb's law for the force between two electric charges and defines magnetic quantities by means of the Biot-Savart law; the electromagnetic system, which originates from Coulomb's law for magnetism; and the Gaussian system, in which the dielectric constant and permeability are taken to be non-dimensional. At present, however, the rationalized MKSA system of units is adopted as the international standard. It uses the **derived units** listed in Table 2 (taken from [2]), where the derived units with proper names in other fields are also listed.

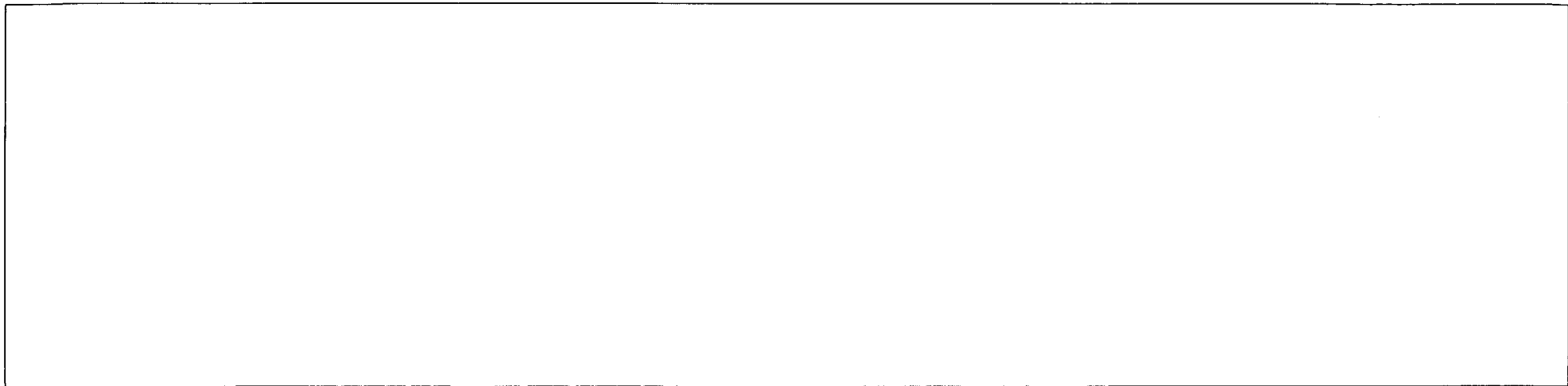
E. Other Units

In the field of photometry, the following definition was adopted in 1948: One candela (cd) (≈ 0.98 old candle) is defined as $1/(6 \times 10^5)$ of the luminous intensity in the direction normal to a plane surface of 1 m^2 area of a black body at the temperature of the solidifying point of platinum. The total luminous flux emanating uniformly in all directions from a source of luminous intensity 1 cd is defined as 4π lumen (lm). One lux (lx) is defined as the illuminance on a surface area of 1 m^2 produced by a luminous flux of 1 cd uniformly incident on the surface. In 1980, the definition was revised as shown in Table 1.

For theoretical purposes, a system of units called the absolute system of units is often used, in which units of mass, length, and time are chosen so that the values of universal constants, such as the universal gravitational constant, speed of light, Planck's constant, and Boltzmann's constant, are equal to 1.

References

- [1] Bureau International des Poids et Mesures, *Le système international d'unités*, 1970, fourth revised edition, 1981.
- [2] R. G. Lerner and G. L. Trigg (eds.), *Encyclopedia of physics*, Addison-Wesley, 1981.



415 (XXI.41) Takagi, Teiji

Teiji Takagi (April 21, 1875–February 28, 1960) was born in Gifu Prefecture, Japan. After graduation from the Imperial University of Tokyo in 1897, he continued his studies in Germany, first with Frobenius in Berlin and then with Hilbert in Göttingen. He returned to Japan in 1901 and taught at the Imperial University of Tokyo until 1936, when he retired. He died in Tokyo of cerebral apoplexy.

Since his student years he had been interested in Kronecker's conjecture on \dagger Abelian extensions of imaginary quadratic number fields. He solved it affirmatively for the case of $\mathbb{Q}(\sqrt{-1})$ while still in Göttingen and presented this result as his doctoral thesis. During World War I, he pursued his research in the theory of numbers in isolation from Western countries. It developed into \dagger class field theory, a beautiful general theory of Abelian extensions of algebraic number fields. This was published in 1920, and was complemented by his 1922 paper on the \dagger reciprocity law of power residues and then by \dagger Artin's general law of reciprocity published in 1927. Besides these arithmetical works, he also published papers on algebraic and analytic subjects and on the foundations of the theories of natural numbers and of real numbers. His book (in Japanese) on the history of mathematics in the 19th century and his *General course of analysis* (also in Japanese) as well as his teaching and research activities at the University exercised great influence on the development of mathematics in Japan.

Reference

- [1] S. Kuroda (ed.), The collected papers of Teiji Takagi, Iwanami, 1973.

416 (XI.16) Teichmüller Spaces

Consider the set \mathbf{M}_g consisting of the conformal equivalence classes of closed Riemann surfaces of genus g . In 1859 Riemann stated, without rigorous proof, that \mathbf{M}_g is parametrized by $m(g)$ ($=0$ if $g=0$, $=1$ if $g=1$, $=3g-3$ if $g\geq 2$) complex parameters (\rightarrow 11 Algebraic Functions). Later, the introduction of a topology and $m(g)$ -dimensional complex structure on \mathbf{M}_g were discussed rigorously in various ways. The following explanation of these methods is due to O. Teichmüller [1, 2], L. V. Ahlfors [3, 4], and L. Bers [5–7]. For the

algebraic-geometric approach \rightarrow 9 Algebraic Curves.

The trivial case $g=0$ is excluded, since \mathbf{M}_0 consists of a single point. Take a closed Riemann surface \mathfrak{R}_0 of genus $g\geq 1$, and consider the pairs (\mathfrak{R}, H) consisting of closed Riemann surfaces \mathfrak{R} of the same genus g and the homotopy classes H of orientation-preserving homeomorphisms of \mathfrak{R}_0 into \mathfrak{R} . Two pairs (\mathfrak{R}, H) and (\mathfrak{R}', H') are defined to be conformally equivalent if the homotopy class $H'H^{-1}$ contains a conformal mapping. The set \mathbf{T}_g consisting of the conformal equivalence classes $\langle \mathfrak{R}, H \rangle$ is called the **Teichmüller space** (with center at \mathfrak{R}_0). Let \mathfrak{S}_g be the group of homotopy classes of orientation-preserving homeomorphisms of \mathfrak{R}_0 onto itself. \mathfrak{S}_g is a transformation group acting on \mathbf{T}_g in the sense that each $\eta \in \mathfrak{S}_g$ induces the transformation $\langle \mathfrak{R}, H \rangle \rightarrow \langle \mathfrak{R}, H\eta \rangle$. It satisfies $\mathbf{T}_g/\mathfrak{S}_g = \mathbf{M}_g$. The set \mathfrak{I}_g of elements of \mathfrak{S}_g fixing every point of \mathbf{T}_g consists only of the unity element if $g\geq 3$ and is a normal subgroup of order 2 if $g=1, 2$. For the remainder of this article we assume that $g\geq 2$. The case $g=1$ can be discussed similarly, and the result coincides with the classical one: \mathbf{T}_1 can be identified with the upper half-plane and $\mathfrak{S}_1/\mathfrak{I}_1$ is the \dagger modular group.

Denote by $B(\mathfrak{R}_0)$ the set of measurable invariant forms $\mu \overline{dz} dz^{-1}$ with $\|\mu\|_\infty < 1$. For every $\mu \in B(\mathfrak{R}_0)$ there exists a pair (\mathfrak{R}, H) for which some $h \in H$ satisfies $h_z = \mu h_z$ (\rightarrow 352 Quasiconformal Mappings). This correspondence determines a surjection $\mu \in B(\mathfrak{R}_0) \mapsto \langle \mathfrak{R}, H \rangle \in \mathbf{T}_g$. Next, if $Q(\mathfrak{R}_0)$ denotes the space of holomorphic quadratic differentials ϕdz^2 on \mathfrak{R}_0 , a mapping $\mu \in B(\mathfrak{R}_0) \mapsto \phi \in Q(\mathfrak{R}_0)$ is obtained as follows: Consider μ on the universal covering space U ($=$ upper half-plane) of \mathfrak{R}_0 . Extend it to U^* ($=$ lower half-plane) by setting $\mu=0$, and let f be a quasiconformal mapping f of the plane onto itself satisfying $f_z = \mu f_{\bar{z}}$. Take the \dagger Schwarzian derivative $\psi = \{f, z\}$ of the holomorphic function f in U^* . The desired ϕ is given by $\phi(z) = \psi(\bar{z})$ on U . It has been verified that two μ induce the same ϕ if and only if the same $\langle \mathfrak{R}, H \rangle$ corresponds to μ . Consequently, an injection $\langle \mathfrak{R}, H \rangle \in \mathbf{T}_g \mapsto \phi \in Q(\mathfrak{R}_0)$ is obtained. Since $Q(\mathfrak{R}_0) = \mathbb{C}^{m(g)}$ by the Riemann-Roch theorem, this injection yields an embedding $\mathbf{T}_g \subset \mathbb{C}^{m(g)}$, where \mathbf{T}_g is shown to be a domain.

As a subdomain of $\mathbb{C}^{m(g)}$, the Teichmüller space is an $m(g)$ -dimensional complex analytic manifold. It is topologically equivalent to the unit ball in real $2m(g)$ -dimensional space and is a bounded \dagger domain of holomorphy in $\mathbb{C}^{m(g)}$.

Let $\{\alpha_1, \dots, \alpha_{2g}\}$ be a 1-dimensional homology basis with integral coefficients in \mathfrak{R}_0 such that the intersection numbers are $(\alpha_i, \alpha_j) = (\alpha_{g+i}, \alpha_{g+j}) = 0$, $(\alpha_i, \alpha_{g+j}) = \delta_{ij}$, $i, j = 1, \dots, g$.

Given an arbitrary $\langle \mathfrak{R}, H \rangle \in \mathbf{T}_g$, consider the \dagger period matrix Ω of \mathfrak{R} with respect to the homology basis $H\alpha_1, \dots, H\alpha_{2g}$ and the basis $\omega_1, \dots, \omega_g$ of \dagger Abelian differentials of the first kind with the property that $\int_{H\alpha_i} \omega_j = \delta_{ij}$. Then Ω is a holomorphic function on \mathbf{T}_g . Furthermore, the analytic structure of the Teichmüller space introduced previously is the unique one (with respect to the topology defined above) for which the period matrix is holomorphic.

\mathfrak{S}_g is a properly discontinuous group of analytic transformations, and therefore \mathbf{M}_g is an $m(g)$ -dimensional normal \dagger analytic space. \mathfrak{S}_g is known to be the whole group of the holomorphic automorphisms of \mathbf{T}_g (Royden [8]); thus \mathbf{T}_g is not a \dagger symmetric space.

To every point τ of the Teichmüller space, there corresponds a Jordan domain $D(\tau)$ in the complex plane in such a way that the fiber space $F_g = \{(\tau, z) | z \in D(\tau), \tau \in \mathbf{T}_g \subset \mathbf{C}^{m(g)}\}$ has the following properties: F_g is a bounded domain of holomorphy of $\mathbf{C}^{m(g)+1}$. It carries a properly discontinuous group \mathfrak{G}_g of holomorphic automorphisms, which preserves every fiber $D(\tau)$ and is such that $D(\tau)/\mathfrak{G}_g$ is conformally equivalent to the Riemann surface corresponding to τ . F_g carries holomorphic functions $F_j(\tau, z)$, $j = 1, \dots, 5g - 5$ such that for every τ the functions F_j/F_1 , $j = 2, \dots, 5g - 5$ restricted to $D(\tau)$ generate the meromorphic function field of the Riemann surface $D(\tau)/\mathfrak{G}_g$.

By means of the \dagger extremal quasiconformal mappings, it can be verified that \mathbf{T}_g is a complete metric space. The metric is called the **Teichmüller metric**, and is known to be a Kobayashi metric.

The Teichmüller space also carries a naturally defined Kähler metric, which for $g = 1$ coincides with the \dagger Poincaré metric if \mathbf{T}_1 is identified with the upper half-plane. The \dagger Ricci curvature, \dagger holomorphic sectional curvature, and \dagger scalar curvature are all negative (Ahlfors [9]).

By means of the quasiconformal mapping f , which we considered previously in order to construct the correspondence $\mu \mapsto \varphi$, it is possible to regard the Teichmüller space as a space of quasi-Fuchsian groups (\rightarrow 234 Kleinian Groups). To the boundary of \mathbf{T}_g , it being a bounded domain in $\mathbf{C}^{m(g)}$, there correspond various interesting Kleinian groups, which are called \dagger boundary groups (Bers [10], Maskit [11]).

The definition of Teichmüller spaces can be extended to open Riemann surfaces \mathfrak{R}_0 and, further, to those with signatures. A number of propositions stated above are valid to these cases as well. In particular, the Teichmüller space for the case where \mathfrak{R}_0 is the unit disk is called the **universal Teichmüller space**. It is a bounded domain of holomorphy in an infinite-

dimensional Banach space and is a symmetric space. Every Teichmüller space is a subspace of the universal Teichmüller space.

References

- [1] O. Teichmüller, *Extremale quasikonforme Abbildungen und quadratische Differentiale*, Abh. Preuss. Akad. Wiss., 1939.
- [2] O. Teichmüller, *Bestimmung der extremalen quasikonformen Abbildung bei geschlossenen orientierten Riemannschen Flächen*, Abh. Preuss. Akad. Wiss., 1943.
- [3] L. V. Ahlfors, *The complex analytic structure of the space of closed Riemann surfaces*, Analytic functions, Princeton Univ. Press, 1960, 45–66.
- [4] L. V. Ahlfors, *Lectures on quasiconformal mappings*, Van Nostrand, 1966.
- [5] L. Bers, *Spaces of Riemann surfaces*. Proc. Intern. Congr. Math., Edinburgh, 1958, 349–361.
- [6] L. Bers, *On moduli of Riemann surfaces*, Lectures at Forschungsinstitut für Mathematik, Eidgenössische Technische Hochschule, Zürich, 1964.
- [7] L. Bers, *Uniformization, moduli, and Kleinian groups*, Bull. London Math. Soc., 4 (1972), 257–300.
- [8] H. L. Royden, *Automorphisms and isometries of Teichmüller spaces*, Advances in the Theory of Riemann Surfaces, Princeton Univ. Press, 1971, 369–383.
- [9] L. V. Ahlfors, *Curvature properties of Teichmüller's space*, J. Analyse Math., 9 (1961), 161–176.
- [10] L. Bers, *On boundaries of Teichmüller spaces and on Kleinian groups I*, Ann. Math., (2) 91 (1970), 570–600.
- [11] B. Maskit, *On boundaries of Teichmüller spaces and on Kleinian groups II*, Ann. Math., (2) 91 (1970), 608–638.

417 (VII.5) Tensor Calculus

A. General Remarks

In a \dagger differentiable manifold with an \dagger affine connection (in particular, in a \dagger Riemannian manifold), we can define an important operator on tensor fields, the operator of covariant differentiation. The **tensor calculus** is a differential calculus on a differentiable manifold that deals with various geometric objects and differential operators in terms of covariant differentiation, and it provides an important tool for studying geometry and analysis on a differentiable manifold.

B. Covariant Differential

Let M be an n -dimensional smooth manifold. We denote by $\mathfrak{F}(M)$ the set of all smooth functions on M and by $\mathfrak{X}_s(M)$ the set of all smooth tensor fields of type (r, s) on M . $\mathfrak{X}_0^1(M)$ is the set of all smooth vector fields on M , and we denote it simply by $\mathfrak{X}(M)$.

In the following we assume that an affine connection ∇ is given on M . Then we can define the **covariant differential** of tensor fields on M with respect to the connection (\rightarrow 80 Connections). We denote the **covariant derivative** of a tensor field K in the direction of a vector field X by $\nabla_X K$ and the covariant differential of K by ∇K . The operator ∇_X maps $\mathfrak{X}_s(M)$ into itself and has the following properties:

- (1) $\nabla_{X+Y} = \nabla_X + \nabla_Y$, $\nabla_{fX} = f\nabla_X$,
- (2) $\nabla_X(K + K') = \nabla_X K + \nabla_X K'$,
- (3) $\nabla_X(K \otimes K') = (\nabla_X K) \otimes K' + K \otimes (\nabla_X K')$,
- (4) $\nabla_X f = Xf$,
- (5) ∇_X commutes with contraction of tensor fields, where K and K' are tensor fields on M , $X, Y \in \mathfrak{X}(M)$ and $f \in \mathfrak{F}(M)$.

The **torsion tensor** T and the **curvature tensor** R of the affine connection ∇ are defined by

$$T(X, Y) = \nabla_X Y - \nabla_Y X - [X, Y],$$

$$R(X, Y)Z = \nabla_X(\nabla_Y Z) - \nabla_Y(\nabla_X Z) - \nabla_{[X, Y]}Z$$

for vector fields X, Y , and Z . The torsion tensor is of type $(1, 2)$, and the curvature tensor is of type $(1, 3)$. Some authors define $-R$ as the curvature tensor. We here follow the convention used in [1–6], while in [7, 8] the sign of the curvature tensor is opposite. The torsion tensor and the curvature tensor satisfy the identities

$$T(X, Y) = -T(Y, X), \quad R(X, Y) = -R(Y, X),$$

$$\begin{aligned} R(X, Y)Z + R(Y, Z)X + R(Z, X)Y \\ = (\nabla_X T)(Y, Z) + (\nabla_Y T)(Z, X) + (\nabla_Z T)(X, Y) \\ + T(T(X, Y), Z) + T(T(Y, Z), X) \\ + T(T(Z, X), Y), \end{aligned}$$

$$\begin{aligned} (\nabla_X R)(Y, Z) + (\nabla_Y R)(Z, X) + (\nabla_Z R)(X, Y) \\ = R(X, T(Y, Z)) + R(Y, T(Z, X)) \\ + R(Z, T(X, Y)). \end{aligned}$$

The last two identities are called the **Bianchi identities**.

The operators ∇_X and ∇_Y for two vector fields X and Y are not commutative in general, and they satisfy the following formula, the **Ricci formula**, for a tensor field K :

$$\nabla_X(\nabla_Y K) - \nabla_Y(\nabla_X K) - \nabla_{[X, Y]}K = R(X, Y) \cdot K,$$

where in the right-hand side $R(X, Y)$ is re-

garded as a derivation of the tensor algebra $\Sigma_{r,s} \mathfrak{X}_s(M)$.

A **moving frame** of M on a neighborhood U is, by definition, an ordered set (e_1, \dots, e_n) of n vector fields on U such that $e_1(p), \dots, e_n(p)$ are linearly independent at each point $p \in U$. For a moving frame (e_1, \dots, e_n) of M on a neighborhood U we define n differential 1-forms $\theta^1, \dots, \theta^n$ by $\theta^i(e_j) = \delta_j^i$, and we call them the **dual frame** of (e_1, \dots, e_n) . For a tensor field K of type (r, s) on M , we define n^{r+s} functions $K_{j_1 \dots j_s}^{i_1 \dots i_r}$ on U by

$$K_{j_1 \dots j_s}^{i_1 \dots i_r} = K(e_{j_1}, \dots, e_{j_s}, \theta^{i_1}, \dots, \theta^{i_r})$$

and call these functions the components of K with respect to the moving frame (e_1, \dots, e_n) .

Since the covariant differentials ∇e_j are tensor fields of type $(1, 1)$, n^2 differential 1-forms ω_j^i are defined by

$$\nabla e_j = \omega_j^i \otimes e_i,$$

where in the right-hand side (and throughout the following) we adopt **Einstein's summation convention**: If an index appears twice in a term, once as a superscript and once as a subscript, summation has to be taken on the range of the index. (Some authors write the above equation as $de_j = \omega_j^i e_i$ or $De_j = \omega_j^i e_i$.) We call these 1-forms ω_j^i the **connection forms** of the affine connection with respect to the moving frame (e_1, \dots, e_n) . The torsion forms Θ^i and the curvature forms Ω_j^i are defined by

$$\Theta^i = d\theta^i + \omega_j^i \wedge \theta^j, \quad \Omega_j^i = d\omega_j^i + \omega_k^i \wedge \omega_j^k.$$

These equations are called the **structure equation** of the affine connection ∇ . If we denote the components of the torsion tensor and the curvature tensor with respect to (e_1, \dots, e_n) by T_{jk}^i and $R_{jkl}^i (= \theta^i(R(e_k, e_l)e_j))$, respectively, then they satisfy the relations

$$\Theta^i = \frac{1}{2} T_{jk}^i \theta^j \wedge \theta^k, \quad \Omega_j^i = \frac{1}{2} R_{jkl}^i \theta^k \wedge \theta^l.$$

Using these forms, the Bianchi identities are written as

$$d\Theta^i + \omega_j^i \wedge \Theta^j = \Omega_j^i \wedge \theta^j,$$

$$d\Omega_j^i + \omega_k^i \wedge \Omega_j^k - \omega_j^k \wedge \Omega_k^i = 0.$$

Let K be a tensor field of type (r, s) on M and $K_{j_1 \dots j_s}^{i_1 \dots i_r}$ be the components of K with respect to (e_1, \dots, e_n) . We define the covariant differential $DK_{j_1 \dots j_s}^{i_1 \dots i_r}$ and the covariant derivative $K_{j_1 \dots j_s, k}^{i_1 \dots i_r}$ by

$$\begin{aligned} DK_{j_1 \dots j_s}^{i_1 \dots i_r} = K_{j_1 \dots j_s, k}^{i_1 \dots i_r} \theta^k = dK_{j_1 \dots j_s}^{i_1 \dots i_r} + \sum_{v=1}^r K_{j_1 \dots j_s}^{i_1 \dots i_{v-1} i_{v+1} \dots i_r} \omega_a^{i_v} \\ - \sum_{v=1}^s K_{j_1 \dots a \dots j_s}^{i_1 \dots i_r} \omega_{j_v}^a, \end{aligned}$$

Then $K_{j_1 \dots j_s, k}^{i_1 \dots i_r}$ are the components of ∇K with respect to the moving frame (e_1, \dots, e_n) . Some authors write $\nabla_k K_{j_1 \dots j_s}^{i_1 \dots i_r}$ instead of $K_{j_1 \dots j_s, k}^{i_1 \dots i_r}$ [5, 6].

Using components, the Bianchi identities are written as

$$\begin{aligned} R_{ijk}^h + R_{jki}^h + R_{kij}^h &= T_{ij, k}^h + T_{jk, i}^h + T_{ki, j}^h \\ &+ T_{ai}^h T_{jk}^a + T_{aj}^h T_{ki}^a + T_{ak}^h T_{ij}^a, \\ R_{ijk, l}^h + R_{ikl, j}^h + R_{ilj, k}^h &= R_{iak}^h T_{jl}^a + R_{iaj}^h T_{lk}^a + R_{ial}^h T_{kj}^a. \end{aligned}$$

The Ricci formula is written as

$$\begin{aligned} K_{j_1 \dots j_s, kl}^{i_1 \dots i_r} - K_{j_1 \dots j_s, lk}^{i_1 \dots i_r} &= \sum_{v=1}^r R_{aik}^{i_v} K_{j_1 \dots a \dots j_s}^{i_1 \dots i_r} \\ &- \sum_{v=1}^s R_{jv, lk}^a K_{j_1 \dots a \dots j_s}^{i_1 \dots i_r} \\ &+ T_{kl}^a K_{j_1 \dots j_s, a}^{i_1 \dots i_r}. \end{aligned}$$

Let (x^1, \dots, x^n) be a local coordinate system defined on a neighborhood U of M . Then $(\partial/\partial x^1, \dots, \partial/\partial x^n)$ is a moving frame of M on U , and we call it the **natural moving frame** associated with the coordinate system (x^1, \dots, x^n) . Components of a tensor field with respect to the natural moving frame $(\partial/\partial x^1, \dots, \partial/\partial x^n)$ are often called components with respect to the coordinate system (x^1, \dots, x^n) . We define an n^3 function Γ_{kj}^i on U by $\omega_j^i = \Gamma_{kj}^i dx^k$, where ω_j^i are the connection forms for the natural moving frame. Γ_{kj}^i are called the coefficients of the affine connection ∇ . The components of the torsion tensor and the curvature tensor with respect to (x^1, \dots, x^n) are given by

$$\begin{aligned} T_{jk}^i &= \Gamma_{jk}^i - \Gamma_{kj}^i, \\ R_{ijk}^h &= \partial_j \Gamma_{ki}^h - \partial_k \Gamma_{ji}^h + \Gamma_{ki}^a \Gamma_{ja}^h - \Gamma_{ji}^a \Gamma_{ka}^h, \end{aligned}$$

where $\partial_i = \partial/\partial x^i$.

With respect to the foregoing coordinate system, the components $K_{j_1 \dots j_s, k}^{i_1 \dots i_r}$ of the covariant differential ∇K of a tensor field K of type (r, s) are given by

$$\begin{aligned} K_{j_1 \dots j_s, k}^{i_1 \dots i_r} &= \partial_j K_{j_1 \dots j_s}^{i_1 \dots i_r} + \sum_{v=1}^r \Gamma_{ja}^{i_v} K_{j_1 \dots a \dots j_s}^{i_1 \dots i_r} \\ &- \sum_{v=1}^s \Gamma_{jv, k}^a K_{j_1 \dots a \dots j_s}^{i_1 \dots i_r}. \end{aligned}$$

C. Covariant Differential of Tensorial Forms

A **tensorial p -form** of type (r, s) on a manifold M is an alternating $\mathfrak{F}(M)$ -multilinear mapping of $\mathfrak{X}(M) \times \dots \times \mathfrak{X}(M)$ to $\mathfrak{X}_s^r(M)$. A tensorial p -form of type $(0, 0)$ is a differential p -form in the usual sense. A tensorial p -form of type $(1, 0)$ is often called a **vectorial p -form**.

If an affine connection ∇ is provided on M , we define the covariant differential of tensorial forms. Let α be a tensorial p -form of type (r, s) .

The covariant differential $D\alpha$ of α is a tensorial $(p+1)$ -form of type (r, s) and is defined by

$$\begin{aligned} (p+1)D\alpha(X_1, \dots, X_{p+1}) &= \sum_{i=1}^{p+1} (-1)^{i-1} \nabla_{X_i}(\alpha(X_1, \dots, \hat{X}_i, \dots, X_{p+1})) \\ &+ \sum_{i < j} (-1)^{i+j} \alpha([X_i, X_j], \\ &X_1, \dots, \hat{X}_i, \dots, \hat{X}_j, \dots, X_{p+1}), \end{aligned}$$

where \hat{X}_i means that X_i is deleted. If α is of type $(0, 0)$, $D\alpha$ coincides with the usual exterior differential $d\alpha$.

The simplest example of a tensorial form is the identity mapping of $\mathfrak{X}(M)$, which will be denoted by θ . Some authors write this vectorial form as dp or dx , where p or x expresses an arbitrary point of a manifold. We call θ the **canonical vectorial form** of M . The torsion tensor T can be regarded as a vectorial 2-form, and we have $2D\theta = T$. The curvature tensor R can be regarded as a tensorial 2-form of type $(1, 1)$, i.e., $(X, Y) \rightarrow R(X, Y) \in \mathfrak{X}_1^1(M)$, and the Bianchi identities are written as $DT = R \wedge \theta$, $DR = 0$, where the exterior product $R \wedge \alpha$ of R and a tensorial p -form α is defined by

$$\begin{aligned} (p+1)(p+2)(R \wedge \alpha)(X_1, \dots, X_{p+2}) &= 2 \sum_{i < j} (-1)^{i+j-1} R(X_i, X_j) \alpha(X_1, \dots, \hat{X}_i, \dots, \hat{X}_j, \\ &\dots, X_{p+2}). \end{aligned}$$

In general, $2D^2\alpha = R \wedge \alpha$ holds for an arbitrary tensorial form α .

Let (e_1, \dots, e_n) be a moving frame of M on a neighborhood U and $\theta^1, \dots, \theta^n$ be its dual frames. A tensorial p -form α of type (r, s) is written as

$$\alpha = \alpha_{j_1 \dots j_s}^{i_1 \dots i_r} \otimes e_{i_1} \otimes \dots \otimes e_{i_r} \otimes \theta^{j_1} \otimes \dots \otimes \theta^{j_s},$$

on U , where the $\alpha_{j_1 \dots j_s}^{i_1 \dots i_r}$ are the usual differential p -forms on U . We call them the components of α with respect to (e_1, \dots, e_n) . Then the components of $D\alpha$, which we denote by $D\alpha_{j_1 \dots j_s}^{i_1 \dots i_r}$, are given by

$$\begin{aligned} D\alpha_{j_1 \dots j_s}^{i_1 \dots i_r} &= d\alpha_{j_1 \dots j_s}^{i_1 \dots i_r} + \sum_{v=1}^r \omega_a^{i_v} \wedge \alpha_{j_1 \dots j_s}^{i_1 \dots a \dots i_r} \\ &- \sum_{v=1}^s \omega_{j_v}^a \wedge \alpha_{j_1 \dots a \dots j_s}^{i_1 \dots i_r}. \end{aligned}$$

Then we have

$$D^2\alpha_{j_1 \dots j_s}^{i_1 \dots i_r} = \sum_{v=1}^r \Omega_a^{i_v} \wedge \alpha_{j_1 \dots j_s}^{i_1 \dots a \dots i_r} - \sum_{v=1}^s \Omega_{j_v}^a \wedge \alpha_{j_1 \dots a \dots j_s}^{i_1 \dots i_r}.$$

This is an expression of $2D^2\alpha = R \wedge \alpha$ in terms of components. The components of the canonical vectorial form θ are the dual forms $\theta^1, \dots, \theta^n$ of (e_1, \dots, e_n) , and we have $D\theta^i = \Theta^i$, which means that the components of $D\theta$ are the torsion forms Θ^i .

D. Tensor Fields on a Riemannian Manifold

Let (M, g) be an n -dimensional Riemannian manifold (\rightarrow 364 Riemannian Manifolds). The fundamental tensor g defines a one-to-one correspondence between vector fields and differential 1-forms. A differential 1-form α which corresponds to a vector field X is defined by $\alpha(Y) = g(X, Y)$ for any vector field Y . This correspondence is naturally extended to a one-to-one correspondence between $\mathfrak{X}_s(M)$ and $\mathfrak{X}'_s(M)$, where $r + s = r' + s'$. Let (e_1, \dots, e_n) be a moving frame of M on a neighborhood U and g_{ij} be the components of g with respect to the moving frame. Let (g^{ij}) be the inverse matrix of the matrix (g_{ij}) . The g^{ij} are the components of a symmetric contravariant tensor field of order 2. Let X^i be the components of a vector field X and α_i be the components of the differential 1-form α corresponding to X . Then X^i and α_i satisfy the relations $\alpha_i = g_{ij}X^j$ and $X^i = g^{ij}\alpha_j$. If K^h_{ij} are the components of a tensor field K of type $(1, 2)$ (here taken for simplicity), then

$$K_{hi} = K^a_{ij}g_{ah}, \quad K^{hi}_j = K^h_{aj}g^{ai},$$

$$K^{hij} = K^h_{ab}g^{ai}g^{bj}, \dots,$$

are the components of a tensor field of type $(0, 3)$, $(2, 1)$, $(3, 0)$, ..., respectively, all of which correspond to K . We call this process of obtaining the components of the corresponding tensor fields from the components of a given tensor field **raising the subscripts and lowering the superscripts** by means of the fundamental tensor g .

On a Riemannian manifold, we use the \dagger Riemannian connection, unless otherwise stated. The covariant derivative with respect to the Riemannian connection is given by

$$\begin{aligned} 2g(\nabla_X Y, Z) &= Xg(Y, Z) + Yg(X, Z) - Zg(X, Y) \\ &\quad + g([X, Y], Z) - g([X, Z], Y) \\ &\quad - g(X, [Y, Z]) \end{aligned}$$

for vector fields X , Y , and Z . The coefficients of the Riemannian connection with respect to a local coordinate system (x^1, \dots, x^n) are usually written as $\{\overset{i}{k}j\}$, called the **Christoffel symbols**, which are given by $\{\overset{i}{k}j\} = g^{ia}(\partial_k g_{ja} + \partial_j g_{ka} - \partial_a g_{kj})/2$. The curvature tensor R of the Riemannian connection satisfies the identities

$$R(X, Y)Z + R(Y, Z)X + R(Z, X)Y = 0,$$

$$(\nabla_X R)(Y, Z) + (\nabla_Y R)(Z, X) + (\nabla_Z R)(X, Y) = 0,$$

$$R(X, Y) = -R(Y, X),$$

$$g(R(X, Y)Z, W) = g(R(Z, W)X, Y)$$

$$= -g(Z, R(X, Y)W),$$

$$g(R(X, Y)Z, W) + g(R(X, Z)W, Y)$$

$$+ g(R(X, W)Y, Z) = 0.$$

In terms of the components, these identities are

$$R^h_{ijk} + R^h_{jki} + R^h_{kij} = 0,$$

$$R^h_{ijk,i} + R^h_{ikt,j} + R^h_{ij,k} = 0,$$

$$R^h_{ijk} = -R^h_{ikj}, \quad R_{hijk} = R_{jkhi} = -R_{ihjk},$$

$$R_{hijk} + R_{hjki} + R_{hkij} = 0,$$

$$\text{where } R_{hijk} = R^a_{ijk}g_{ah}.$$

The \dagger Ricci tensor S of the Riemannian manifold is a tensor field of type $(0, 2)$ defined by

$$S(X, Y) = \text{trace of the mapping } Z \rightarrow R(Z, X)Y$$

for vector fields X and Y . The components S_{ji} of the Ricci tensor are given by $S_{ji} = R^a_{jai}$. The \dagger scalar curvature k of the Riemannian manifold M is a scalar on M defined by $k = g^{ji}S_{ji}$. The Ricci tensor and the scalar curvature satisfy the identities

$$S(X, Y) = S(Y, X) \quad \text{or} \quad S_{ji} = S_{ij},$$

$$S_{ij,k} - S_{ik,j} = R^a_{ikj,a}, \quad 2g^{jk}S_{ij,k} = \partial_i k.$$

For a moving frame of a Riemannian manifold, it is convenient to use an **orthonormal moving frame**. A moving frame (e_1, \dots, e_n) is orthonormal if e_1, \dots, e_n satisfy $g(e_i, e_j) = \delta_{ij}$. Since the components of the fundamental tensor with respect to an orthonormal moving frame are δ_{ij} , raising or lowering the indices does not change the values of the components. Some authors write all the indices as subscripts. Also they write the dual 1-forms, the connection forms, and the curvature forms as θ_i , ω_{ji} , and Ω_{ji} , respectively, instead of θ^i , ω^i_j , and Ω^i_j . With respect to an orthonormal moving frame, the connection forms ω^i_j and the curvature forms Ω^i_j satisfy

$$\omega^j_j + \omega^i_i = 0 \quad \text{and} \quad \Omega^j_j + \Omega^i_i = 0.$$

On a Riemannian manifold, the divergence of a vector field and the operators d , δ , and Δ on differential forms (\rightarrow 194 Harmonic Integrals) can be expressed by using the covariant derivatives with respect to the Riemannian connection.

If X^i are the components of a vector field X with respect to a local coordinate system (x^1, \dots, x^n) , the divergence $\text{div } X$ of X is given by $\text{div } X = X^i_{,i}$.

Let α be a differential p -form on M . α is written locally in the form $\alpha = (1/p!)\alpha_{i_1 \dots i_p} dx^{i_1} \wedge \dots \wedge dx^{i_p}$, where the coefficients $\alpha_{i_1 \dots i_p}$ are skew-symmetric in all the indices. We call $\alpha_{i_1 \dots i_p}$ the components of α with respect to the coordinate system. Since α is regarded as an alternating tensor field of type $(0, p)$, we can define the covariant differential $\nabla \alpha$ of α . Then the components of $d\alpha$, $\delta\alpha$, and $\Delta\alpha$ are

given by

$$\begin{aligned} (d\alpha)_{i_1 \dots i_{p+1}} &= \sum_{v=1}^{p+1} (-1)^{v-1} \alpha_{i_1 \dots \widehat{i_v} \dots i_{p+1}, i_v}, \\ (\delta\alpha)_{i_1 \dots i_{p-1}} &= -g^{ab} \alpha_{a i_1 \dots i_{p-1}, b}, \\ (\Delta\alpha)_{i_1 \dots i_p} &= -g^{ab} \left[\alpha_{i_1 \dots i_p, ab} - \sum_{v=1}^p S_{i, a} \alpha_{i_1 \dots b \dots i_p} \right. \\ &\quad \left. - \sum_{v < w} R_{a i_v i_w}^c \alpha_{i_1 \dots b \dots c \dots i_p} \right]. \end{aligned}$$

For a smooth function f and a differential 1-form β we have

$$\Delta f = -\frac{1}{\sqrt{g}} \tilde{e}_i (g^{ij} \sqrt{g} \partial_j f),$$

$$(\Delta\beta)_i = -g^{ab} [\beta_{i, ab} - S_{ia} \beta_b],$$

where $g = \det(g_{ij})$.

E. Van der Waerden–Bortolotti Covariant Differential

Let E be a finite dimensional smooth $^+$ vector bundle over a smooth manifold M and $\Gamma(E)$ be an $\mathfrak{F}(M)$ -module of all smooth sections of E . A connection ∇' in E is a mapping of $\mathfrak{X}(M) \times \Gamma(E)$ to $\Gamma(E)$ such that

- (1) $\nabla'_X(\xi + \eta) = \nabla'_X \xi + \nabla'_X \eta$,
- (2) $\nabla'_X(f\xi) = Xf \cdot \xi + f \nabla'_X \xi$,
- (3) $\nabla'_{X+Y} \xi = \nabla'_X \xi + \nabla'_Y \xi$,
- (4) $\nabla'_{fX} \xi = f \nabla'_X \xi$,

for $X, Y \in \mathfrak{X}(M)$, $\xi, \eta \in \Gamma(E)$, and $f \in \mathfrak{F}(M)$. $\nabla'_X \xi$ is called the covariant derivative of ξ in the direction X .

An element K of $\mathfrak{X}^s(M) \otimes \Gamma(E)$ is called a **tensor field of type (r, s) with values in E** (or simply an E -valued tensor field of type (r, s)). K can be regarded as an $\mathfrak{F}(M)$ -linear mapping of $\mathfrak{X}^s(M)$ to $\Gamma(E)$ or an $\mathfrak{F}(M)$ -multilinear mapping of $\mathfrak{X}(M) \times \dots \times \mathfrak{X}(M)$ to $\mathfrak{X}_0^s(M) \otimes \Gamma(E)$. For a given $\xi \in \Gamma(E)$, a mapping $X \rightarrow \nabla'_X \xi$ defines a tensor field of type $(0, 1)$ with values in E which we call the covariant differential of ξ , denoted by $\nabla' \xi$.

The curvature tensor R' of ∇' is a tensor field of type $(0, 2)$ with values in $E^* \otimes E$ (E^* is the dual vector bundle of E), and is defined by

$$R'(X, Y)\xi = \nabla'_X(\nabla'_Y \xi) - \nabla'_Y(\nabla'_X \xi) - \nabla'_{[X, Y]}\xi$$

for any vector fields X and Y and any $\xi \in \Gamma(E)$.

If an affine connection ∇ is given on M , we can define the **van der Waerden–Bortolotti covariant derivative** $\bar{\nabla}_X K$ for ∇ and ∇' of a tensor field K of type (r, s) with values in E . It is defined by

$$(\bar{\nabla}_X K)(S) = \nabla'_X(K(S)) - K(\nabla_X S)$$

for any $S \in \mathfrak{X}^s(M)$. If we regard $\xi \in \Gamma(E)$ as an E -

valued tensor field of type $(0, 0)$, we have $\bar{\nabla}_X \xi = \nabla'_X \xi$. The covariant derivative $\bar{\nabla}_X R'$ of the curvature tensor R' of ∇' is a tensor field of type $(0, 2)$ with values in $E^* \otimes E$ is defined by

$$\begin{aligned} (\bar{\nabla}_X R')(Y, Z)\xi &= \nabla'_X(R'(Y, Z)\xi) - R'(\nabla_X Y, Z)\xi \\ &\quad - R'(Y, \nabla_X Z)\xi - R'(Y, Z)\nabla'_X \xi. \end{aligned}$$

The Bianchi identity is written as

$$\begin{aligned} (\bar{\nabla}_X R')(Y, Z) + (\bar{\nabla}_Y R')(Z, X) + (\bar{\nabla}_Z R')(X, Y) \\ = R'(X, T(Y, Z)) + R'(Y, T(Z, X)) \\ + R'(Z, T(X, Y)), \end{aligned}$$

where T is the torsion tensor of ∇ . The Ricci formula is given by

$$\begin{aligned} (\bar{\nabla}_X(\bar{\nabla}_Y K))(S) - (\bar{\nabla}_Y(\bar{\nabla}_X K))(S) - (\bar{\nabla}_{[X, Y]}K)(S) \\ = R'(X, Y) \cdot K(S) - K(R(X, Y) \cdot S), \end{aligned}$$

where R is the curvature tensor of ∇ , $K \in \mathfrak{X}^s(M) \otimes \Gamma(E)$ and $S \in \mathfrak{X}^s(M)$.

In the following we assume that the fiber of E is of finite dimension m . A moving frame of E on a neighborhood U of M is an ordered set (ξ_1, \dots, ξ_m) of local sections ξ_1, \dots, ξ_m on U such that $\xi_1(p), \dots, \xi_m(p)$ are linearly independent at each point p of U . Let (e_1, \dots, e_n) be a moving frame of M on U . Then an E -valued tensor field K of type (r, s) is locally written as

$$K_{j_1 \dots j_s}^{i_1 \dots i_r} e_{i_1} \otimes \dots \otimes e_{i_r} \otimes \theta^{j_1} \otimes \dots \otimes \theta^{j_s} \otimes \xi_\alpha,$$

where $\theta^1, \dots, \theta^n$ are the dual 1-forms of (e_1, \dots, e_n) . The $n^r \times m$ functions $K_{j_1 \dots j_s}^{i_1 \dots i_r}$ on U are called the components of K with respect to (e_1, \dots, e_n) and (ξ_1, \dots, ξ_m) . We define the connection forms ω_β^α of the connection ∇' by $\nabla' \xi_\beta = \omega_\beta^\alpha \otimes \xi_\alpha$. Then the curvature forms Ω_β^α are defined by

$$\Omega_\beta^\alpha = d\omega_\beta^\alpha + \omega_\lambda^\alpha \wedge \omega_\beta^\lambda = \frac{1}{2} R_{\beta\mu}^\alpha \theta^j \wedge \theta^i,$$

where $R_{\beta\mu}^\alpha$ are the components of the curvature tensor R' , i.e., $R'(e_j, e_i)\xi_\beta = R_{\beta\mu}^\alpha \xi_\alpha$.

For a given tensor field K of type (r, s) with values in E , the mapping $X \rightarrow \bar{\nabla}_X K$ defines a tensor field $\bar{\nabla} K$ of $(r, s+1)$ with values in E which we call the van der Waerden–Bortolotti covariant differential of K . Then if $K_{j_1 \dots j_s}^{i_1 \dots i_r}$ are the components of K with respect to (e_1, \dots, e_n) and (ξ_1, \dots, ξ_m) , the components $K_{j_1 \dots j_s, k}^{i_1 \dots i_r}$ of $\bar{\nabla} K$ are given by

$$\begin{aligned} K_{j_1 \dots j_s, k}^{i_1 \dots i_r} \theta^k = dK_{j_1 \dots j_s}^{i_1 \dots i_r} + \sum_{v=1}^r K_{j_1 \dots j_s}^{i_1 \dots i_{v-1} i_{v+1} \dots i_r} \omega_a^{i_v} \\ - \sum_{v=1}^s K_{j_1 \dots a \dots j_s}^{i_1 \dots i_r} \omega_{j_v}^a + K_{j_1 \dots j_s}^{i_1 \dots i_r \beta} \omega_\beta^\alpha. \end{aligned}$$

Let f be a smooth mapping of M into a smooth manifold M' . The differential f_* (or df) can be regarded as a tensor field of type $(0, 1)$ with values in $f^*T(M')$. Assume that M (resp. M') has a Riemannian metric g (resp. g'). We denote the Riemannian connection of M by ∇ .

From the Riemannian connection of M' a connection ∇' in $f^*T(M')$ can be defined. Let (y^1, \dots, y^m) be a local coordinate system of M' on a neighborhood V and (x^1, \dots, x^n) be a local coordinate system on a neighborhood U of M such that $f(U) \subset V$. Put $\xi_\alpha(p) = (\partial/\partial y^\alpha)(f(p))$ for a point $p \in U$. Then (ξ_1, \dots, ξ_m) is a moving frame of $f^*T(M')$. The components of f_* with respect to $(\partial/\partial x^1, \dots, \partial/\partial x^n)$ and (ξ_1, \dots, ξ_m) are given by $f^\alpha(p) = (\partial y^\alpha/\partial x^i)(p)$. The Laplacian Δf of the mapping f is a tensor field of type $(0, 0)$ with values in $f^*T(M')$ and is defined by $(\Delta f)^\alpha = g^{ij}f_{i,j}^\alpha$. If $\Delta f = 0$, the mapping f is called a harmonic mapping (\rightarrow 195 Harmonic Mappings).

F. Tensor Fields on a Submanifold

Consider an n -dimensional smooth manifold M immersed in an $(n+m)$ -dimensional Riemannian manifold (\bar{M}, \bar{g}) . If we denote the immersion $M \rightarrow \bar{M}$ by f , then $g = f^*\bar{g}$ is a Riemannian metric on M , and we denote its Riemannian connection by ∇ . The induced bundle $f^*T(\bar{M})$ splits into the sum of the tangent bundle $T(M)$ of M and the normal bundle $T^\perp(M)$. The Riemannian connection on \bar{M} induces connections in $f^*T(\bar{M})$ and in $T^\perp(M)$ which are denoted by $\bar{\nabla}$ and ∇^\perp , respectively. The van der Waerden–Bortolotti covariant derivative for ∇ and ∇^\perp is denoted by $\bar{\nabla}$.

For vector fields X and Y on M , the tangential part of $\bar{\nabla}_X Y$ (here we regard Y as a section of $f^*T(\bar{M})$) is $\nabla_X Y$, and we denote the normal part of $\bar{\nabla}_X Y$ by $h(X, Y)$. Then h is a symmetric tensor field of type $(0, 2)$ with values in $T^\perp(M)$, and we call h the **second fundamental tensor** of the immersion f . For $\xi \in \Gamma(T^\perp(M))$, the tangential part of $\bar{\nabla}_X \xi$ (here ξ is also regarded as a section of $f^*T(\bar{M})$) is denoted by $-A_\xi X$ and the normal part of $\bar{\nabla}_X \xi$ is $\nabla_X^\perp \xi$. Thus we have

$$\bar{\nabla}_X Y = \nabla_X Y + h(X, Y), \quad \bar{\nabla}_X \xi = -A_\xi X + \nabla_X^\perp \xi.$$

h and A are related by

$$\bar{g}(h(X, Y), \xi) = g(A_\xi X, Y).$$

We have the following formulas, called the equations of Gauss, Codazzi, and Ricci:

$$\begin{aligned} \bar{g}(\bar{R}(X, Y)Z, W) &= g(R(X, Y)Z, W) \\ &\quad + \bar{g}(h(X, Z), h(Y, W)) \\ &\quad - \bar{g}(h(X, W), h(Y, Z)), \\ \bar{g}(\bar{R}(X, Y)Z, \xi) &= \bar{g}((\bar{\nabla}_X h)(Y, Z), \xi) \\ &\quad - \bar{g}((\bar{\nabla}_Y h)(X, Z), \xi), \\ \bar{g}(\bar{R}(X, Y)\xi, \eta) &= \bar{g}(R^\perp(X, Y)\xi, \eta) \\ &\quad + g([A_\xi, A_\eta]X, Y), \end{aligned}$$

for $X, Y, Z, W \in X(M)$ and $\xi, \eta \in \Gamma(T^\perp(M))$, where R, \bar{R} , and R^\perp are the curvature tensors of $\nabla, \bar{\nabla}$, and ∇^\perp , respectively.

For the manifold M immersed in \bar{M} , we use a moving frame $(e_1, \dots, e_n, \xi_1, \dots, \xi_m)$ such that (e_1, \dots, e_n) is an orthonormal moving frame of M on a neighborhood U and (ξ_1, \dots, ξ_m) is a moving frame of $T^\perp(M)$ on U with $\bar{g}(\xi_\alpha, \xi_\beta) = \delta_{\alpha\beta}$. Then we can define the connection forms ω_j^i for ∇ and ω_β^α for ∇^\perp . If we extend $(e_1, \dots, e_n, \xi_1, \dots, \xi_m)$ to an orthonormal moving frame $(\bar{e}_1, \dots, \bar{e}_{n+m})$ of \bar{M} such that $\bar{e}_i(p) = e_i(p)$ ($i = 1, \dots, n$) and $\bar{e}_{n+\alpha}(p) = \xi_\alpha(p)$ ($\alpha = 1, \dots, m$) for $p \in U$, then the restriction $f^*\bar{\theta}^A$ and $f^*\omega_B^A$ of the dual 1-forms and the connection forms of \bar{M} with respect to $(\bar{e}_1, \dots, \bar{e}_{n+m})$ satisfy the relations

$$\begin{aligned} f^*\bar{\theta}^i &= \theta^i, \quad f^*\bar{\theta}^{n+\alpha} = 0, \quad f^*\omega_j^i = \omega_j^i, \\ f^*\omega_{n+\alpha}^{n+\beta} &= \omega_\beta^\alpha, \quad f^*\omega_i^{n+\alpha} = \sum_j h_{ij}^\alpha \theta^j, \end{aligned}$$

where h_{ij}^α are the components of the second fundamental tensor h with respect to $(e_1, \dots, e_n, \xi_1, \dots, \xi_m)$.

The components $h_{ij,k}^\alpha$ of the covariant differential $\bar{\nabla}h$ of h are defined by

$$h_{ij,k}^\alpha \theta^k = dh_{ij}^\alpha - h_{aj}^\alpha \omega_i^a - h_{ia}^\alpha \omega_j^a + h_{ij}^\beta \omega_\beta^\alpha.$$

In terms of the components, the equations of Gauss, Codazzi, and Ricci are given by

$$\bar{R}_{hijk} = R_{hijk} + \sum_\alpha (h_{ij}^\alpha h_{hk}^\alpha - h_{ik}^\alpha h_{hj}^\alpha),$$

$$\bar{R}_{ijk}^\alpha = h_{ik,j}^\alpha - h_{ij,k}^\alpha,$$

$$\bar{R}_{\beta jk}^\alpha = R^\perp \alpha_{\beta jk} - \sum_a (h_{ja}^\beta h_{ak}^\alpha - h_{ja}^\alpha h_{ak}^\beta).$$

Let (x^1, \dots, x^n) be a local coordinate system on a neighborhood U of M and (y^1, \dots, y^{n+m}) be a local coordinate system on a neighborhood V of \bar{M} such that $f(U) \subset V$. Regarding the differential f_* of the immersion f as a tensor field of type $(0, 1)$ with values in $f^*T(\bar{M})$, we denote the components of f_* with respect to (x^1, \dots, x^n) and (y^1, \dots, y^{n+m}) by B_i^A ($i = 1, \dots, n; A = 1, \dots, n+m$). Then we have $B_i^A = \partial y^A / \partial x^i$. We denote by ∇' the van der Waerden–Bortolotti covariant derivative for ∇ and $\bar{\nabla}$. Then the components $B_{i,j}^A$ of $\nabla' f_*$ are given by

$$B_{i,j}^A = \partial_j B_i^A - \{ \begin{smallmatrix} a \\ ji \end{smallmatrix} \} B_i^a + B_j^C B_i^B \{ \begin{smallmatrix} A \\ CB \end{smallmatrix} \},$$

where $\partial_j = \partial/\partial x^j$, $\{ \begin{smallmatrix} h \\ ji \end{smallmatrix} \}$, and $\{ \begin{smallmatrix} A \\ CB \end{smallmatrix} \}$ are the Christoffel symbols of the Riemannian metrics g and \bar{g} , respectively.

Let (ξ_1, \dots, ξ_m) be an orthonormal moving frame of $T^\perp(M)$ on U and ξ_α^A be the components of ξ_α with respect to (y^1, \dots, y^{n+m}) . Then we have

$$B_{i,j}^A = h_{ij}^\alpha \xi_\alpha^A,$$

where h_{ij} are the components of the second

fundamental tensor with respect to $(\partial/\partial x^1, \dots, \partial/\partial x^n)$ and (ξ_1, \dots, ξ_m) .

A tensor field K with values in $T^\perp(M)$ can be regarded as a tensor field with values in $f^*T(M)$, and $\tilde{\nabla}K$ is the normal component of ∇K . For example, if we regard the second fundamental tensor h as a tensor field with values in $f^*T(\bar{M})$, the components of h with respect to the coordinates (x^1, \dots, x^n) and (y^1, \dots, y^{n+m}) are equal to $B_{i,j}^A$, and we have

$$h_{ij,k}^z = B_{i,j,k}^A \xi_\alpha^{B\bar{\alpha}} \vartheta_{AB}.$$

References

- [1] E. Cartan, *Leçon sur la géométrie des espaces de Riemann*, Gauthier-Villars, second edition, 1946; English translation, *Geometry of Riemannian Spaces*, Math-Sci Press, 1983.
- [2] B.-Y. Chen, *Geometry of submanifolds*, Dekker, 1973.
- [3] S. Helgason, *Differential geometry, Lie groups, and symmetric spaces*, Academic Press, 1978.
- [4] S. Kobayashi and K. Nomizu, *Foundations of differential geometry*, Interscience, I, 1963; II, 1969.
- [5] J. A. Schouten, *Ricci-calculus*, Springer, second edition, 1954.
- [6] K. Yano, *Integral formulas in Riemannian geometry*, Dekker, 1970.
- [7] G. de Rham, *Variétés différentiables*, Actualités Sci. Ind., Hermann, second edition, 1960.
- [8] L. P. Eisenhart, *Riemannian geometry*, Princeton Univ. Press, second edition, 1949.
- [9] R. L. Bishop and S. I. Goldberg, *Tensor analysis on manifolds*, Macmillan, 1968.

418 (IX.20) Theory of Singularities

A. Introduction

Let f_1, f_2, \dots, f_r be \dagger holomorphic functions defined in an open set U of the complex space \mathbb{C}^n . Let X be the analytic set $f_1^{-1}(0) \cap \dots \cap f_r^{-1}(0)$. Let $z_0 \in X$, and let g_1, \dots, g_s be a system of generators of the ideal $\mathcal{I}(X)_{z_0}$ of the germs of the holomorphic functions which vanish identically on a neighborhood of z_0 in X . z_0 is called a **simple point** of X if the matrix $(\partial g_i / \partial z_j)$ attains its maximal rank, say k , at $z = z_0$. In this case, X is a \dagger complex manifold of dimension $n - k$ near z_0 . Otherwise, z_0 is called a **singular point** of X .

B. Resolution of Singularities

Let X be a complex analytic space, and let Y be its singular locus. A **resolution of the singularity** of X is a pair of a complex manifold \tilde{X} and a proper surjective holomorphic mapping $\pi: \tilde{X} \rightarrow X$ such that the restriction $\pi|_{\tilde{X} - \pi^{-1}(Y)}$ is biholomorphic and $\tilde{X} - \pi^{-1}(Y)$ is dense in \tilde{X} . H. Hironaka proved that there exists a resolution for any X such that $\pi^{-1}(Y)$ is a divisor in \tilde{X} with only \dagger normal crossings [16, 17].

Suppose that a compact connected analytic subset \tilde{Y} of a complex manifold \tilde{X} has a \dagger strongly pseudoconvex neighborhood in \tilde{X} . Then the contraction \tilde{X}/\tilde{Y} naturally has a structure of a \dagger normal complex analytic variety such that the projection $\tilde{X} \rightarrow \tilde{X}/\tilde{Y}$ is a resolution of \tilde{X}/\tilde{Y} (H. Grauert [14]).

C. Two-Dimensional Singularities

Let X be a normal 2-dimensional analytic space. Then the singular points of X are discrete.

Among the resolutions of X , there exists a unique resolution $\pi: \tilde{X} \rightarrow X$ with the following universal property: For any resolution $\pi': \tilde{X}' \rightarrow X$, there exists a unique mapping $\rho: \tilde{X}' \rightarrow \tilde{X}$ with $\pi' = \pi \circ \rho$. This resolution is called the **minimal resolution**.

Let $\pi: \tilde{X} \rightarrow X$ be a resolution of a singular point x of X , and let A_i ($i = 1, \dots, m$) be the irreducible components of $\pi^{-1}(x)$. The matrix $(A_i \cdot A_j)$ of the \dagger intersection numbers is known to be negative definite (P. Du Val [12]).

The resolution $\pi: \tilde{X} \rightarrow X$ is called **good** if (i) each A_i is nonsingular, (ii) $A_i \cap A_j$ ($i \neq j$) is at most one point and the intersection is transverse and (iii) no three A_i 's meet at a point. For a given good resolution $\pi: \tilde{X} \rightarrow X$, we associate a diagram in which the vertices v_i ($i = 1, \dots, m$) correspond to A_i ($i = 1, \dots, m$) and v_i and v_j are joined by a segment if and only if $A_i \cap A_j \neq \emptyset$.

The **geometric genus** $p_g(X, x)$ of a singular point $x \in X$ is the dimension of the \dagger stalk at x of the first direct image sheaf $R^1 \pi_* \mathcal{O}_{\tilde{X}}$, where $\pi: \tilde{X} \rightarrow X$ is a resolution of $x \in X$ and $\mathcal{O}_{\tilde{X}}$ is the \dagger structure sheaf of \tilde{X} . The definition is independent of the choice of the resolution, and $p_g(X, x)$ is a finite integer.

Among the positive cycles of the form $Z = \sum_{i=1}^n n_i A_i$ (i.e., $n_i \geq 0$) such that $Z \cdot A_i < 0$ for each $i = 1, \dots, m$, there exists a smallest one Z_0 , which is called the **fundamental cycle** [3].

(1) **Rational singularities**. A singular point x of X is called **rational** if $p_g(X, x) = 0$. (The singularity (X, x) is also called rational even when $\dim X \geq 3$ if the direct image sheaf $R^i \pi_* \mathcal{O}_{\tilde{X}} = 0$ for $i > 0$.)

For a rational singularity $x \in X$, the multiplicity of X at x equals $-Z_0^2$ and the local embedding dimension of X at x is $-Z_0^2 + 1$. Hence a rational singularity with multiplicity 2, which is called a **rational double point**, is a hypersurface singularity. The following weighted homogeneous polynomials (\rightarrow Section D) give the complete list of the defining equations up to analytic isomorphism:

$$\begin{aligned} A_n: x^{n+1} + y^2 + z^2, \\ \text{weights } (1/(n+1), 1/2, 1/2), \quad n \geq 1; \\ D_n: x^{n-1} + xy^2 + z^2, \\ \text{weights } (1/(n-1), (n-2)/2(n-1), 1/2), \quad n \geq 4; \\ E_6: x^4 + y^3 + z^2, \\ \text{weights } (1/4, 1/3, 1/2); \\ E_7: x^3y + y^3 + z^2, \\ \text{weights } (2/9, 1/3, 1/2); \\ E_8: x^5 + y^3 + z^2, \\ \text{weights } (1/5, 1/3, 1/2), \end{aligned}$$

where the labels appearing at the left are given according to the coincidence of the diagram of the respective minimal resolutions and the \dagger Dynkin diagrams. Rational double points have many different characterizations [11].

The generic part of the singular locus of the unipotent variety of a \dagger complex simple Lie group G (=the orbit of the subregular \dagger unipotent elements in G) is locally expressed as the product of a rational double point and a polydisk. The \dagger universal deformation of a rational double point and its \dagger simultaneous resolution are constructed by restricting the following diagram on a transverse slice to the subregular unipotent orbit (Brieskorn [7]; [34]):

$$\begin{array}{ccc} Y & \longrightarrow & G \\ \downarrow & & \downarrow \\ T & \longrightarrow & T/W \end{array}$$

where T is a \dagger Cartan subgroup of G with the action of the Weyl group W , $G \rightarrow T/W$ is the quotient mapping by the \dagger adjoint action of G and $Y = \{(x, B) | x \in G \text{ and } B \text{ is a } \dagger\text{Borel subgroup of } G \text{ with } x \in B\}$, and other morphisms are defined naturally so that the diagram commutes. Here, $Y \rightarrow T$ is the simultaneous resolution of the morphism $G \rightarrow T/W$.

(2) **Quotient singularities.** A singular point $x \in X$ is called a **quotient singularity** if there exists a neighborhood of x which is analytically isomorphic to an orbit space U/G , where U is a neighborhood of 0 in \mathbb{C}^2 and G is a finite group of analytic automorphisms of U with the unique fixed point 0. The quotient singularities are rational, and their resolutions

have been well studied [6]. U/G has a rational double point at 0 if and only if G is conjugate to a nontrivial finite subgroup of $SU(2)$.

(3) **Elliptic singularities.** The singularity (X, x) is called **minimally elliptic** if $p_g(X, x) = 1$ and (X, x) is Gorenstein [23]. The following are examples of minimally elliptic singularities.

A singular point $x \in X$ is called **simply elliptic** if the exceptional set A of the minimal resolution is a smooth \dagger elliptic curve [33]. When $A^2 = -1, -2, -3$, (X, x) is a hypersurface singularity given by the following weighted homogeneous polynomials:

$$\begin{aligned} \tilde{E}_6: x^3 + y^3 + z^3 + axyz, \\ \text{weights } (1/3, 1/3, 1/3), \quad A^2 = -3; \\ \tilde{E}_7: x^4 + y^4 + z^2 + axyz, \\ \text{weights } (1/4, 1/4, 1/2), \quad A^2 = -2; \\ \tilde{E}_8: x^6 + y^3 + z^2 + axyz, \\ \text{weights } (1/6, 1/3, 1/2), \quad A^2 = -1, \end{aligned}$$

(4) **Cusp singularities.** A singular point $x \in X$ is called a **cusp singularity** if the exceptional set of the minimal resolution is either a single rational curve with a \dagger node or a cycle of smooth rational curves. Cusp singularities appear as the boundary of \dagger Hilbert modular surfaces [18]. The hypersurface cusp singularities are given by the polynomials

$$T_{p,q,r}: x^p + y^q + z^r + axyz,$$

where $1/p + 1/q + 1/r < 1$ and $a \neq 0$.

D. The Milnor Fibration for Hypersurface Singularities

Let V be an analytic set in \mathbb{C}^N , and take a point $z_0 \in V$. Let $S_\varepsilon = S(z_0, \varepsilon)$ be a $(2N-1)$ -dimensional sphere in \mathbb{C}^N with center z_0 and radius $\varepsilon > 0$, and let $K_\varepsilon = V \cap S_\varepsilon$. If ε is sufficiently small, the topological type of the pair $(S_\varepsilon, K_\varepsilon)$ is independent of ε [27]. By virtue of this fact, the study of singular points constitutes an important aspect of the application of topology to the theory of functions of several complex variables.

A singular point z_0 of V is said to be **isolated** if, for some open neighborhood W of z_0 in \mathbb{C}^N , $W \cap V - \{z_0\}$ is a smooth submanifold of $W - \{z_0\}$. In that case, K_ε is a closed smooth submanifold of S_ε , and the diffeomorphism type of $(S_\varepsilon, K_\varepsilon)$ is independent of (sufficiently small) $\varepsilon > 0$. So far, the topological study of such singular points has been primarily focused on isolated singularities. When V is a plane curve, that is, $N = 2$ and $r = 1$, all the singular points of V are isolated, and the submanifold K_ε of the 3-sphere S_ε can be described as an iterated torus link, where type numbers are

completely determined by the \dagger Puiseux expansion of the defining equation f of V at the point z_0 [5]. In 1961, D. Mumford, using a resolution argument, showed that if an algebraic surface V is \dagger normal at z_0 and if the closed 3-manifold K_ε is simply connected, then K_ε is diffeomorphic to the 3-sphere and z_0 is nonsingular [29]. The following theorem in the higher-dimensional case is due to E. Brieskorn [8] (1966):

Every \dagger homotopy $(2n-1)$ -sphere ($n \neq 2$) that is a boundary of a $\dagger\pi$ -manifold is diffeomorphic to the K_ε of some complex hypersurface defined by an equation of the form $f(z) = z_1^{a_1} + \dots + z_{n+1}^{a_{n+1}} = 0$ at the origin in \mathbf{C}^{n+1} , provided that $n \neq 2$. The hypersurface of this type is called the **Brieskorn variety**. Inspired by Brieskorn's method, J. W. Milnor developed topological techniques for the study of hypersurface singularities and obtained results such as the **Milnor fibering theorem**, which can be briefly stated as follows:

Suppose that V is defined by a single equation $f(z) = 0$ in the neighborhood of $z_0 \in \mathbf{C}^{n+1}$. Then there is an associated smooth \dagger fiber bundle $\varphi: S_\varepsilon - K_\varepsilon \rightarrow S^1$, where $\varphi(z) = f(z)/|f(z)|$ for $z \in S_\varepsilon - K_\varepsilon$. The fiber $F = \varphi^{-1}(p)$ ($p \in S^1$) has the homotopy type of a finite CW-complex of dimension n , and K_ε is $(n-2)$ -connected.

Suppose that z_0 is an isolated critical point of f . Then F has the homotopy type of a \dagger bouquet of spheres of dimension n [27]. The **Milnor number** $\mu(f)$ of f is defined by the n th Betti number of F , and it is equal to $\dim_{\mathbf{C}} \mathcal{O}_{\mathbf{C}^{n+1}, z_0} / (\partial f / \partial z_1, \dots, \partial f / \partial z_{n+1})$, where $\mathcal{O}_{\mathbf{C}^{n+1}, z_0}$ is the ring of the germs of analytic functions of $n+1$ variables at $z = z_0$. The **Milnor monodromy** h_* is the automorphism of $H_n(F)$ that is induced by the action of the canonical generator of the fundamental group of the base space S^1 . The \dagger Lefschetz number of h_* is zero if z^0 is a singular point of V . Let $\Delta(t)$ be the characteristic polynomial of h_* . Then K_ε is a homology sphere if and only if $\Delta(1) = \pm 1$ [27]. It is known that $\Delta(t)$ is a product of \dagger cyclotomic polynomials.

The diffeomorphism class of $(S_\varepsilon, K_\varepsilon)$ is completely determined by the congruence class of the linking matrix $L(e_i, e_j)$ ($1 \leq i, j \leq \mu(f)$), where $e_1, \dots, e_{\mu(f)}$ is an integral basis of $H_n(F)$ and $L(e_i, e_j)$ is the \dagger linking number [21, 10].

The Milnor fibration is also described in the following way. Let $E(\varepsilon, \delta)$ be the intersection of $f^{-1}(D_\delta^*)$ and $B(\varepsilon)$, the open disk of radius ε and center z_0 , where D_δ^* is $\{\eta \in \mathbf{C} \mid 0 < |\eta| < \delta\}$. The restriction of f to $E(\varepsilon, \delta)$ is a \dagger locally trivial fibration over D_δ^* if δ is sufficiently smaller than ε [27].

Let $f(z)$ be an analytic function; suppose that $f(0) = 0$ and let $\sum_{p \in \mathbf{N}^{n+1}} a_p z^p$ be the Taylor expansion of f at $z = 0$. Let $\Gamma_+(f)$ be the con-

vex hull of the union of $\{p + (\mathbf{R}^+)^{n+1}\}$ for $p \in \mathbf{N}^{n+1} \subset \mathbf{R}^{n+1}$ with $a_p \neq 0$, where $\mathbf{R}^+ = \{x \in \mathbf{R} \mid x \geq 0\}$, and let $\Gamma(f)$ be the union of compact faces of $\Gamma_+(f)$. We call $\Gamma(f)$ the **Newton boundary** of f in the coordinates z_1, \dots, z_{n+1} . For a closed face Δ of $\Gamma(f)$ of any dimension, let $f_\Delta(z) = \sum_{p \in \Delta} a_p z^p$. We say that f has a **non-degenerate Newton boundary** if $(\partial f_\Delta / \partial z_1, \dots, \partial f_\Delta / \partial z_{n+1})$ is a nonzero vector for any $z \in (\mathbf{C}^*)^{n+1}$ and any $\Delta \in \Gamma(f)$. Suppose that f has a non-degenerate Newton boundary and 0 is an isolated critical point of f . Then the Milnor fibration of f is determined by $\Gamma(f)$ and $\mu(f)$, and the characteristic polynomial can be explicitly computed by $\Gamma(f)$ [22, 38].

$f(z)$ is called **weighted homogeneous** if there exist positive rational numbers r_1, \dots, r_{n+1} , which are called **weights**, such that $a_p = 0$ if $\sum_{i=1}^{n+1} p_i r_i \neq 1$. An analytic function $f(z)$ with an isolated critical point at 0 is weighted homogeneous in suitable coordinates if and only if f belongs to the ideal $(\partial f / \partial z_1, \dots, \partial f / \partial z_{n+1})$ (K. Saito [32]). Suppose that $f(z)$ is a weighted homogeneous polynomial with an isolated critical point at 0. Then the Milnor fibration of f is uniquely determined by the weights, and $\mu(f) = \prod_{i=1}^{n+1} \left(\frac{1}{r_i} - 1 \right)$. The surface $f^{-1}(0)$ for $n = 2$ is a rational double point if and only if $\sum_{i=1}^3 r_i > 1$.

E. Unfolding Theory

An **unfolding** of a germ of an analytic function $f(z)$ at 0 is a germ of an analytic function $F(z, t)$, where $t \in \mathbf{C}^m$ (m is finite) such that $F(z, 0) = f(z)$. We assume that f has an isolated critical point at 0. Among all the unfoldings of f , there exists a universal one, in a suitable sense, that is unique up to a local analytic isomorphism. It is called the **universal unfolding** of f [36, 37, 26] (\rightarrow 51 Catastrophe Theory). Explicitly it can be given by $F(z, t) = f(z) + t_1 \varphi_1(z) + \dots + t_\mu \varphi_\mu(z)$, where $\varphi_i(z)$ ($i = 1, \dots, \mu$) are holomorphic functions which form a \mathbf{C} -basis of the Jacobi ring $\mathcal{O}_{\mathbf{C}^{n+1}, 0} / (\partial f / \partial z_1, \dots, \partial f / \partial z_{n+1})$ ($\mu = \mu(f)$).

In the universal unfolding $F(z, t)$ of f , the set of points (z_0, t_0) such that $F(z, t_0)$ has an isolated critical point at z_0 with the Milnor number $\mu(f)$ and $F(z_0, t_0) = 0$ forms an analytic set at $(z, t) = 0$. The **modulus number** of f is the dimension of this set at 0. This set is sometimes called the **μ -constant stratum**. Let g be a germ of an analytic function. g is said to be **adjacent** to f (denoted by $f \rightarrow g$), if there exists a sequence of points $(z(m), t(m))$ in $\mathbf{C}^{n+1} \times \mathbf{C}^\mu$ that converges to the origin such that the term of $F(z, t(m))$ at $z(m)$ is equivalent to g . Adjacency relations are important for the

understanding of the degeneration phenomena of functions. The unfolding theory can be considered in exactly the same way as that for the germ of a real-valued smooth function that is finitely determined [36, 26].

The germs of analytic functions with modulus number 0, 1, and 2 are called **simple**, **unimodular**, and **bimodular**, respectively. They were classified by V. I. Arnold [1] (\rightarrow Appendix A, Table 5.V). Simple germs correspond to the equations for the rational double points, and unimodular germs define simply elliptic singularities or cusp singularities. Every unimodular or bimodular germ defines a singularity with $p_g = 1$.

F. Picard-Lefschetz Theory

Let $f(z)$ be a holomorphic function such that $f(0) = 0$ and 0 is an isolated critical point with the Milnor number μ . Let $F(z, t)$ be a universal unfolding of f at 0. Let $f: E(\varepsilon, \delta) \rightarrow D_\delta^*$ be the Milnor fibration of f by the second description in Section D. There exists a positive number r and a codimension 1 analytic subset Δ (called the **bifurcation set**) of $B'(r)$, the open disk of radius r with the center 0 in the parameter space \mathbb{C}^u , such that for any $t_0 \in B'(r) - \Delta$, $f_{t_0} = F|_{B(\varepsilon) \times t_0}$ has μ different nondegenerate critical points in $B(\varepsilon)$. Let p_1, \dots, p_μ be the critical points of f_{t_0} . For each p_i , one can choose local coordinates (y_1, \dots, y_{n+1}) so that $f_{t_0}(y) = f_{t_0}(p_i) + y_1^2 + \dots + y_{n+1}^2$. Such an f_{t_0} is called a **Morsification** of f .

Let B_i be a small disk with center p_i in \mathbb{C}^{n+1} . Then for any q_i which is near enough to $f_{t_0}(p_i)$, the intersection $f_{t_0}^{-1}(q_i) \cap B_i$ is diffeomorphic to the tangent disk bundle of the sphere S^n . The **vanishing cycle** e_i is the corresponding n -dimensional homology class of $f_{t_0}^{-1}(q_i) \cap B_i$. (We fix q_i .) The self-intersection number of e_i is given by

$$\langle e_i, e_i \rangle = \begin{cases} 2(-1)^{n(n-1)/2}, & n \text{ even,} \\ 0, & n \text{ odd.} \end{cases}$$

For a sufficiently small $t_0 \in B'(r) - \Delta$, one has the following: (i) $|f_{t_0}(p_i)| < \delta$; (ii) the restriction of f_{t_0} to E is a fiber bundle over D' , where $D' = \{w \in \mathbb{C} \mid |w| \leq \delta, \text{ and } w \neq f_{t_0}(p_i) \text{ for } i = 1, \dots, \mu\}$ and $E = f_{t_0}^{-1}(D') \cap B(\varepsilon)$; (iii) the restriction of the above fibration to $\{w \mid |w| = \delta\}$ is equivalent to the restriction of the Milnor fibration of f to $\{w \mid |w| = \delta\}$. Let w_0 be a fixed point of D' , and let $F = f_{t_0}^{-1}(w_0) \cap E$. Then F is diffeomorphic to the Milnor fiber of f . Let l_i be a simple path from w_0 to q_i , and let γ_i be the loop $|w - f_{t_0}(p_i)| = |q_i - f_{t_0}(p_i)|$. We suppose that the union of the l_i is contractible to w_0 . By parallel translation of the vanishing cycle e_i along l_i , we consider $e_i \in H_n(F)$. The collection $\{e_i \mid i =$

$1, \dots, \mu\}$ is an integral basis of $H_n(F)$, which is called a **strongly distinguished basis** (Fig. 1).

Now let h_i be the linear transformation of $H_n(F)$ that is induced by the parallel translation along $l_i \gamma_i l_i^{-1}$. The **Picard-Lefschetz formula** says that

$$h_i(e) = e - (-1)^{n(n-1)/2} \langle e, e_i \rangle \cdot e_i \text{ for } e \in H_n(F).$$

Here $\langle \cdot, \cdot \rangle$ is the intersection number in $H_n(F)$. For n even, h_i is a \dagger reflection.

The Milnor monodromy h_* of f is equal to the composition $h_1 \dots h_\mu$ under a suitable ordering of the h_i . The subgroup of the group of linear isomorphisms of $H_n(F)$ generated by h_1, \dots, h_μ is called the **total monodromy group**.

When f is a simple germ and $n \equiv 2 \pmod{4}$, the total monodromy group is isomorphic to the \dagger Weyl group of the corresponding Dynkin diagram. Even-dimensional simple singularities are the only ones for which the monodromy group is finite. These are also characterized as the singularities with definite intersection forms.

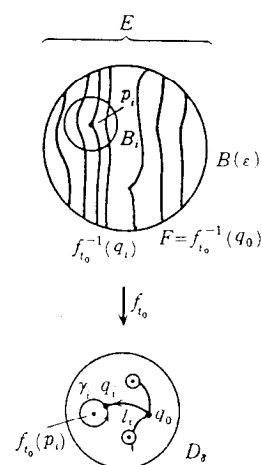


Fig. 1

G. Stratification Theory

The notion of **Whitney stratification** was first introduced by H. Whitney to study the singularities of analytic varieties [39] and was developed by R. Thom for the general case [37].

Let X and Y be submanifolds of the space \mathbb{R}^n . We say that the pair (X, Y) satisfies the **Whitney condition (b)** at a point $y \in Y$ if the following holds: Let $x_i (i = 1, 2, \dots)$ and $y_i (i = 1, 2, \dots)$ be sequences in X and Y , respectively, that converge to y . Suppose that the tangent space $T_{x_i} X$ converges to a plane T in the corresponding Grassmannian space and the secant $\overline{x_i y_i}$ converges to a line L . Then $L \subset T$. We say that (X, Y) satisfies the **Whitney condition (b)** if it satisfies the Whitney con-

dition (b) at any point $y \in Y$. Let h be a local diffeomorphism of a neighborhood of y . One can see that $(h(X), h(Y))$ satisfies the Whitney condition (b) at $h(y)$ if (X, Y) satisfies it at y . Thus the Whitney condition can be considered for a pair of submanifolds X and Y of a manifold M using a local coordinate system. Let S be a subset of a manifold M , and let \mathcal{S} be a family of submanifolds of M . \mathcal{S} is called a **Whitney prestratification** of S if \mathcal{S} is a locally finite disjoint cover of S satisfying the following: (i) For any $X \in \mathcal{S}$, the frontier $\bar{X} - X$ is a union of $Y \in \mathcal{S}$; (ii) for any pair (X, Y) ($X, Y \in \mathcal{S}$), the Whitney condition (b) is satisfied. A submanifold X in \mathcal{S} is called a **stratum**. There exists a canonical partial order in \mathcal{S} that is defined by $X < Y$ if and only if $X \subset \bar{Y} - Y$.

Let V be an analytic variety, and let \mathcal{S} be an analytic stratification of V that satisfies the frontier condition (i). Then there exists a Whitney prestratification \mathcal{S}' that is finer than \mathcal{S} (Whitney [39]).

For a given Whitney prestratification \mathcal{S} , one can construct the following **controlled tubular neighborhood system**: For each $X \in \mathcal{S}$, a \dagger tubular neighborhood $|T_X|$ of X in M and the projection $\pi_X: |T_X| \rightarrow X$ and a tubular function $\rho_X: |T_X| \rightarrow \mathbf{R}^+$ (=the square of a norm under the identification of $|T_X|$ with the \dagger normal disk bundle of X) are given such that the commutation relations

$$\pi_X \cdot \pi_Y(m) = \pi_X(m), \quad \rho_X \pi_Y(m) = \rho_X(m) \\ \text{for } m \in M, X < Y.$$

are satisfied whenever both sides are defined.

By virtue of this, the notions of vector fields and their integral curves can be defined on a Whitney prestratified set so that several important results on a differentiable manifold can be generalized to the case of stratified sets. For example, the following is **Thom's first isotopy lemma**: Let M and P be differentiable manifolds, and let (S, \mathcal{S}) be a Whitney prestratified subset of M . Let $f: S \rightarrow P$ be a continuous mapping that is the restriction of a differentiable mapping from M to P . Suppose that the restriction of f to each stratum X of \mathcal{S} is a proper submersion onto P . Then $f: S \rightarrow P$ is a fiber bundle [37].

H. b -Functions

Let $f(z)$ be a germ of an analytic function in \mathbf{C}^{n+1} with $f(0)=0$. The **b -function** of f at 0 is the monic polynomial $b_f(s)$ of lowest degree among all polynomials $b(s)$ with the following property [4, 20]: There exists a differential operator $P(z, \partial/\partial z, s)$, which is a polynomial in s , such that $b(s)f^*(z) = P(z, \partial/\partial z, s)f^{s+1}(z)$. Since $b_f(s)$ is always

418 Ref. Theory of Singularities

divisible by $s+1$, we define $\tilde{b}_f(s) = b_f(s)/(s+1)$. All the roots of $\tilde{b}_f(s)=0$ are negative rational numbers (M. Kashiwara [20]). When f has an isolated critical point at 0, the set $\{\exp(2\pi i \alpha) | \alpha \text{ is a root of } b_f(s)=0\}$ coincides with the set of eigenvalues of the Milnor monodromy [25].

The name " b -function" is due to M. Sato. He first introduced it in the study of \dagger prehomogeneous vector spaces. Some authors call it the **Bernstein (Bernshtein) polynomial**.

I. Hyperplane Sections

Let V be an algebraic variety of complex dimension k in the complex projective space \mathbf{P}^n . Let L be a hyperplane that contains the singular points of V . Then the \dagger relative homotopy group $\pi_i(V, V \cap L)$ is zero for $i < k$. Thus the same assertion is true for the \dagger relative homology groups (S. Lefschetz [24]; [28]).

Let f be a holomorphic function defined in the neighborhood of $0 \in \mathbf{C}^{n+1}$ and $f(0)=0$. Let H be the hypersurface $f^{-1}(0)$. There exists a \dagger Zariski open subset U of the space $(=\mathbf{P}^n)$ of hyperplanes such that for each $L \in U$, there exists a positive number ε such that $\pi_i(B(r) - H, (B(r) - H) \cap L) = 0$ for $i < n$ and $0 < r \leq \varepsilon$, where $B(r)$ is a disk of radius r (D. T. Lê and H. Hamm [15]). This implies the following theorem of Zariski: Let V be a hypersurface of \mathbf{P}^n , and let \mathbf{P}^2 be a general plane in \mathbf{P}^n . Then the fundamental group of $\mathbf{P}^n - V$ is isomorphic to the fundamental group of $\mathbf{P}^2 - C$, where $C = V \cap \mathbf{P}^2$. The fundamental group of $\mathbf{P}^2 - C$ is an Abelian group if C is a nodal curve [9, 13].

Suppose that f has an isolated critical point at 0. Let $\mu^{(n+1)}$ be the Milnor number $\mu(f)$. Take a generic hyperplane L . The Milnor number of $f|_L$ is well defined, and we let $\mu^{(n)} = \mu(f|_L)$. Similarly one can define $\mu^{(i)}$ of f and let $\mu^* = (\mu^{(n+1)}, \mu^{(n)}, \dots, \mu^{(1)})$. Let $f_t(z)$ be a deformation of f . Each f_t has an isolated critical point at 0, and t is a point of a disk D of the complex plane. Let $W = \{(z, t) | f_t(z) = 0\}$ and $D' = \{0\} \times D$. $W - D'$ and D' satisfy the Whitney condition (b) if and only if $\mu^*(f_t)$ is invariant under the deformation [35]. The Whitney condition (b) implies topological triviality of the deformation.

References

- [1] V. I. Arnol'd, Local normal forms of functions, *Inventiones Math.*, 35 (1976), 87–109.
- [2] V. I. Arnol'd, Normal forms for functions near degenerate critical points, the Weyl group of A_k, D_k, E_k and Lagrangian singularities, *Functional Anal. Appl.*, 6 (1972), 254–272. (Original in Russian, 1972.)

- [3] M. Artin, On isolated rational singularities of surfaces, *Amer. J. Math.*, 88 (1966), 129–136.
- [4] I. N. Bernshtein, The analytic continuation of generalized functions with respect to parameter, *Functional Anal. Appl.*, 6 (1972), 273–285. (Original in Russian, 1972.)
- [5] K. Brauer, Zur Geometrie der Funktionen zweier komplexen Veränderlichen III, *Abh. Math. Sem. Hamburg*, 6 (1928), 8–54.
- [6] E. Brieskorn, Rationale Singularitäten Komplexer Flächen, *Inventiones Math.*, 4 (1968), 336–358.
- [7] E. Brieskorn, Singular elements of semi-simple algebraic groups, *Actes Congrès Int. Math.*, 1970, vol. 2, 279–284.
- [8] E. Brieskorn, Beispiele zur Differentialtopologie, *Inventiones Math.*, 2 (1966), 1–14.
- [9] P. Deligne, Le groupe fondamental du complément d'une courbe plane n'ayant que des points doubles ordinaires est abélien, *Sém. Bourbaki*, no. 543, 1979–1980.
- [10] H. Durfee, Fibered knots and algebraic singularities, *Topology*, 13 (1974), 47–59.
- [11] A. H. Durfee, Fifteen characterizations of rational double points and simple critical points, *Enseignement Math.*, 25 (1979), 131–163.
- [12] P. Du Val, On isolated singularities of surfaces which do not affect the condition of adjunction I–III, *Proc. Cambridge Philos. Soc.*, 30 (1933/34), 453–465, 483–491.
- [13] W. Fulton, On the fundamental group of the complement of a node curve, *Ann. Math.*, (2) 111, (1980), 407–409.
- [14] H. Grauert, Über Modifikationen und exzeptionelle analytische Mengen, *Math. Ann.*, 146 (1962), 331–368.
- [15] H. Hamm and D. T. Lê, Un théorème de Zariski du type de Lefschetz, *Ann. Sci. Ecole Norm. Sup.*, 3, 1973.
- [16] H. Hironaka, Resolution of singularities of an algebraic variety over a field of characteristic zero, *Ann. Math.*, (2) 79 (1964), 109–326.
- [17] H. Hironaka, Bimeromorphic smoothing of a complex analytic space, *Acta Math. Vietnamica*, 2, no. 2 (1977), 106–168.
- [18] F. Hirzebruch, Hilbert modular surfaces, *Enseignement Math.*, 19 (1973), 183–281.
- [19] P. Holm, Real and complex singularities, *Sijthoff and Noordhoff*, 1976.
- [20] M. Kashiwara, B -functions and holonomic systems, *Inventiones Math.*, 38 (1976), 33–54.
- [21] M. Kato, A classification of simple spinable structure on a 1-connected Alexander manifold, *J. Math. Soc. Japan*, 26 (1974), 454–463.
- [22] A. G. Kouchnirenko, Polyèdre de Newton et nombres de Milnor, *Inventiones Math.*, 32 (1976), 1–31.
- [23] H. Laufer, On minimally elliptic singularities, *Amer. J. Math.*, 99 (1977), 1257–1295.
- [24] S. Lefschetz, *L'analyse situs et la géométrie algébrique*, Gauthiers-Villars, 1924.
- [25] B. Malgrange, Le polynôme de Bernstein d'une singularité isolée, *Lecture notes in math.* 459, Springer, 1976, 98–119.
- [26] J. N. Mather, Stability of C^∞ -mappings. I, *Ann. Math.*, (2) 87 (1968), 89–104; II, (2) 89 (1969), 254–291; III, *Publ. Math. Inst. HES*, 35 (1970), 301–336; IV, *Lecture notes in math.* 192, Springer, 1971, 207–253.
- [27] J. W. Milnor, Singular points of complex hypersurfaces, *Ann. math. studies* 61, Princeton Univ. Press, 1968.
- [28] J. W. Milnor, Morse theory, *Ann. math. studies* 51, Princeton Univ. Press, 1963.
- [29] D. Mumford, The topology of normal singularities of an algebraic surface and a criterion for simplicity, *Publ. Math. Inst. HES*, 9 (1961), 5–22.
- [30] F. Pham, Formules de Picard-Lefschetz généralisées et ramification des intégrales, *Bull. Soc. Math. France*, 93 (1965), 333–367.
- [31] E. Picard and S. Simart, *Traité des fonctions algébriques de deux variables I*, Gauthier-Villars, 1897.
- [32] K. Saito, Quasihomogene isolierte Singularitäten von Hyperflächen, *Inventiones Math.*, 14 (1971), 123–142.
- [33] K. Saito, Einfach elliptische Singularitäten, *Inventiones Math.*, 23 (1974), 289–325.
- [34] P. Slodowy, Simple singularities and simple algebraic groups, *Lecture notes in math.* 815, Springer, 1980.
- [35] B. Teissier, Cycles evanescents, sections planes et conditions de Whitney, *Astérisque*, 78 (1973), 285–362.
- [36] R. Thom, La stabilité topologique des applications polynomiales, *Enseignement Math.*, 8 (1962), 24–33.
- [37] R. Thom, Ensembles et morphismes stratifiés, *Bull. Amer. Math. Soc.*, 75 (1969), 240–284.
- [38] A. N. Varchenko, Zeta function of the monodromy and Newton's diagram, *Inventiones Math.*, 37 (1976), 253–262.
- [39] H. Whitney, Tangents to an analytic variety, *Ann. Math.*, (2) 81 (1964), 496–549.
- [40] O. Zariski, Algebraic surfaces, *Erg. math.* 61, Springer, 1971.

419 (XX.18) Thermodynamics

A. Basic Concepts and Postulates

Thermodynamics traditionally focuses its attention on a particular class of states of a

given system called (thermal) equilibrium states, although a more recent extension, called the thermodynamics of irreversible processes, deals with certain nonequilibrium states. In a simple system, an **equilibrium state** is completely specified (up to the shape of the volume it occupies) by the volume V (a positive real number), the **mole numbers** N_1, \dots, N_r (nonnegative reals) of its chemical components, and the **internal energy** U (real). (More variables might be needed if the system were, e.g., inhomogeneous, anisotropic, electrically charged, magnetized, chemically not inert, or acted on by electric, magnetic, or gravitational fields.) This means that any of the quantities associated with equilibrium states (called **thermodynamical quantities**) of a simple system under consideration is a function of V , N_1, \dots, N_r , and U .

When n copies of the same state are put next to each other and the dividing walls are removed, V , N_1, \dots, N_r , and U for the new state will be n times the old values of these variables under the assumptions that each volume is sufficiently large and that the effects of the boundary walls can be neglected. Thermodynamical quantities behaving in this manner are called **extensive**. Those that are invariant under the foregoing procedure are called **intensive**. More precisely, the thermodynamic variables are defined by homogeneity of degree 1 and 0 as functions of V , N_1, \dots, N_r , and U .

By a shift of the position of the boundary (called an **adiabatic wall** if energy and chemical substances do not move through it) or by transport of energy through the boundary (called a **diathermal wall** if this is allowed) or by transport of chemical components through the boundary (called a **permeable membrane**) (in short, by thermodynamical processes), these variables can change their values. If these shifts or transports are not permitted (especially for a composite system consisting of several simple systems, at its boundary with the outside), the system is called **closed**. Otherwise it is called **open**.

Those equilibrium states that do not undergo any change when brought into contact with each other across an immovable and impermeable diathermal wall (called a **thermal contact**) form an equivalence class. This is sometimes called the **0th law of thermodynamics**. The equivalence class, called the **temperature** of states belonging to it, is an intensive quantity.

The force needed to keep a movable wall at rest, divided by the area of the wall, is called the **pressure**. It is another intensive quantity. For a (slow) change of the volume by an amount dV under a constant pressure P , mechanical work of amount $-PdV$ is done on

the system. Together with a possible change of the internal energy, say of amount dU , the amount

$$\delta Q = dU - PdV \quad (1)$$

of energy is somehow gained (if it is positive) or lost (if it is negative) by the system. This amount of energy is actually transported from or to a neighboring system through diathermal walls so that the total energy for a bigger closed composite system is conserved. This is called the **first law of thermodynamics**, and δQ is called the **heat gain** or **loss** by the system.

If two states of different temperatures T_1 and T_2 are brought into thermal contact, energy is transferred from one, say T_1 , to the other (called heat transfer). This defines a binary class relation denoted by $T_1 > T_2$. The Clausius formulation of the **second law of thermodynamics** says that it is impossible to make a positive heat transfer from a state of lower temperature to another state of higher temperature without another change elsewhere. By considering a certain composite system, one reaches the conclusion that there exists a labeling of temperatures by positive real numbers T , called the **absolute temperature**, for which the following is an exact differential:

$$\delta Q/T = (dU - PdV)/T = dS. \quad (2)$$

The integral S is an extensive quantity, called the **entropy**. Furthermore, the sum of the entropies of component simple systems in an isolated composite system is nondecreasing during any thermodynamic process, and the following **entropy maximum principle** holds: An isolated composite system reaches an equilibrium at those values of extensive parameters that maximize the sum of the entropies of component simple systems (for constant total energy and volume and within the set of allowed states under a given constraint).

A relation expressing the entropy of a given system as a function of the extensive parameters (specifying equilibrium states) is known as the **fundamental relation** of the system. If it is given as a continuous and differentiable homogeneous function of V , N_1, \dots, N_r , and U and is monotone increasing in U for fixed V , N_1, \dots, N_r , then one can develop the thermodynamics of the system based on the above entropy maximum principle. A relation expressing an intensive parameter as a function of some other independent variables is called an **equation of state**.

Another postulate, which is much less frequently used, is the **Nernst postulate** or the **third law of thermodynamics**, which says that the entropy vanishes at the vanishing absolute temperature.

B. Various Coefficients and Relationships

The partial derivative $\partial/\partial x$ of a function $f(x, y, \dots)$ with respect to the variable x with the variables y, \dots fixed is denoted by $(\partial f/\partial x)_{y, \dots}$. We abbreviate N_1, \dots, N_r as N in the following.

If the fundamental relation is written as $U = U(V, N_1, \dots, N_r, S)$ (instead of S being represented as a function of the other quantities), then (2) implies

$$(\partial U/\partial S)_{V, N} = T, \quad (\partial U/\partial V)_{S, N} = -P.$$

The other first-order partial derivatives of U are

$$\mu_j = (\partial U/\partial N_j)_{V, N_1, \dots, \hat{N}_j, \dots, N_r, S},$$

with μ_j called the **chemical potential** (or electrochemical potential) of the j th component.

If a system is surrounded by an adiabatic wall (i.e., the system is thermally isolated) and goes through a gradual reversible change (**quasistatic adiabatic process**), then the entropy has to stay constant. If a system is in thermal contact through a diathermal wall with a large system (called the **heat bath**) whose temperature is assumed to remain unchanged during the thermal contact, then the temperature of the system itself remains constant (an **isothermal process**). The decrease of the volume per unit increase of pressure under the latter circumstance is called the **isothermal compressibility** and is given by

$$\kappa_T = -V^{-1}(\partial V/\partial P)_{T, N}.$$

Under constant pressure, the increase of the volume per unit increase of the temperature is called the **coefficient of thermal expansion** and is given by

$$\alpha = V^{-1}(\partial V/\partial T)_{P, N}.$$

Under constant pressure, the amount of (quasistatic) heat transfer into the system per mole required to produce a unit increase of temperature is called the **specific heat at constant pressure** and is given by

$$c_P = N^{-1}T(\partial S/\partial T)_{P, N},$$

where $N = N_1 + \dots + N_r$. The same quantity under constant volume is called the **specific heat at constant volume** and is given by

$$c_V = N^{-1}T(\partial S/\partial T)_{V, N}.$$

The positivity of c_V is equivalent to the convexity of energy as a function of entropy for fixed values of V and N .

Because of the first-order homogeneity of an extensive quantity as a function of other extensive variables, one can derive an **Euler**

relation, such as

$$U = TS - PV + \mu_1 N_1 + \dots + \mu_r N_r,$$

for a simple system. Its differential form implies the following **Gibbs-Duhem relation**:

$$SdT - VdP + N_1 d\mu_1 + \dots + N_r d\mu_r = 0.$$

Because of the identity

$$\left(\frac{\partial}{\partial x}\right)\left(\frac{\partial f}{\partial y}\right) = \left(\frac{\partial}{\partial y}\right)\left(\frac{\partial f}{\partial x}\right),$$

there arise relationships among second derivatives, known as the **Maxwell relations**:

$$(\partial T/\partial V)_{S, N} = -(\partial P/\partial S)_{V, N},$$

$$(\partial V/\partial S)_{P, N} = (\partial T/\partial P)_{S, N},$$

$$(\partial S/\partial V)_{T, N} = (\partial P/\partial T)_{V, N},$$

$$(\partial S/\partial P)_{T, N} = -(\partial V/\partial T)_{P, N}.$$

By computing the Jacobian of transformations of variables, further relations can be obtained.

For example,

$$c_P = c_V + N^{-1}TV\alpha^2/\kappa_T.$$

C. Legendre Transform and Variational Principles

The **Legendre transform** of a function $f(x_1, \dots, y_1, \dots)$ relative to the variables x is given by

$$g(p_1, \dots, y_1, \dots) = f - \sum_j x_j p_j$$

as a function of the variables $p_j = \partial f/\partial x_j$ and y . The original variables x can be recovered as $-x_j = \partial g/\partial p_j$.

In terms of Legendre transforms, the entropy maximum principle can be reformulated in various forms:

Energy minimum principle: For given values of the total entropy and volume, the equilibrium is reached at those values of unconstrained parameters that minimize the total energy. This principle is applicable in reversible processes where the total entropy stays constant.

Helmholtz free energy minimum principle: For given values of the temperature (equal to that of a heat bath in thermal contact with the system) and the total volume, the equilibrium is reached at those values of the unconstrained parameters that minimize the total Helmholtz free energy, where the **Helmholtz free energy** for a simple system is defined as a function of T, V, N_1, \dots, N_r by

$$F = U - TS,$$

$$dF = -SdT - PdV + \mu_1 dN_1 + \mu_r dN_r.$$

Enthalpy minimum principle: For given values of the pressure and the total entropy,

the equilibrium is reached at those values of unconstrained parameters that minimize the total enthalpy, where the **enthalpy** for a simple system is defined as a function of S, P, N_1, \dots, N_r by

$$H = U + PV,$$

$$dH = TdS + VdP + \mu_1 dN_1 + \dots + \mu_r dN_r.$$

Gibbs free energy minimum principle: For constant temperature and pressure, the equilibrium is reached at those values of unconstrained parameters that minimize the total Gibbs free energy, where the **Gibbs free energy** for a simple system is given as a function of T, P, N_1, \dots, N_r by

$$G = U - TS + PV,$$

$$dG = -SdT + VdP + \mu_1 dN_1 + \dots + \mu_r dN_r.$$

References

[1] M. W. Zemansky, Heat and thermodynamics, McGraw-Hill, 1951.
[2] H. B. Callen, Thermodynamics, Wiley, 1960.
[3] J. W. Gibbs, The collected works of J. Willard Gibbs, vol. 1, Thermodynamics, Yale Univ. Press, 1948.
[4] M. Planck, Treatise on thermodynamics, Dover, 1945. (Original in German, seventh edition, 1922.)
[5] E. A. Guggenheim, Thermodynamics, North-Holland, 1949.
[6] F. C. Andrews, Thermodynamics: Principles and applications, Wiley, 1971.

420 (XX.8)
Three-Body Problem

A. n -Body Problem and Classical Integrals

In the n -body problem, we study the motions of n particles $P_i(x_i, y_i, z_i)$ ($i = 1, 2, \dots, n$) with arbitrary masses m_i (> 0) following ^{*}Newton's law of motion,

$$m_i \frac{d^2 w_i}{dt^2} = \frac{\partial U}{\partial w_i}, \quad i = 1, 2, \dots, n, \tag{1}$$

where w_i is any one of x_i, y_i , or z_i ,

$$U = \sum_{i \neq j} k^2 m_i m_j / r_{ij},$$

with k^2 the gravitation constant, and

$$r_{ij} = \sqrt{(x_i - x_j)^2 + (y_i - y_j)^2 + (z_i - z_j)^2}.$$

Although the one-body and two-body problems have been completely solved, the prob-

420 C
Three-Body Problem

lem has not been solved for $n > 2$. The **three-body problem** is well known and is important both in celestial mechanics and in mathematics. For $n > 3$ the problem is called the **many-body problem**.

The equations (1) have the so-called ten classical integrals, that is, the **energy integral** $\sum_i (m_i/2)((\dot{x}_i)^2 + (\dot{y}_i)^2 + (\dot{z}_i)^2) - U = \text{constant}$ ($\dot{w} = dw/dt$), six **integrals of the center of mass** $\sum_i m_i \dot{w}_i = \text{constant}$, $\sum_i m_i w_i = (\sum_i m_i \dot{w}_i)t + \text{constant}$, and three **integrals of angular momentum** $\sum_i m_i (u_i \dot{w}_i - w_i \dot{u}_i) = \text{constant}$ ($u \neq w$). Using these integrals and eliminating the time t and the ascending node by applying Jacobi's method, the order of the equations (1) can be reduced to $6n - 12$. H. Bruns proved that algebraic integrals cannot be found except for the classical integrals, and H. Poincaré showed that there is no other single-valued integral (Bruns, *Acta Math.*, 11 (1887); Poincaré [2, I, ch. 5]). These results are called **Poincaré-Bruns theorems**. Therefore we cannot hope to obtain general solutions for the equations (1) by quadrature. General solutions for $n \geq 3$ have not been discovered except for certain specific cases.

B. Particular Solutions

Let r_i be the position vector of the particle P_i with respect to the center of mass of the n -body system. A configuration $r \equiv \{r_1, \dots, r_n\}$ of the system is said to form a **central figure** (or **central configuration**) if the resultant force acting on each particle P_i is proportional to $m_i r_i$, where each proportionality constant is independent of i . The proportionality constant is uniquely determined as $-U/\sum_{i=1}^n m_i r_i^2$ by the configuration of the system. A configuration r is a central figure if and only if r is a ^{*}critical point of the mapping $r \mapsto U^2(r) \sum_{i=1}^n m_i r_i^2$ [5, 6]. A rotation of the system, in planar central figure, with appropriate angular velocity is a particular solution of the planar n -body problem.

Particular solutions known for the three-body problem are the **equilateral triangle solution** of Lagrange and the **straight line solution** of Euler. They are the only solutions known for the case of arbitrary masses, and their configuration stays in the central figure throughout the motion.

C. Domain of Existence of Solutions

The solutions for the three-body problem are analytic, except for the collision case, i.e., the case where $\min r_{ij} = 0$, in a strip domain enclosing the real axis of the t -plane (Poincaré, P.

Painlevé). K. F. Sundman proved that when two bodies collide at $t = t_0$, the solution is expressed as a power series in $(t - t_0)^{1/3}$ in a neighborhood of t_0 , and the solution which is real on the real axis can be uniquely and analytically continued across $t = t_0$ along the real axis. When all three particles collide, the total angular momentum f with respect to the center of mass must vanish (and the motion is planar) (**Sundman's theorem**); so under the assumption $f \neq 0$, introducing $s = \int^t (U + 1) dt$ as a new independent variable and taking it for granted that any binary collision is analytically continued, we see that the solution of the three-body problem is analytic on a strip domain $|\operatorname{Im} s| < \delta$ containing the real axis of the s -plane. The conformal mapping

$$\omega = (\exp(\pi s / 2\delta) - 1) / (\exp(\pi s / 2\delta) + 1)$$

maps the strip domain onto the unit disk $|\omega| < 1$, where the coordinates of the three particles w_k , their mutual distances r_{kl} , and the time t are all analytic functions of ω and give a complete description of the motion for all real time (Sundman, *Acta Math.*, 36 (1913); Siegel and Moser [7]).

When a triple collision occurs at $t = t_0$, G. Biscocini, Sundman, H. Block, and C. L. Siegel showed that as $t \rightarrow t_0$, (i) the configuration of the three particles approaches asymptotically the Lagrange equilateral triangle configuration or the Euler straight line configuration, (ii) the collision of the three particles takes place in definite directions, and (iii) in general the triple-collision solution cannot be analytically continued beyond $t = t_0$.

D. Final Behavior of Solutions

Suppose that the center of mass of the three-body system is at rest. The motion of the system was classified by J. Chazy into seven types according to the asymptotic behavior when $t \rightarrow +\infty$, provided that the angular momentum f of the system is different from zero. In terms of the † order of the three mutual distances r_{ij} (for large t) these types are defined as follows:

- (i) H^+ : **Hyperbolic motion**. $r_{ij} \sim t$.
- (ii) HP^+ : **Hyperbolic-parabolic motion**. $r_{13}, r_{23} \sim t$ and $r_{12} \sim t^{2/3}$.
- (iii) HE^+ : **Hyperbolic-elliptic motion**. $r_{13}, r_{23} \sim t$ and $r_{12} < a$ ($a = \text{finite}$).
- (iv) P^+ : **Parabolic motion**. $r_{ij} \sim t^{2/3}$.
- (v) PE^+ : **Parabolic-elliptic motion**. $r_{13}, r_{23} \sim t^{2/3}$ and $r_{12} < a$.
- (vi) L^+ : **Lagrange-stable motion or bounded motion**. $r_{ij} < a$.
- (vii) OS^+ : **Oscillating motion**. $\overline{\lim}_{t \rightarrow \infty} \sup r_{ij} = \infty$, $\underline{\lim}_{t \rightarrow \infty} \sup r_{ij} < \infty$.

Define H^- , HE^- , etc. analogously but with $t \rightarrow -\infty$. There are three classes for each of the motions HP, HE, and PE, depending on which of the three bodies separates from the other two bodies and recedes to infinity, denoted by HP_i , HE_i , PE_i ($i = 1, 2, 3$), respectively. The energy constant h is positive for H^- and HP^- motion, zero for P^- motion, and negative for PE^- , L^- , and OS^- motion. For HE^- motion, h may be positive, zero, or negative.

We say that a **partial capture** takes place when the motion is H^- for $t \rightarrow -\infty$ and HE_i^+ for $t \rightarrow +\infty$ (for $h > 0$), and a **complete capture** when the motion is HE_i^- for $t \rightarrow -\infty$ and L^+ for $t \rightarrow +\infty$ (for $h < 0$). We say also that an **exchange** takes place when HE_i^- for $t \rightarrow -\infty$ and HE_j^+ for $t \rightarrow +\infty$ ($t \neq j$). The probability of complete capture in the domain $h < 0$ is zero (J. Chazy, G. A. Merman).

E. Perturbation Theories

The radius of convergence in the s -plane for Sundman's solution is too small and the convergence is too slow in the ω -plane to make it possible to compute orbits of celestial bodies, and for that purpose a perturbation method is usually adopted. When the masses m_2, \dots, m_n are negligibly small compared with m_1 for the n -body problem, the motion of the n th body is derived as the solution of the two-body problem for m_1 and m_n by assuming $m_2 = \dots = m_{n-1} = 0$ as a first approximation, and then the deviations of the true orbit from the ellipse are derived as † perturbations. In the **general theory of perturbations** the deviations are derived theoretically by developing a disturbing function, whereas in the **special theory of perturbations** they are computed by numerical integration. In general perturbation theory, problems concerning convergence of the solution are important, and it becomes necessary to simplify the disturbing function in dealing with the actual relations among celestial bodies. Specific techniques have to be developed in order to compute perturbations for lunar motion, motions of characteristic asteroids, and motions of satellites (e.g., the system of the Sun, Jupiter, and Jovian satellites).

F. The Restricted Three-Body Problem

Since the three-body problem is very difficult to handle mathematically, mathematical interest has been concentrated on the **restricted three-body problem** (in particular, the planar problem) since Hill studied lunar theory in the 19th century. For the restricted three-body problem, the third body, of zero mass, cannot have any influence on the motion of the other

two bodies, which are of finite masses and which move uniformly on a circle around the center of mass. In the planar case, let us choose units so that the total mass, the angular velocity of the two bodies about their center of mass, and the gravitation constant are all equal to 1, and let (q_1, q_2) be the coordinates of the third body with respect to a rotating coordinate system chosen in such a way that the origin is at the center of mass and the two bodies of finite masses μ and $1-\mu$ are always fixed on the q_1 -axis. Then the equations of motion for the third body are given by a Hamiltonian system:

$$\frac{dq_i}{dt} = \frac{\partial H}{\partial p_i}, \quad \frac{dp_i}{dt} = -\frac{\partial H}{\partial q_i}, \quad i = 1, 2, \quad (2)$$

with

$$H = \frac{1}{2}(p_1^2 + p_2^2) + q_2 p_1 - q_1 p_2 - U(q_1, q_2),$$

$$U = \frac{1-\mu}{\sqrt{(q_1+\mu)^2 + q_2^2}} + \frac{\mu}{\sqrt{(q_1+\mu-1)^2 + q_2^2}}.$$

The equations (2) have the energy integral $H(p, q) = \text{constant}$, called **Jacobi's integral**. Siegel showed that there is no other algebraic integral, and it can be proved by applying Poincaré's theorem that there is no other single-valued integral. Regularization of the two singular points for the equations (2) and solutions passing through the singular points were studied by T. Levi-Civita, and solutions tending to infinity were studied by B. O. Koopman.

After reducing the number of variables by means of the Jacobi integral, the equations (2) give rise to a flow in a 3-dimensional manifold of which the topological type was clarified by G. D. Birkhoff (*Rend. Circ. Mat. Palermo*, 39 (1915)). Since this flow has an invariant measure, the equations have been studied topologically, and important results for the restricted three-body problem, particularly on periodic solutions, have been obtained.

G. Stability of Equilateral Triangular Solutions

Suppose that the origin $q_i = p_i = 0$ is an equilibrium point for an autonomous Hamiltonian system with two degrees of freedom:

$$\frac{dq_i}{dt} = \frac{\partial H}{\partial p_i}, \quad \frac{dp_i}{dt} = -\frac{\partial H}{\partial q_i}, \quad i = 1, 2,$$

with the Hamiltonian H being analytic at the origin. When the eigenvalues of the corresponding linearized system are purely imaginary and distinct, denoted by $\pm\lambda_1$, $\pm\lambda_2$, and $\lambda_1 k_1 + \lambda_2 k_2 \neq 0$ for $0 < |k_1| + |k_2| \leq 4$ (where k_i is an integer), we can find suitable coordinates

ξ_i, η_i so that the Hamiltonian H takes the form

$$H = \lambda_1 \xi_1 + \lambda_2 \xi_2 + \frac{1}{2}(c_{11} \xi_1^2 + 2c_{12} \xi_1 \xi_2 + c_{22} \xi_2^2) + H_5 + \dots$$

with $\xi_i \equiv \xi_i \eta_i$ and real c_{ij} . It is necessary that $\eta_i = \sqrt{-1} \bar{\xi}_i$ for the solutions to be real. In addition, if the condition

$$D \equiv c_{11} \lambda_2^2 - 2c_{12} \lambda_1 \lambda_2 + c_{22} \lambda_1^2 \neq 0$$

is satisfied, then the origin is a †stable equilibrium point of the original system (V. I. Arnol'd, J. Moser) [7].

For Lagrange equilateral triangular solutions of the planar restricted three-body problem, the eigenvalues λ of the linearized system derived from (2) are given as roots of the equation $\lambda^4 + \lambda^2 + (27/4)\mu(1-\mu) = 0$ and are purely imaginary if $\mu(1-\mu) < 1/27$. Applying the Arnol'd-Moser result, A. M. Leontovich and A. Deprit and Bartholomé showed that the Lagrange equilibrium points are stable for μ such that $0 < \mu < \mu_0$, where μ_0 is the smaller root of $27\mu(1-\mu) = 1$, excluding three values: μ_1, μ_2 at which $\lambda_1 k_1 + \lambda_2 k_2 = 0$ $|k_1| + |k_2| \leq 4$ and μ_3 at which $D = 0$.

Arnol'd proved that if the masses m_2, \dots, m_n are negligibly small in comparison with m_1 , the motion of the n -body system is †quasi-periodic for the majority of initial conditions for which the eccentricities and inclinations of the osculating ellipses are small.

References

- [1] Y. Hagihara, *Celestial mechanics. I, Dynamical principles and transformation theory*, MIT Press, 1970; *II, Perturbation theory*, pts. 1, 2, MIT 1972; *III, Differential equations in celestial mechanics*, pts. 1, 2, Japan Society for the Promotion of Science, 1974; *IV, Periodic and quasi-periodic solutions*, pts. 1, 2, Japan Society, 1975; *V, Topology of the three-body problem*, pts. 1, 2, Japan Society, 1976.
- [2] H. Poincaré, *Les méthodes nouvelles de la mécanique céleste I-III*, Gauthier-Villars, 1892-1899.
- [3] E. T. Whittaker, *A treatise on the analytical dynamics of particles and rigid bodies*, Cambridge Univ. Press, fourth edition, 1937.
- [4] H. Happel, *Das Dreikörperproblem*, Koehler, 1941.
- [5] A. Wintner, *The analytical foundations of celestial mechanics*, Princeton Univ. Press, 1947.
- [6] R. Abraham and J. E. Marsden, *Foundations of mechanics*, Benjamin, second edition, 1978.
- [7] C. L. Siegel and J. Moser, *Lectures on celestial mechanics*, Springer, 1971.

[8] V. M. Alekseev (Alexeyev), Sur l'allure finale de mouvement dans le problème des trois corps, Actes Congrès Intern. Math., 1970, vol. 2, 893–907.

[9] V. Szebehely, Theory of orbits, Academic Press, 1967.

421 (XVIII.11) Time Series Analysis

A. Time Series

A time series is a sequence of observations ordered in time. Here we assume that measurements are quantitative and the times of measurements are equally spaced. We consider this sequence to be a realization of a stochastic process X_t (\rightarrow 407 Stochastic Processes). Usually time series analysis means a statistical analysis based on samples drawn from a stationary process (\rightarrow 395 Stationary Processes) or a related process. In what follows we denote the sample by $\mathbf{X} = (X_1, X_2, \dots, X_T)$.

B. Statistical Inference of the Autocorrelation

Let us assume X_t (t an integer) to be real-valued and weakly stationary (\rightarrow 395 Stationary Processes) and for simplicity $EX_t = 0$ and consider the estimation of the **autocorrelation** $\rho_h = R_h/R_0$ of time lag h , where $R_t = EX_t X_{t+1}$. We denote the **sample autocovariance** of time lag h as

$$\tilde{R}_h = \frac{1}{T-|h|} \sum_{t=1}^{T-|h|} X_t X_{t+|h|},$$

and define the **serial correlation coefficient** of time lag h by $\tilde{\rho}_h = \tilde{R}_h/\tilde{R}_0$. It can be shown that the joint distribution of $\{\sqrt{T}(\tilde{\rho}_h - \rho_h) \mid 1 \leq h \leq H\}$ tends to an H -dimensional \dagger normal (Gaussian) distribution with mean vector $\mathbf{0}$, if one assumes that X_t is expressed as $X_t = \sum_{j=-\infty}^{\infty} b_j \xi_{t-j}$, where $\sum_{j=-\infty}^{\infty} |b_j| < +\infty$, $\sum_{j=-\infty}^{\infty} |j|^{1/2} b_j^2 < +\infty$, and the ξ_t are independently and identically distributed random variables with $E\xi_t = 0$ and $E\xi_t^4 < +\infty$.

When X_t is an autoregressive process of order K (\rightarrow Section D) and also a \dagger Gaussian process, it can be shown that the asymptotic distribution of $\{\sqrt{T}(\tilde{\rho}_h - \rho_h) \mid 1 \leq h \leq K\}$ as $T \rightarrow \infty$ is equal to the asymptotic distribution of $\{\sqrt{T}(\hat{\rho}_h - \rho_h) \mid 1 \leq h \leq K\}$, where $\hat{\rho}_h$ is the \dagger maximum likelihood estimator of ρ_h . In general, it is difficult to obtain the maximum likelihood estimator of ρ_h . The statistical properties of other estimators of ρ_h , e.g., an estimator constructed by using $\text{sgn}(X_t)$ ($\text{sgn}(y)$ means

1 ($y > 0$), 0 ($y = 0$), -1 ($y < 0$)) have also been investigated.

Testing hypotheses concerning autocorrelation can be carried out by using the above results. Let us now consider the problem of testing the hypothesis that X_t is a \dagger white noise. Assume that X_t is a Gaussian process and that a white noise with $EX_t^2 = \sigma^2$ exists, and define $\tilde{C}_h = \sum_{t=1}^T (X_t - \bar{X})(X_{t+h} - \bar{X})$ and $\tilde{\gamma}_h = \tilde{C}_h/\tilde{C}_0$ for $h \geq 0$, where $X_{T+j} = X_j$ and $\bar{X} = \sum_{t=1}^T X_t/T$. Then the probability density function of $\tilde{\gamma}_1$ can be obtained and it can be shown that

$$P(\tilde{\gamma}_1 > \gamma) = \sum_{j=1}^m (\lambda_j - \gamma)^{(T-3)/2} \frac{1}{\Lambda_j}, \quad \lambda_{m+1} \leq \gamma \leq \lambda_m,$$

where $\lambda_j = \cos 2\pi j/T$ and

$$\Lambda_j = \prod_{\substack{k=1 \\ (k \neq j)}}^{(T-1)/2} (\lambda_j - \lambda_k), \quad T = 3, 5, \dots,$$

$$\Lambda_j = \prod_{\substack{k=1 \\ (k \neq j)}}^{T/2-1} (\lambda_j - \lambda_k) \sqrt{1 + \lambda_j}, \quad T = 4, 6, \dots,$$

$$1 \leq m \leq (T-3)/2 \quad \text{if } T \text{ is odd,}$$

$$1 \leq m \leq T/2 - 1 \quad \text{if } T \text{ is even.}$$

This can be used to obtain a test of significance.

C. Statistical Inference of the Spectrum

To find the periodicities of a real-valued \dagger weakly stationary process X_t with mean 0, the statistic, called the **periodogram**,

$$I_T(\lambda) = \left| \frac{1}{\sqrt{T}} \sum_{t=1}^T X_t e^{-2\pi i t \lambda} \right|^2$$

is used. If X_t is expressed as

$$X_t = \sum_{l=1}^L \{m_l \cos 2\pi \lambda_l t + m'_l \sin 2\pi \lambda_l t\} + Y_t,$$

where $\{m_l\}$, $\{m'_l\}$, and $\{Y_t\}$ are mutually independent random variables with $Em_l = Em'_l = 0$ and $V(m_l) = V(m'_l) = \sigma_l^2$ and $\{Y_t\}$ is independent and identically distributed with means 0 and finite variances σ^2 , the distribution of $I_T(\lambda)$ converges to a distribution with finite mean and finite variance at $\lambda \neq \pm \lambda_l$ for $1 \leq l \leq L$ when T tends to infinity. On the other hand, the magnitude of $I_T(\lambda)$ is of the order of T at $\lambda = \pm \lambda_l$, $1 \leq l \leq L$. This means that we can find the periodicities of X_t by using $I_T(\lambda)$. When $X_t = Y_t$, we find that the distribution of $2I_T(\lambda)/\sigma^2$ (when $\lambda \neq 0, \pm 1/2$) or $I_T(\lambda)/\sigma^2$ (when $\lambda = 0$ or $\pm 1/2$) tends to the $\dagger\chi^2$ distribution with degrees of freedom 2 or 1, respectively, and $I(\mu_1), I(\mu_2), \dots, I(\mu_M)$ are asymptotically independent random variables for $0 \leq |\mu_1| < |\mu_2| < \dots < |\mu_M| \leq 1/2$ when $T \rightarrow \infty$. Applying this result, we can test for periods in the data.

Let $f(\lambda)$ be the spectral density function of a real-valued weakly stationary process X_t . In general, the variance of $|\sum_{t=1}^T X_t e^{2\pi i t \lambda} / \sqrt{T}|$ does not tend to 0 as T tends to infinity; hence $I_T(\lambda)$ cannot be used as a good estimator for the spectral density. To obtain an estimate of $f(\lambda)$, several estimators defined by using weight functions have been proposed by several authors. Let $W_T(\lambda)$ be a weight function defined on $(-\infty, \infty)$, and construct a statistic $\tilde{f}(\lambda) = \int_{-1/2}^{1/2} I_T(\mu) W_T(\lambda - \mu) d\mu$. Let us use $\tilde{f}(\mu)$ for the estimation of $f(\lambda)$. $W_T(\lambda)$ is called a **window**. An important class of $W_T(\lambda)$ is as follows. Let $W(\lambda)$ be continuous, $W(\lambda) = W(-\lambda)$, $W(0) = 1$, $|W(\lambda)| < 1$, and $\int_{-\infty}^{\infty} W(\lambda)^2 d\lambda < +\infty$, and let H be a positive integer depending on T such that $H \rightarrow \infty$ and $H/T \rightarrow 0$ as $T \rightarrow \infty$. Putting $w_j = W(j/H)$, we define $W_T(\lambda)$ by $W_T(\lambda) = \sum_{j=-1}^{T-1} w_j e^{-2\pi i j \lambda}$. Then $\tilde{f}(\lambda)$ can be expressed as $\tilde{f}(\lambda) = \sum_{h=-1}^{T-1} \tilde{R}_h w_h e^{-2\pi i h \lambda}$, where $\tilde{R}_h = \sum_{t=1}^{T-h} X_{t+h} X_t / T$ for $h \geq 0$ and $\tilde{R}_h = \sum_{t=|h|+1}^T X_{t+h} X_t / T$ for $h < 0$. Let X_t be stationary to the fourth order (\rightarrow 395 Stationary Processes) and satisfy

$$\sum_{h=-\infty}^{\infty} |R_h| < +\infty,$$

$$\sum_{h,l,p=-\infty}^{\infty} |C_{o,h,l,p}| < +\infty,$$

where $C_{o,h,l,p}$ is the fourth-order joint cumulant of X_t, X_{t+h}, X_{t+l} , and X_{t+p} . Then we have

$$\lim_{T \rightarrow \infty} \frac{T}{H} V(\tilde{f}(0)) = 2f(0)^2 \int_{-\infty}^{\infty} W(\lambda)^2 d\lambda,$$

$$\lim_{T \rightarrow \infty} \frac{T}{H} V(\tilde{f}(\pm 1/2)) = 2f(1/2)^2 \int_{-\infty}^{\infty} W(\lambda)^2 d\lambda,$$

$$\lim_{T \rightarrow \infty} \frac{T}{H} V(\tilde{f}(\lambda)) = f(\lambda)^2 \int_{-\infty}^{\infty} W(\lambda)^2 d\lambda,$$

$$\lambda \neq 0, \pm 1/2,$$

$$\lim_{T \rightarrow \infty} \frac{T}{H} \text{Cov}(\tilde{f}(\lambda), \tilde{f}(\mu)) = 0, \quad \lambda \neq \mu. \quad (1)$$

$\{w_h\}$ or $W_T(\lambda)$ should have an optimality, e.g., to minimize the mean square error of $\tilde{f}(\lambda)$. But, generally, it is difficult to obtain such a $\{w_h\}$ or $W_T(\lambda)$.

Several authors have proposed specific types of windows. The following are some examples: (i) (Bartlett) $w_h = (1 - |h|/H)$ for $|h| \leq H$ and $w_h = 0$ for $|h| > H$; (ii) (Tukey) $w_h = \sum_{i=-\infty}^{\infty} a_i \cos(\pi i h / H)$ for $|h| \leq H$ and $w_h = 0$ for $|h| > H$, where the a_i are constants such that $\sum_{i=-\infty}^{\infty} |a_i| < +\infty$, $\sum_{i=-\infty}^{\infty} a_i = 1$ and $a_i = a_{-i}$. The Hanning and Hamming windows are $a_0 = 0.50$, $a_1 = a_{-1} = 0.25$, and $a_i = 0$ for $|i| \geq 2$ and $a_0 = 0.54$, $a_1 = a_{-1} = 0.23$, and $a_i = 0$ for $|i| \geq 2$, respectively [2]. Let $X_t = \sum_{j=-\infty}^{\infty} b_j \varepsilon_{t-j}$, where $\sum_{j=-\infty}^{\infty} |b_j| < +\infty$ and the ε_t are independently and identically distributed random variables

with $E\varepsilon_t = 0$ and $E\varepsilon_t^4 < +\infty$. Let $\{\lambda_j | 1 \leq j \leq M\}$ be arbitrary real numbers such that $0 \leq \lambda_1 < \lambda_2 < \dots < \lambda_M \leq 1/2$, where M is an arbitrary positive integer. Then the joint distribution of $\{\sqrt{T/H}(\tilde{f}(\lambda_v) - E\tilde{f}(\lambda_v)) | 1 \leq v \leq M\}$ tends to the normal distribution with means 0 and covariance matrix Σ , which is defined by (1). Let us assume, furthermore, that $\lim_{x \rightarrow 0} (1 - w(x))/|x|^q = C$ and $\sum_{h=-\infty}^{\infty} |h|^p |R_h| < +\infty$, where C, q , and p are some positive constants satisfying the following conditions: (i) when $p \geq q$, $H^q/T \rightarrow 0$ ($p \geq 1$) and $H^{q+1-p}/T \rightarrow 0$ ($p \geq 1$) as $T \rightarrow \infty$ and $\lim_{T \rightarrow \infty} T/H^{2q+1}$ is finite; (ii) when $p < q$, $H^p/T \rightarrow 0$ ($p \geq 1$) and $H/T \rightarrow 0$ ($p \leq 1$) as $T \rightarrow \infty$ and $\lim_{T \rightarrow \infty} T/H^{2p+1} = 0$. Then $\sqrt{T/H}(\tilde{f}(\lambda_v) - E\tilde{f}(\lambda_v))$ in the results above can be replaced by $\sqrt{T/H}(\tilde{f}(\lambda_v) - f(\lambda_v))$.

Estimation of higher-order spectra, particularly the bispectrum, has also been discussed. Let X_t be a weakly stationary process with mean 0, and let its spectral decomposition be given by $X_t = \int_{-1/2}^{1/2} e^{2\pi i t \lambda} dZ(\lambda)$ (\rightarrow 395 Stationary Processes). We assume that X_t is a weakly stationary process of degree 3 and put $R_{h_1, h_2} = EX_t X_{t+h_1} X_{t+h_2}$ for any integers h_1 and h_2 . Then we have

$$R_{h_1, h_2} = \int_{-1/2}^{1/2} \int_{-1/2}^{1/2} e^{2\pi i (h_1 \lambda_1 + h_2 \lambda_2)} dF(\lambda_1, \lambda_2).$$

Symbolically, $dF(\lambda_1, \lambda_2) = EdZ(\lambda_1) dZ(\lambda_2) dZ(-\lambda_1 - \lambda_2)$. If $F(\lambda_1, \lambda_2)$ is absolutely continuous with respect to the Lebesgue measure of \mathbf{R}^2 and $\partial F(\lambda_1, \lambda_2) / \partial \lambda_1 \partial \lambda_2 = f(\lambda_1, \lambda_2)$, we call $f(\lambda_1, \lambda_2)$ the **bispectral density function**. When X_t is Gaussian, $R_{h_1, h_2} = 0$ and $f(\lambda_1, \lambda_2) = 0$ for any h_1, h_2 and any λ_1, λ_2 . $f(\lambda_1, \lambda_2)$ can be considered to give a kind of measure of the departure from a Gaussian process or a kind of nonlinear relationship among waves of different frequencies. We can construct an estimator for $f(\lambda_1, \lambda_2)$ by using windows as in the estimation of a spectral density [3].

D. Statistical Analysis of Parametric Models

When we assume merely that X_t is a stationary process and nothing further, then X_t contains infinite-dimensional unknown parameters. In this case, it may be difficult to develop a satisfactory general theory for statistical inference about X_t . But in most practical applications of time series analysis, we can safely assume at least some of the time dependences to be known. For this reason, we can often use a model with finite-dimensional parameters. This means, mainly, that the moments (usually, second-order moments) or the spectral density are assumed to be expressible in terms of finite-dimensional parameters. As examples of such

models, autoregressive models, moving average models, and autoregressive moving average models are widely used.

A process X_t is called an **autoregressive process** of order K if X_t satisfies a difference equation $\sum_{k=0}^K a_k X_{t-k} = \xi_t$, where the a_k are constants, $a_0 = 1$, $a_K \neq 0$, and the ξ_t are mutually uncorrelated with $E\xi_t = 0$ and $V(\xi_t) = \sigma_\xi^2 > 0$. We usually assume that X_t is a weakly stationary process with $EX_t = 0$. We sometimes use the notation $AR(K)$ to express a weakly stationary and autoregressive process of order K . Let $\{\xi_t\}$ be as above. If X_t is expressed as $X_t = \sum_{l=0}^L b_l \xi_{t-l}$, where the b_l are constants, $b_0 = 1$ and $b_L \neq 0$, X_t is called a **moving average process** of order L ($MA(L)$ process). Furthermore, if X_t is weakly stationary with $EX_t = 0$ and expressed as $\sum_{k=0}^K a_k X_{t-k} = \sum_{l=0}^L b_l \xi_{t-l}$ with $a_0 = 1$, $b_0 = 1$, and $a_K b_L \neq 0$, then X_t is called an **autoregressive moving average process** of order (K, L) ($ARMA(K, L)$ process). Let $A(Z)$ and $B(Z)$ be two polynomials of Z such that $A(Z) = \sum_{k=0}^K a_k Z^{K-k}$ and $B(Z) = \sum_{l=0}^L b_l Z^{L-l}$, and let $\{\alpha_k | 1 \leq k \leq K\}$ and $\{\beta_l | 1 \leq l \leq L\}$ be the solutions of the associated polynomial equations $A(Z) = 0$ and $B(Z) = 0$, respectively, we assume that $|\alpha_k| < 1$ for $1 \leq k \leq K$ and $|\beta_l| < 1$ for $1 \leq l \leq L$. This condition implies that X_t is purely nondeterministic. Let the observed sample be $\{X_t | 1 \leq t \leq T\}$. If we assume that X_t is Gaussian and an $ARMA(K, L)$ process, we can show that the †maximum likelihood estimators $\{\hat{a}_k\}$ and $\{\hat{b}_l\}$ of $\{a_k\}$ and $\{b_l\}$ are †consistent and asymptotically efficient when $T \rightarrow \infty$ ("asymptotically efficient" means that the covariance matrix of the distribution of the estimators is asymptotically equal to the inverse of the information matrix) [5] (\rightarrow 399 Statistical Estimation D). Furthermore, if X_t is an $AR(K)$ process, the joint distribution of $\{\sqrt{T}(\hat{a}_k - a_k) | 1 \leq k \leq K\}$ tends to a K -dimensional normal distribution with means 0, and this distribution is the same as the one to which the distribution of the †least-square estimators $\{\hat{a}_k\}$ minimizing $Q = \sum_{t=K+1}^T (X_t + \sum_{k=1}^K a_k X_{t-k})^2$ tends when $T \rightarrow \infty$. If X_t is a $MA(L)$ or $ARMA(K, L)$ process ($L \geq 1$), the likelihood equations are complicated and cannot be solved directly. Many approximation methods have been proposed to obtain the estimates.

When X_t is an $AR(K)$ process with $|\alpha_k| < 1$ for $1 \leq k \leq K$, R_h satisfies $\sum_{k=0}^K a_k R_{h-k} = 0$ for $h \geq 1$. These are often called the **Yule-Walker equations**. R_h can be expressed as $R_h = \sum_{j=1}^K C_j \alpha_j^h$ if the α_k are distinct and $a_K \neq 0$, where $\{C_j\}$ are constants and determined by R_h for $0 \leq h \leq K-1$. When X_t is an $ARMA(K, L)$ process, $\sum_{k=0}^K a_k R_{h-k} = 0$ for $h \geq L+1$, and the C_j of $R_h = \sum_{j=1}^K C_j \alpha_j^h$ are determined by $\{R_h | 0 \leq h \leq \max(K, L)\}$.

The spectral density is expressed as $f(\lambda) = \sigma_\xi^2 |B(e^{2\pi i \lambda})|^2 / |A(e^{2\pi i \lambda})|^2$. If X_t is Gaussian, the maximum likelihood estimator of $f(\lambda)$ is asymptotically equal to the statistic obtained by replacing σ_ξ^2 , $\{b_l\}$, and $\{a_k\}$ in $f(\lambda)$ with $\hat{\sigma}_\xi^2$, $\{\hat{b}_l\}$, and $\{\hat{a}_k\}$, respectively, where $\hat{\sigma}_\xi^2$ is the maximum likelihood estimator of σ_ξ^2 , when $T \rightarrow \infty$.

When we analyze a time series and intend to fit an $ARMA(K, L)$ model, we have to determine the values of K and L . For $AR(K)$ models, many methods have been proposed to determine the value of K . Some examples are: (i) (Quenouille) Let $(Z^K A(1/Z))^2 = \sum_{j=0}^{2K} A_j Z^j$, and $G_K = \sum_{j=0}^{2K} \hat{A}_j (\hat{R}_j / \hat{R}_0)$, where \hat{A}_j is obtained by replacing $\{a_k\}$ in A_j by $\{\hat{a}_k\}$, and we construct the statistic $\chi_f^2 = \sum_{j=1}^f G_{K+j}$. Then χ_f^2 has a $\dagger\chi^2$ distribution asymptotically with f degrees of freedom under the assumption that $K \geq K_0$, where K_0 is the true order, as $T \rightarrow \infty$. Using this fact, we can determine the order of an AR model. (ii) (Akaike) We consider choosing an order K satisfying $K_L \leq K \leq K_M$, where K_L and K_M are minimum order and maximum order, respectively, specified a priori. Then we construct the statistic $AIC(K) = (T - K) \log \hat{\sigma}_\xi^2(K) + 2K$, where

$$\hat{\sigma}_\xi^2(K) = \sum_{t=K+1}^T (X_t + \hat{a}_1 X_{t-1} + \dots + \hat{a}_K X_{t-K})^2 / T$$

and $\{\hat{a}_k | 1 \leq k \leq K\}$ are the least square estimators of the autoregressive coefficients of an $AR(K)$ model fitting X_t . Calculate $AIC(K)$ for $K = K_L, K_L + 1, \dots, K_M$. If $AIC(K)$ has the minimum value at $K = \hat{K}$, we determine the order to be \hat{K} [6] (\rightarrow 403 Statistical Models F). Parzen proposed another method by using the criterion autoregressive transfer function (CAT). Here $CAT(K) = 1 - \hat{\sigma}^2(\infty) / \hat{\sigma}_\xi^2(K) + K/T$, where $\hat{\sigma}_\xi^2(K) = (T/(T-K)) \hat{\sigma}_\xi^2(K)$ and $\hat{\sigma}^2(\infty)$ is an estimator of $\sigma^2(\infty) = \exp(\int_{-1/2}^{1/2} \log f(\lambda) d\lambda)$ [7]. (iii) We can construct a test statistic for the null hypothesis $AR(K)$ against the alternative hypothesis $AR(K+1)$ (Jenkins) or use a multiple decision procedure (T. W. Anderson [8]).

Not much is known about the statistical properties of the above methods, and few comparisons have been made among them.

Another parametric model is an exponential model for the spectrum. The spectral density is expressed by $f(\lambda) = C^2 \exp\{2 \sum_{k=1}^K \theta_k \cos(2\pi k \lambda)\}$, where the θ_k and C are constants.

We now discuss some general theories of estimation for finite-dimensional-parameter models. Let X_t be a real-valued Gaussian process of mean 0 and of spectral density $f(\lambda)$ which is continuous and positive in $[-1/2, 1/2]$, and let the moving average representation of X_t be $X_t = \sum_{l=0}^\infty b_l \xi_{t-l}$, where ξ_t is a white noise and $\sigma_\xi^2 = E\xi_t^2$. We assume that $f(\lambda)/\sigma_\xi^2 = g(\lambda)$

depends only on M parameters $\theta = (\theta_1, \theta_2, \dots, \theta_M)'$ which are independent of σ_ξ^2 . Then the logarithm of the likelihood function can be approximated by $-(1/2) \{T \log 2\pi\sigma_\xi^2 + \mathbf{X}' \Sigma_T^{-1}(\theta) \mathbf{X} / \sigma_\xi^2\}$ by ignoring the lower-order terms in T , where $\sigma_\xi^2 \Sigma_T(\theta)$ is the covariance matrix of \mathbf{X} . Usually, it is difficult to find an explicit expression for each element of $\Sigma_T^{-1}(\theta)$. Another approximation for the logarithm of the likelihood function is given by

$$-\frac{T}{2} \int_{-1/2}^{1/2} \left[\log f(\lambda) + \frac{I_T(\lambda)}{f(\lambda)} \right] d\lambda.$$

Under mild conditions on the regularity of $g(\lambda)$, the estimators $\hat{\theta} = (\hat{\theta}_1, \hat{\theta}_2, \dots, \hat{\theta}_M)$ and $\hat{\sigma}_\xi^2$, obtained as the solutions of the likelihood equations, are consistent and asymptotically normal as T tends to infinity. This means that the distribution of $\sqrt{T}(\hat{\sigma}_\xi^2 - \sigma_\xi^2)$ is asymptotically normal and $\sqrt{T}(\hat{\sigma}_\xi^2 - \sigma_\xi^2)$ and $\sqrt{T}(\hat{\theta} - \theta)$ are asymptotically independent. The asymptotic distribution of $\sqrt{T}(\hat{\theta} - \theta)$ is the normal distribution $N(\mathbf{0}, \Gamma^{-1})$, where the (k, l) -component Γ_{kl} of Γ is given by

$$\Gamma_{kl} = \frac{1}{2} \int_{-1/2}^{1/2} \left(\frac{\partial \log g(\lambda)}{\partial \theta_k} \cdot \frac{\partial \log g(\lambda)}{\partial \theta_l} \right)_\theta d\lambda.$$

E. Statistical Analysis of Multiple Time Series

Let $\mathbf{X}_t = (X_t^{(1)}, X_t^{(2)}, \dots, X_t^{(p)})'$ be a complex-valued weakly stationary process with $E\mathbf{X}_t = \mathbf{0}$ and $E\mathbf{X}_t \bar{\mathbf{X}}_s' = \mathbf{R}_{t-s}$. \mathbf{R}_{t-s} is the $p \times p$ matrix whose (k, l) -component is $R_{t-s}^{(k, l)} = E X_t^{(k)} \bar{X}_s^{(l)}$. We discuss the case when t is an integer. \mathbf{R}_h has the spectral representation

$$\mathbf{R}_h = \int_{-1/2}^{1/2} e^{2\pi i h \lambda} dF(\lambda),$$

where $F(\lambda)$ is a $p \times p$ matrix and $F(\lambda_1) - F(\lambda_2)$, $\lambda_1 \geq \lambda_2$, is Hermitian nonnegative. Let $f^{k, l}(\lambda)$ be the (k, l) -component of the spectral density matrix $f(\lambda)$, i.e., $F_a(\lambda) = \int_{-1/2}^{1/2} f(\mu) d\mu$, of the absolutely continuous part in the Lebesgue decomposition of $F(\lambda)$. The function $f^{k, l}(\lambda)$ for $k \neq l$ is called the **cross spectral density function**. $f^{k, l}(\lambda)$ represents a kind of correlation between the wave of frequency λ included in $X_t^{(k)}$ and the one included in $X_t^{(l)}$.

Let $\mathbf{X}_t = (X_t^{(1)}, X_t^{(2)}, \dots, X_t^{(p)})'$ and $\mathbf{Y}_t = (Y_t^{(1)}, Y_t^{(2)}, \dots, Y_t^{(q)})'$ be two complex-valued weakly stationary processes with $E\mathbf{X}_t = \mathbf{0}$, $E\mathbf{Y}_t = \mathbf{0}$, $E\mathbf{X}_t \bar{\mathbf{X}}_s' = \mathbf{R}_{t-s}^X$, $E\mathbf{Y}_t \bar{\mathbf{Y}}_s' = \mathbf{R}_{t-s}^Y$ and $E\mathbf{X}_t \bar{\mathbf{Y}}_s' = \mathbf{R}_{t-s}^{XY}$. We assume $\mathbf{Y}_t = \sum_{s=-\infty}^{\infty} \mathbf{A}_s \mathbf{X}_{t-s}$, where \mathbf{A}_s is a $q \times p$ matrix whose components are constants depending on s . Put $A(\lambda) = \sum_{s=-\infty}^{\infty} \mathbf{A}_s e^{-2\pi i s \lambda}$. $A(\lambda)$ should exist in the sense of mean square convergence with respect to the spectral distribution function F for \mathbf{X}_t .

The function $A(\lambda)$ is called the **matrix frequency response function**.

As a measure of the strength of association between $X_t^{(k)}$ and $X_t^{(l)}$ at frequency λ , we introduce the quantity $\gamma^{k, l}(\lambda) = |f^{k, l}(\lambda)|^2 / f^{k, k}(\lambda) f^{l, l}(\lambda)$. $\gamma^{k, l}(\lambda)$ is called the **coherence**. Let $X_t^{(k)} = \sum_{s=-\infty}^{\infty} a_s^{k, l} X_{t-s}^{(l)} + \eta_t$, where η_t is a weakly stationary process with mean 0 and uncorrelated with $X_s^{(l)}$, $-\infty < s < \infty$. If $E|\eta_t|^2 = 0$, $\gamma^{k, l}(\lambda) = 1$. If $E|\sum_{s=-\infty}^{\infty} a_s^{k, l} X_{t-s}^{(l)}|^2 = 0$, $\gamma^{k, l}(\lambda) = 0$. Generally, we have $0 \leq \gamma^{k, l}(\lambda) \leq 1$.

For the estimation of $F(\lambda)$, $A(\lambda)$, and $\gamma^{k, l}(\lambda)$, the theories have been similar to those for the estimation of the spectral density of a scalar time series. For example, an estimator of $f(\lambda)$ is given [11] in the form

$$\hat{f}(\lambda) = \sum_{h=-(T-1)}^{T-1} \tilde{\mathbf{R}}_h \mathbf{w}_h e^{-2\pi i h \lambda},$$

where

$$\tilde{\mathbf{R}}_h = \sum_{t=1}^{T-|h|} \mathbf{X}_{t+|h|} \bar{\mathbf{X}}_t' / T$$

and the \mathbf{w}_h are the same as in Section C.

We can define an autoregressive, moving average, or autoregressive moving average process in a similar way as for a scalar time series. The a_k and b_l in Section D should be replaced by $p \times p$ matrices and the associated polynomial equations $A(Z) = 0$ and $B(Z) = 0$ should be understood in the vector sense [11]. There are problems with determining the coefficients uniquely or identifying an ARMA(K, L) model, and these problems have been discussed to some extent.

F. Statistical Inference of the Mean Function

Let X_t be expressed as $X_t = m_t + Y_t$, where m_t is a real-valued deterministic function of t and Y_t is a real-valued weakly stationary process with mean 0 and spectral distribution function $F(\lambda)$. This means that $E X_t = m_t$. We consider the case when $m_t = \sum_{j=1}^M C_j \varphi_t^{(j)}$, where $\mathbf{C} = (C_1, C_2, \dots, C_M)'$ is a vector of unknown coefficients and $\boldsymbol{\varphi}_t = (\varphi_t^{(1)}, \varphi_t^{(2)}, \dots, \varphi_t^{(M)})'$ is a set of known (regression) functions.

Let us construct linear unbiased estimators $\{\tilde{C}_j = \sum_{t=1}^T \gamma_{jt} X_t \mid 1 \leq j \leq M\}$ for the coefficients C_j , where the γ_{jt} are known constants. Put $\boldsymbol{\Phi} = (\boldsymbol{\varphi}_1, \boldsymbol{\varphi}_2, \dots, \boldsymbol{\varphi}_T)'$. Then the least squares estimator of \mathbf{C} is given by $\hat{\mathbf{C}} = (\boldsymbol{\Phi}' \boldsymbol{\Phi})^{-1} \boldsymbol{\Phi}' \mathbf{X}$ when $\boldsymbol{\Phi}' \boldsymbol{\Phi}$ is nonsingular. Let $\boldsymbol{\Sigma}$ be the covariance matrix of \mathbf{X} . Then the best linear unbiased estimator is $\hat{\mathbf{C}}^* = (\boldsymbol{\Phi}' \boldsymbol{\Sigma}^{-1} \boldsymbol{\Phi})^{-1} \boldsymbol{\Phi}' \boldsymbol{\Sigma}^{-1} \mathbf{X}$. We put $\|\boldsymbol{\varphi}^{(j)}\|_T^2 = \sum_{t=1}^T (\varphi_t^{(j)})^2$ and assume that $\lim_{T \rightarrow \infty} \|\boldsymbol{\varphi}^{(j)}\|_T^2 = \infty$, $\lim_{T \rightarrow \infty} \|\boldsymbol{\varphi}^{(j)}\|_T^2 / \|\boldsymbol{\varphi}^{(j)}\|_T^2 = 1$ for $1 \leq j \leq M$ and any fixed h and assume the existence of $\psi_h^{(j, k)} = \lim_{T \rightarrow \infty} \sum_{t=1}^T \varphi_t^{(j, h)} \varphi_t^{(k)} / \|\boldsymbol{\varphi}^{(j)}\|_T \|\boldsymbol{\varphi}^{(k)}\|_T$ for $1 \leq j, k \leq M$. We also assume

that $F(\lambda)$ is absolutely continuous and $F'(\lambda) = f(\lambda)$ is positive and piecewise continuous. Let ψ_h be the $M \times M$ matrix whose (j, k) -component is $\psi_h^{(j, k)}$. Then ψ_h can be represented by

$$\psi_h = \int_{-1/2}^{1/2} e^{2\pi i h \lambda} dG(\lambda),$$

where $G(\lambda) - G(\mu)$ is a nonnegative definite matrix for $\lambda > \mu$. Assume that $\psi_0 = G(1/2) - G(-1/2)$ is nonsingular and put $H(\lambda) = \psi_0^{-1/2} G(\lambda) \psi_0^{-1/2}$, and for any set S , $H(S) = \int_S H(d\lambda)$. Suppose further that S_1, S_2, \dots, S_q are q sets such that $H(S_j) > 0$, $\sum_{j=1}^q H(S_j) = I$, $H(S_j)H(S_k) = 0$, $j \neq k$, and for any j there is no subset $S'_j \subset S_j$ such that $H(S'_j) > 0$, $H(S_j - S'_j) > 0$ and $H(S'_j)H(S_j - S'_j) = 0$. We have $q \leq M$. It can be shown that the spectrum of the regression can be decomposed into such disjoint sets S_1, \dots, S_q . Then we can show that \hat{C} is asymptotically efficient in the sense that the asymptotic covariance matrix of \hat{C} is equivalent to that of \hat{C}^* if and only if $f(\lambda)$ is constant on each of the elements S_j . Especially, if $\psi_t^{(j)} = t^j e^{2\pi i t \mu_j}$, \hat{C} is asymptotically efficient.

G. Nonstationary Models

It is difficult to develop a statistical theory for a general class of nonstationary time series, but some special types of nonstationary processes have been investigated more or less in detail. Let X_t (t an integer) be a real-valued stochastic process and ∇ be the backward difference operator defined by $\nabla X_t = X_t - X_{t-1}$ and $\nabla^d X_t = \nabla(\nabla^{d-1} X_t)$ for $d \geq 2$. We assume that X_t is defined for $t \geq t_0$ (t_0 a finite integer), and $EX_t^2 < +\infty$. For analyzing a nonstationary time series, Box and Jenkins introduced the following model: For a positive integer d , $Y_t = \nabla^d X_t$, $t \geq t_0 + d$, is stationary and is an autoregressive moving average process of order (K, L) for $t \geq t_0 + d + \max(K, L)$. They called such an X_t an **autoregressive integrated moving average process** of order (K, d, L) and denoted it by $ARIMA(K, d, L)$. The word "integrated" means a kind of summation; in fact, X_t can be expressed as a sum of the weakly stationary process Y_t , i.e.,

$$\begin{aligned} X_t = & X_0 + (\nabla X_0)t + (\nabla^2 X_0) \left(\sum_{s_2=1}^t \sum_{s_1=1}^{s_2} \right) + \dots \\ & + (\nabla^{d-1} X_0) \left(\sum_{s_{d-1}=1}^t \dots \sum_{s_1=1}^{s_{d-1}} \right) \\ & + \sum_{s_d=1}^t \sum_{s_{d-1}=1}^{s_d} \dots \sum_{s_1=1}^{s_{d-1}} Y_{s_1}, \end{aligned}$$

when $t_0 = -d + 1$. Using this model, methods of forecasting and of model identification and estimation can be discussed [13].

Another nonstationary model is based on the concept of evolutionary spectra [14]. In this approach, spectral distribution functions are taken to be time-dependent. Let X_t be a complex-valued stochastic process (t an integer) with $EX_t = 0$ and $R_{t,s} = EX_t \bar{X}_s$. In the following, we write simply \int for $\int_{-1/2}^{1/2}$. We now restrict our attention to the class of X_t for which there exist functions $\{u_t(\lambda)\}$ defined on $[-1/2, 1/2]$ such that $R_{t,s}$ can be expressed as $R_{t,s} = \int u_t(\lambda) \bar{u}_s(\lambda) d\mu(\lambda)$, where $\mu(\lambda)$ is a measure. $u_t(\lambda)$ should satisfy $\int |u_t(\lambda)|^2 d\mu(\lambda) < +\infty$. Then X_t admits a representation of the form $X_t = \int u_t(\lambda) dZ(\lambda)$, where $Z(\lambda)$ is a process with orthogonal increments and $E|dZ(\lambda)|^2 = d\mu(\lambda)$. If $u_t(\lambda)$ is expressed as $u_t(\lambda) = \gamma_t(\lambda) e^{2\pi i \theta(\lambda)t}$ and $\gamma_t(\lambda)$ is of the form $\gamma_t(\lambda) = \int e^{2\pi i t w} d\Gamma_\lambda(w)$ with $|d\Gamma_\lambda(w)|$ having the absolute maximum at $w = 0$, we call $u_t(\lambda)$ an oscillatory function and X_t an oscillatory process. The evolutionary power spectrum $dF_t(\lambda)$ is defined by $dF_t(\lambda) = |\gamma_t(\lambda)|^2 d\mu(\lambda)$.

Other models, such as an autoregressive model whose coefficients vary with time or whose associated polynomial has roots outside the unit circle, have also been discussed.

References

- [1] E. J. Hannan, Time series analysis, Methuen, 1960.
- [2] R. B. Blackman and J. W. Tukey, The measurement of power spectra from the point of view of communications engineering, Dover, 1959.
- [3] D. R. Brillinger, Time series: Data analysis and theory, Holt, Rinehart and Winston, 1975.
- [4] M. S. Bartlett, An introduction to stochastic processes with special reference to methods and applications, Cambridge Univ. Press, second edition, 1966.
- [5] P. Whittle, Estimation and information in stationary time series, Ark. Mat. 2 (1953), 423-434.
- [6] H. Akaike, Information theory and an extension of the maximum likelihood principle, 2nd Int. Symp. Information Theory, B. N. Petrov and F. Csaki (eds.), Akadémiai Kiadó, 1973, 267-281.
- [7] E. Parzen, Some recent advances in time series modeling, IEEE Trans. Automatic Control, AC-19 (1974), 723-730.
- [8] T. W. Anderson, The statistical analysis of time series, Wiley, 1971.
- [9] P. Whittle, Hypothesis testing in time series analysis, Almqvist and Wiksell, 1951.
- [10] H. B. Mann and A. Wald, On the statistical treatment of linear stochastic difference equations, Econometrica, 11 (1943), 173-220.

- [11] E. J. Hannan, Multiple time series, Wiley, 1970.
- [12] U. Grenander and M. Rosenblatt, Statistical analysis of stationary time series, Wiley, 1957.
- [13] G. E. P. Box and G. M. Jenkins, Time series analysis: Forecasting and control, Holden-Day, revised edition, 1976.
- [14] M. B. Priestley, Evolutionary spectra and nonstationary processes, J. Roy. Statist. Soc., ser. B, 27 (1965), 204–237.

422 (IV.7) Topological Abelian Groups

A. Introduction

A commutative topological group is called a **topological Abelian group**. Throughout this article, except in Section L, all topological groups under consideration are locally compact Hausdorff topological Abelian groups and are simply called groups (\rightarrow 423 Topological Groups).

B. Characters

A **character** of a group is a continuous function $\chi(x)$ ($x \in G$) that takes on as values complex numbers of absolute value 1 and satisfies $\chi(xy) = \chi(x)\chi(y)$. Equivalently, χ is a 1-dimensional and therefore an irreducible * unitary representation of G . Conversely any irreducible unitary representation of G is 1-dimensional. Indeed, for a topological Abelian group, the set of its characters coincides with the set of its irreducible unitary representations. If the product of two characters χ, χ' is defined by $\chi\chi'(x) = \chi(x)\chi'(x)$, then the set of all characters forms the **character group** $C(G)$ of G . With * compact-open topology, $C(G)$ itself becomes a locally compact topological Abelian group.

C. The Duality Theorem

For a fixed element x of G , $\chi(x)$ ($\chi \in C(G)$) is a character of $C(G)$, namely, an element of $CC(G)$. Denote this character of $C(G)$ by $x(\chi)$, and consider the correspondence $G \ni x \rightarrow x(\chi)$. That this correspondence is one-to-one follows from the fact that any locally compact G has * sufficiently many irreducible unitary representations (\rightarrow 437 Unitary Representations) and the fact that if G is an Abelian group, then any irreducible unitary representation of G is a character of G . Furthermore, any character

of $C(G)$ is given as one of the $x(\chi)$; indeed, by this correspondence, we have $G \cong CC(G)$ (**Pontryagin's duality theorem**).

By the duality theorem, each of G and $C(G)$ is isomorphic to the character group of the other. In this sense, G and $C(G)$ are said to be **dual** to each other.

D. Correspondence between Subgroups

Let $G, G' = C(G)$ be groups that are dual to each other. Given a closed subgroup g of G , the set of all χ' such that $\chi'(x) = 1$ for all x in g forms a closed subgroup of G' , usually denoted by (G', g) . The definition of (G, g') is similar. Then $g \leftrightarrow (G', g) = g'$ gives a one-to-one correspondence between the closed subgroups of G and those of G' . If $g_1 \supset g_2$, then g_1/g_2 and $(G', g_2)/(G', g_1)$ are dual to each other. If the group operations of G, G' are written in additive form, with 0 for the identity, then $x(\chi') = 1$ is written as $x(\chi') = 0$. In this sense, (G', g) is called the **annihilator** (or **annulator**) of g .

E. The Structure Theorem

Let \mathfrak{A} be the set of all groups (more precisely, of all locally compact Hausdorff topological Abelian groups). If $G_1, G_2 \in \mathfrak{A}$, then the direct product $G_1 \times G_2 \in \mathfrak{A}$, and if $G \in \mathfrak{A}$ and H is a closed subgroup of G , then $H \in \mathfrak{A}$ and $G/H \in \mathfrak{A}$. In addition, if H is a closed subgroup of a group G such that $H \in \mathfrak{A}$ and $G/H \in \mathfrak{A}$, then $G \in \mathfrak{A}$. In other words, \mathfrak{A} is closed under the operations of forming direct products, closed subgroups, quotient groups, and * extensions by members of \mathfrak{A} . Furthermore, the operation C that assigns to each element of \mathfrak{A} its dual element is a reflexive correspondence of \mathfrak{A} onto \mathfrak{A} , and if $G \supset H$, the annihilator $(C(G), H)$ of H is a closed subgroup of $C(G)$. Also, $C(G/H) \cong (C(G), H)$, $C(H) \cong C(G)/(C(G), H)$. Furthermore, $C(G_1 \times G_2) \cong C(G_1) \times C(G_2)$. Finally, $H = (G, (C(G), H))$ (**reciprocity of annihilators**).

Typical examples of groups in \mathfrak{A} are the additive group \mathbf{R} of real numbers, the additive group \mathbf{Z} of rational integers, the 1-dimensional * torus group $\mathbf{T} = \mathbf{R}/\mathbf{Z}$, and finite Abelian groups \mathbf{F} . The torus group \mathbf{T} is also isomorphic to the multiplicative group $U(1)$ of complex numbers of absolute value 1. The direct product \mathbf{R}^n of n copies of \mathbf{R} is the **vector group** of dimension n , and the direct product \mathbf{T}^n of n copies of \mathbf{T} is the **torus** (or **torus group**) of dimension n (or **n -torus**). Both \mathbf{T}^n and \mathbf{F} are compact, while \mathbf{R}^n and \mathbf{Z}^n are not. We have $C(\mathbf{R}) = \mathbf{R}$, $C(\mathbf{T}) = \mathbf{Z}$, $C(\mathbf{Z}) = \mathbf{T}$. Any finite Abelian group \mathbf{F} is isomorphic to its character group $C(\mathbf{F})$. The direct product of a finite

Topological Abelian Groups

number of copies of \mathbf{R} , \mathbf{T} , \mathbf{Z} , and a finite Abelian group \mathbf{F} , namely, a group of the form $\mathbf{R}^l \times \mathbf{T}^m \times \mathbf{Z}^n \times \mathbf{F}$, is called an **elementary topological Abelian group**.

Any group in \mathfrak{A} is isomorphic to the direct product of a vector group of some dimension and the extension of a compact group by a discrete group (the **structure theorem**). Hence, if the effect of the operation C is explicitly known, then the problem of finding the structure of groups in \mathfrak{A} is reduced to the problem concerning discrete groups alone. For the structure of groups in \mathfrak{A} , the following theorem is known: If $G \in \mathfrak{A}$ is generated by a compact neighborhood of the identity e , then G is isomorphic to the direct product of a compact subgroup K and a group of the form $\mathbf{R}^n \times \mathbf{Z}^m$ (n, m are nonnegative integers). Then any compact subgroup of G is contained in K , which is the unique maximal compact subgroup of G . A group $G \in \mathfrak{A}$ generated by a compact neighborhood of e is the \dagger projective limit of elementary topological Abelian groups. L. S. Pontryagin first proved a structure theorem of this type and then the duality theorem.

F. Compact Elements

An element a of a group $G \in \mathfrak{A}$ is called a **compact element** if the cyclic group $\{a^n | n \in \mathbf{Z}\}$ generated by a is contained in a compact subset of G . The set C_0 of all compact elements of G is a closed subgroup of G , and the quotient group G/C_0 does not contain any compact element other than the identity. In particular, if G is generated by a compact neighborhood of the identity, then C_0 coincides with the maximal compact subgroup K of G . Let C_0 be the set of all compact elements of a group $G \in \mathfrak{A}$. The annihilator $(C(G), C_0)$ is a connected component of the character group $C(G)$ of G . If G is a discrete group, then a compact element of G is an element of G of finite order.

G. Compact Groups and Discrete Groups

Suppose that two groups $G, X \in \mathfrak{A}$ are dual to each other. Then one group is compact if and only if the other group is discrete. By the duality theorem, the properties of a compact Abelian group G can be stated, in principle, through the properties of the discrete Abelian group $C(G)$. The following are a few such examples. Let G be a compact Abelian group. Then its \dagger dimension is equal to the \dagger rank of the discrete Abelian group $C(G)$. A subgroup Y of a discrete Abelian group X is called a **divisible**

subgroup if the quotient group X/Y contains no element of finite order other than the identity. A compact Abelian group G is locally connected if and only if any finite subset of the character group $C(G)$ is contained in some divisible subgroup of $C(G)$ generated by a finite number of elements. Hence if a compact locally connected Abelian group G has an \dagger open basis consisting of a countable number of open sets, then G is of the form $\mathbf{T}^a \times \mathbf{F}$, where \mathbf{F} is a finite Abelian group and \mathbf{T}^a is the direct product of an at most countable number of 1-dimensional torus groups \mathbf{T} .

H. Dual Decomposition into Direct Products

Let G be a compact or discrete Abelian group, and let $\mathfrak{M} = \{H_\alpha | \alpha \in A\}$ be a family of closed subgroups of G . Let $\Delta(\mathfrak{M}) = \bigcap_{\alpha \in A} H_\alpha$, and denote by $\Sigma(\mathfrak{M})$ the smallest closed subgroup of G containing $\bigcup_{\alpha \in A} H_\alpha$. Then, with $\Omega = \{(C(G), H_\alpha) | \alpha \in A\}$, the relations $\Delta(\Omega) = (C(G), \Sigma(\mathfrak{M}))$ and $\Sigma(\Omega) = (C(G), \Delta(\mathfrak{M}))$ hold. Furthermore, suppose that G is decomposed into the direct product $G = \prod_{\alpha \in A} H_\alpha$, and for each $\alpha \in A$ put $K_\alpha = \Sigma(\mathfrak{M} - \{H_\alpha\})$, $X_\alpha = (C(G), K_\alpha)$. Then X_α is the character group of H_α , and $C(G)$ can be decomposed into the direct product $C(G) = \prod_{\alpha \in A} X_\alpha$. This decomposition of $C(G)$ into a direct product is called the **dual direct product decomposition** corresponding to the decomposition $G = \prod_{\alpha \in A} H_\alpha$.

I. Orthogonal Group Pairs

Suppose that for two groups G, G' there exists a mapping $(x, x') \rightarrow xx'$ of the Cartesian product $G \times G'$ into the set $U(1)$ of all complex numbers of absolute value 1 such that

$$(x_1 x_2) x' = (x_1 x') (x_2 x'),$$

$$x(x'_1 x'_2) = (xx'_1)(xx'_2).$$

Then G, G' are said to form a **group pair**. Suppose that G, G' form a group pair, and consider xx' to be a function $x(x')$ in x' . If two functions $x_1(x')$ and $x_2(x')$ coincide only when $x_1 = x_2$ and the same is true when the roles of G and G' are interchanged, then G, G' are said to form an **orthogonal group pair**. If G is a compact Abelian group, G' is a discrete Abelian group, and G, G' form an orthogonal group pair, then G, G' are dual to each other.

J. Commutative Lie Groups

An elementary topological Abelian group $\mathbf{R}^l \times \mathbf{T}^m \times \mathbf{Z}^n \times \mathbf{F}$ is a commutative \dagger Lie group. Conversely, any commutative Lie group G

generated by a compact neighborhood of the identity is isomorphic to an elementary topological Abelian group. In particular, any connected commutative Lie group G is isomorphic to $\mathbf{R}^l \times \mathbf{T}^m$ for some l and m . A closed subgroup H of the vector group \mathbf{R}^n of dimension n is isomorphic to $\mathbf{R}^p \times \mathbf{Z}^q$ ($0 \leq p + q \leq n$). More precisely, there exists a basis a_1, \dots, a_n of the vector group \mathbf{R}^n such that $H = \{\sum_{i=1}^p x_i a_i + \sum_{j=p+1}^n n_j a_j \mid x_i \in \mathbf{R}, n_j \in \mathbf{Z}\}$. Hence the quotient groups of \mathbf{R}^n that are \ast separated topological groups are all isomorphic to groups of the form $\mathbf{R}^l \times \mathbf{T}^m$ ($0 \leq l + m \leq n$). Any closed subgroup of the torus group \mathbf{T}^n of dimension n is isomorphic to a group of the form $\mathbf{T}^p \times \mathbf{F}$ ($0 \leq p \leq n$), where \mathbf{F} is a finite Abelian group. Hence the quotient groups of \mathbf{T}^n that are separated topological groups are all isomorphic to \mathbf{T}^m ($0 \leq m \leq n$). A \ast regular linear transformation of the linear space \mathbf{R}^n is a continuous automorphism of the vector group \mathbf{R}^n , and in fact, any continuous automorphism of \mathbf{R}^n is given by a regular linear transformation. Indeed, the group of all continuous automorphisms of \mathbf{R}^n is isomorphic to the \ast general linear group $GL(n, \mathbf{R})$ of degree n . Any continuous automorphism of the torus group $\mathbf{T}^n = \mathbf{R}^n/\mathbf{Z}^n$ of dimension n is given by a regular linear transformation φ of \mathbf{R}^n such that $\varphi(\mathbf{Z}^n) = \mathbf{Z}^n$. Hence the group of continuous automorphisms of \mathbf{T}^n is isomorphic to the multiplicative group of all $n \times n$ matrices, with determinant ± 1 and with entries in the set of rational integers.

K. Kronecker's Approximation Theorem

Let H be a subgroup of a group $G \in \mathfrak{A}$ (not necessarily closed). Then $(G, (C(G), H))$ coincides with the closure \bar{H} of H . In particular, H is \ast dense in G if and only if the annihilator $(C(G), H)$ consists of the identity alone. Now let $G = \mathbf{R}^n$ and let H be the subgroup of \mathbf{R}^n generated by $\theta = (\theta_1, \dots, \theta_n) \in \mathbf{R}^n$ and the natural \ast basis $e_1 = (1, 0, \dots, 0), \dots, e_n = (0, \dots, 0, 1)$ of \mathbf{R}^n . Then H is dense in \mathbf{R}^n if and only if $(\mathbf{R}^n, H) = \{0\}$; that is, $\theta_1, \dots, \theta_n, 1$ are linearly independent over the rational number field \mathbf{Q} (**Kronecker's approximation theorem**). This theorem implies that the torus group \mathbf{T}^n of dimension n has a cyclic subgroup and a 1-parameter subgroup that are both dense in \mathbf{T}^n .

L. Linear Topology

Consider the discrete topology in a field Ω . Suppose that an Ω -module G has a topology that satisfies \ast Hausdorff's separation axiom and is such that a base for the neighborhood

system of the zero element 0 consists of Ω -submodules, and suppose that G together with this topology constitutes a topological Abelian group. Then this topology is called a **linear topology**. If a linear topology is restricted to a Ω -submodule, then it is also a linear topology. If G is of finite rank, then any linear topology is the discrete topology. The discrete topology on G is a linear topology. Let H be a Ω -submodule. Then the subset $V = H + g$ of G obtained by translating H by an element g of G is called a **linear variety** in G . If V is a linear variety, then \bar{V} is also a linear variety. If Ω -modules G, G' have linear topologies, a homomorphism of G into G' is always assumed to be open and continuous with respect to these topologies. A linear variety V in G is said to be **linearly compact** if, for any system $\{V_\alpha\}$ of linear varieties closed in V with the \ast finite intersection property, we have $\bigcap_\alpha V_\alpha \neq \emptyset$. In this case V is closed in G . If linearly compact Ω -submodules can be chosen as a base for the neighborhood system of the zero element of G , we say that G is **locally linearly compact**. The set $C_\Omega(G)$ of homomorphisms of an Ω -module G with linear topology into Ω is also an Ω -module. For any linearly compact Ω -submodule H of G , let $U(H) = \{\chi \mid \chi(g) = 0, g \in H\}$. Then, with $\{U(H)\}$ as a base for the neighborhood system, a linear topology can be introduced in $C_\Omega(G)$. According as G is discrete, linearly compact, or locally linearly compact, $C_\Omega(G)$ is linearly compact, discrete, or locally linearly compact. Let G, H be Ω -modules each of which has a linear topology, and let $\varphi: G \ni g \rightarrow \varphi_g \in C_\Omega(H)$, $\psi: H \ni h \rightarrow \psi_h \in C_\Omega(G)$ be homomorphisms such that $\varphi_g(h) = \psi_h(g)$. Then if one of φ, ψ is an isomorphism, so is the other. This is an analog of the Pontryagin duality theorem and is called the **duality theorem for Ω -modules**. In particular, a linearly compact Ω -module is the direct sum of 1-dimensional spaces (S. Lefschetz [3]).

References

- [1] L. S. Pontryagin, Topological groups, first edition, Princeton Univ. Press, 1939; second edition, Gordon & Breach, 1966 (translation of the 1954 Russian edition).
- [2] A. Weil, L'intégration dans les groupes topologiques et ses applications, Actualités Sci. Ind., Hermann, 1940.
- [3] S. Lefschetz, Algebraic topology, Amer. Math. Soc. Colloq. Publ., 1942.
- [4] W. Rudin, Fourier analysis on groups, Interscience, 1962.
- [5] E. Hewitt and K. A. Ross, Abstract harmonic analysis, Springer, I, 1963; II, 1970.

423 (IV.6) Topological Groups

A. Definitions

If a * group G has the structure of a * topological space such that the mapping $(x, y) \rightarrow xy$ (product) of the Cartesian product $G \times G$ into G and the mapping $x \rightarrow x^{-1}$ (inverse) of G into G are both continuous, then G is called a **topological group**. The group G without a topological structure is called the **underlying group** of the topological group G , and the topological space G is called the **underlying topological space** of the topological group G . Let G, G' be topological groups. A mapping f of G into G' is called an **isomorphism** of the topological group G onto the topological group G' if f is a * isomorphism of the underlying group G onto the underlying group G' and also a * homeomorphism of the underlying topological space G onto the underlying topological space G' . Two topological groups are said to be **isomorphic** if there exists an isomorphism of one onto the other.

B. Neighborhood Systems

Let \mathfrak{N} be the * neighborhood system of the identity e of a topological group G . Namely, \mathfrak{N} consists of all subsets of G each of which contains an open set containing the element e . Then \mathfrak{N} satisfies the following six conditions: (i) If $U \in \mathfrak{N}$ and $U \subset V$, then $V \in \mathfrak{N}$. (ii) If $U, V \in \mathfrak{N}$, then $U \cap V \in \mathfrak{N}$. (iii) If $U \in \mathfrak{N}$, then $e \in U$. (iv) For any $U \in \mathfrak{N}$, there exists a $W \in \mathfrak{N}$ such that $WW = \{xy \mid x, y \in W\} \subset U$. (v) If $U \in \mathfrak{N}$, then $U^{-1} \in \mathfrak{N}$. (vi) If $U \in \mathfrak{N}$ and $a \in G$, then $aUa^{-1} \in \mathfrak{N}$. Conversely, if a nonempty family \mathfrak{N} of subsets of a group G satisfies conditions (i)–(vi), then there exists a * topology \mathfrak{D} of G such that \mathfrak{N} is the neighborhood system of e and G is a topological group with this topology. Moreover, such a topology is uniquely determined by \mathfrak{N} . * Left translation $x \rightarrow ax$ and * right translation $x \rightarrow xa$ in a topological group G are homeomorphisms of G onto G ; thus if \mathfrak{N} is the neighborhood system of the identity e , then $a\mathfrak{N} = \mathfrak{N}a$ is the neighborhood system of a , where $a\mathfrak{N} = \{aU \mid U \in \mathfrak{N}\}$.

If the underlying topological space of a topological group G is a * Hausdorff space, G is called a **T_2 -topological group (Hausdorff topological group or separated topological group)**. If the underlying topological space of a topological group G is a * T_0 -topological space, then, as is easily seen, it is a * T_1 -topological space. If it is a T_1 -topological space, then by the fact that the topology may be defined by a

* uniformity, it is a * completely regular space, hence, in particular, a Hausdorff space (\rightarrow Section G). Thus a topological group whose underlying topological space is a T_0 -topological space is a T_2 -topological group.

C. Direct Product of Topological Groups

Consider a family $\{G_\alpha\}_{\alpha \in A}$ of topological groups. The Cartesian product $G = \prod_{\alpha \in A} G_\alpha$ of the underlying groups of G_α is a topological group with the * product topology of the underlying topological spaces of G_α . This topological group $G = \prod_{\alpha \in A} G_\alpha$ is called the **direct product** of topological groups G_α ($\alpha \in A$).

D. Subgroups

Let H be a subgroup of the underlying group of a topological group G . Then H is a topological group with the topology of a * topological subspace of G (* relative topology). This topological group H is called a **subgroup** of G . A subgroup that is a closed (open) set is called a **closed (open) subgroup**. Any open subgroup is also a closed subgroup. For any subgroup H of a topological group G , the closure \bar{H} of H is also a subgroup. If H is a normal subgroup, so is \bar{H} . If H is commutative, so is \bar{H} . In a T_2 -topological group G , the * centralizer $C(M) = \{x \in G \mid xm = mx \ (m \in M)\}$ of a subset M of G is a closed subgroup of G . In particular, the * center $C = C(G)$ of a T_2 -topological group is a closed normal subgroup.

E. Quotient Spaces

Given a subgroup H of a topological group G , let $G/H = \{aH \mid a \in G\}$ be the set of * left cosets, and let p be the canonical surjection $p(a) = aH$ of G onto G/H . Consider the * quotient topology on G/H , namely, the strongest topology such that p is a continuous mapping. Since a subset A of G/H is open when $p^{-1}(A)$ is an open set of G , p is also a * open mapping. The set G/H with this topology is called the **left quotient space (or left coset space)** of G by H . The **right quotient space (or right coset space)** $H \backslash G = \{Ha \mid a \in G\}$ is defined similarly. The quotient space G/H is discrete if and only if H is an open subgroup of G . The quotient space is a Hausdorff space if and only if H is a closed subgroup. If G/H and H are both * connected, then G itself is connected. If G/H and H are both * compact, then G is compact. If H is a closed subgroup of G and $G/H, H$ are both * locally compact, then G is locally compact.

Suppose that H is a normal subgroup of a topological group G . Then the quotient group

G/H is a topological group with the topology of the quotient space G/H . This topological group is called the **quotient group** of the topological group G by the normal subgroup H .

F. Connectivity

The \ast connected component G_0 containing the identity e of a topological group G is a closed normal subgroup of G . The connected component that contains an element $a \in G$ is the coset $aG_0 = G_0a$. G_0 is called the **identity component** of G . The quotient group G/G_0 is \ast totally disconnected. A connected topological group G is generated by any neighborhood U of the identity. Namely, any element of G can be expressed as the product of a finite number of elements in U . Totally disconnected (in particular, discrete) normal subgroups of a connected topological group G are contained in the center of G .

G. Uniformity

Let \mathfrak{N}_0 be the neighborhood system of the identity of a topological group G , and let $U_i = \{(x, y) \in G \times G \mid y \in xU\}$ for $U \in \mathfrak{N}_0$. Then a \ast uniformity having $\{U_i \mid U \in \mathfrak{N}_0\}$ as a base is defined on G . This uniformity is called the **left uniformity** of G . Left translation $x \rightarrow ax$ of G is \ast uniformly continuous with respect to the left uniformity. The **right uniformity** is defined similarly by $U_r = \{(x, y) \mid y \in Ux\}$. These two uniformities do not necessarily coincide. The mapping $x \rightarrow x^{-1}$ is a \ast uniform isomorphism of G considered as a uniform space with respect to the left uniformity onto the same group G considered as a uniform space with respect to the right uniformity. A topological group G is thus a \ast uniform space under a uniformity \ast compatible with its topology, and hence it is a completely regular space if the underlying topological space is a T_1 -space.

H. Completeness

If a topological group G is \ast complete with respect to the left uniformity, then it is also complete with respect to the right uniformity, and conversely. In this case the topological group G is said to be **complete**. A locally compact T_2 -topological group is complete. If a T_2 -topological group G is isomorphic to a dense subgroup of a complete T_2 -topological group \hat{G} , then \hat{G} is called the **completion** of G , and G is said to be **completable**. A T_2 -topological group G is not always completable. For a T_2 -topological group G to be completable it is necessary and sufficient that any \ast Cauchy filter

of G considered as a uniform space with respect to the left uniformity is mapped to a Cauchy filter of the same uniform space G under the mapping $x \rightarrow x^{-1}$. Then the completion \hat{G} of G is uniquely determined up to isomorphism. A commutative T_2 -topological group always has a completion \hat{G} , and \hat{G} is also commutative. If each point of a T_2 -topological group G has a \ast totally bounded neighborhood, there exists a completion \hat{G} , and \hat{G} is locally compact.

I. Metrization

If a \ast metric can be introduced in a T_2 -topological group G so that the metric gives the topology of G , then G is said to be **metrizable**. For a T_2 -topological group G to be metrizable it is necessary and sufficient that G satisfy the \ast first axiom of countability. Then the metric can be chosen so that it is **left invariant**, i.e., invariant under left translation. Similarly, it can be chosen so that it is right invariant. In particular, the topology of a compact T_2 -topological group that satisfies the first axiom of countability can be given by a metric that is both left and right invariant.

J. Isomorphism Theorems

Let G and G' be topological groups. If a homomorphism f of the underlying group of G into the underlying group of G' is a continuous mapping of the underlying topological space of G into that of G' , f is called a **continuous homomorphism**. If f is a continuous open mapping, f is called a **strict morphism** (or **open continuous homomorphism**). A continuous homomorphism of a \ast paracompact locally compact topological group onto a locally compact T_2 -topological group is an open continuous homomorphism.

A topological group G' is said to be **homomorphic** to a topological group G if there exists an open continuous homomorphism f of G onto G' . Let N denote the kernel $f^{-1}(e)$ of f . Then the quotient group G/N is isomorphic to G' , with G/N and G' both considered as topological groups (**homomorphism theorem**). Let f be an open continuous homomorphism of a topological group G onto a topological group G' , and let H' be a subgroup of G' . Then $H = f^{-1}(H')$ is a subgroup of G , and the mapping φ defined by $\varphi(gH) = f(g)H'$ is a homeomorphism of the quotient space G/H onto G'/H' . In particular, if H' is a normal subgroup, then H is also a normal subgroup and φ is an isomorphism of the quotient group G/H onto G'/H' as topological groups (**first isomorphism theorem**). Let H and N be subgroups of a topo-

logical group G such that $HN = NH$. Then the canonical mapping $f: h(H \cap N) \rightarrow hN$ of the quotient space $H/H \cap N$ to HN/N is a continuous bijection but not necessarily an open mapping. In particular, if N is a normal subgroup of the group HN , then f is a continuous homomorphism. In addition, if f is an open mapping, the quotient groups $H/H \cap N$ and HN/N are isomorphic as topological groups (**second isomorphism theorem**). For example, f is an open mapping (1) if N is compact or (2) if G is locally compact, HN and N are closed subgroups of G , and H is the union of a countable number of compact subsets. Let H be a subgroup of a topological group G and N be a normal subgroup of G such that $H \supset N$. Then the canonical mapping of the quotient space $(G/N)/(H/N)$ onto G/H is a homeomorphism. In particular, if H is also a normal subgroup, the quotient groups $(G/N)/(H/N)$ and G/H are isomorphic as topological groups (**third isomorphism theorem**).

K. The Projective Limit

Let $\{G_\alpha\}_{\alpha \in A}$ be a family of topological groups indexed by a \ast -directed set A , and suppose that if $\alpha \leq \beta$, there exists a continuous homomorphism $f_{\alpha\beta}: G_\beta \rightarrow G_\alpha$ such that $f_{\alpha\gamma} = f_{\alpha\beta} \circ f_{\beta\gamma}$ if $\alpha \leq \beta \leq \gamma$. Then the collection $\{G_\alpha, f_{\alpha\beta}\}$ of the family $\{G_\alpha\}_{\alpha \in A}$ of topological groups together with the family $\{f_{\alpha\beta}\}$ of mappings is called a **projective system** of topological groups. Consider the direct product $\prod_{\alpha \in A} G_\alpha$ of topological groups $\{G_\alpha\}$, and denote by G the set of all elements $x = \{x_\alpha\}_{\alpha \in A}$ of $\prod G_\alpha$ that satisfy $x_\alpha = f_{\alpha\beta}(x_\beta)$ for $\alpha \leq \beta$. Then G is a subgroup of $\prod G_\alpha$. The topological group G obtained in this way is called the **projective limit** of the projective system $\{G_\alpha, f_{\alpha\beta}\}$ of topological groups and is denoted by $G = \varprojlim G_\alpha$. If each G_α is a T_2 -topological (resp. complete) group, then G is also a T_2 -topological (complete) group.

Now consider another projective system $\{G'_\alpha, f'_{\alpha\beta}\}$ of topological groups indexed by the same A , and consider continuous homomorphisms $u_\alpha: G_\alpha \rightarrow G'_\alpha$ such that $u_\alpha \circ f_{\alpha\beta} = f'_{\alpha\beta} \circ u_\beta$ for $\alpha \leq \beta$. Then there exists a unique continuous homomorphism u of $G = \varprojlim G_\alpha$ into $G' = \varprojlim G'_\alpha$ such that for any $\alpha \in A$, $u_\alpha \circ f_\alpha = f'_\alpha \circ u$ holds, where $f_\alpha(f'_\alpha)$ is the restriction to $G(G')$ of the projection of $\prod G_\alpha$ ($\prod G'_\alpha$) onto $G_\alpha(G'_\alpha)$. The homomorphism u is called the **projective limit** of the family $\{u_\alpha\}$ of continuous homomorphisms and is denoted by $u = \varprojlim u_\alpha$. Let G be a T_2 -topological group, and let $\{H_\alpha\}_{\alpha \in A}$ be a decreasing sequence ($H_\alpha \supset H_\beta$ for $\alpha \leq \beta$) of closed normal subgroups of G . Consider the quotient group G/H_α , and let $f_{\alpha\beta}$ be the canonical mapping $gH_\beta \rightarrow gH_\alpha$ of G_β to G_α for $\alpha \leq \beta$.

Then $\{G_\alpha, f_{\alpha\beta}\}$ is a projective system of topological groups. Let f_α be the projection of G onto $G_\alpha = G/H_\alpha$, and let $f = \varprojlim f_\alpha$. Now assume that any neighborhood of the identity of G contains some H_α and that some H_α is complete. Then $f = \varprojlim f_\alpha$ is an isomorphism of G onto $\varprojlim G/H_\alpha$ as topological groups. (For a general discussion of the topological groups already discussed \rightarrow [1, 4].)

L. Locally Compact Groups

For the rest of this article, all topological groups under consideration are assumed to be T_2 -topological groups. The identity component G_0 of a locally compact group G is the intersection of all open subgroups of G . In particular, any neighborhood of the identity of a totally disconnected locally compact group contains an open subgroup. A totally disconnected compact group is a projective limit of finite groups with discrete topology.

A T_1 -topological space L is called a **local Lie group** if it satisfies the following six conditions:

- (i) There exist a nonempty subset M of $L \times L$ and a continuous mapping $\mu: M \rightarrow L$, called **multiplication** ($\mu(a, b)$ is written as ab).
- (ii) If $(a, b), (ab, c), (b, c), (a, bc)$ are all in M , then $(ab)c = a(bc)$.
- (iii) L contains an element e , called the **identity**, such that $L \times \{e\} \subset M$ and $ae = a$ for all $a \in L$.
- (iv) There exists a nonempty open subset N of L and a continuous mapping $\nu: N \rightarrow L$ such that $\nu(a) = e$ for all $a \in N$.
- (v) There exist a neighborhood U of e in L and a homeomorphism f of U into a neighborhood V of the origin in the Euclidean space \mathbb{R}^n .
- (vi) Let D be the open subset of $V \times V$ defined by $D = \{(x, y) \in V \times V \mid (f^{-1}(x), f^{-1}(y)) \in M, f^{-1}(x), f^{-1}(y) \in U\}$. Then the function $F: D \rightarrow V$ defined by $F(x, y) = f\mu(f^{-1}(x), f^{-1}(y))$ is of \ast -class C^ω .

For any neighborhood U of the identity e of a connected locally compact group G , there exist a compact normal subgroup K and a subset L that is a local Lie group under the \ast -induced topology and the group operations of G such that the product LK is a neighborhood of e contained in U . Furthermore, under $(l, k) \rightarrow lk$, LK is homeomorphic to the product space $L \times K$. Any compact subgroup of a connected locally compact group G is contained in a maximal compact subgroup, and maximal compact subgroups of G are \ast -conjugate. For a maximal compact subgroup K of G , there exists a finite number of subgroups H_1, \dots, H_r of G , each of which is isomorphic to the additive group of real numbers such that $G = KH_1 \dots H_r$, and the mapping $(k, h_1, \dots, h_r) \rightarrow kh_1 \dots h_r$ is a homeomorphism of the direct product $K \times H_1 \times \dots \times H_r$ onto G . Any locally compact group has a left-invariant positive

measure and a right-invariant positive measure, which are uniquely determined up to constant multiples (\rightarrow 225 Invariant Measures). Using these measures, the theory of harmonic analysis on the additive group \mathbf{R} of real numbers can be extended to that on G (\rightarrow 69 Compact Groups; 192 Harmonic Analysis; 422 Topological Abelian Groups; 437 Unitary Representations).

M. Locally Euclidean Groups

Suppose that each point of a topological group G has a neighborhood homeomorphic to an open set of a given Euclidean space. Then G is called a **locally Euclidean group**. If the underlying topological space of a topological group has the structure of a * real analytic manifold such that the group operation $(x, y) \rightarrow xy^{-1}$ is a real analytic mapping, then G is called a * **Lie group**. A Lie group is a locally Euclidean group.

N. Hilbert's Fifth Problem

Hilbert's fifth problem asks if every locally Euclidean group is a Lie group (\rightarrow 196 Hilbert). This problem was solved affirmatively in 1952; it was proved that any * locally connected finite-dimensional locally compact group is a Lie group (D. Montgomery and L. Zippin [3]). In connection with this, the relation between Lie groups and general locally compact groups has been studied, and the following results have been obtained: A necessary and sufficient condition for a locally compact group to be a Lie group is that there exist a neighborhood of the identity e that does not contain any subgroup (or any normal subgroup) other than $\{e\}$. A locally compact group has an open subgroup that is the projective limit of Lie groups. Hilbert's fifth problem is closely related to the following problem: Find the conditions for a * topological transformation group operating * effectively on a manifold to be a Lie group (\rightarrow 431 Transformation Groups).

O. Covering Groups

Let \mathfrak{G} be the collection of all * arcwise connected and * locally arcwise connected T_2 -topological groups. Suppose that $G^* \in \mathfrak{G}$ is a * covering space of $G \in \mathfrak{G}$ and the * covering mapping $f: G^* \rightarrow G$ is an open continuous homomorphism, with G^* and G considered as topological groups. Then G^* (or, more precisely, (G^*, f)) is called a **covering group** of G . Then the kernel $f^{-1}(e) = D$ of f is a discrete

Topological Groups

subgroup contained in the center of G^* , and G^*/D and G , considered as topological groups, are isomorphic to each other. Let $\pi_1(G)$ be the * fundamental group of G . The natural homomorphism $f^*: \pi_1(G^*) \rightarrow \pi_1(G)$ induced by f is an injective homomorphism, and if we identify $\pi_1(G^*)$ with the subgroup $f^*(\pi_1(G^*))$ of $\pi_1(G)$, we have $D \cong \pi_1(G)/\pi_1(G^*)$. Conversely, if D is any discrete subgroup contained in the center of $G^* \in \mathfrak{G}$, then G^* is a covering group of $G = G^*/D$. For any covering space (G^*, f) of $G \in \mathfrak{G}$, a multiplication law can be introduced in G^* so that G^* is a topological group belonging to \mathfrak{G} and (G^*, f) is a covering group of G . In particular, any $G \in \mathfrak{G}$ has a * simply connected covering group (\tilde{G}, φ) . Then for any covering group (G^*, f) of G , there exists a homomorphism $f^*: \tilde{G} \rightarrow G^*$, and (\tilde{G}, f^*) is a covering group of G^* . Furthermore, $\varphi = f \circ f^*$. Hence, in particular, any simply connected covering group of G is isomorphic to \tilde{G} , with G and \tilde{G} considered as topological groups. This simply connected covering group (\tilde{G}, φ) is called the **universal covering group**.

Let G and G' be topological groups, and let e and e' be their identities. A homeomorphism f of a neighborhood U of e onto a neighborhood U' of e' is called a **local isomorphism** of G to G' if it satisfies the following two conditions: (i) If a, b, ab are all contained in U , then $f(ab) = f(a)f(b)$. (ii) Let $f^{-1} = g$, then if $a', b', a'b' \in U'$, $g(a'b') = g(a')g(b')$ holds. If there exists a local isomorphism of G to G' , we say that G and G' are **locally isomorphic**. If G^* is a covering group of G , then G^* and G are locally isomorphic. For two topological groups G and G' to be locally isomorphic it is necessary and sufficient that the universal covering groups of G and G' be isomorphic. For two connected Lie groups to be locally isomorphic it is necessary and sufficient that their * Lie algebras be isomorphic.

Let f be a mapping of a neighborhood U of the identity of a topological group G into a group H such that if a, b, ab are all contained in U , then $f(ab) = f(a)f(b)$. Then f is called a **local homomorphism** of G into H and U is called its **domain**. A local homomorphism of a simply connected group $G \in \mathfrak{G}$ into a group H can be extended to a homomorphism of G into H if the domain is connected [2, 4].

P. Topological Rings and Fields

If a ring R has the structure of a topological group such that $(x, y) \rightarrow x + y$ (sum) and $(x, y) \rightarrow xy$ (product) are both continuous mappings of $R \times R$ into R , then R is called a **topological ring**. If a topological ring K is a field (not necessarily commutative) such that $x \rightarrow x^{-1}$

Topological Groups

(inverse element) is a continuous mapping of $K^* = K - \{0\}$ into K^* , then K is called a **topological field**. Let us assume that K is a topological field that is a locally compact Hausdorff space and is not discrete. If K is connected, then K is a \dagger division algebra of finite rank over the field \mathbf{R} of real numbers; hence it is isomorphic to the field \mathbf{R} of real numbers, the field \mathbf{C} of complex numbers, or the \dagger quaternion field \mathbf{H} . If K is not connected, then K is totally disconnected and is isomorphic to a division algebra of finite rank over the $\dagger p$ -adic number field \mathbf{Q}_p or a division algebra of finite rank over the \dagger formal power series field with coefficients in a finite field [4].

For various important classes of topological groups — 69 Compact Groups; 249 Lie Groups; 422 Topological Abelian Groups; 424 Topological Linear Spaces.

References

- [1] N. Bourbaki, *Eléments de mathématique*, Topologie générale, ch. 3, *Actualités Sci. Ind.*, 1143c, Hermann, third edition, 1960; English translation, *General topology*, pt. 1, Addison-Wesley, 1966.
- [2] C. Chevalley, *Theory of Lie groups I*, Princeton Univ. Press, 1946.
- [3] D. Montgomery and L. Zippin, *Topological transformation groups*, Interscience, 1955.
- [4] L. S. Pontryagin, *Topological groups*, first edition, Princeton Univ. Press, 1939, second edition, Gordon & Breach, 1966. (Second English edition translated from the second Russian edition, 1954.)

424 (XII.5) Topological Linear Spaces

A. Definition

A \dagger linear space E over the real or complex number field K is said to be a **topological linear space**, **topological vector space**, or **linear topological space** if E is a \dagger topological space and the basic operations $x + y$ and αx ($x, y \in E$, $\alpha \in K$) in the linear space are continuous as mappings of $E \times E$ and $K \times E$, respectively, into E . The coefficient field K may be a general \dagger topological field, although it is usually assumed to be the real number field \mathbf{R} or the complex number field \mathbf{C} , and accordingly E is called a **real topological linear space** or a **complex topological linear space**. Topological linear spaces are generalizations of \dagger normed linear spaces and play an important role in the study

of \dagger function spaces, such as the \dagger space of distributions, that are not \dagger Banach spaces.

Each topological linear space E is equipped with a \dagger uniform topology in which translations of the neighborhoods of zero form a \dagger uniform family of neighborhoods, and the addition $x + y$ and the multiplication αx by a scalar α are uniformly continuous relative to this uniform topology. In particular, if for each $x \neq 0$ there is a neighborhood of the origin that does not contain x , then E satisfies the \dagger separation axiom T_1 and hence is a \dagger completely regular space. The \dagger completion \hat{E} of E is also a topological linear space.

We assume in this article that K is the real or complex number field and E is a topological linear space over K satisfying the axiom of T_1 -spaces. Then E is finite-dimensional if and only if E has a \dagger totally bounded neighborhood of zero. The topology of E is \dagger metrizable if and only if it satisfies the \dagger first countability axiom.

B. Linear Functional

A K -valued function $f(x)$ on E is said to be a **linear functional** if it satisfies (i) $f(x + y) = f(x) + f(y)$ and (ii) $f(\alpha x) = \alpha f(x)$. A linear functional that is continuous relative to the topologies of E and K is said to be a continuous linear functional. (Sometimes continuous linear functionals are simply called linear functionals, while abstract linear functionals are called **algebraic linear functionals**.) The following three statements are equivalent for linear functionals $f(x)$: (i) $f(x)$ is continuous; (ii) the half-space $\{x \in E \mid \operatorname{Re} f(x) > 0\}$ is open; (iii) the hyperplane $\{x \in E \mid f(x) = 0\}$ is closed.

C. The Hahn-Banach Theorem

A linear functional $f(x)$ defined on a linear subspace F of E can be extended to a continuous linear functional on E if and only if there exists an open \dagger convex neighborhood V of the origin in E that is disjoint with $\{x \in F \mid f(x) = 1\}$. Furthermore, if $f(x)$ can be extended, at least one extension $f(x)$ never takes the value 1 on V (**Hahn-Banach theorem**).

D. Dual Spaces

The set E' of all continuous linear functionals on E is called the **dual space** of E . It is often denoted by E^* and is also called the **conjugate space** or **adjoint space**. It forms a linear space when $f + g$ and αf ($f, g \in E'$, $\alpha \in K$) are defined by $(f + g)(x) = f(x) + g(x)$ and $(\alpha f)(x) = \alpha(f(x))$ for $x \in E$.

E. Locally Convex Spaces

A topological linear space is said to be **locally convex** if it has a family of convex sets as a \dagger base of the neighborhood system of 0. It follows from the Hahn-Banach theorem that for each $x \neq 0$ in a locally convex space E there is a continuous linear functional f such that $f(x) \neq 0$. A subset M of E is said to be **circled** if M contains $\alpha M = \{\alpha x \mid x \in M\}$ whenever $|\alpha| \leq 1$. A set that is both circled and convex is called **absolutely convex**. In a locally convex space, a family of absolutely convex and closed sets can be chosen as a base of the neighborhood system of the origin. Let A and B be subsets of E . A is said to **absorb** B if there is an $\alpha > 0$ such that $\alpha A \supset B$. A set V that absorbs every point $x \in E$ is called **absorbing**. Neighborhoods of 0 are absorbing.

F. Seminorms

A real-valued function $p(x)$ on E is said to be a **seminorm** (or **pseudonorm**) if it satisfies (i) $0 \leq p(x) < +\infty$ ($x \in E$); (ii) $p(x+y) \leq p(x) + p(y)$; and (iii) $p(\alpha x) = |\alpha|p(x)$. The relation $V = \{x \mid p(x) \leq 1\}$ gives a one-to-one correspondence between seminorms $p(x)$ and absolutely convex absorbing sets V whose intersection with any line through the origin is closed. In terms of seminorms, the Hahn-Banach theorem states: Let E be a linear space on which a seminorm $p(x)$ is given. If a linear functional $f(x)$ defined on a linear subspace F of E satisfies $|f(x)| \leq p(x)$ on F , then $f(x)$ can be extended to the whole space E in such a way that the inequality holds on E .

The topology of a locally convex space is determined by the family of continuous seminorms on it. Conversely, if there is a family of seminorms $\{p_\lambda(x)\}$ ($\lambda \in \Lambda$) on a linear space E over K that satisfies (iv) $p_\lambda(x) = 0$ for all λ implies $x = 0$, then there exists on E the weakest locally convex topology that renders the seminorms continuous. This topology is called the locally convex topology determined by $\{p_\lambda(x)\}$.

We assume that E is a locally convex space whose topology is determined by the family of seminorms $\{p_\lambda(x)\}$ ($\lambda \in \Lambda$). Then a \dagger net x_ν of E converges to x if and only if $p_\lambda(x_\nu - x) \rightarrow 0$ for all $\lambda \in \Lambda$. If F is a locally convex space whose topology is determined by the family of seminorms $\{q_\mu(y)\}$, then a necessary and sufficient condition for a linear mapping $u: E \rightarrow F$ to be continuous is that for every $q_\mu(y)$ there exist a finite number of $\lambda_1, \dots, \lambda_n \in \Lambda$ and a constant C such that $q_\mu(u(x)) \leq C(p_{\lambda_1}(x) + \dots + p_{\lambda_n}(x))$ ($x \in E$).

A set is said to be **bounded** if it is absorbed

by every neighborhood of zero. When the topology of E is determined by the family $\{p_\lambda(x)\}$ of seminorms a set B is bounded if and only if every p_λ is bounded on B . Totally bounded sets are bounded. The unit ball in a normed space is bounded. Conversely, a locally convex space is normable if it has a bounded neighborhood of 0. A locally convex space is called **quasicomplete** if every bounded closed set is complete. Since Cauchy sequences $\{x_n\}$ are totally bounded, all Cauchy sequences converge in a quasicomplete space (i.e., the space is sequentially complete).

G. Pairing of Linear Spaces

Let E and F be linear spaces over the same field K . A K -valued function $B(x, y)$ ($x \in E$, $y \in F$) on $E \times F$ is called a **bilinear functional** or **bilinear form** if for each fixed $y \in F$ (resp. $x \in E$), it is a linear functional of x (resp. y). When a bilinear functional $\langle x, y \rangle$ on $E \times F$ is given so that $\langle x, y \rangle = 0$ for all $y \in F$ (all $x \in E$) implies $x = 0$ ($y = 0$), then E and F are said to form a (separated) **pairing** relative to the **inner product** $\langle x, y \rangle$. A locally convex space E and its dual space E' form a pairing relative to the natural inner product $\langle x, x' \rangle = x'(x)$ ($x \in E$, $x' \in E'$).

H. Weak Topologies

When E and F form a pairing relative to an inner product $\langle x, y \rangle$, the locally convex topology on E determined by the family of seminorms $\{|\langle x, y \rangle| \mid y \in F\}$ is called the **weak topology (relative to the pairing $\langle E, F \rangle$)** and is denoted by $\sigma(E, F)$. A net x_ν in E is said to **converge weakly** if it converges in the weak topology. When E and E' are a locally convex space and its dual space, $\sigma(E, E')$ is called the **weak topology** of E , and $\sigma(E', E)$ the **weak* topology** of E' . The weak topology on a locally convex space E is weaker than the original topology on E . Consequently, a weakly closed set is closed. If the set is convex, the converse holds, and hence a convex closed set is weakly closed. Also, boundedness is preserved if we replace the original topology by the weak topology. Thus a weakly bounded set is bounded.

Let E and F form a pairing relative to $\langle x, y \rangle$, and let A be a subset of E . Then the set A° of points $y \in F$ satisfying $\operatorname{Re} \langle x, y \rangle \geq -1$ for all $x \in A$ is called the **polar** of A (relative to the pairing). If A is absolutely convex, A° is also absolutely convex and is the set of points y such that $|\langle x, y \rangle| \leq 1$ for all $x \in A$. If A is a convex set containing zero, its (weak) closure is equal to the **bipolar** $A^{\circ\circ} = (A^\circ)^\circ$ (**bipolar theorem**). In general, let A be a subset of a

Topological Linear Spaces

topological linear space E . We call the smallest closed convex set containing A the **closed convex hull** of A . If E is locally convex, the bipolar $A^{\circ\circ}$ relative to E' coincides with the closed convex hull of $A \cup \{0\}$.

A subset B of the dual space E' is **equi-continuous** on E if and only if it is contained in the polar V° of a neighborhood V of 0 in E . Also, V° is weak*-compact in E' (**Banach-Alaoglu theorem**).

I. Barreled Spaces and Bornological Spaces

An absorbing absolutely convex closed set in a locally convex space E is called a **barrel**. In a sequentially complete space (hence in a quasi-complete space also), a barrel absorbs every bounded set. A locally convex space is said to be **barreled** if each barrel is a neighborhood of 0. A locally convex space is said to be **quasi-barreled** (or **evaluable**) if each barrel that absorbs every bounded set is a neighborhood of 0. Furthermore, a locally convex space is said to be **bornological** if each absolutely convex set that absorbs every bounded set is a neighborhood of 0. Bornological spaces are quasi-barreled. However, they are not necessarily barreled. Furthermore, barreled spaces are not necessarily bornological. A metrizable locally convex space, i.e., a space whose topology is determined by a countable number of seminorms, is bornological. A complete metrizable locally convex space is called a **locally convex Fréchet space** (**(F)-space** or simply **Fréchet space**). To distinguish it from Fréchet space as in 37 Banach Spaces, it is sometimes called a **Fréchet space in the sense of Bourbaki**. (F)-spaces are bornological and barreled.

A continuous linear mapping $u: E \rightarrow F$ of one locally convex space into another maps each bounded set of E to a bounded set in F . Conversely, if E is bornological, then each linear mapping that maps every bounded sequence to a bounded set is continuous.

J. The Banach-Steinhaus Theorem

In the dual space of a barreled space E , each (weak*-)bounded set is equicontinuous. Thus if a sequence of continuous linear mappings u_n of E into a locally convex space F converges at each point of E , then u_n converges uniformly on each totally bounded set of E , and the limit linear mapping is continuous (**Banach-Steinhaus theorem**).

K. The S -Topology

Let E and F be paired linear spaces relative to the inner product $\langle x, y \rangle$. When a family S

of (weakly) bounded sets of F generates a dense subspace of F , the family of seminorms $\{\sup_{y \in B} |\langle x, y \rangle| \mid B \in S\}$ determines a locally convex topology on E . This is called the **S -topology** or **topology of uniform convergence on members of S** , because $x_\nu \rightarrow x$ in the S -topology is equivalent to the uniform convergence of $\langle x_\nu, y \rangle \rightarrow \langle x, y \rangle$ on each $B \in S$. The space E with the S -topology is denoted by E_S . The weak topology is the same as the topology of pointwise convergence. The S -topology in which S is the family of all bounded sets in F is called the **strong topology** and is denoted by $\beta(E, F)$. The dual space E' of a locally convex space E is usually regarded as a locally convex space with the strong topology $\beta(E', E)$. It is called the **strong dual space**. The topology of a locally convex space E is that of uniform convergence on equicontinuous sets of E' . The topology of a barreled space E coincides with the strong topology $\beta(E, E')$.

L. Grothendieck's Criterion of Completeness

Let E and F be paired spaces, and let S be a family of absolutely convex bounded sets of F such that: (i) the sets of S generate F ; (ii) if $B_1, B_2 \in S$, then there is a $B_3 \in S$ such that $B_3 \supset B_1$ and $B_3 \supset B_2$. Then E_S is complete if and only if each algebraic linear functional $f(y)$ on F that is weakly continuous on every $B \in S$ is expressed as $f(y) = \langle x, y \rangle$ for some $x \in E$. When E_S is not complete, the space of all linear functionals satisfying this condition gives the completion \hat{E}_S of E_S .

M. Mackey's Theorem

Let E, F , and S satisfy the same conditions as in Section L. Then the dual space of E_S is equal to the union of the weak completions of λB , where $\lambda > 0$ and $B \in S$ (**Mackey's theorem**).

N. The Mackey Topology

When E and F form a pairing, the topology on E of uniform convergence on convex weakly compact sets of F is called the **Mackey topology** and is denoted by $\tau(E, F)$. The dual space of E endowed with a locally convex topology T coincides with F if and only if T is stronger than the weak topology $\sigma(E, F)$ and weaker than the Mackey topology $\tau(E, F)$ (**Mackey-Arens theorem**). A locally convex space is said to be a **Mackey space** if the topology is equal to the Mackey topology $\tau(E, E')$. Every quasi-barreled space is a Mackey space.

O. Reflexivity

Let E be a locally convex space. The dual space E'' of the dual space E' equipped with the strong topology contains the original space E . We call E **semireflexive** if $E'' = E$, and **reflexive** if in addition the topology of E coincides with the strong topology $\beta(E, E')$. E is semireflexive if and only if every bounded weakly closed set of E is weakly compact. E is reflexive if and only if E is semireflexive and (quasi)barreled.

A barreled space in which every bounded closed set is compact is called a **Montel space** or **(M)-space**. (M)-spaces are reflexive, and their strong dual spaces are also (M)-spaces.

Many of the function spaces that appear in applications are (F)-spaces or their dual spaces. For these spaces detailed consequences of the countability axiom are known [7, 8]. A convex set C in the dual space E' of an (F)-space E is weak*-closed if and only if for every neighborhood V of 0 in E , $C \cap V^\circ$ is weak*-closed (**Kreĭn-Shmul'yan theorem**). The strong dual space E' of an (F)-space E is (quasi)barreled if and only if it is bornological. In particular, the dual space of a reflexive (F)-space is bornological.

P. (DF)-Spaces

A locally convex space is called a **(DF)-space** if it satisfies: (i) There is a countable base of bounded sets (i.e., every bounded set is included in one of them); (ii) if the intersection V of a countable number of absolutely convex closed neighborhoods of 0 absorbs every bounded set, then V is also a neighborhood of 0. The dual space of an (F)-space is a (DF)-space, and the dual space of a (DF)-space is an (F)-space. A linear mapping of a (DF)-space E into a locally convex space F is continuous if and only if its restriction to every bounded set of E is continuous. A quasicomplete (DF)-space is complete.

Q. Bilinear Mappings

A bilinear mapping $b(x, y)$ on locally convex spaces E and F ($x \in E, y \in F$) to a locally convex space G is said to be **separately continuous** if for each fixed $y \in F$ ($x \in E$) it is continuous as a function of x (y). The linear mappings obtained from $b(x, y)$ by fixing x (y) are denoted by $b_x(y)$ ($b_y(x)$). We call $b(x, y)$ **hypocontinuous** if for each bounded set B of E and B' of F , $\{b_x(y) | x \in B\}$ and $\{b_y(x) | y \in B'\}$ are equicontinuous. A continuous bilinear mapping is hypocontinuous. However, the converse is

not always true. A separately continuous bilinear mapping is not necessarily hypocontinuous. If both E and F are barreled, however, then every separately continuous mapping is hypocontinuous. If E is an (F)-space and F is metrizable, then every separately continuous bilinear mapping is continuous. Similarly, if both E and F are (DF)-spaces, then every hypocontinuous bilinear mapping is continuous.

R. Tensor Products

It is possible to introduce many topologies in the tensor product $E \otimes F$ of locally convex spaces E and F . The **projective topology** (or **topology π**) is defined to be the strongest topology such that the natural bilinear mapping $E \times F \rightarrow E \otimes F$ is continuous. The dual space of $E \otimes_\pi F$ is identified with the space $B(E, F)$ of all continuous bilinear functionals on $E \times F$. The completion of $E \otimes_\pi F$ is denoted by $E \hat{\otimes} F$. The **topology of biequicontinuous convergence** (or **topology ϵ**) is defined to be the topology of uniform convergence on sets $V^\circ \times U^\circ$, where V and U are neighborhoods of 0 in E and F , respectively, considering the elements of $E \otimes F$ as linear functionals on $E' \otimes F'$ by the natural pairing of $E \otimes F$ and $E' \otimes F'$. The completion of $E \otimes_\epsilon F$ is denoted by $E \hat{\otimes}_\epsilon F$. The dual space of $E \otimes_\epsilon F$ coincides with the subspace $J(E, F)$ of $B(E, F)$ composed of the union of the absolute convex hulls of the products $V^\circ \otimes U^\circ$ of equicontinuous sets. The elements of $J(E, F)$ are called **integral bilinear functionals**.

Closely related to $E \hat{\otimes}_\epsilon F$ is L. Schwartz's ϵ **tensor product** $E \epsilon F$ [12]. (They coincide if E and F are complete and if E or F has the ϵ -approximation property.) $E \epsilon F$ can be regarded as (i) a space of vector-valued functions if E is a space of functions and F is an abstract locally convex space, especially a space of functions of two variables if E and F are, respectively, spaces of functions of one variable, and (ii) a space of operators $G \rightarrow F$ if E is the dual space G' of a locally convex space G .

S. Nuclear Spaces

Let E be a locally convex space, V be an absolutely convex closed neighborhood of the origin, and $p(x)$ be the seminorm corresponding to V . Then we denote by E_V the normed space with norm $p(x)$ obtained from E by identifying the two elements x and y with $p(x - y) = 0$. If $U \subset V$, then a natural linear mapping $\phi_{U,V}: E_U \rightarrow E_V$ is defined.

A locally convex space E is said to be a

nuclear space (resp. **Schwartz space** or simply **(S)-space**) if for each absolutely convex closed neighborhood V of 0 there is another U such that $\varphi_{U,V}$ is a † nuclear operator (resp. † compact operator) as an operator of E_U into the completion of E_V . A nuclear space or (S)-space is an (M)-space if it is quasicomplete and quasibarreled. A locally convex space E is a nuclear space if and only if the topologies π and ε coincide on the tensor product $E \otimes F$ with any locally convex space F . Accordingly, it follows that $B(E, F) = J(E, F)$. This can be regarded as a generalization of Schwartz's **kernel theorem**, which says that every separately continuous bilinear functional on $\mathcal{D}_x \times \mathcal{D}_y$ is represented by an integral with kernel in \mathcal{D}'_{xy} . The theory of topological tensor products and nuclear spaces is due to Grothendieck [9].

A locally convex space E is a nuclear (F)-space if and only if E is isomorphic to a closed subspace of $C^\infty(-\infty, \infty)$ (T. Kōmura and Y. Kōmura, 1966). An example of a nuclear (F)-space without basis is known (B. S. Mityagin and N. M. Zobin, 1974).

T. Gel'fand Triplet

Let H and L be Hilbert spaces. If L is a dense subspace of H and the injection $L \rightarrow H$ is a † Hilbert-Schmidt operator, then $H = H'$ is regarded as a dense subspace of L' and the injection $H' \rightarrow L'$ is a Hilbert-Schmidt operator. In this case, (L, H, L') is called a **Gel'fand triplet** (or a **rigged Hilbert space**).

A subset of H is called a cylindrical set if it is expressed in the form $P_F^{-1}(B)$ by the orthogonal projection P_F onto a finite-dimensional subspace F and a Borel subset B of F . If a finitely additive positive measure μ with $\|\mu\|_1 = 1$ defined on the cylindrical sets of H satisfies (i) μ is countably additive on cylindrical sets for a fixed F and (ii) for any $\varepsilon > 0$ there exists a $\delta > 0$ such that $\|x\| < \delta$ implies $\mu\{y \in H \mid |\langle x, y \rangle| \geq 1\} < \varepsilon$, then μ is the restriction of a countably additive measure $\tilde{\mu}$ defined on the Borel subsets of L' (**Minlos's theorem**, 1959).

Let T be a self-adjoint operator in H . Then T has a natural extension \tilde{T} in L' and almost every continuous spectrum λ of T has an associated eigenvector x_λ in L' : $\tilde{T}x_\lambda = \lambda x_\lambda$, $x_\lambda \in L'$.

U. The Extreme Point Theorem

Let A be a subset of a linear space E . A point $x \in A$ is said to be an **extreme point** if x is an extreme point of any real segment containing x and contained in A . If A is a compact convex subset of a locally convex space E , A is the convex closed hull of (i.e., smallest convex

closed set containing) the set of its extreme points (**Krein-Milman theorem**). In applications it is important to know whether every point of A is represented uniquely as an integral of extreme points. For a metrizable convex compact subset A of a locally convex space E , the following two conditions are equivalent (**Choquet's theorem**): (i) A is a **simplex**, i.e., if we put $\tilde{A} = \{(\lambda x, \lambda) \mid x \in A, \lambda > 0\} \subset E \times \mathbf{R}^1$, the vector space $\tilde{A} - \tilde{A}$ becomes a † lattice with positive cone \tilde{A} ; (ii) for any $x \in A$ there exists a unique positive measure μ on A with $\|\mu\|_1 = 1$ such that $l(x) = \int_A l(y) d\mu(y)$ ($l \in E'$) and the support of μ is contained in the set of extreme points of A .

V. Weakly Compact Set

A subset of a quasicomplete locally convex space is relatively weakly compact if and only if every sequence in the set has a weak accumulation point (**Eberlein's theorem**). If E is a metrizable locally convex space, every weakly compact set of E is weakly sequentially compact (**Shmul'yan's theorem**). If E is a quasicomplete locally convex space, the convex closed hull of any weakly compact subset is weakly compact (**Krein's theorem**). If E is not quasicomplete, this is not necessarily true.

W. Permanence

Each subspace, quotient space, direct product, direct sum, projective limit, and inductive limit (of a family) of locally convex spaces has a unique natural locally convex topology. These spaces, except for quotient spaces and inductive limits, are separated, and a quotient space E/A is separated if and only if the subspace A is closed. The limit of a sequence $E_1 \subset E_2 \subset \dots$ is said to be a **strictly inductive limit** if E_n has the induced topology as a subspace of E_{n+1} . If E is a strictly inductive limit of a sequence E_n such that E_n is closed in E_{n+1} or if E is the inductive limit of a sequence $E_1 \subset E_2 \subset \dots$ such that the mapping $E_n \rightarrow E_{n+1}$ maps a neighborhood of 0 to a relatively weakly compact set, then E is separated and each bounded set of E is the image of a bounded set in some E_n . If $E = \bigcup E_n$ is the strictly inductive limit of the sequence $\{E_n\}$, then the topology of E_n coincides with the relative topology of $E_n \subset E$. The strictly inductive limit of a sequence of (F)-spaces is called an **(LF)-space**.

Any complete locally convex space (resp. any locally convex space) is (resp. a dense linear subspace of) the projective limit of Banach spaces. Every (F)-space E is the projective limit of a sequence of Banach spaces $E_1 \leftarrow E_2 \leftarrow \dots$. In particular, E is said to be a **count-**

ably normed space if the mappings $E \rightarrow E_n$ are one-to-one and $\|x\|_n \leq \|x\|_{n+1}$ for all $x \in E$ with E considered as a subspace E_n . We call E a **countably Hilbertian space** if, in particular, the E_n are * Hilbert spaces. An (F)-space with at least one continuous norm is a nuclear space if and only if it is a countably Hilbertian space such that the mappings $E_{n+1} \rightarrow E_n$ are Hilbert-Schmidt operators or nuclear operators.

A locally convex space is bornological if and only if it is the inductive limit of normed spaces. A locally convex space is said to be **ultrabornological** if it is the inductive limit of Banach spaces, or in particular, if it is quasicomplete and bornological.

Properties of spaces, such as being complete, quasicomplete, semireflexive, or having every bounded closed set compact, are inherited by closed subspaces, direct products, projective limits, direct sums, and strictly inductive limits formed from the original spaces, and properties of spaces, such as being Mackey, quasibarreled, barreled, and bornological, are inherited by quotient spaces, direct sums, inductive limits, and direct products formed from the spaces. (For direct products of high power of bornological spaces, unsolved problems still exist concerning the inheritance of properties.) Quotient spaces of (F)-spaces are (F)-spaces, but quotient spaces of general complete spaces are not necessarily complete. There are examples of a Montel (F)-space whose quotient space is not reflexive and a Montel (DF)-space whose closed subspace is neither a Mackey space nor a (DF)-space. The property of being a Schwartz space or a nuclear space is inherited by the completions, subspaces, quotient spaces of closed subspaces, direct products, projective limits, direct sums of countable families, and inductive limits of countable families formed from such spaces. Tensor products of nuclear spaces are nuclear spaces. Y. Kōmura gave an example of a non-complete space that is quasicomplete, bornological, and nuclear (and hence a Montel space).

X. The Open Mapping Theorem and the Closed Graph Theorem

Let E and F be topological linear spaces. The statement that every continuous linear mapping of E onto F is open is called the **open mapping theorem** (or **homomorphism theorem**), and the statement that every linear mapping of F into E is continuous if its graph is closed in $F \times E$ is called the **closed graph theorem**. These theorems hold if both E and F are complete and metrizable (S. Banach).

A locally convex space is said to be B-

complete (or **fully complete**) if a subspace C of E' is weak*-closed whenever $C \cap V^\circ$ is weak*-closed for every neighborhood V of 0 in E . (F)-spaces and the dual spaces of reflexive (F)-spaces are B-complete. B-complete spaces are complete, and closed subspaces and quotient spaces by closed subspaces of B-complete spaces are B-complete. If E is B-complete and F is barreled, then the open mapping theorem and the closed graph theorem hold (V. Pták).

Both theorems hold also if F is ultrabornological and E is a locally convex space obtained from a family of (F)-spaces after a finite number of operations of taking closed subspaces, quotient spaces by closed spaces, direct products of countable families, projective limits of countable families, direct sums of countable families, and inductive limits of countable families. This was conjectured by Grothendieck and proved by W. Słowiński (1961) and D. A. Raikov. Later, L. Schwartz, A. Martineau, M. De Wilde, W. Robertson, and M. Nakamura simplified the proof and enlarged the class of spaces E [15].

(LF)-spaces, the dual spaces of Schwartz (F)-spaces, and the space \mathcal{D}' of distributions are examples of spaces E described in the previous paragraph.

Y. Nonlocally Convex Spaces

The space L_p for $0 < p < 1$ shows that nonlocally convex spaces are meaningful in functional analysis. Recently, the Banach-Steinhaus theorem, closed graph theorems, etc. have been investigated for nonlocally convex topological linear spaces [13].

Z. Diagram of Topological Linear Spaces

The spaces in Fig. 1 are all locally convex spaces over the real number field or the complex number field and satisfy the separation axiom T_1 . The notation $A \rightarrow B$ means that spaces with property A have property B . Main properties of dual spaces are listed in Table 1.

References

- [1] N. Bourbaki, *Éléments de mathématique, Espaces vectoriels topologiques*, Actualités Sci. Ind., Hermann, 1189a, 1966; 1229b, 1967; 1230a, 1955.
- [2] A. Grothendieck, *Espaces vectoriels topologiques*, Lecture notes, São Paulo, 1954.
- [3] G. Köthe, *Topologische lineare Räume I*, Springer, second edition, 1966; English translation, *Topological vector spaces I, II*, Springer, 1969, 1979.

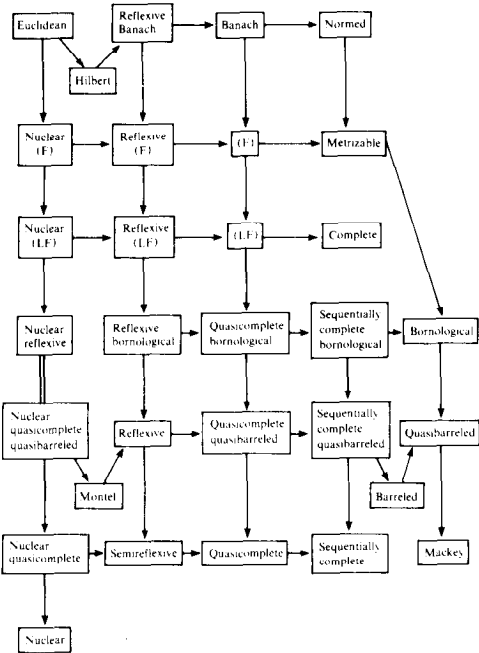


Fig. 1
Topological linear spaces.

Table 1

E	E'
Semireflexive	Barreled
Reflexive	Reflexive
Quasibarreled	Quasicomplete
Bornological	Complete
Reflexive, (F)	Bornological
(F)	(DF)
(DF)	(F)
(M)	(M)
Nuclear, (LF) or (DF)	Nuclear, reflexive
Complete, (S)	Ultrabornological

[4] I. M. Gel'fand et al., Generalized functions. II, Functions and generalized function spaces (with G. E. Shilov), Academic Press, 1968; IV, Applications of harmonic analysis (with N. Ya. Vilenkin), Academic, 1964. (Original in Russian, 1958–1966.)

[5] A. Pietsch, Nukleare lokalkonvexe Räume, Berlin, 1965.

[6] A. P. Robertson and W. Robertson, Topological vector spaces, Cambridge Univ. Press, 1964.

[7] J. Dieudonné and L. Schwartz, La dualité dans les espaces (\mathfrak{F}) et $(\mathfrak{L}\mathfrak{F})$, Ann. Inst. Fourier, 1 (1949), 61–101.

[8] A. Grothendieck, Sur les espaces (F) et (DF), Summa Brasil. Math., 3 (1954), 57–123.

[9] A. Grothendieck, Produits tensoriels topologiques et espaces nucléaires, Mem. Amer. Math. Soc., 1955.

[10] G. Choquet, Le théorème de représentation intégrale dans les ensembles convexes compacts, Ann. Inst. Fourier, 10 (1960), 333–344.

[11] J. L. Kelley, I. Namioka, et al., Linear topological spaces, Van Nostrand, 1963.

[12] L. Schwartz, Théorie des distributions à valeurs vectorielles I, Ann. Inst. Fourier, 7 (1957), 1–141.

[13] N. Adasch, B. Ernst, and D. Keim, Topological vector spaces: The theory without convexity conditions, Lecture notes in math. 639, Springer, 1978.

[14] F. Trèves, Topological vector spaces, distributions and kernels, Academic Press, 1967.

[15] M. De Wilde, Closed graph theorems and webbed spaces, Pitman, 1978.

425 (II.16)
Topological Spaces

A. Introduction

Convergence and continuity, as well as the algebraic operations on real numbers, are fundamental notions in analysis. In an abstract space too, it is possible to provide an additional structure so that convergence and continuity can be defined and a theory analogous to classical analysis can be developed. Such a structure is called a **topological structure** (for a precise definition, → Section B). There are several ways of giving a topology to a space. One method is to axiomatize the notion of convergence (M. Fréchet [1], 1906; → 87 Convergence). However, defining a topology in terms of either a neighborhood system (due to F. Hausdorff [3], 1914), a closure operation (due to C. Kuratowski, *Fund. Math.*, 3 (1922)), or a family of open sets is more common.

B. Definition of a Topology

Let X be a set. A **neighborhood system** for X is a function \mathfrak{U} that assigns to each point x of X , a family $\mathfrak{U}(x)$ of subsets of X subject to the following axioms (U):

- (1) $x \in U$ for each U in $\mathfrak{U}(x)$.
- (2) If $U_1, U_2 \in \mathfrak{U}(x)$, then $U_1 \cap U_2 \in \mathfrak{U}(x)$.
- (3) If $U \in \mathfrak{U}(x)$ and $U \subset V$, then $V \in \mathfrak{U}(x)$.
- (4) For each U in $\mathfrak{U}(x)$, there is a member W of $\mathfrak{U}(x)$ such that $U \in \mathfrak{U}(y)$ for each y in W .

A **system of open sets** for a set X is a family \mathfrak{O} of subsets of X satisfying the following axioms (O):

- (1) $X, \emptyset \in \mathfrak{D}$.
- (2) If $O_1, O_2 \in \mathfrak{D}$, then $O_1 \cap O_2 \in \mathfrak{D}$.
- (3) If $O_\lambda \in \mathfrak{D}$ ($\lambda \in \Lambda$), then $\bigcup_{\lambda \in \Lambda} O_\lambda \in \mathfrak{D}$.

A **system of closed sets** for a space X is a family \mathfrak{F} of subsets of X satisfying the following axioms (F):

- (1) $X, \emptyset \in \mathfrak{F}$.
- (2) If $F_1, F_2 \in \mathfrak{F}$, then $F_1 \cup F_2 \in \mathfrak{F}$.
- (3) If $F_\lambda \in \mathfrak{F}$ ($\lambda \in \Lambda$), then $\bigcap_{\lambda \in \Lambda} F_\lambda \in \mathfrak{F}$.

A **closure operator** for a space X is a function that assigns to each subset A of X , a subset A^a of X satisfying the following axioms (C):

- (1) $\emptyset^a = \emptyset$.
- (2) $(A \cup B)^a = A^a \cup B^a$.
- (3) $A \subset A^a$.
- (4) $A^a = A^{aa}$.

An **interior operator** for a space X is a function that assigns to each subset A of X a subset A^i of X satisfying the following axioms (I):

- (1) $X^i = X$.
- (2) $(A \cap B)^i = A^i \cap B^i$.
- (3) $A^i \subset A$.
- (4) $A^{ii} = A^i$.

Any one of these five structures for a set X , i.e., a structure satisfying any one of (U), (O), (F), (C), or (I), determines the four other structures in a natural way. For instance, assume that a system of open sets \mathfrak{D} satisfying (O) is given. In this case, each member of \mathfrak{D} is called an **open set**. A subset U of X is called a **neighborhood** of a point x in X provided that there is an open set O such that $x \in O \subset U$. If $\mathfrak{U}(x)$ is the family of all neighborhoods of x , the function $x \rightarrow \mathfrak{U}(x)$ satisfies (U). The complement of an open set in X is called a **closed set**. The family \mathfrak{F} of all closed sets satisfies (F). Given a subset A of X , the intersection A^a of the family of all closed sets containing A is called the **closure** of A , and each point of A^a is called an **adherent point** of A . The closure A^a is the smallest closed set containing A , and the function $A \rightarrow A^a$ satisfies (A). The closure A^a is also denoted by \bar{A} or $\text{Cl } A$. Dually, there is a largest open subset A^i of A . The set A^i (also denoted by A° or $\text{Int } A$) is called the **interior** of A , and each point of A° is called an **interior point** of A . The closure and interior are related by $A^\circ = X - (\overline{X - A})$ and $\bar{A} = X - (X - A)^\circ$. The correspondence $A \rightarrow A^\circ$ satisfies (I). Conversely, open sets can be characterized variously as follows:

$$\begin{aligned} A \text{ is open} &\Leftrightarrow A \in \mathfrak{U}(x) \text{ for each } x \text{ in } A \\ &\Leftrightarrow X - A \in \mathfrak{F} \\ &\Leftrightarrow \overline{(X - A)} = X - A \\ &\Leftrightarrow A^\circ = A. \end{aligned}$$

When a structure satisfying (U), (F), (C), or (I) is given, one of the four characterizations of open sets can be used to define a system of

open sets satisfying (O) and hence the other structure.

A **topological structure** or simply a **topology** for a space X is any of these five structures for X . If two topologies τ_1 and τ_2 for X give rise to identical systems of open sets, then τ_1 and τ_2 are considered to be identical. For this reason "topology" frequently means simply "system of open sets" in the literature. A **topological space** is a set X provided with a topology τ and is denoted by (X, τ) or simply X when there is no ambiguity.

C. Examples

(1) **Discrete Topology.** Let X be a set, and let the system \mathfrak{D} of open sets be the family of all subsets of X . The resulting topology is called the **discrete topology**, and X with the discrete topology is a **discrete topological space**. In this space, $\bar{A} = A^\circ = A$ for each subset A , and A is a neighborhood of each of its points.

(2) **Trivial Topology.** The **trivial** (or **indiscrete**) **topology** for a set X is defined by the system of open sets which consists of X and \emptyset only. If $A \subsetneq X$, then $A^\circ = \emptyset$, and if $A \neq \emptyset$, then $\bar{A} = X$. Each point of X has only one neighborhood, X itself.

(3) **Metric Topology.** Let (X, ρ) be a * metric space, i.e., a set X provided with a * metric ρ . For a positive number ε , the ε -neighborhood of a point x is defined to be the set $U_\varepsilon(x) = \{y \mid y \in X, \rho(x, y) < \varepsilon\}$. Let $\mathfrak{U}(x)$ be the family of all sets V such that $U_\varepsilon(x) \subset V$ for some ε ; then the assignment $x \rightarrow \mathfrak{U}(x)$ satisfies (U) and hence defines a topology. This topology is the **metric topology** for the metric space (X, ρ) .

(4) **Order Topology.** Let X be a set * linearly ordered by \leq . For each point x in X , let $\mathfrak{U}(x)$ be the family of all subsets U such that $x \in \{y \mid a < y < b\} \subset U$ for some a, b . The function $x \rightarrow \mathfrak{U}(x)$ satisfies (U) and defines the **order topology** for the linearly ordered set X .

(5) **Convergence and Topology.** We can define the notion of convergence in a topological space, and conversely we can define a topology using convergence as a primitive notion (\rightarrow 87 Convergence). In particular, for a metric space, the metric topology can be defined in terms of convergent sequences (\rightarrow 273 Metric Spaces).

D. Generalized Topological Spaces

When a space X is equipped with a closure operator that does not satisfy all of (C), the

space is called a **generalized topological space** by some authors. Topological implications of each axiom in (C) have been investigated for such spaces.

E. Local Bases

Let X be a topological space, and let x be a point of X . A collection $\mathcal{U}_0(x)$ of neighborhoods of x is called a **base for the neighborhood system (fundamental system of neighborhoods)** of a point x or **local base at x** if each neighborhood of x contains a member of $\mathcal{U}_0(x)$. Let $\{\mathcal{U}_0(x) | x \in X\}$ be a system of local bases; then the system has the following properties (\mathcal{U}_0):

- (1) For each V in $\mathcal{U}_0(x)$, $x \in V \subset X$.
- (2) If $V_1, V_2 \in \mathcal{U}_0(x)$, then there is a V_3 in $\mathcal{U}_0(x)$ such that $V_3 \subset V_1 \cap V_2$.
- (3) For each V in $\mathcal{U}_0(x)$, there exists a $W \subset V$ in $\mathcal{U}_0(x)$ such that for each y in W , V contains some member of $\mathcal{U}_0(y)$.

Conversely, suppose that $\{\mathcal{U}_0(x) | x \in X\}$ is a system satisfying (\mathcal{U}_0). For each x in X , let $\mathcal{U}(x)$ consist of all subsets V of X such that $V \supset U$ for some U in $\mathcal{U}_0(x)$. Then the system $\{\mathcal{U}(x) | x \in X\}$ satisfies (\mathcal{U}) and therefore defines a topology for X . This topology is called the topology determined by the system $\{\mathcal{U}_0(x) | x \in X\}$.

For instance, in a metric space X , the set of ε -neighborhoods of x ($\varepsilon > 0$) is a local base at x with respect to the metric topology. In an arbitrary topological space, the collection of all open sets containing x , i.e., the **open neighborhoods** of x , is a local base at x .

Two systems satisfying (\mathcal{U}_0) are called **equivalent** if they determine the same topology. For systems $\{\mathcal{U}_0(x) | x \in X\}$ and $\{\mathcal{V}_0(x) | x \in X\}$ to be equivalent it is necessary and sufficient that for each x in X each member of $\mathcal{U}_0(x)$ contain a member of $\mathcal{V}_0(x)$ and each member of $\mathcal{V}_0(x)$ contain a member of $\mathcal{U}_0(x)$.

Sometimes the word "neighborhood" stands for a member of a local base or for an open neighborhood. However, this convention is not used here.

F. Bases and Subbases

A family \mathfrak{D}_0 of open sets of a topological space X is called a **base for the topology (base for the space, or open base)** if each open set is the union of a subfamily of \mathfrak{D}_0 . A base \mathfrak{D}_0 for the topology of a topological space X has the following properties (\mathcal{O}_0):

- (1) $\bigcup \mathfrak{D}_0 = X$.
- (2) If $W_1, W_2 \in \mathfrak{D}_0$ and $x \in W_1 \cap W_2$, then there is a W_3 in \mathfrak{D}_0 such that $x \in W_3 \subset W_1 \cap W_2$.

Conversely, if a family \mathfrak{D}_0 of subsets of a set

X satisfies (\mathcal{O}_0), then \mathfrak{D}_0 is a base for a unique topology. A member of \mathfrak{D}_0 is called a **basic open set**.

A family \mathfrak{D}_{00} of open sets of a topological space X is a **subbase for the topology (or sub-base for the space)** if the family of all finite intersections of members of \mathfrak{D}_{00} is a base for the topology. If \mathfrak{D}_{00} a subbase for the topology of a topological space X , then $\bigcup \mathfrak{D}_{00} = X$. Conversely, if \mathfrak{D}_{00} is a family of subsets of a set X such that $\bigcup \mathfrak{D}_{00} = X$, then the family of all finite intersections of members of \mathfrak{D}_{00} is a base for a unique topology τ . A subset of X is open for τ if and only if it is the union of a family of finite intersections of members of \mathfrak{D}_{00} . The system of open sets relative to τ is said to be **generated** by the family \mathfrak{D}_{00} . Thus any family of sets defines a topology for its union.

A set \mathfrak{F} of subsets of a topological space is called a **network** if for each point x and its neighborhood U there is a member $F \in \mathfrak{F}$ such that $x \in F \subset U$ (A. V. Arkhangel'skii, 1959). If all $F \in \mathfrak{F}$ are required to be open, the network \mathfrak{F} is exactly an open base.

G. Continuous Mappings

A mapping f on a topological space X into a topological space Y is called **continuous** at a point a of X if it satisfies one of the following equivalent conditions:

- (1) For each neighborhood V of $f(a)$, there is a neighborhood U of a such that $f(U) \subset V$. (1')
- For each neighborhood V of $f(a)$, the inverse image $f^{-1}(V)$ is a neighborhood of a .
- (2) For an arbitrary subset A of X such that $a \in \bar{A}$, $f(a) \in \overline{f(A)}$.

Continuity can also be defined in terms of convergence (\rightarrow 87 Convergence).

If f is continuous at each point of X , f is said to be **continuous**. Continuity of f is equivalent to each of the following conditions:

- (1) For each open subset O of Y , the inverse image $f^{-1}(O)$ is open in X .
- (1') The inverse image under f of each member of a subbase for the topology of Y is open in X .
- (2) For each closed subset F of Y , the inverse image $f^{-1}(F)$ is closed.
- (3) For each subset A of X , $f(\bar{A}) \subset \overline{f(A)}$.

The image $f(X)$ of X under a continuous mapping f is called a **continuous image** of X . Let X, Y , and Z be topological spaces, and let $f: X \rightarrow Y$ and $g: Y \rightarrow Z$ be mappings. If f is continuous at a point a of X and g is continuous at $f(a)$, then the composite mapping $g \circ f: X \rightarrow Z$ is continuous at the point a . Hence if f and g are continuous, so is $g \circ f$.

When a continuous mapping $f: X \rightarrow Y$ is

†bijective and f^{-1} is continuous, the mapping f is called a **homeomorphism** (named by H. Poincaré, 1895) or **topological mapping**. Two topological spaces X and Y are **homeomorphic**, $X \approx Y$, if there is a homeomorphism $f: X \rightarrow Y$.

The relation of being homeomorphic is an equivalence relation. A property which, when held by a topological space, is also held by each space homeomorphic to it is a **topological property** or **topological invariant**. The problem of deciding whether or not given spaces are homeomorphic is called the **homeomorphism problem**.

A mapping $f: X \rightarrow Y$ is called **open** (resp. **closed**) if the image under f of each open (resp. closed) subset of X is open (closed) in Y . A continuous bijection that is either open or closed is a homeomorphism.

A continuous surjection $f: X \rightarrow Y$ is called a **quotient mapping** if $U \subset Y$ is open whenever $f^{-1}(U)$ is open (\rightarrow Section L). If moreover $f|f^{-1}(S)$ is quotient for each $S \subset Y$ as a mapping from the subspace (\rightarrow Section J) $f^{-1}(S)$ onto the subspace S , then f is called a **hereditarily quotient mapping**. Open or closed continuous mappings are hereditarily quotient mappings.

H. Comparison of Topologies

When a set X is provided with two topologies τ_1 and τ_2 and the identity mapping: $(X, \tau_1) \rightarrow (X, \tau_2)$ is continuous, the topology τ_1 is said to be **stronger** (**larger** or **finer**) than the topology τ_2 , τ_2 is said to be **weaker** (**smaller** or **coarser**) than τ_1 , and the notation $\tau_1 \geq \tau_2$ or $\tau_2 \leq \tau_1$ is used. Let \mathfrak{O}_i , \mathfrak{F}_i , \mathfrak{U}_i , and a_i be the system of open sets, system of closed sets, neighborhood system, and closure operation for X relative to the topology τ_i ($i = 1, 2$), respectively. Then each of the following is equivalent to the statement $\tau_1 \geq \tau_2$:

- (1) $\mathfrak{O}_1 \supset \mathfrak{O}_2$.
- (2) $\mathfrak{F}_1 \supset \mathfrak{F}_2$.
- (3) For each x in X , $\mathfrak{U}_1(x) \supset \mathfrak{U}_2(x)$.
- (4) $A^{a_1} \subset A^{a_2}$ for each subset A of X .

Let S be the family of all topologies for X . Then S is ordered by the relation \geq . The discrete topology is the strongest topology for X . If $\{\tau_\lambda | \lambda \in \Lambda\}$ is a subfamily of S , then among the topologies stronger than each τ_λ , there is a weakest one $\tau_1 = \sup\{\tau_\lambda | \lambda \in \Lambda\}$. Similarly, among the topologies weaker than each τ_λ , there is a strongest one $\tau_2 = \inf\{\tau_\lambda | \lambda \in \Lambda\}$. In fact, let \mathfrak{O}_λ be the family of all open sets relative to τ_λ ; then the system of open sets for τ_1 is generated by $\bigcup_{\lambda \in \Lambda} \mathfrak{O}_\lambda$, and the system of open sets for τ_2 is precisely $\bigcap_{\lambda \in \Lambda} \mathfrak{O}_\lambda$. The family S is therefore a \dagger complete lattice.

I. Induced Topology

Let f be a mapping from a set X into a topological space Y . Then the family $\{f^{-1}(O) | O \text{ is open in } Y\}$ satisfies axioms (O) and defines a topology for X . This topology is called the **topology induced by f** (or simply **induced topology**), and it is characterized as the weakest one among the topologies for X relative to which the mapping f is continuous.

J. Subspaces

Let (X, τ) be a topological space and M be a subset of X . The topology for M induced by the inclusion mapping $f: M \rightarrow X$, i.e., the mapping f defined by $f(x) = x$ for each x in M , is called the **relativization** of τ to M or the **relative topology**. The set M provided with the relative topology is called a **subspace** of the topological space (X, τ) . Topological terms, when applied to a subspace, are frequently preceded by the adjective "relative" to avoid ambiguity. Thus a **relative neighborhood** of a point x in M is a set of the form $M \cap U$, where U is a neighborhood of x in X . A **relatively open** (**relatively closed**) set in M is a set of the form $M \cap A$, where A is open (closed) in X . For a subset A of M , the relative closure of A in M is $M \cap \bar{A}$, where \bar{A} is the closure of A in X . A mapping $f: X \rightarrow Y$ is called an **embedding** if f is a homeomorphism from X to the subspace $f(X)$, and in this case X is said to be **embedded** into Y . A property P is said to hold **locally** on a topological space X if each point x of X has a neighborhood U such that the property P holds on the subspace U . A subset A of X is **locally closed** if for each point x of X , there exists a neighborhood V of x such that $V \cap A$ is relatively closed in V . A subset of X is locally closed if and only if it can be represented as $O \cap F$, where O is open and F is closed in X .

K. Product Spaces

Let X be a set, and for each member λ of an index set Λ , let f_λ be a mapping of X into a topological space X_λ . Then there is a weakest topology for X that makes each f_λ continuous. In fact, this topology is $\sup\{\tau_\lambda\}$, where τ_λ is the topology for X induced by f_λ . In particular, let $\{X_\lambda | \lambda \in \Lambda\}$ be a family of topological spaces, and let X be the Cartesian product $\prod_{\lambda \in \Lambda} X_\lambda$. Then the weakest topology for X such that each projection $\text{pr}_\lambda: X \rightarrow X_\lambda$ is continuous is called the **product topology** or **weak topology**. The Cartesian product $\prod_{\lambda \in \Lambda} X_\lambda$ equipped with the product topology is called the **product topological space** or simply the **product space**.

or **direct product** of the family $\{X_\lambda | \lambda \in \Lambda\}$ of topological spaces. If \mathfrak{D} is the family of all open subsets of X_λ , the union $\bigcup_\lambda \text{pr}_\lambda^{-1}(\mathfrak{D}_\lambda)$ is a subbase for the product topology. If $x = \{x_\lambda\}$ is a point of X , then sets of the type $\bigcap_{j=1}^n \text{pr}_j^{-1}(U_j) = \prod_{\lambda \neq \lambda_1, \dots, \lambda_n} X_\lambda \times U_1 \times \dots \times U_n$ form a local base at x for the product topology, where $\lambda_1, \dots, \lambda_n \in \Lambda$ and U_j is a neighborhood of x_{λ_j} . Each projection $\text{pr}_\lambda: X \rightarrow X_\lambda$ is continuous and open, and a mapping f from a topological space Y into the product space $\prod_\lambda X_\lambda$ is continuous if and only if $\text{pr}_\lambda \circ f: Y \rightarrow X_\lambda$ is continuous for each λ . Given a family $\{f_\lambda\}$ of continuous mappings $f_\lambda: X_\lambda \rightarrow Y_\lambda$, the product mapping $\prod_\lambda f_\lambda: \prod_\lambda X_\lambda \rightarrow \prod_\lambda Y_\lambda$ is continuous with respect to the product topologies.

For the Cartesian product $\prod_\lambda X_\lambda$ of a family $\{X_\lambda | \lambda \in \Lambda\}$ of topological spaces, there is another topology called the **box topology** (or **strong topology**). A base for the box topology is the family of all sets $\prod_\lambda O_\lambda$, where O_λ is open in X_λ for each λ . For a point $x = \{x_\lambda\}$, the family of all sets of the form $\prod_\lambda U_\lambda$ is a local base at x relative to the box topology, where U_λ is a neighborhood of x_λ for each λ . With respect to the box topology, each projection $\text{pr}_\lambda: \prod_\lambda X_\lambda \rightarrow X_\lambda$ is continuous and open, and the product mapping $\prod f_\lambda: \prod_\lambda X_\lambda \rightarrow \prod_\lambda Y_\lambda$ of a family $\{f_\lambda\}$ of continuous mappings $f_\lambda: X_\lambda \rightarrow Y_\lambda$ is continuous. For a finite product of topological spaces, the product topology agrees with the box topology, but for an arbitrary product the product topology is weaker than the box topology. For the Cartesian product of topological spaces the usual topology considered is the product topology rather than the box topology.

L. Quotient Spaces

Let f be a mapping of a topological space X onto a set Y . The **quotient topology** for Y (relative to the mapping f) is the strongest topology for Y such that f is continuous. A subset O of Y is open relative to the quotient topology if and only if $f^{-1}(O)$ is open. Given an equivalence relation \sim on a topological space X , the quotient set $Y = X/\sim$ provided with the quotient topology relative to the projection $\varphi: X \rightarrow Y$ is called the **quotient topological space** (or simply **quotient space**). A mapping f from the quotient space $Y = X/\sim$ into a topological space is continuous if and only if $f \circ \varphi$ is continuous.

A **partition** of a space X is a family $\{A_\lambda | \lambda \in \Lambda\}$ of pairwise disjoint subsets of X such that $\bigcup_\lambda A_\lambda = X$. A partition $\{A_\lambda\}$ of a topological space X determines an equivalence relation \sim on X such that the family $\{A_\lambda\}$ is precisely

the family of all equivalence classes under \sim , and therefore the partition determines the quotient space $Y = X/\sim$. This space is called the **identification space** of X by the given partition. Each member A_λ of the partition can be regarded as a point of Y , and the projection $\varphi: X \rightarrow Y$ satisfies $\varphi(x) = A_\lambda$ whenever $x \in A_\lambda$. A partition $\{A_\lambda | \lambda \in \Lambda\}$ of a topological space is called **upper semicontinuous** if for each A_λ and each open set U containing A_λ , there is an open set V such that $A_\lambda \subset V \subset U$, and V is the union of members of $\{A_\lambda | \lambda \in \Lambda\}$. A partition $\{A_\lambda | \lambda \in \Lambda\}$ is upper semicontinuous if and only if the projection $\varphi: X \rightarrow Y = \{A_\lambda | \lambda \in \Lambda\}$ is a closed mapping.

M. Topological Sums

Let X be a set, and for each member λ of an index set Λ , let f_λ be a mapping of a topological space X_λ to X . Then the family $\{O \subset X | f_\lambda^{-1}(O) \text{ is open for any } \lambda\}$ satisfies the axioms of the open sets. This topology τ is characterized as the strongest one for X that makes each f_λ continuous. A mapping g on X with τ to a topological space Y is continuous if and only if $g \circ f_\lambda: X_\lambda \rightarrow Y$ is continuous for each $\lambda \in \Lambda$. The simplest is the case where X is the disjoint union of X_λ and f_λ is the inclusion mapping. Then we call the topological space X the **direct sum** or the **topological sum** of $\{X_\lambda\}$ and denote it by $\oplus X_\lambda$ or $\coprod X_\lambda$. More generally let the set X be the union of topological spaces $\{X_\lambda\}_{\lambda \in \Lambda}$ such that for each λ and $\mu \in \Lambda$ the relative topologies of $X_\lambda \cap X_\mu$ from X_λ and X_μ coincide. Then we call the topology τ the **weak topology** with respect to $\{X_\lambda\}$. If $X_\lambda \cap X_\mu$ is closed (resp. open) in X_μ for any μ , then X_λ is closed (resp. open) in X and the original topology of X_λ coincides with the relative topology. If, moreover, for each subset Γ of Λ , $F = \bigcup_{\lambda \in \Gamma} X_\lambda$ is closed and the weak topology of F with respect to $\{X_\lambda\}_{\lambda \in \Gamma}$ coincides with the relative topology induced by τ , then X with τ is said to have the **hereditarily weak topology** with respect to $\{X_\lambda\}$ (or to be **dominated** by $\{X_\lambda\}$). A topological space has the hereditarily weak topology with respect to any locally finite closed covering, and every CW-complex (\rightarrow 70 Complexes) has the hereditarily weak topology with respect to the covering of all finite subcomplexes.

When $\{X_n\}$ is an increasing sequence of topological spaces such that each X_n is a subspace of X_{n+1} , then the union $X = \bigcup X_n$ with the weak topology is called the **inductive limit** of $\{X_n\}$ and is denoted by $\varinjlim X_n$. Each X_n may again be regarded as a subspace of X .

N. Baire Spaces

For a subset A of a topological space X , the set $X - \bar{A}$ is called the **exterior** of A , and the set $\bar{A} \cap \overline{X - A}$ is called the **boundary** of A , denoted by $\text{Bd } A$, $\text{Fr } A$, or ∂A . A point belonging to the exterior (boundary) of A is an **exterior point** (**boundary point** or **frontier point**) of A . If the closure of A is X , then A is said to be **dense** in X . When $X - A$ is dense in X , i.e., when the interior of A is empty, A is called a **boundary set** (or **border set**), and if the closure \bar{A} is a boundary set, A is said to be **nowhere dense**. The union of a countable family of nowhere dense sets is called a **set of the first category** (or **meager set**). A set that is not of the first category is called a **set of the second category** (or **nonmeager set**). The complement of a set of the first category is called a **residual set**. In the space \mathbf{R} of real numbers, the set \mathbf{Q} of all rational numbers is of the first category, and the set $\mathbf{R} - \mathbf{Q}$ of all irrational numbers is of the second category. Both \mathbf{Q} and $\mathbf{R} - \mathbf{Q}$ are dense in X and hence are boundary sets. The union of a finite family of nowhere dense sets is nowhere dense, and the union of a countable family of sets of the first category is also of the first category. A subset A of X is nowhere dense in X if and only if for each open set O , $O \cap A$ is not dense in O .

A topological space X is called a **Baire space** (Baire, 1899) if each subset of X of the first category has an empty interior. Each of the following conditions is necessary and sufficient for a space X to be a Baire space:

- (1) Each nonempty open subset of X is of the second category.
- (2) If F_1, F_2, \dots is a sequence of closed subsets of X such that the union $\bigcup_{n=1}^{\infty} F_n$ has an interior point, then at least one F_n has an interior point.
- (3) If O_1, O_2, \dots is a sequence of dense open subsets of X , then the intersection $\bigcap_{n=1}^{\infty} O_n$ is dense in X .

An open subset of a Baire space is a Baire space for the relative topology. A topological space that is homeomorphic to a complete metric space (\rightarrow 436 Uniform Spaces I) is a Baire space (**Baire-Hausdorff theorem**). A locally compact Hausdorff space (\rightarrow Section V) is also a Baire space. The class of Čech-complete completely regular spaces (\rightarrow Section T) includes both of these spaces, but there are also Baire spaces that are not in the class. A subset A of a topological space is said to satisfy **Baire's condition** or to have the **Baire property** if there exist an open set O and sets P_1, P_2 of the first category such that $A = (O \cup P_1) - P_2$. A \ast Borel set satisfies Baire's condition.

O. Accumulation Points

A point x is called an **accumulation point**, or a **cluster point** of a subset A of a topological space X if $x \in \overline{A - \{x\}}$. The set of all accumulation points of a set A is called the **derived set** of A and is denoted by A' or A^d . A point x belongs to A' if and only if each neighborhood of x contains a point of A other than x itself. A point belonging to the set $A^s = A - A'$ is called an **isolated point** of A , and a set A consisting of isolated points only, i.e., $A = A^s$, is said to be **discrete**. If each nonempty subset of A contains an isolated point, then A is said to be **scattered**; and if A does not possess an isolated point, i.e., $A \subset A'$, then A is said to be **dense in itself**. The largest subset of A which is dense in itself is called the **kernel** of A . If $A = A'$, then A is called a **perfect set**.

If x is an accumulation point of A , then for each neighborhood U of x , $U \cap (A - \{x\}) \neq \emptyset$. Furthermore, it is possible to classify an accumulation point of A according to the \ast cardinality of $U \cap (A - \{x\})$. A point x is called a **condensation point** of a set A if for each neighborhood U of x , the set $U \cap A$ is uncountable. A point x is a **complete accumulation point** of A if for each neighborhood U of x , the set $U \cap A$ has the same cardinality as A .

P. Countability Axioms

A topological space X satisfies the **first countability axiom** if each point x of X has a countable local base (F. Hausdorff [3]). Metric spaces satisfy the first countability axiom. In fact, the family of $(1/n)$ -neighborhoods ($n = 1, 2, \dots$) of a point is a local base of the point. The topology of a topological space that satisfies the first countability axiom is completely determined by convergent sequences. For instance, the closure of a subset A of such a space consists of all limits of sequences in A (\rightarrow 87 Convergence). A topological space X is said to satisfy the **second countability axiom** or to be **perfectly separable** if there is a countable base for the topology. \ast Euclidean spaces satisfy the second countability axiom. If X contains a countable dense subset, X is said to be **separable**. A space that satisfies the second countability axiom satisfies the first and is also a separable Lindelöf space (\rightarrow Section S). However, the converse is not true. Each of the following properties is independent of the others: separability, the first countability axiom, and the Lindelöf property. If a metric space is separable, then it satisfies the second countability axiom. There are metric spaces that are not separable.

Q. Separation Axioms

Topological spaces that are commonly encountered usually satisfy some of the following separation axioms.

(T_0) **Kolmogorov's axiom.** For each pair of distinct points, there is a neighborhood of one point of the pair that does not contain the other.

(T_1) **The first separation axiom or Fréchet's axiom.** For each pair x, y of distinct points, there are neighborhoods U of x and V of y such that $x \notin V$ and $y \notin U$.

Axiom (T_1) can be restated as follows:

(T_1') For each point x of the space, the singleton $\{x\}$ is closed.

(T_2) **The second separation axiom or Hausdorff's axiom** [3]. For each pair x, y of distinct points of the space X , there exist disjoint neighborhoods of x and y .

Axiom (T_2) is equivalent to the following:

(T_2) In the product space $X \times X$ the diagonal set Δ is closed.

(T_3) **The third separation axiom or Vietoris's axiom** (*Monatsh. Math. Phys.*, 31 (1921)).

Given a point x and a subset A such that $x \notin \bar{A}$, there exist disjoint open sets O_1 and O_2 such that $x \in O_1$ and $A \subset O_2$. (In this case, the sets $\{x\}$ and A are said to be **separated** by open sets.)

Axiom (T_3) can be restated as (T_3') or (T_3''):

(T_3') For each point x of the space, there is a local base at x consisting of closed neighborhoods of x .

(T_3'') An arbitrary closed set and a point not belonging to it can be separated by open sets.

(T_4) **The fourth separation axiom or Tietze's first axiom** (*Math. Ann.*, 88 (1923)). Two disjoint closed sets F_1 and F_2 can be separated by open sets, i.e., there exist disjoint open sets O_1 and O_2 such that $F_1 \subset O_1$ and $F_2 \subset O_2$.

(T_5) **Tietze's second axiom.** Whenever two subsets A_1 and A_2 satisfy $A_1 \cap \bar{A}_2 = \bar{A}_1 \cap A_2 = \emptyset$, A_1 and A_2 can be separated by open sets.

It is easily seen that (T_5) \Rightarrow (T_4), (T_0) and (T_3) \Rightarrow (T_2), (T_4) and (T_1) \Rightarrow (T_3). Axiom (T_4) is equivalent to each of (T_4') and (T_4''):

(T_4') Whenever F_1 and F_2 are disjoint closed subsets, there exists a continuous function f on the space into the interval $[0, 1]$ such that f is identically 0 on F_1 and 1 on F_2 .

(T_4'') Each real-valued continuous function defined on a closed subspace can be extended to a real-valued continuous function on the entire space.

The implications (T_4) \Rightarrow (T_4') and (T_4) \Rightarrow (T_4'') are known as **Uryson's lemma** (*Math. Ann.*, 94 (1925)) and the **Tietze extension theorem** (*J. Reine Angew. Math.*, 145 (1915)), respectively. In addition, there are two more related axioms:

($T_{3\frac{1}{2}}$) **Tikhonov's separation axiom.** For each closed subset F and each point x not in F , there is a real-valued continuous function f on the space such that $f(x) = 0$ and f is identically 1 on F .

(T_6) (N. Vedenisov). For each closed subset F , there is a real-valued continuous function f on the space such that $F = \{x \mid f(x) = 0\}$.

Axioms (T_5) and (T_6) are equivalent to the following (T_5') and (T_6'), respectively:

(T_5') Each subspace satisfies (T_4)

(T_6') X satisfies (T_4) and each closed set is a $^*G_\delta$ -set.

The following implications are valid: ($T_{3\frac{1}{2}}$) \Rightarrow (T_3), (T_6) \Rightarrow (T_5), (T_4) and (T_1) \Rightarrow ($T_{3\frac{1}{2}}$).

Table 1 gives a classification of topological spaces by the separation axioms. Each line represents a special case of the preceding line.

A * metrizable space is perfectly normal, but the converse is false (for metrization theorems \rightarrow 273 Metric Spaces). Among the spaces satisfying the second countability axiom, regular spaces are normal (**Tikhonov's theorem**, *Math. Ann.*, 95 (1925)) and metrizable (**Tikhonov-Uryson theorem**; P. Uryson, *Math. Ann.*, 94 (1925)).

Table 2 shows whether various topological properties are preserved in subspaces, product spaces, and quotient spaces. The topological properties considered are T_1 , T_2 = Hausdorff, T_3 = regular, CR = completely regular, T_4 = normal, T_5 = completely normal, M = metrizable, C_1 = first axiom of countability, C_Π = second axiom of countability, C = compact, S = separable, and L = Lindelöf. Each position is filled with \bigcirc or \times according as the property (say, P) listed at the head of the column is preserved or not in the sort of space listed on the left obtained from space(s) all having property P.

R. Coverings

A family $\mathfrak{M} = \{M_\lambda\}_{\lambda \in \Lambda}$ of subsets of a set X is called a **covering** of a subset A of X if $A \subset \bigcup_\lambda M_\lambda$. If \mathfrak{M} is finite (countable), it is called a **finite covering** (**countable covering**). An **open (closed) covering** is a covering consisting of open (closed) sets.

A family \mathfrak{M} of subsets of a topological space X is said to be **locally finite** if for each point x of X , there is a neighborhood of x which intersects only a finite number of members of \mathfrak{M} . If moreover $\{\bar{M}_\lambda\}_{\lambda \in \Lambda}$ is disjoint, then \mathfrak{M} is called **discrete**. \mathfrak{M} is called **star-finite** if each member of \mathfrak{M} intersects only a finite number of members of \mathfrak{M} . A **σ -locally finite** or **σ -discrete** family of subsets of X is respectively the union of a countable number of locally finite or discrete families of subsets of X . A covering \mathfrak{M}

Table 1. Separation Axioms

Axioms	Spaces Satisfying the Axioms
(T ₀)	T ₀ -space (Kolmogorov space)
(T ₁)	T ₁ -space (Kuratowski space)
(T ₂)	T ₂ -space (Hausdorff space, separated space)
(T ₀) and (T ₃)	T ₃ -space (regular space)
(T ₁) and (T _{3½})	Completely regular space (Tikhonov space)
(T ₁) and (T ₄)	T ₄ -space (normal space)
(T ₁) and (T ₅)	T ₅ -space (completely normal space, hereditarily normal space)
(T ₁) and (T ₆)	T ₆ -space (perfectly normal space)

Table 2. Topological Properties and Spaces

Space	T ₁	T ₂	T ₃	CR	T ₄	T ₅	M	C _I	C _{II}	C	S	L
Subspace	○	○	○	○	×	○	○	○	○	×	×	×
Closed subspace	○	○	○	○	○	○	○	○	○	○	×	○
Open subspace	○	○	○	○	×	○	○	○	○	×	○	×
Product	○	○	○	○	×	×	×	×	×	○	×	×
Countable product	○	○	○	○	×	×	○	○	○	○	○	×
Quotient space	×	×	×	×	×	×	×	×	×	○	○	○

is called **point-finite** if each infinite number of members of \mathfrak{M} has an empty intersection. A covering \mathfrak{M} is a **refinement** of a covering \mathfrak{N} (written $\mathfrak{M} < \mathfrak{N}$) if each member of \mathfrak{M} is contained in a member of \mathfrak{N} . The **order** of the covering \mathfrak{M} is the least integer r such that any subfamily of \mathfrak{M} consisting of $r + 1$ members has an empty intersection.

Let \mathfrak{M} be a covering of X , and let A be a subset of X . The **star** of A relative to \mathfrak{M} , denoted by $S(A, \mathfrak{M})$, is the union of all members of \mathfrak{M} whose intersection with A is nonempty. Let \mathfrak{M}^Δ denote the family $\{S(\{x\}, \mathfrak{M})\}_{x \in X}$ and \mathfrak{M}^* the family $\{S(M, \mathfrak{M})\}_{M \in \mathfrak{M}}$. Then \mathfrak{M}^Δ and \mathfrak{M}^* are coverings of X , and $\mathfrak{M} < \mathfrak{M}^\Delta < \mathfrak{M}^* < \mathfrak{M}^{\Delta\Delta}$. A covering \mathfrak{M} is a **star refinement** of a covering \mathfrak{N} if $\mathfrak{M}^* < \mathfrak{N}$, and \mathfrak{M} is a **barycentric refinement** (or **Δ -refinement**) of \mathfrak{N} if $\mathfrak{M}^\Delta < \mathfrak{N}$.

A sequence $\mathfrak{M}_1, \mathfrak{M}_2, \dots$ of open coverings of a topological space is called a **normal sequence** if $\mathfrak{M}_{n+1}^\Delta < \mathfrak{M}_n$ for $n = 1, 2, \dots$, and an open covering \mathfrak{M} is said to be a **normal covering** if there is a normal sequence $\mathfrak{M}_1, \mathfrak{M}_2, \dots$ such that $\mathfrak{M}_1 < \mathfrak{M}$. The **support** (or **carrier**) of a real-valued function f on a topological space X is defined to be the closure of the set $\{x \mid f(x) \neq 0\}$. Let $\{f_x\}_{x \in A}$ be a family of continuous nonnegative real-valued functions on a topological space X , and for each α in A , let C_α be the support of f_α . The family $\{f_x\}_{x \in A}$ is called a **partition of unity** if the family $\{C_\alpha\}_{\alpha \in A}$ is locally finite and $\sum_\alpha f_\alpha(x) = 1$ for each x in X . If the covering $\{C_\alpha\}_{\alpha \in A}$ is a refinement of a covering \mathfrak{M} , the family $\{f_x\}_{x \in A}$ is called a **partition of unity subordinate to the covering \mathfrak{M}** . A partition of unity subordinate to a covering \mathfrak{M} exists only if \mathfrak{M} is a normal covering (\rightarrow Section X). If ρ is a continuous pseudometric on a T_1 -space X , then define a covering M_n for

each natural number n by $M_n = \{U(x; 2^{-n})\}_{x \in X}$, where $U(x; \varepsilon) = \{y \mid \rho(x, y) < \varepsilon\}$. Then the sequence $\mathfrak{M}_1, \mathfrak{M}_2, \dots$ is a normal sequence of open coverings. Conversely, given a normal sequence $\mathfrak{M}_1, \mathfrak{M}_2, \dots$ of open coverings of X , there exists a continuous pseudometric ρ such that $\rho(x, y) \leq 2^{-n}$ whenever $x \in S(y, \mathfrak{M}_n)$, and $\rho(x, y) \geq 2^{-n-1}$ whenever $x \notin S(y, \mathfrak{M}_n)$. If in addition for each x the family $\{S(x, \mathfrak{M}_n) \mid n = 1, 2, \dots\}$ is a local base at x , then the metric topology of ρ agrees with the topology of X .

S. Compactness

If each open covering of a topological space X admits a finite open covering as its refinement, the space X is called **compact**; if each open covering of X admits a countable open refinement, X is said to be a **Lindelöf space** (P. Alexandrov and P. Uryson, *Verh. Akad. Wetensch.*, Amsterdam, 19 (1929)); if each open covering of X admits a locally finite open refinement, X is called **paracompact** (J. Dieudonné, *J. Math. Pures Appl.*, 23 (1944)); and if each open covering of X admits a star-finite open refinement, X is said to be **strongly paracompact** (C. H. Dowker, *Amer. J. Math.*, 69 (1947)) or to have the **star-finite property** (K. Morita, *Math. Japonicae*, 1 (1948)). The space X is compact (Lindelöf) if for each open covering \mathfrak{M} of X , there is a finite (countable) subfamily of \mathfrak{M} whose union is X .

The following properties for a topological space X are equivalent: (1) The space X is compact. (2) If a family $\{F_\lambda\}_{\lambda \in \Lambda}$ of closed subsets of X has the **finite intersection property**, i.e., each finite subfamily of $\{F_\lambda\}_{\lambda \in \Lambda}$ has non-empty intersection, then $\bigcap_\lambda F_\lambda \neq \emptyset$. (3) Each

infinite subset of X has a complete accumulation point. (4) Each \dagger net has a convergent \dagger subnet. (5) Each \dagger universal net and each \dagger ultrafilter converge.

If a subset A of X is compact for the relative topology, A is called a **compact** subset. A subset A of X is said to be **relatively compact** if the closure of A in X is a compact subset. A closed subset of a compact topological space is compact, and a compact subset of a Hausdorff space is closed. A continuous image of a compact space is compact, each continuous mapping of a compact space into a Hausdorff space is a closed mapping, and a continuous bijection of a compact space onto a Hausdorff space is a homeomorphism. The product space of a family $\{X_\lambda\}_{\lambda \in \Lambda}$ of topological spaces is compact if and only if each factor space is compact (**Tikhonov's product theorem**, *Math. Ann.*, 102 (1930)). A compact Hausdorff space is normal. A compact Hausdorff space is metrizable if and only if it satisfies the second countability axiom. A metric space or a \dagger uniform space is compact if and only if it is \dagger totally bounded and \dagger complete. A subset of a Euclidean space is compact if and only if it is closed and bounded. In a discrete space only finite subsets are compact. The cardinality of a compact Hausdorff space with the first countability axiom cannot exceed the power of the continuum (Arkhangel'skii).

There are a number of conditions related to compactness. A topological space is **sequentially compact** if each sequence in X has a convergent subsequence. A space X is **countably compact** (M. Fréchet [1]) if each countable open covering of X contains a finite subfamily that covers X . A space X is **pseudocompact** (E. Hewitt, 1948) if each continuous real-valued function on X is bounded. Some authors use *compact* and *bicompact* for what we call countably compact and compact, respectively. N. Bourbaki [9] uses *compact* and *quasicompact* instead of compact Hausdorff and compact, respectively. A T_1 -space is countably compact if and only if each infinite set possesses an accumulation point. If X is countably compact, then X is pseudocompact, and if X is normal, the converse also holds. If a \dagger complete uniform space is pseudocompact, then it is compact. A space satisfying the second countability axiom is compact if and only if it is sequentially compact. If X is sequentially compact, then X is countably compact, and if X satisfies the first countability axiom, the converse is true.

T. Compactification

A **compactification** of a topological space X consists of a compact space Y and a homeo-

morphism of X onto a dense subspace X_1 of Y . We can always regard X as a dense subspace of a compactification Y . If X is completely regular, then there is a Hausdorff compactification Y such that each bounded real-valued continuous function on X can be extended continuously to Y . Such a compactification is unique up to homeomorphism; it is called the **Stone-Čech compactification** of X (E. Čech, *Ann. Math.*, 38 (1937); M. H. Stone, *Trans. Amer. Math. Soc.*, 41 (1937)) and is denoted by $\beta(X)$. Let $\{f_\lambda\}_{\lambda \in \Lambda}$ be the set of all continuous functions on a completely regular space X into the closed interval $I = [0, 1]$. Then a continuous mapping φ of X into a **parallelootope** $I^\Lambda = \prod_\lambda I_\lambda$ ($I_\lambda = I$) is defined by $\varphi(x) = \{f_\lambda(x)\}_{\lambda \in \Lambda}$, and the mapping φ is a homeomorphism of X onto the subspace $\varphi(X)$ of I^Λ (**Tikhonov's embedding theorem**, *Math. Ann.*, 102 (1930)). The closure $\overline{\varphi(X)}$ of $\varphi(X)$ in I^Λ is the Stone-Čech compactification of X . The natural mapping $\beta(X_1 \times X_2) \rightarrow \beta(X_1) \times \beta(X_2)$ is a homeomorphism if and only if $X_1 \times X_2$ is pseudocompact (I. Glicksberg, 1959).

For a topological space X , let ∞ be a point not in X , and define a topology on the union $X \cup \{\infty\}$ as follows: A subset U of $X \cup \{\infty\}$ is open if and only if either $\infty \notin U$ and U is open in X , or $\infty \in U$ and $X - U$ is a compact closed subset of X . The topological space $X \cup \{\infty\}$ thus obtained is compact, and if X is not already compact, the space $X \cup \{\infty\}$ is a compactification of X called the **one-point compactification** of X (P. S. Aleksandrov, *C. R. Acad. Sci. Paris*, 178 (1924)). The one-point compactification of a Hausdorff space is not necessarily Hausdorff. The one-point compactification of the n -dimensional Euclidean space \mathbf{R}^n is homeomorphic to the n -dimensional sphere S^n .

A completely regular space X is a $\dagger G_\delta$ -set in the Stone-Čech compactification $\beta(X)$ if and only if it is a G_δ -set in any Hausdorff space Y which contains X as a dense subspace. Then X is said to be **Čech-complete**.

U. Absolutely Closed Spaces

A Hausdorff space X is said to be **absolutely closed** (or **H-closed**; P. Aleksandrov and P. Uryson, 1929) if X is closed in each Hausdorff space containing it. A compact Hausdorff space is absolutely closed. A Hausdorff space is absolutely closed if and only if for each open covering $\{N_\lambda\}_{\lambda \in \Lambda}$ of X , there is a finite subfamily of $\{N_\lambda\}_{\lambda \in \Lambda}$ that covers X . The product space of a family of absolutely closed spaces is absolutely closed. Each Hausdorff space is a dense subset of an absolutely closed space. Similarly, a regular space X is said to be **r-**

closed if X is closed in each regular space containing it (N. Weinberg, 1941).

V. Locally Compact Spaces

A topological space X is said to be **locally compact** if each point of X has a compact neighborhood (P. Aleksandrov and P. Uryson, 1929). A * uniform space X is said to be **uniformly locally compact** if there is a member U of the * uniformity such that $U(x)$ is compact for each x in X (\rightarrow 436 Uniform Spaces). A noncompact space X is locally compact and Hausdorff if and only if the one-point compactification of X is Hausdorff, and this is the case if and only if X is homeomorphic to an open subset of a compact Hausdorff space. A locally compact Hausdorff space is completely regular, and for each point of the space, the family of all of its compact neighborhoods forms a local base at the point. A locally closed, hence open or closed, subset of a locally compact Hausdorff space is also locally compact for the relative topology. If a subspace A of a Hausdorff space X is locally compact, then A is a locally closed subset of X . The Euclidean space \mathbf{R}^n is locally compact, and hence each **locally Euclidean space**, i.e., a space such that each point admits a neighborhood homeomorphic to a Euclidean space, is locally compact. A topological space is called **σ -compact** if it can be expressed as the union of at most countably many compact subsets.

W. Proper (Perfect) Mappings

A mapping f of a topological space X into a topological space Y is said to be **proper** (N. Bourbaki [9]) (or **perfect** [14]) if it is continuous and for each topological space Z , the mapping $f \times 1: X \times Z \rightarrow Y \times Z$ is closed, where $(f \times 1)(x, z) = (f(x), z)$. A continuous mapping $f: X \rightarrow Y$ is proper if and only if it is closed and $f^{-1}(\{y\})$ is compact for each y in Y . Another necessary and sufficient condition is that if $\{x_\alpha\}_\alpha$ is a * net in X such that its image $\{f(x_\alpha)\}$ converges to $y \in Y$, then a subnet of $\{x_\alpha\}$ converges to an $x \in f^{-1}(y)$ in X . A continuous mapping of a compact space into a Hausdorff space is always proper. For a compact Hausdorff space X , a quotient space Y is Hausdorff if and only if the canonical projection $\varphi: X \rightarrow Y$ is proper.

For a continuous mapping f of a locally compact Hausdorff space X into a locally compact Hausdorff space Y , the following three conditions are equivalent: (1) f is proper. (2) For each compact subset K of Y , the inverse image $f^{-1}(K)$ is compact. (3) If $X \cup \{x_\infty\}$

Topological Spaces

and $Y \cup \{y_\infty\}$ are the one-point compactifications of X and Y , then the extension f_1 of f such that $f_1(x_\infty) = y_\infty$ is continuous.

The composition of two proper mappings is proper and the direct product of an arbitrary number of proper mappings is proper.

X. Paracompact Hausdorff Spaces

A paracompact Hausdorff space (often called simply a paracompact space) is normal. For a Hausdorff space X , the following five conditions are equivalent: (1) X is paracompact. (2) X is **fully normal** (J. W. Tukey [8]), i.e., each open covering of X admits an open barycentric refinement. (3) Each open covering has a partition of unity subordinate to it. (4) Each open covering is refined by a closed covering $\{F_\alpha | \alpha \in A\}$ that is **closure-preserving**, i.e., $\bigcup \{F_\beta | \beta \in B\}$ is closed for each $B \subset A$. (5) Each open covering $\{U_\alpha | \alpha \in A\}$ has a **cushioned refinement** $\{V_\alpha | \alpha \in A\}$, i.e., $\text{Cl}(\bigcup \{V_\beta | \beta \in B\}) \subset \bigcup \{U_\beta | \beta \in B\}$ for each $B \subset A$. The implication (1) \rightarrow (2) is **Dieudonné's theorem**. The implication (2) \rightarrow (1) is **A. H. Stone's theorem** (1948), from which it follows that each metric space is paracompact. The implications (5) \rightarrow (4) \rightarrow (1) is **Michael's theorem** (1959, 1957).

For normal spaces, the following weaker versions of (2) and (3) hold: A T_1 -space X is normal if and only if each finite open covering of X admits a finite open star refinement (or finite open barycentric refinement). For each locally finite open covering of a normal space, there is a partition of unity subordinate to it.

For a regular space X the following three conditions are equivalent: (1) X is paracompact. (2) Each open covering of X is refined by a σ -discrete open covering. (3) Each open covering of X is refined by a σ -locally finite open covering. **Tamano's product theorem**: For a completely regular space X to be paracompact it is necessary and sufficient that $X \times \beta(X)$ be normal (1960).

For a * connected locally compact space X , the following conditions are equivalent: (1) X is paracompact. (2) X is σ -compact. (3) In the one-point compactification $X \cup \{\infty\}$, the point ∞ admits a countable local base. (4) There is a locally finite open covering $\{U_\lambda\}_{\lambda \in \Lambda}$ of X such that \bar{U}_λ is compact for each λ . (5) X is the union of a sequence $\{U_n\}$ of open sets such that \bar{U}_n is compact and $\bar{U}_n \subset U_{n+1}$ ($n = 1, 2, \dots$). (6) X is strongly paracompact.

Every * F_σ -set of a paracompact Hausdorff space is paracompact (Michael, 1953). When a T_1 -space X has the hereditarily weak topology with respect to a closed covering $\{F_\lambda\}$, then X is paracompact Hausdorff (normal, completely normal or perfectly normal) if and only if each

F_λ is (Morita, 1954; Michael, 1956). In particular, every CW-complex is paracompact and perfectly normal (Morita, 1953).

Y. Normality and Paracompactness of Direct Products

A topological space X is discrete if $X \times Y$ is normal for any normal space Y (M. Atsugi and M. Rudin, 1978). There are a paracompact Lindelöf space X and a separable metric space Y such that the product $X \times Y$ is not normal (Michael, 1963). The following are conditions under which the products are normal or paracompact. Let m be an infinite \aleph -cardinal number. A topological space X is called m -paracompact if every open covering consisting of at most m open sets admits a locally finite open covering as its refinement. When m is countable, it is called **countably paracompact**. If X has an open base of at most m members, m -paracompact means paracompact. The following conditions are equivalent for a topological space X : (1) X is normal and countably paracompact; (2) The product $X \times Y$ is normal and countably paracompact for any compact metric space Y ; (3) $X \times I$ is normal, where $I = [0, 1]$ (C. H. Dowker, 1951). Rudin (1971) constructed an example of a collection-wise normal space (\rightarrow Section AA) that is not countably paracompact. When m is general the following conditions are equivalent: (1) X is normal and m -paracompact; (2) If Y is a compact Hausdorff space with an open base consisting of at most m sets, then $X \times Y$ is normal and m -paracompact; (3) $X \times I^m$ is normal; (4) $X \times \{0, 1\}^m$ is normal (Morita, 1961). In particular, the product $X \times Y$ of a paracompact Hausdorff space X and a compact Hausdorff space Y is paracompact (Dieudonné, 1944).

A topological space X is called a **P-space** if it satisfies the following conditions: Let Ω be an arbitrary set and $\{G(\alpha_1, \dots, \alpha_i) \mid \alpha_1, \dots, \alpha_i \in \Omega, i = 1, 2, \dots\}$ be a family of open sets such that $G(\alpha_1, \dots, \alpha_i) \subset G(\alpha_1, \dots, \alpha_i, \alpha_{i+1})$. Then there is a family of closed sets $\{F(\alpha_1, \dots, \alpha_i) \mid \alpha_1, \dots, \alpha_i \in \Omega, i = 1, 2, \dots\}$ such that $F(\alpha_1, \dots, \alpha_i) \subset G(\alpha_1, \dots, \alpha_i)$ and that if $\bigcup_{i=1}^{\infty} G(\alpha_1, \dots, \alpha_i) = X$ for a sequence $\{\alpha_i\}$, then $\bigcup_{i=1}^{\infty} F(\alpha_1, \dots, \alpha_i) = X$. Perfectly normal spaces, countably compact spaces, Čech-complete paracompact spaces and σ -compact regular spaces are P-spaces. Normal P-spaces are countably paracompact. A Hausdorff space X is a normal (resp. paracompact) P-space if and only if the product $X \times Y$ is normal (resp. paracompact) for any metric space Y (Morita, *Math. Ann.*, 154 1964).

The product $X \times Y$ of locally compact Hausdorff spaces X and Y is a locally compact Hausdorff space. If, in this case, X and Y are paracompact, then so is the product. If the direct product space $\prod_\lambda X_\lambda$ of metric spaces is normal, then X_λ are compact except for at most countably many λ , and hence the product space is paracompact (A. H. Stone, 1948).

A class \mathcal{C} of topological spaces is called **countably productive** if for a sequence X_i of members of \mathcal{C} their product $\prod X_i$ is again a member of \mathcal{C} . The classes of (complete) (separable) metric spaces form such examples. The class of paracompact and Čech-complete spaces is countably productive (Z. Frolík, 1960). A topological space X is called a **p-space** if it is completely regular and there is a sequence \mathfrak{M}_i of families of open sets in the Stone-Čech compactification $\beta(X)$ such that, for each point $x \in X$, $x \in \bigcap S(x, \mathfrak{M}_i) \subset X$ (Arkhangel'skii, 1963). X is called an **M-space** if there is a normal sequence \mathfrak{M}_i of open coverings of X such that if $K_1 \supset K_2 \supset \dots$ is a sequence of nonempty closed sets and $K_i \subset S(x, \mathfrak{M}_i)$, $i = 1, 2, \dots$, for an $x \in X$, then $\bigcap K_i \neq \emptyset$ (Morita, 1963). The class of paracompact p-spaces and that of paracompact Hausdorff M-spaces are the same and are countably productive. For a covering \mathfrak{F} of X and an $x \in X$ we set $C(x, \mathfrak{F}) = \bigcap \{F \mid x \in F \in \mathfrak{F}\}$. X is called a **Σ -space** if X admits a sequence \mathfrak{F}_i of locally finite closed coverings such that if $K_1 \supset K_2 \supset \dots$ is a sequence of nonempty closed sets and $K_i \subset C(x, \mathfrak{F}_i)$, $i = 1, 2, \dots$, for an $x \in X$, then $\bigcap K_i \neq \emptyset$ (K. Nagami, 1969). Σ -spaces are P-spaces. The class of all paracompact Σ -spaces is also countably productive. Among the above classes each one is always wider than its predecessors. Yet the product $X \times Y$ of a paracompact Hausdorff P-space X and a paracompact Hausdorff Σ -space Y is paracompact. Other examples of countably productive classes are the Suslin spaces and the Luzin spaces (\rightarrow Section CC) introduced by Bourbaki (1958), the **stratifiable spaces** by J. G. Ceder (1961) and C. J. R. Borges (1966), the **\aleph_0 -spaces** by Michael (1966) and the **σ -spaces** by A. Okuyama (1967).

Z. Strongly Paracompact Spaces

Regular Lindelöf spaces are strongly paracompact. Conversely, if a connected regular space is strongly paracompact, then it is a Lindelöf space (Morita, 1948). Hence a connected non-separable metric space is not strongly paracompact. Paracompact locally compact Hausdorff spaces and uniformly locally compact Hausdorff spaces are strongly paracompact.

These classes of spaces coincide under suitable \dagger uniform structures.

AA. Collectionwise Normal Spaces

A Hausdorff space X is called a **collectionwise normal space** if for each discrete collection $\{F_\alpha | \alpha \in A\}$ of closed sets of X there exists a disjoint collection $\{U_\alpha | \alpha \in A\}$ of open sets with $F_\alpha \subset U_\alpha$ ($\alpha \in A$) (R. H. Bing, 1951). If X satisfies an analogous condition for the case where each F_α is a singleton, X is called a **collectionwise Hausdorff space**. Paracompact Hausdorff spaces are collectionwise normal (Bing). Every point-finite open covering of a collectionwise normal space has a locally finite open refinement (Michael, Nagami).

A topological space X is called a **developable space** if it admits a sequence \mathcal{U}_i , $i = 1, 2, \dots$, of open coverings such that, for each point $x \in X$, $\{S(x, \mathcal{U}_i) | i = 1, 2, \dots\}$ forms a base for the neighborhood system of x (R. L. Moore, 1916). A regular developable space is called a **Moore space**. The question of whether or not every normal Moore space is metrizable is known as the **normal Moore space problem** (\rightarrow 273 Metric Spaces K). Collectionwise normal Moore spaces are metrizable (Bing).

BB. Real-Compact Spaces

A completely regular space X is called **real-compact** if X is complete under the smallest \dagger uniformity such that each continuous real-valued function on X is uniformly continuous (\rightarrow 422 Uniform Spaces). This notion was introduced by E. Hewitt (*Trans. Amer. Math. Soc.*, 64 (1948)) under the name of **Q-space**, and independently by L. Nachbin (*Proc. International Congress of Mathematicians*, Cambridge, Mass., 1950).

A Lindelöf space is real-compact. If X_1 and X_2 are real-compact spaces such that the rings $C(X_1)$ and $C(X_2)$ of continuous real-valued functions on X_1 and X_2 are isomorphic, then X_1 and X_2 are homeomorphic (Hewitt). If X is real-compact, then X is homeomorphic to a closed subspace of the product space of copies of the space of real numbers, and conversely.

CC. Images and Inverse Images of Topological Spaces

Each continuous mapping $f: X \rightarrow Y$ is decomposed into the product $i \circ h \circ p$ of continuous mappings $p: X \rightarrow X/\sim$, $h: X/\sim \rightarrow f(X)$ and $i: f(X) \rightarrow Y$, where \sim is the equivalence relation such that $x_1 \sim x_2$ if and only if $f(x_1) = f(x_2)$.

The mapping f is open (resp. closed) if and only if these mappings are all open (resp. closed). Then h is a homeomorphism. The image of a paracompact Hausdorff space under a closed continuous mapping is paracompact (Michael, 1957).

Let $f: X \rightarrow Y$ be a perfect surjection. Then Y is called a **perfect image** of X and X a **perfect inverse image** of Y . If, in this case, one of X and Y satisfies a property such as being compact, locally compact, σ -compact, Lindelöf, or countably compact, then the other also satisfies the property. When X and Y are completely regular, the same is true with regard to Čech completeness. Properties such as regularity, normality, complete normality, perfect normality, and the second countability axiom are preserved in perfect images; but complete regularity and strong paracompactness are not. Perfect images of metric spaces are also metrizable (S. Hanai and Morita, A. H. Stone, 1956). Conversely, perfect inverse images of paracompact spaces are paracompact. If a Hausdorff space is a perfect inverse image of a regular space (resp. k -space; \rightarrow below), then it is a regular space (resp. k -space). Every paracompact Čech-complete space is a perfect inverse image of a \dagger complete metric space (Z. Frolik, 1961). A completely regular space is a paracompact p -space if and only if it is a perfect inverse image of a metric space (Arkhangel'skiĭ, 1963). A mapping $f: X \rightarrow Y$ is called **quasi-perfect** if it is closed and continuous and the inverse image $f^{-1}(y)$ of each point $y \in Y$ is countably compact. A topological space X is an M -space if and only if there is a quasi-perfect mapping from X onto a metric space Y (Morita, 1964). Let $f: X \rightarrow Y$ be a quasi-perfect surjection. If one of X and Y is a Σ -space, then the other is also a Σ -space (Nagami, 1969).

A topological space X is called a **Fréchet-Uryson space** (or a **Fréchet space**) if the closure of an arbitrary set $A \subset X$ is the set of all limits of sequences in A (Arkhangel'skiĭ, 1963). X is called a **sequential space** if $A \subset X$ is closed whenever A contains all the limits of sequences in A (S. P. Franklin, 1965). X is called a **k' -space** if the closure of an arbitrary set A is the set of all points adherent to the intersection $A \cap K$ for a compact set K in X (Arkhangel'skiĭ, 1963). X is called a **k -space** if $A \subset X$ is closed whenever $A \cap K$ is closed in K for any compact set K (\rightarrow Arkhangel'skiĭ, *Trudy Moskov. Mat. Obshch.*, 13 (1965)). Spaces satisfying the first countability axiom are Fréchet-Uryson spaces. The Fréchet-Uryson spaces (resp. sequential spaces) are characterized as the images under hereditarily quotient (resp. quotient) mappings of metric spaces or locally compact metric

spaces. Similarly the k' -spaces (resp. k -spaces) coincide with the images under hereditarily quotient (resp. quotient) mappings of locally compact spaces. The image of a metric space under a closed continuous mapping is called a **Lashnev space**. Any subspace of a Fréchet-Uryson space is a Fréchet-Uryson space. Conversely, a Hausdorff space is a Fréchet-Uryson space if any of its subspaces is a k -space. Čech-complete spaces are k -spaces. A Hausdorff space is called a **Suslin space** (resp. **Luzin space**) if it is the image under a continuous surjection (resp. continuous bijection) of a complete separable metric space (Bourbaki [9]; also \rightarrow 22 Analytic Sets).

In Figs. 1, 2, and 3, the relationships be-

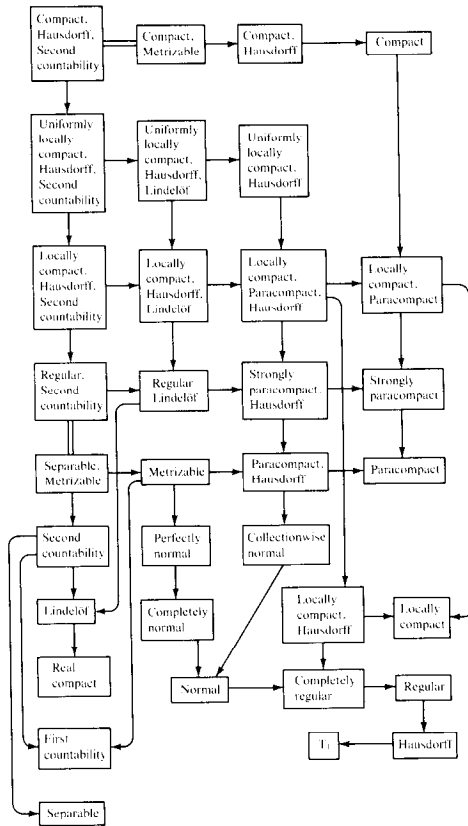


Fig. 1

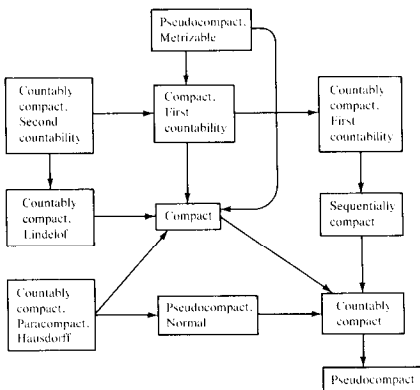


Fig. 2

tween the various properties are indicated by the arrows.

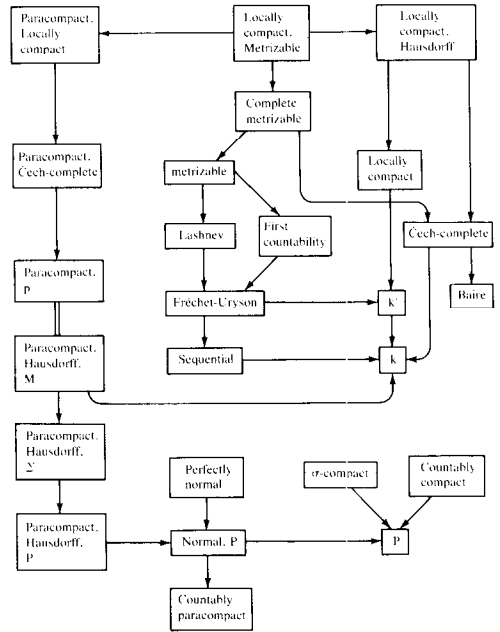


Fig. 3

References

- [1] M. Fréchet, Sur quelques points du calcul fonctionnel, Rend. Circ. Mat. Palermo, 22 (1906), 1–74.
- [2] F. Riesz, Stetigkeitsbegriff und abstrakte Mengenlehre, Atti del IV Congr. Intern. Mat. Roma 1908, II, 18–24, Rome, 1909.
- [3] F. Hausdorff, Grundzüge der Mengenlehre, Teubner, 1914.
- [4] M. Fréchet, Les espaces abstraits, Gauthier-Villars, 1928.
- [5] C. Kuratowski, Topologie, I, Monograf. Mat., I, 1933, revised edition, 1948; II, 1950.
- [6] W. Sierpiński, Introduction to general topology, University of Toronto Press, 1934.
- [7] P. S. Aleksandrov (Alexandroff) and H. Hopf, Topologie I, Springer, 1935.
- [8] J. W. Tukey, Convergence and uniformity in topology, Ann. Math. Studies, Princeton Univ. Press, 1940.
- [9] N. Bourbaki, Eléments de mathématique, III. Topologie générale, 1940–1949; new edition, 1970–1974; Hermann, English translation, General topology, Addison-Wesley, 1966.
- [10] J. L. Kelley, General topology, Van Nostrand, 1955.
- [11] J. Dugundji, Topology, Allyn and Bacon, 1966.
- [12] J. Nagata, Modern general topology, North-Holland, 1968.
- [13] R. Engelking, General topology, Polish Scientific Publ., 1977.

- [14] P. S. Aleksandrov, Some results in the theory of topological spaces, obtained within the last twenty-five years, *Russian Math. Surveys*, 16, no. 2 (1960), 23–83.
- [15] A. V. Arkhangel'skii, Mappings and spaces, *Russian Math. Surveys*, 21, no. 4 (1966), 115–162.
- [16] Y. Kodama and K. Nagami, *Theory of topological spaces* (in Japanese), Iwanami, 1974.
- [17] L. Steen and J. Seebach, Jr., *Counterexamples in topology*, Holt, Rinehart and Winston, 1970.
- [18] L. Gillman and M. Jerison, *Rings of continuous functions*, Van Nostrand, 1960.
- [19] S. A. Gaal, *Point set topology*, Academic Press, 1964.
- [20] E. Čech, *Topological spaces*, Interscience, 1966.
- [21] R. C. Walker, *The Stone-Čech compactification*, Springer, 1974.
- [22] R. A. Alò and H. L. Shapiro, *Normal topological spaces*, Cambridge Univ. Press, 1974.
- [23] M. D. Weir, *Hewitt-Nachbin spaces*, North-Holland, 1975.

426 (IX.1) Topology

The term **topology** means a branch of mathematics that deals with topological properties of geometric figures or point sets. A classical result in topology is the Euler relation on polyhedra: Let α_0 , α_1 , and α_2 be the numbers of vertices, edges, and faces of a polyhedron homeomorphic to the 2-dimensional sphere; then $\alpha_0 - \alpha_1 + \alpha_2 = 2$ (*Euler-Poincaré formula for the 2-dimensional case; actually, the formula was known to Descartes). It is one of the earliest results in topology. In 1833, C. F. Gauss used integrals to define the notion of *linking numbers of two closed curves in a space (→ 99 Degree of Mapping). It was in J. B. Listing's classical work *Vorstudien zur Topologie* (1847) that the term *topology* first appeared in print.

In the 19th century, B. Riemann published many works on function theory in which topological methods played an essential role. He solved the homeomorphism problem for compact surfaces (→ 410 Surfaces); his result is basic in the theory of algebraic functions. In the same period, mathematicians began to study topological properties of n -dimensional polyhedra. E. Betti considered the notion of *homology. H. Poincaré, however, was the first to recognize the importance of a topological

approach to analysis in general; he defined the *homology groups of a complex [1]. He obtained the famous *Poincaré duality theorem and defined the *fundamental group. He considered *polyhedra as the basic objects in topology, and deduced topological properties utilizing *complexes obtained from polyhedra by *simplicial decompositions. He thus constructed a branch of topology known as **combinatorial topology**.

In its beginning stages combinatorial topology dealt only with polyhedra. In the late 1920s, however, it became possible to apply combinatorial methods to general *compact spaces. P. S. Alexandrov introduced the concept of approximation of a *compact metric space by an inverse sequence of complexes and the definition of homology groups for these spaces. His idea had a precursor in the notion of *simplicial approximations of continuous mappings, which was introduced by L. E. J. Brouwer in 1911. In 1932, E. Čech defined homology groups for arbitrary spaces utilizing the *inductive limit of the homology groups of polyhedra; and *Čech cohomology groups for arbitrary spaces were also defined. S. Eilenberg established *singular (co)homology theory using *singular chain complexes (1944). The axiomatic approach to (co)homology theory is due to Eilenberg and Steenrod, who gave axioms for (co)homology theory in a most comprehensive way and unified various (co)homology theories (1945) (→ 201 Homology Theory).

The approach using algebraic methods has progressed extensively in connection with the development of homology theory. This branch is called **algebraic topology**. In the 1920s and 1930s, a number of remarkable results in algebraic topology, such as the *Alexander duality theorem, the *Lefschetz fixed-point theorem, and the *Hopf invariant, were obtained. In the late 1930s, W. Hurewicz developed the theory of higher-dimensional *homotopy groups (→ 153 Fixed-Point Theorems, 201 Homology Theory, 202 Homotopy Theory). J. H. C. Whitehead introduced the concept of *CW complexes and proved an algebraic characterization of the homotopy equivalence of CW complexes. N. Steenrod developed *obstruction theory utilizing *squaring operations in the cohomology ring (1947). Subsequently, the theory of *cohomology operations was introduced (→ 64 Cohomology Operations, 305 Obstructions). The theory of *spectral sequences for *fiber spaces was originated by J. Leray (1945) and J.-P. Serre (1951) and was successfully applied to cohomology operations and homotopy theory by H. Cartan and Serre (1954) (→ 148 Fiber Spaces, 200 Homological Algebra). The study of the combinatorial

structures of polyhedra and \dagger piecewise linear mappings has flourished since 1940 in the works of Whitehead, S. S. Cairns, and others. S. Smale and, independently, J. Stallings solved the \dagger generalized Poincaré conjecture in 1960. The \dagger Hauptvermutung in combinatorial topology was solved negatively in 1961 by B. Mazur and J. Milnor. E. C. Zeeman proved the unknottedness of codimension 3 (1962). The recent development of the theory in conjunction with progress in \dagger differential topology is notable. The Hauptvermutung for combinatorial manifolds was solved in 1969 by Kirby, Siebenmann, and Wall. In particular, there exist different combinatorial structures on tori of dimension ≥ 5 , and there are topological manifolds that do not admit any combinatorial structure (\rightarrow 65 Combinatorial Manifolds, 114 Differential Topology, 235 Knot Theory).

The global theory of differentiable manifolds started from the algebraic-topological study of \dagger fiber bundles and \dagger characteristic classes in the 1940s. R. Thom's fundamental theorem of \dagger cobordism (1954) was obtained through extensive use of cohomology operations and homotopy groups. Milnor (1956) showed that the sphere S^7 may have differentiable structures that are essentially distinct from each other by using \dagger Morse theory and the \dagger index theorem of Thom and Hirzebruch. These results led to the creation of a new field, \dagger differential topology (\rightarrow 56 Characteristic Classes, 114 Differential Topology).

Since 1959, A. Grothendieck, M. F. Atiyah, F. Hirzebruch, and J. F. Adams have developed \dagger K-theory, which is a generalized cohomology theory constructed using stable classes of \dagger vector bundles (\rightarrow 237 K-Theory).

\dagger Knot theory, an interesting branch of topology, was one of the classical branches of topology and is now studied in connection with the theory of low-dimensional manifolds (\rightarrow 235 Knot Theory).

On the other hand, G. Cantor established general set theory in the 1870s and introduced such notions as \dagger accumulation points, \dagger open sets, and \dagger closed sets in Euclidean space. The first important generalization of this theory was the concept of \dagger topological space, which was proposed by M. Fréchet and developed by F. Hausdorff at the beginning of the 20th century. The theory subsequently became a new field of study, called **general topology** or **set-theoretic topology**. It deals with the topological properties of point sets in a Euclidean or topological space without reference to polyhedra. There has been a remarkable development of the theory since about 1920, notably by Polish mathematicians S. Janiszewski, W. Sierpiński, S. Mazurkiewicz, C. Kuratow-

ski, and others. The contributions of R. L. Moore, G. T. Whyburn, and K. Menger are also important (\rightarrow 382 Shape Theory, 425 Topological Spaces).

Topology is not only a foundation of various theories, but is also itself one of the most important branches of mathematics. It consists of \dagger homology theory, \dagger homotopy theory, \dagger differential topology, \dagger combinatorial manifolds, \dagger K-theory, \dagger transformation groups, \dagger theory of singularities, \dagger foliations, \dagger dynamical systems, \dagger catastrophe theory, etc. It continues to develop in interaction with other branches of mathematics (\rightarrow 51 Catastrophe Theory, 126 Dynamical Systems, 154 Foliations, 418 Theory of Singularities, 431 Transformation Groups).

References

- [1] H. Poincaré, *Analysis situs*, J. Ecole Polytech., (2) 1 (1895), 1–121. (Oeuvres, Gauthier-Villars, 1953, vol. 6, 193–288.)
- [2] P. S. Aleksandrov and H. Hopf, *Topologie I*, Springer, 1935 (Chelsea, 1965).
- [3] S. Lefschetz, *Algebraic topology*, Amer. Math. Soc. Colloq. Publ., 1942.
- [4] S. Eilenberg and N. E. Steenrod, *Foundations of algebraic topology*, Princeton Univ. Press, 1952.

427 (IX.12) Topology of Lie Groups and Homogeneous Spaces

A. General Remarks

Among various topological structures of \dagger Lie groups and \dagger homogeneous spaces, the structures of their \dagger (co)homology groups and \dagger homotopy groups are of special interest. Let G/H be a homogeneous space, where G is a Lie group and H is its closed subgroup. Then $(G, G/H, H)$ is a \dagger fiber bundle, where G/H is the base space and H is the fiber. Thus homology and homotopy theory of fiber bundles (\dagger spectral sequences and \dagger homotopy exact sequences) can be applied. The \dagger cellular decomposition of \dagger Stiefel manifolds, \dagger Grassmann manifolds, and \dagger Kähler homogeneous spaces are known. Concerning \dagger symmetric Riemannian spaces, we have various interesting methods, such as the use of invariant differential forms in connection with real cohomology rings and the use of \dagger Morse theory in order to establish relations between the diagrams of symmetric Riemannian spaces G/H and homological properties of their \dagger loop spaces and

some related homogeneous spaces [4, 5]. Lie groups can be regarded as special cases of homogeneous spaces or symmetric spaces, although their group structures are of particular importance. A connected Lie group is homeomorphic to the product of one of its compact subgroups and a Euclidean space (*Cartan-Mal'tsev-Iwasawa theorem). Hence the topological structure of a connected Lie group is essentially determined by the topological structures of its compact subgroups.

B. Homology of Compact Lie Groups

Let G be a connected compact Lie group. Since G is an tH -space whose multiplication is given by its group multiplication h , $H^*(G; k)$ and $H_*(G; k)$ are dual t Hopf algebras for any coefficient field k . Also, $H^*(G; k)$ is isomorphic as a t graded algebra to the tensor product of t elementary Hopf algebras (\rightarrow 203 Hopf Algebras), but no factor of the tensor product is isomorphic to a polynomial ring because G is a finite t polyhedron. In particular, if $k = \mathbf{R}$ (the field of real numbers), then $H^*(G; \mathbf{R}) \cong \bigwedge_{\mathbf{R}}(x_1, \dots, x_l)$ (the exterior (Grassmann) algebra over \mathbf{R} with generators x_1, \dots, x_l of odd degrees). Here we can choose generators x_i such that $h^*(x_i) = 1 \otimes x_i + x_i \otimes 1$, $1 \leq i \leq l$. The x_i that satisfy this property are said to be **primitive**. Since in this case the t comultiplication h^* is commutative, the multiplication h_* is also commutative and the Hopf algebra $H_*(G; \mathbf{R})$ is an exterior algebra generated by elements y_i having the same degree as x_i ($i = 1, \dots, l$). When the characteristic of the coefficient field k is nonzero, h_* need not be commutative.

The dimension of a t maximal torus of a connected compact Lie group G is independent of the choice of the maximal torus and is called the **rank** of G . The rank of G coincides with the number l of generators of $H^*(G; \mathbf{R})$. E. Cartan studied $H^*(G; \mathbf{R})$ by utilizing invariant differential forms. The cohomology theory of Lie algebras originated from the method he used in his study. $H^*(G; \mathbf{R})$ is invariant under t local isomorphisms of groups G . For t classical compact simple Lie groups G , R. Brauer calculated $H^*(G; \mathbf{R})$, while C.-T. Yen and C. Chevalley calculated $H^*(G; \mathbf{R})$ for t exceptional compact simple Lie groups (\rightarrow Appendix A, Table 6.IV). The degrees of the generators have group-theoretic meaning. Suppose that the degree of the i th generator is $2m_i - 1$, $1 \leq i \leq l$, and that $m_1 \leq m_2 \leq \dots \leq m_l$. When G is simple, there is a relation $m_i + m_{l-i+1} = \text{constant}$ (Chevalley's duality). We have a proof for this property that does not use classification.

The cohomology groups $H^*(G; \mathbf{Z}_p)$ (where p

is a prime and $\mathbf{Z}_p = \mathbf{Z}/p\mathbf{Z}$) have been determined as graded algebras for all compact simple Lie groups by A. Borel, S. Araki, and P. Baum and W. Browder (\rightarrow Appendix A, Table 6.IV).

C. Cohomology of Classifying Spaces

Let (E_G, B_G, G) be a t universal bundle of a connected compact Lie group G and p a prime or zero. Suppose that the integral cohomology of G has no p -torsion (no torsion when $p = 0$). Then $H^*(G; \mathbf{Z}_p) = \bigwedge_{\mathbf{Z}_p}(x'_1, \dots, x'_l)$ ($H^*(G; \mathbf{Z}) = \bigwedge_{\mathbf{Z}}(x'_1, \dots, x'_l)$ when $p = 0$), an exterior algebra with $\deg x'_i = 2m_i - 1$, $1 \leq i \leq l$, and the generators x'_i can be chosen to be t transgressive in the spectral sequence of the universal bundle. Let y_1, \dots, y_l be their transgression images. Then $\deg y_i = 2m_i$, $1 \leq i \leq l$, and the cohomology of the t classifying space B_G over \mathbf{Z}_p (resp. \mathbf{Z}) is the polynomial algebra with generators y_1, \dots, y_l . Let T be a maximal torus of G . Then $B_T = E_G/T$ is a classifying space of T , the t Weyl group $W = N(T)/T$ of G with respect to T operates on B_T by t right translations, and $H^*(T; \mathbf{Z})$ has no torsion and is an exterior algebra with l generators of degree 1. Thus $H^*(B_T; \mathbf{Z}) = \mathbf{Z}[u_1, \dots, u_l]$, $\deg u_i = 2$. Let I_W be the subalgebra of $H^*(B_T; \mathbf{Z})$ consisting of W -invariant polynomials, and let ρ be the projection of the bundle $(B_T, B_G, G/T)$. Then under the assumption that G has no p -torsion (no torsion), the cohomology mapping ρ^* over $\mathbf{Z}_p(\mathbf{Z})$ is monomorphic, and $\rho^*: H^*(B_G; \mathbf{Z}_p) \cong I_W \otimes \mathbf{Z}_p$ ($H^*(B_G; \mathbf{Z}) \cong I_W$) [1]. In the case of real coefficients, we have $H^*(B_G; \mathbf{R}) \cong I_W \otimes \mathbf{R}$ for all G , and m_1, \dots, m_l are the degrees of generators of the ring I_W of W -invariant polynomials.

Example (1) $G = U(n)$: $l = n$ and G has no torsion. W operates on $H^*(B_T; \mathbf{Z})$ as the group of all permutations of generators u_1, \dots, u_n . Thus generators of I_W are the t elementary symmetric polynomials $\sigma_1, \dots, \sigma_n$ of u_1, \dots, u_n . Let c_1, \dots, c_n be the t universal Chern classes; then $\rho^*(c_i) = \sigma_i$ and $H^*(B_{U(n)}; \mathbf{Z}) = \mathbf{Z}[c_1, \dots, c_n]$.

Example (2) $G = SO(n)$: $l = [n/2]$ and G has no p -torsion for $p \neq 2$. W operates on $H^*(B_T; \mathbf{Z})$ as the group generated by the permutations of generators u_1, \dots, u_l and by the transformations $\sigma(u_i) = e_i u_i$, $e_i = \pm 1$, where the number of u_i for which $e_i = -1$ is arbitrary for odd n and even for even n . Thus the generators of I_W are the elementary symmetric polynomials $\sigma'_1, \dots, \sigma'_l$ of u_1^2, \dots, u_l^2 for odd n and $\sigma'_1, \dots, \sigma'_{l-1}$ and $u_1 \dots u_l$ for even n . Let p_1, \dots, p_l be the t universal Pontryagin classes and χ be the t universal Euler-Poincaré class in the case of even n . Then $\rho^*(p_i) = \sigma'_i$ and $\rho^*(\chi) = u_1 \dots u_l$ for integral cohomology. Denote the mod p

Topology of Lie Groups, Homogeneous Spaces

reduction of p_i and χ by \bar{p}_i and $\bar{\chi}$, respectively. Then $H^*(B_{SO(2l+1)}; \mathbb{Z}_p) = \mathbb{Z}_p[\bar{p}_1, \dots, \bar{p}_l]$ and $H^*(B_{SO(2l)}; \mathbb{Z}_p) = \mathbb{Z}_p[\bar{p}_1, \dots, \bar{p}_{l-1}, \bar{\chi}]$ ($p = 0$ or > 2).

Example (3) $G = O(n)$: If we use the subgroup Q consisting of all diagonal matrices instead of T , then we can make a similar argument for \mathbb{Z}_2 -cohomology. Since $Q \cong (\mathbb{Z}_2)^n$, $H^*(B_Q; \mathbb{Z}_2) = \mathbb{Z}_2[v_1, \dots, v_n]$ ($\mathbb{Z}_2[v_1, \dots, v_n]$ is a polynomial ring with $\deg v_i = 1$), and $W_2 = N(Q)/Q$ operates on B_Q by right translations and on $H^*(B_Q; \mathbb{Z}_2)$ as the group of all permutations of v_1, \dots, v_n . Let I_{W_2} be the subalgebra of $H^*(B_Q; \mathbb{Z}_2)$ consisting of all W_2 -invariant polynomials. Then I_{W_2} is a polynomial ring generated by the elementary symmetric polynomials $\sigma_1'', \dots, \sigma_n''$ of v_1, \dots, v_n . The projection $\rho_2: B_Q \rightarrow B_{O(n)}$ induces a monomorphic cohomology mapping ρ_2^* over \mathbb{Z}_2 , and $\rho_2^*: H^*(B_{O(n)}; \mathbb{Z}_2) \cong I_{W_2}$. Let w_1, \dots, w_n be the \dagger universal Stiefel-Whitney classes. Then $\rho_2^*(w_i) = \sigma_i''$ and $H^*(B_{O(n)}; \mathbb{Z}_2) = \mathbb{Z}_2[w_1, \dots, w_n]$ [2].

D. Grassmann Manifolds

The following manifolds are called **Grassmann manifolds**: The manifold $M_{n+m,n}(\mathbb{R})$ consisting of all n -subspaces of \mathbb{R}^{n+m} ; the manifold $\tilde{M}_{n+m,n}(\mathbb{R})$ consisting of all oriented n -subspaces of \mathbb{R}^{n+m} ; and the manifold $M_{n+m,n}(\mathbb{C})$ consisting of all complex n -subspaces of \mathbb{C}^{n+m} . These are expressed as quotient spaces as follows: $M_{n+m,n}(\mathbb{R}) = O(n+m)/O(n) \times O(m)$, $\tilde{M}_{n+m,n}(\mathbb{R}) = SO(n+m)/SO(n) \times SO(m)$, and $M_{n+m,n}(\mathbb{C}) = U(n+m)/U(n) \times U(m)$. They admit cellular decompositions by \dagger Schubert varieties from which their cohomologies can be computed (\rightarrow 56 Characteristic Classes). $M_{n+m,n}(\mathbb{R})$ and $\tilde{M}_{n+m,n}(\mathbb{R})$ have no p -torsion for $p \neq 2$, and $M_{n+m,n}(\mathbb{C})$ has no torsion. These spaces are m -, m -, and $(2m+1)$ -classifying spaces of $O(n)$, $SO(n)$, and $U(n)$, respectively. Hence their cohomologies are isomorphic to those of B_G ($G = O(n)$, $SO(n)$, $U(n)$) in dimensions $< m$, $< m$, and $\leq 2m$, respectively; and they are polynomial rings generated by suitable universal characteristic classes in low dimensions.

E. Cohomologies of Homogeneous Spaces G/U (Rank $G = \text{Rank } U$)

Let G be a compact connected Lie group and U a closed subgroup of G with the same rank as G . Denote the degrees of generators of $H^*(G; \mathbb{R})$ and $H^*(U; \mathbb{R})$ by $2m_1 - 1, \dots, 2m_l - 1$, and $2n_1 - 1, \dots, 2n_l - 1$, respectively. Then the real-coefficient \dagger Poincaré polynomial P_0 of the homogeneous space G/U is given by $P_0(G/U, t) = \prod_i (1 - t^{2m_i}) / (1 - t^{2n_i})$ (G. Hirsch). When G , U , and G/U have no p -torsion, the same formula

is valid for the \mathbb{Z}_p -coefficient Poincaré polynomial [1]. When U is the \dagger centralizer of a torus, G/U has a complex analytic cellular decomposition [3]. Hence G/U has no torsion in this case. This was proved by R. Bott and H. Samelson by utilizing Morse theory [5] (\rightarrow 279 Morse Theory). The case $U = T$ has also been studied.

F. Homotopy Groups of Compact Lie Groups

The \dagger fundamental group $\pi_1(G)$ of a compact Lie group G is Abelian. Furthermore, $\pi_2(G) = 0$. If we apply Morse theory to G , the variational completeness of G can be utilized to show that the loop space ΩG has no torsion and that its odd-dimensional cohomologies vanish [4]. Consequently, when G is non-Abelian and simple, we have $\pi_3(G) \cong \mathbb{Z}$. A \dagger periodicity theorem on \dagger stable homotopy groups of classical groups proved by Bott is used in K -theory (\rightarrow 202 Homotopy Theory; 237 K -Theory). (For explicit forms of homotopy groups \rightarrow Appendix A, Table 6.VI).

Homotopy groups of Stiefel manifolds are used to define characteristic classes by \dagger obstruction cocycles (\rightarrow 147 Fiber Bundles; Appendix A, Table 6.VI).

References

- [1] A. Borel, Sur la cohomologie des espaces fibrés principaux et des espaces homogènes de groupes de Lie compacts, *Ann. Math.*, (2) 57 (1953), 115–207.
- [2] A. Borel, La cohomologie mod 2 de certains espaces homogènes, *Comment. Math. Helv.*, 27 (1953), 165–197.
- [3] A. Borel, Kählerian coset spaces of semi-simple Lie groups, *Proc. Nat. Acad. Sci. US*, 40 (1954), 1147–1151.
- [4] R. Bott, An application of the Morse theory to the topology of Lie-groups, *Bull. Soc. Math. France*, 84 (1956), 251–281.
- [5] R. Bott and H. Samelson, Applications of the theory of Morse to symmetric spaces, *Amer. J. Math.*, 80 (1958), 964–1029.

428 (XIII.17) Total Differential Equations

A. Pfaff's Problem

A **total differential equation** is an equation of the form

$$\omega = 0, \quad (1)$$

where ω is a * differential 1-form $\sum_{i=1}^n a_i(x) dx_i$ on a manifold X . A submanifold M of X is called an **integral manifold** of (1) if each vector ξ of the * tangent vector space $T_x(M)$ of M at every point x on M satisfies $\omega(\xi)=0$. We denote the maximal dimension of integral manifolds of (1) by $m(\omega)$. J. F. Pfaff showed that $m(\omega) \geq (n-1)/2$ for any ω . The problem of determining $m(\omega)$ for a given form ω is called **Pfaff's problem**. This problem was solved by G. Frobenius, J. G. Darboux, and others as follows: Form an * alternating matrix

$$(a_{ij})_{1 \leq i, j \leq n} \quad (2)$$

from the coefficients of the * exterior derivative of ω ,

$$d\omega = \frac{1}{2} \sum_{i=1}^n \sum_{j=1}^n a_{ij}(x) dx_i \wedge dx_j,$$

where $a_{ij} = \partial a_j / \partial x_i - \partial a_i / \partial x_j$. Suppose that the rank of (2) is $2t$. Then the rank of the matrix

$$\begin{pmatrix} a_{ij} & -a_i \\ a_j & 0 \end{pmatrix}_{1 \leq i, j \leq n}$$

is $2t$ or $2t+2$. In the former case $m(\omega) = n-t$, and ω can be expressed in the form

$$\sum_{i=1}^t u_{2i-1} du_{2i}$$

by choosing a suitable coordinate system (u_1, \dots, u_n) . In the latter case $m(\omega) = n-t-1$, and ω can be expressed in the form

$$\sum_{i=1}^t u_{2i-1} du_{2i} + du_{2t+1}$$

by choosing a suitable coordinate system (u_1, \dots, u_n) . This theorem is called **Darboux's theorem**.

A 1-form ω is called a **Pfaffian form**, and equation (1) is called a **Pfaffian equation**. A system of equations $\omega_i = 0$ ($1 \leq i \leq s$) for 1-form ω_i is called a **system of Pfaffian equations** or a **system of total differential equations** [6, 12, 26].

B. Systems of Differential Forms and Systems of Partial Differential Equations

Let Ω be a system of differential forms ω_i^p , $0 \leq p \leq n$, $1 \leq i \leq v_p$, on X , where ω_i^p is a p -form on X . A submanifold M of X is called an **integral manifold** of $\Omega=0$ if for each p ($0 \leq p \leq \dim M$), any p -dimensional subspace E_p of $T_x(M)$ satisfies $\omega_i^p(E_p) = 0$ ($1 \leq i \leq v_p$) at every point x on M . Denote the maximal dimension of integral manifolds of $\Omega=0$ by $m(\Omega)$. The problem of determining $m(\Omega)$ for a given system Ω is called the **generalized Pfaff problem**, and will be explained in later sections. By fixing a local coordinate system of X and

dividing it into two systems (x_1, \dots, x_r) and (y_1, \dots, y_m) ($m = n-r$), we can consider the problem of finding an integral manifold of $\Omega=0$ defined by

$$y_\alpha = y_\alpha(x_1, \dots, x_r), \quad 1 \leq \alpha \leq m.$$

This problem can be reduced to solving a system of partial differential equations of the first order on the submanifold N with the local coordinate system (x_1, \dots, x_r) .

Consider a system of partial differential equations $\Phi=0$ of order l :

$$\varphi_\lambda(x_i, y_\alpha, p_\beta^{j_1 \dots j_r}) = 0, \quad 1 \leq \lambda \leq s, \quad (3)$$

with $1 \leq i \leq r$, $1 \leq \alpha, \beta \leq m$, $j_1 + \dots + j_r \leq l$, where

$$p_\beta^{j_1 \dots j_r} = \frac{\partial^{j_1 + \dots + j_r} y_\beta}{\partial x_1^{j_1} \dots \partial x_r^{j_r}}. \quad (4)$$

A submanifold defined by $y_\alpha = y_\alpha(x_1, \dots, x_r)$, $1 \leq \alpha \leq m$, is called a **solution** of $\Phi=0$ if it satisfies (3) identically. The problem of determining whether a given system $\Phi=0$ has a solution was solved by C. Riquier, who showed that any system can be prolonged either to a passive orthonomic system or to an incompatible system by a finite number of steps. A system of partial differential equations is called a **prolongation** of another system if the former contains the latter and they have the same solution. A **passive orthonomic system** is one whose general solution can be parametrized by an infinite number of arbitrary constants. A solution containing parameters is called a **general solution** if by specifying the parameters we can obtain a solution of the * Cauchy problem for any initial data. A system (3) is said to be **incompatible** if it implies a nontrivial relation $f(x_1, \dots, x_r) = 0$ among the x_i .

The problem of solving a system $\Phi=0$ of partial differential equations can be reduced to that of finding integral manifolds of a system of differential forms Σ as follows: Let J^l be a manifold with the local coordinate system

$$(x_i, y_\alpha, p_\beta^{j_1 \dots j_r}; \quad 1 \leq i \leq r, \quad 1 \leq \alpha, \beta \leq m, \quad j_1 + \dots + j_r \leq l),$$

and Σ be a system of 0-forms φ_λ ($1 \leq \lambda \leq s$) and 1-forms

$$dy_\alpha - \sum_{i=1}^r p_\alpha^i dx_i,$$

$$dp_\beta^{j_1 \dots j_r} - \sum_{k=1}^r p_\beta^{j_1 \dots j_k+1 \dots j_r} dx_k$$

($1 \leq \alpha, \beta \leq m$, $j_1 + \dots + j_r < l$). Then an integral manifold of $\Sigma=0$ of the form

$$y_\alpha = y_\alpha(x_1, \dots, x_r), \quad 1 \leq \alpha \leq m,$$

$$p_\beta^{j_1 \dots j_r} = p_\beta^{j_1 \dots j_r}(x_1, \dots, x_r),$$

$$1 \leq \beta \leq m, \quad j_1 + \dots + j_r \leq l,$$

Total Differential Equations

gives a solution $y_\alpha = y_\alpha(x_1, \dots, x_r)$, $1 \leq \alpha \leq m$, of $\Phi = 0$, and y_β and $p_\beta^{j_1, \dots, j_r}$ satisfy (4).

Conversely, a solution $y_\alpha = y_\alpha(x_1, \dots, x_r)$, $1 \leq \alpha \leq m$, of $\Phi = 0$ gives an integral manifold of $\Sigma = 0$ if we define $p_\beta^{j_1, \dots, j_r}(x_1, \dots, x_r)$ by (4) [23, 24, 26].

C. Systems of Partial Differential Equations of First Order with One Unknown Function

Consider a system of independent \dagger vector fields on N :

$$L_\lambda = \sum_{i=1}^r b_{\lambda i}(x) \frac{\partial}{\partial x_i}, \quad 1 \leq \lambda \leq s.$$

We solve a system of inhomogeneous equations

$$L_\lambda y - f_\lambda(x)y - g_\lambda(x) = 0, \quad 1 \leq \lambda \leq s, \quad (5)$$

for a given system of $f_\lambda(x)$ and $g_\lambda(x)$. The system (5) is called a **complete system** if each of the expressions

$$[L_\lambda, L_\mu]y - (L_\lambda f_\mu - L_\mu f_\lambda)y - (f_\mu g_\lambda - f_\lambda g_\mu) - (L_\lambda g_\mu - L_\mu g_\lambda), \quad 1 \leq \lambda < \mu \leq s, \quad (6)$$

is a linear combination of the left-hand sides of (5), where $[L_\lambda, L_\mu]$ means the \dagger commutator of L_λ and L_μ . This condition is called the **complete integrability condition** for (5). Suppose that the homogeneous system

$$L_\lambda y = 0, \quad 1 \leq \lambda \leq s, \quad (7)$$

is complete. Then it has a system of \dagger functionally independent solutions y_1, \dots, y_{r-s} , and any solution y of (8) is a function of them: $y = \psi(y_1, \dots, y_{r-s})$. If the inhomogeneous system (5) is complete, then the homogeneous system (7) is complete. This notion of a complete system is due to Lagrange and was extended to a system of nonlinear equations by Jacobi as follows (\rightarrow 324 Partial Differential Equations of First Order C).

Consider a system of nonlinear equations

$$F_\lambda(x_1, \dots, x_r, y, p_1, \dots, p_r) = 0, \quad 1 \leq \lambda \leq s, \quad (8)$$

where $p_i = \partial y / \partial x_i$. The system (8) is called an **involutory system** if each of $[F_\lambda, F_\mu]$, $1 \leq \lambda < \mu \leq s$, is a linear combination of F_1, \dots, F_s . Here \dagger Lagrange's bracket $[F, G]$ is defined by

$$[F, G] = \sum_{i=1}^r \frac{\partial F}{\partial p_i} \left(\frac{\partial G}{\partial x_i} + p_i \frac{\partial G}{\partial y} \right) - \sum_{i=1}^r \frac{\partial G}{\partial p_i} \left(\frac{\partial F}{\partial x_i} + p_i \frac{\partial F}{\partial y} \right).$$

Suppose that the system (8) is involutory and F_1, \dots, F_s are functionally independent. Then, in general, we can solve the following \dagger Cauchy problem for an $(r-s)$ -dimensional submani-

fold N_{r-s} of N : Given a function f on N_{r-s} , find a solution y of (8) satisfying $y = f$ on N_{r-s} . We can construct a solution by integrating a system of ordinary differential equations called a \dagger characteristic system of differential equations. Hence the solution of these problems may be carried out in the C^∞ -category (\rightarrow 322 Partial Differential Equations (Methods of Integration) B) [7, 11].

D. Frobenius's Theorem

Let X be a \dagger differentiable manifold of class C^∞ and Ω be a system of independent 1-forms ω_i , $1 \leq i \leq s$, on X . Then the system of Pfaffian equations $\Omega = 0$ is called a **completely integrable system** if at every point x of X ,

$$d\omega_i = \sum_{j=1}^s \theta_{ij} \wedge \omega_j, \quad 1 \leq i \leq s,$$

for 1-forms θ_{ij} on a neighborhood of x . Suppose that $\Omega = 0$ is completely integrable. Then at every point x of X , there exists a local coordinate system $(f_1, \dots, f_s, x_{s+1}, \dots, x_n)$ in a neighborhood U of x for which a tangent vector ξ of X at $z \in U$ satisfies $\omega_i(\xi) = 0$, $1 \leq i \leq s$, if and only if $\xi f_i = 0$, $1 \leq i \leq s$. In this case, each of the df_i is a linear combination of $\omega_1, \dots, \omega_s$, and conversely, each of the ω_i is a linear combination of df_1, \dots, df_s . In general, a function f for which df is a linear combination of $\omega_1, \dots, \omega_s$ is called a **first integral** of $\Omega = 0$.

The theorem of the previous paragraph is called **Frobenius's theorem**, which can be stated in the dual form as follows: Let $D(X)$ be a \dagger subbundle of the \dagger tangent bundle $T(X)$ over X . The mapping $X \ni x \rightarrow D_x(X)$ is called a **distribution** on X . It is said to be an **involutive distribution** if at every point x of X we can find a system of independent vector fields L_i ($1 \leq i \leq s$) on a neighborhood U of x such that the $L_i(z)$ ($1 \leq i \leq s$) form a basis of $D_z(X)$ at every $z \in U$ and satisfy $[L_i, L_j] \equiv 0$ (L_1, \dots, L_s), $1 \leq i < j \leq s$, on U . A connected submanifold M of X is called an **integral manifold** of $D(X)$ if $T_x(M) = D_x(X)$ at every point x of M . Suppose that $D(X)$ gives an involutive distribution on X . Then every point x of X is in a maximal integral manifold M that contains any integral manifold including x as a submanifold.

E. Cartan-Kähler Existence Theorems

Let X be a \dagger real analytic manifold. Denote the \dagger sheaf of rings of differential forms on X by $\Lambda(X)$ and its subsheaf of $\mathcal{O}(X)$ -modules of p -forms on X by $\Lambda_p(X)$, $1 \leq p \leq n$, where $\mathcal{O}(X)$ is the sheaf of rings of 0-forms on X . A subsheaf of ideals Σ is called a **differential ideal** if it is generated by Σ_p , $0 \leq p \leq n$, and contains $d\Sigma$,

where $\Sigma_p = \Sigma \cap \Lambda_p(X)$. Consider a differential ideal Σ on X . Denote the t Grassmann manifold of p -dimensional subspaces of $T_x(X)$ with origin $x \in X$ by $G_p(x)$, and the Grassmann manifold $\bigcup_{x \in X} G_p(x)$ over X by $G_p(X)$. An element E_p of $G_p(X)$ is called a p -dimensional **contact element** with **origin** x . An element E_p of $G_p(X)$ is called an **integral element** of Σ_p if $\omega(E_p) = 0$ at x for any p -form ω in Σ ; furthermore, E_p is called an integral element of Σ if any element E_q contained in E_p , $0 \leq q \leq p$, is an integral element of Σ_q . In particular, 0-dimensional and 1-dimensional integral elements are called **integral points** and **integral vectors**, respectively. It can be proved that an element E_p is an integral element of Σ if and only if it is an integral element of Σ_p . The **polar element** $H(E_p)$ of an integral element E_p with origin x is defined as the subspace of $T_x(X)$ consisting of all vectors that generate with E_p an integral element of Σ . Let $(\Sigma_p)^0$, $0 \leq p \leq n$, be the subsheaf of $\mathcal{O}(X)$ -modules in $\mathcal{O}(G_p(X))$ consisting of all 0-forms

$$\sum_{1 \leq i_1 < \dots < i_p \leq n} a_{i_1 \dots i_p} z_{i_1 \dots i_p}$$

on $G_p(X)$ derived from a p -form

$$\sum_{1 \leq i_1 < \dots < i_p \leq n} a_{i_1 \dots i_p} dx_{i_1} \wedge \dots \wedge dx_{i_p} \in \Sigma_p,$$

where $z_{i_1 \dots i_p}$ is the t Grassmann coordinate of E_p . An integral element E_p^0 is called a **regular integral element** if the following two conditions are satisfied: (i) $(\Sigma_p)^0$ is a regular local equation of $I\Sigma_p$ at E_p^0 , where $I\Sigma_p$ is the set of all integral elements of Σ_p ; (ii) $\dim H(E_p^0) = \text{constant}$ around E_p^0 on $I\Sigma_p$. This definition, due to E. Kähler, is different from that given by E. Cartan [4].

Here, in general, a subsheaf Φ of $\mathcal{O}(X)$ is called a **regular local equation** of $I\Phi$ at an integral point x_0 if there exists a neighborhood U of x_0 and t cross sections ϕ_1, \dots, ϕ_s of Φ on U that satisfy the following two conditions: (i) $d\phi_1, \dots, d\phi_s$ are linearly independent at every x on U ; (ii) a point x of U is an integral point of Φ if and only if $\phi_1(x) = \dots = \phi_s(x) = 0$.

First existence theorem. Suppose that we are given a p -dimensional integral manifold M with a regular integral element $T_x(M)$ at a point x on M . Suppose further that there exists a submanifold F of X containing M such that $\dim F = n - t_{p+1}$, $\dim(T_x(F) \cap H(E_p)) = p + 1$, where $E_p = T_x(M)$ and $t_{p+1} = \dim H(E_p) - p - 1$. Then around x there exists a unique integral manifold N such that $\dim N = p + 1$ and $F \supset N \supset M$.

This theorem is proved by integrating a system of partial differential equations of Cauchy-Kovalevskaya type. E. Cartan [2–4] also tried to obtain an existence theorem by integrating a system of ordinary differential equations.

A chain of integral elements $E_0 \subset E_1 \subset \dots \subset E_r$ is called a **regular chain** if each of E_p ($0 \leq p < r$) is a regular integral element. For a regular chain $E_0 \subset E_1 \subset \dots \subset E_r$, define t_{p+1} by $t_{p+1} = \dim H(E_p) - p - 1$, $0 \leq p < r$, and define s_p by $s_p = t_p - t_{p+1} - 1$ ($0 \leq p < r$), $s_r = t_r$, where $t_0 = \dim I\Sigma_0$. Then we have $s_p \geq 0$ ($0 \leq p \leq r$), $s_0 + \dots + s_r = t_0 - r$, and we can take a local coordinate system $(x_1, \dots, x_r, y_1, \dots, y_m)$, $m = n - r$, around E_0 that satisfies the following four conditions:

- (i) $I\Sigma_0$ is defined by $y_{t_0-r+1} = \dots = y_m = 0$;
- (ii) $H(E_p) = \left\{ \frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_r}, \frac{\partial}{\partial y_{s_0+\dots+s_{p-1}+1}}, \dots, \frac{\partial}{\partial y_{t_0-r}} \right\}$, $0 \leq p < r$;
- (iii) $E_p = \left\{ \frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_p} \right\}$, $1 \leq p \leq r$;
- (iv) $E_0 = (0, \dots, 0, 0, \dots, 0)$.

The integers s_0, \dots, s_r are called the **characters** of the regular chain $E_0 \subset \dots \subset E_r$.

Second existence theorem. Suppose that a chain of integral elements $E_0 \subset \dots \subset E_r$ is regular, and take a local coordinate system satisfying (i)–(iv). Consider a system of initial data

$$\begin{aligned} &f_1, \dots, f_{s_0}, \\ &f_{s_0+1}(x_1), \dots, f_{s_0+s_1}(x_1), \\ &f_{s_0+s_1+1}(x_1, x_2), \dots, f_{s_0+s_1+s_2}(x_1, x_2), \\ &\dots \\ &f_{s_0+\dots+s_{r-1}+1}(x_1, \dots, x_r), \dots, f_{t_0-r}(x_1, \dots, x_r). \end{aligned}$$

Then if their values and derivatives of the first order are sufficiently small, there exists a unique integral manifold defined by $y_\alpha = y_\alpha(x_1, \dots, x_r)$, $y_\beta = 0$, $1 \leq \alpha \leq t_0 - r < \beta \leq m$, such that

$$y_\alpha(x_1, \dots, x_p, 0, \dots, 0) = f_\alpha(x_1, \dots, x_p),$$

$$s_0 + \dots + s_{p-1} < \alpha \leq s_0 + \dots + s_p, \quad 0 \leq p \leq r.$$

This theorem is proved by successive application of the first existence theorem. These two theorems are called the **Cartan-Kähler existence theorems**. Σ is said to be **involutive** at an integral element E_r if there exists a regular chain $E_0 \subset \dots \subset E_r$. An integral manifold possessing a tangent space at which Σ is involutive is called an **ordinary integral manifold** or **ordinary solution** of Σ . An integral manifold that does not possess such a tangent space is called a **singular integral manifold** or **singular solution** of Σ .

Cartan's definition of ordinary and regular integral elements is as follows: An integral point E_0^0 is an ordinary integral point if Σ_0 is a regular local equation of $I\Sigma_0$ at E_0^0 . An ordi-

nary integral point E_0^0 is a regular integral point if $\dim H(E_0)$ is constant on $I\Sigma_0$ around E_0^0 . Inductively, an integral element E_p^0 is called an **ordinary integral element** if $(\Sigma_p)^0$ is a regular local equation of $I\Sigma_p$ at E_p^0 and E_p^0 contains a regular integral element E_{p-1}^0 . An ordinary integral element E_p^0 is a regular integral element (in the sense of Cartan) if $\dim H(E_p)$ is constant on $I\Sigma_p$ around E_p^0 . It can be proved that Σ is involutive at an integral element E_r if and only if E_r is an ordinary integral element of Σ . An integral manifold possessing a tangent space that is a regular integral element of Σ is called a **regular integral manifold** or **regular solution** of Σ . Let m_{p+1} be the minimal dimension of $H(E_p)$, where E_p varies over the set of p -dimensional ordinary integral elements, and g be an integer such that $m_p \geq p$ ($1 \leq p \leq g$) and $m_{g+1} = p$. Then this integer g is called the **genus** of Σ . It is the maximal dimension of ordinary integral manifolds of Σ . However, in general, it is not the maximal dimension of integral manifolds of Σ .

D. C. Spencer and others have been trying to obtain an existence theorem in the C^∞ -category analogous to that of Cartan and Kähler. (For a system of linear partial differential equations \rightarrow [2, 4, 11, 13, 25, 27].)

F. Involutive Systems of Partial Differential Equations

To give a definition of an involutive system of partial differential equations, we define an involutive subspace of $\text{Hom}(V, W)$, where V and W are finite-dimensional vector spaces over the real number field \mathbf{R} . Let A be a subspace of $\text{Hom}(V, W)$. For a system of vectors v_1, \dots, v_p in V , $A(v_1, \dots, v_p)$ denotes the subspace of A that annihilates v_1, \dots, v_p . Let g_p be the minimal dimension of $A(v_1, \dots, v_p)$ as (v_1, \dots, v_p) varies, where $0 \leq p \leq r = \dim V$. A basis (v_1, \dots, v_r) of V is called a **generic basis** if it satisfies $g_p = \dim A(v_1, \dots, v_p)$ for each p . There exists a generic basis for any A . Let $W \otimes S^2(V^*)$ be the subspace of $\text{Hom}(V, \text{Hom}(V, W))$ consisting of all elements ξ satisfying $\xi(u)v = \xi(v)u$ for any u and v in V . Then the prolongation pA of A is defined by $pA = \text{Hom}(V, A) \cap W \otimes S^2(V^*)$. For any basis (v_1, \dots, v_r) of V , we have the inequality

$$\dim pA \leq \sum_{p=0}^r \dim A(v_1, \dots, v_p).$$

The subspace A is called an **involutive subspace** of $\text{Hom}(V, W)$ if $\dim pA = \sum_{p=0}^r g_p$. This notion of an involutive subspace was obtained by V. W. Guillemin and S. Sternberg [13].

A triple $(X, N; \pi)$ consisting of two manifolds X, N and a projection π from X onto N is called a **fibered manifold** if the t differential π_*

is surjective at every point of X . Take the set of all mappings f from a domain in N to X satisfying $\pi \circ f = \text{identity}$ for a fibered manifold $(X, N; \pi)$. Then an t l -jet $j_x^l(f)$ is an equivalence class under the equivalence relation defined as follows: $j_x^l(f) = j_u^l(g)$ if and only if $x = u$, $f(x) = g(u)$, and

$$\frac{\partial^{i_1+\dots+i_r} f}{\partial x_1^{i_1} \dots \partial x_r^{i_r}}(x) = \frac{\partial^{i_1+\dots+i_r} g}{\partial x_1^{i_1} \dots \partial x_r^{i_r}}(u),$$

$i_1 + \dots + i_r \leq l$, where (x_1, \dots, x_r) is a local coordinate system of N around $x = u$ (\rightarrow 105 Differentiable Manifolds X).

Denote the space of all l -jets of a fibered manifold $(X, N; \pi)$ by $J^l(X, N; \pi)$ or simply J^l . Then a subsheaf of ideals Φ in $\mathcal{O}(J^l)$ is called a **system of partial differential equations of order l** on N . A point z of J^l is called an integral point of Φ if $\varphi(z) = 0$ for all $\varphi \in \Phi$. The set of all integral points of Φ is denoted by $I\Phi$. Let π^l be the natural projection of J^l onto J^{l-1} . Then at a point z of J^l , we can identify $\text{Ker } \pi_*^l$ with $\text{Hom}(T_x(N), \text{Ker } \pi_*^l)$, where $x = \pi \pi^1 \dots \pi^{l-1} z$. The principal part $C_z(\Phi)$ of Φ is defined as the subspace of $\text{Ker } \pi_*^l$ that annihilates Φ . The **prolongation** $p\Phi$ of Φ is defined as the system of order $l+1$ on N generated by Φ and $\partial_k \Phi$, $1 \leq k \leq \dim N$, where ∂_k is the formal derivative with respect to a coordinate x_k of N :

$$(\partial_k \varphi)(j_x^{l+1}(f)) = \frac{\partial}{\partial x_k} \varphi(j_x^l(f)), \quad \varphi \in \mathcal{O}(J^l).$$

Let w be an integral point of $p\Phi$ and z be $\pi^{l+1} w$. Then we have the identity

$$pC_z(\Phi) = C_w(p\Phi).$$

The following definition of an involutive system is due to M. Kuranishi [19]: Φ is involutive at an integral point z if the following two conditions are satisfied: (i) Φ is a regular local equation of $I\Phi$ at z ; (ii) there exists a neighborhood U of z in J^l such that $(\pi^{l+1})^{-1} U \cap I(p\Phi)$ forms a fibered manifold with base $U \cap I\Phi$ and projection π^{l+1} .

A system of partial differential equations is said to be **involutive** (or **involutory**) if it has an integral point at which it is involutive. Fix a system of independent variables (y_1, \dots, y_N) in X . Then a system of differential forms is said to be **involutive** (or **involutory**) if it has an integral element at which it is involutive and $dy_1 \wedge \dots \wedge dy_N \neq 0$. It can be proved that these two definitions of involutive system are equivalent [19, 25].

G. Prolongation Theorems

Cartan gave a method of prolongation by which we can obtain an involutive system from a given system with two independent

variables, if it has a solution. He proposed the following problem: For any $r > 2$, construct a method of prolongation by which we can obtain an involutive system from a given system with r independent variables, if it has a solution. To solve this problem, Kuranishi prolonged a given system Φ successively to $p^t\Phi$, $t = 1, 2, 3, \dots$, and proved the following theorem: Suppose that there exists a sequence of integral points z^t of $p^t\Phi$ with $\pi^{t+1}z^t = z^{t-1}$, $t = 1, 2, 3, \dots$, that satisfies the following two conditions for each t : (i) $p^t\Phi$ is a regular local equation of $I(p^t\Phi)$ at z^t ; (ii) there exists a neighborhood V^t of z^t in $I(p^t\Phi)$ such that $\pi^{t+1}V^t$ contains a neighborhood of z^{t-1} in $I(p^{t-1}\Phi)$ and forms a fibered manifold $(V^t, \pi^{t+1}V^t; \pi^{t+1})$. Then $p^t\Phi$ is involutive at z^t for a sufficiently large integer t .

This prolongation theorem gives a powerful tool to the theory of infinite Lie groups. However, if we consider a system of partial differential equations of general type, there exist examples of systems that cannot be prolonged to an involutive system by this prolongation, although they have a solution. To improve Kuranishi's prolongation theorem, M. Matsuda [22] defined the prolongation of the same order by $p_0\Phi = p\Phi \cap \mathcal{O}(J^l)$ for a system Φ of order l . This is a generalization of the classical method of completion given by Lagrange and Jacobi. Applying this prolongation successively to a given system Φ , we have $\Psi = \bigcup_{\sigma=1}^{\infty} p_0^\sigma\Phi$. Define the p_* -operation by $p_* = \bigcup_{\sigma=1}^{\infty} p_0^\sigma p$. Then applying this prolongation successively to Ψ , we have the following theorem: suppose that there exists a sequence of integral points z^t of $p_*^t\Psi$ with $\pi^{t+1}z^t = z^{t-1}$, $t = 1, 2, 3, \dots$, that satisfies the following two conditions for each t : (i) $p_*^t\Psi$ is a regular local equation of $I(p_*^t\Psi)$ at z^t ; (ii) $\dim pC(p_*^t\Psi)$ is constant around z^t on $I(p_*^t\Psi)$. Then $p_*^t\Psi$ is involutive at z^t for a sufficiently large integer t .

To prove this theorem Matsuda applied the following theorem obtained by V. W. Guillemin, S. Sternberg, and J.-P. Serre [25, appendix]: suppose that we are given a subspace A_0 of $\text{Hom}(V, W)$ and subspaces A_t of $\text{Hom}(V, A_{t-1})$ satisfying $A_t \subset pA_{t-1}$, $t = 1, 2, 3, \dots$. Then A_t is an involutive subspace of $\text{Hom}(V, A_{t-1})$ for a sufficiently large integer t . Thus Cartan's problem was solved affirmatively. To the generalized Pfaff problem these prolongation theorems give another solution, which differs from that obtained by Riquier.

H. Pfaffian Systems in the Complex Domain

Consider a linear system of Pfaffian equations

$$du_i = \sum_{k=1}^n \sum_{j=1}^m a_{ij}^k(x) u_j dx_k, \quad i = 1, \dots, m,$$

where $x = (x_1, \dots, x_n)$ is a local coordinate of a complex manifold X and a_{ij}^k are meromorphic functions on X . If we put $u = (u_1, \dots, u_m)$ and $A^k(x) = (a_{ij}^k(x))$, $k = 1, \dots, n$, the system is written as

$$du = \left(\sum_{k=1}^n A^k(x) dx_k \right) u. \quad (9)$$

System (9) is completely integrable if and only if

$$\frac{\partial A^j}{\partial x^i} - \frac{\partial A^i}{\partial x^j} = [A^i, A^j], \quad j, l = 1, \dots, n.$$

Suppose that (9) is completely integrable. If the $A^k(x)$ are holomorphic at $x^0 = (x_1^0, \dots, x_n^0) \in X$, there exists for any $u^0 \in \mathbb{C}^m$ one and only one solution of (9) that is holomorphic at x^0 and satisfies $u(x^0) = u^0$. This implies that the solution space of (9) is an m -dimensional vector space; the basis of this space is called a fundamental system of solutions. Therefore any solution is expressible as a linear combination of a fundamental system of solutions and can be continued analytically in a domain where the $A^k(x)$ are holomorphic. A subvariety of X that is the pole set of at least one of the $A^k(x)$ is called a singular locus of (9), and a point on a singular locus is called a singular point.

R. Gérard has given a definition of regular singular points and an analytic expression of a fundamental system of solutions around a regular singular point, and he studied systems of Fuchsian type [8; also 9, 30].

Let $\Omega = \sum_{k=1}^n A^k(x) dx_k$. Then the system (9) can be rewritten as

$$(d - \Omega)u = 0.$$

If we consider a local coordinate (x, u) of a fiber bundle over X , the operator $d - \Omega$ induces a meromorphic linear connection ∇ over X . Starting from this point of view, P. Deligne [5] introduced several important concepts and obtained many results.

The first results for irregular singular points were obtained by Gérard and Y. Sibuya [10], and H. Majima [20] studied irregular singular points of mixed type.

The systems of partial differential equations that are satisfied by the hypergeometric functions of several variables are equivalent to linear systems of Pfaffian equations [1]. This means that such systems of partial differential equations are holonomic systems. M. Kashiwara and T. Kawai [15] studied holonomic systems with regular singularities from the standpoint of microlocal analysis. Special types of holonomic systems were investigated by T. Terada [28] and M. Yoshida [29].

Consider a system of Pfaffian equations

$$\omega_j = 0, \quad j = 1, \dots, r, \quad (10)$$

where $\omega_j = \sum_{k=1}^n a_{jk}(x) dx_k$ and $x = (x_1, \dots, x_n)$. Suppose that a_{jk} are holomorphic in a domain D of C^n and that $d\omega_j \wedge \omega_1 \wedge \dots \wedge \omega_r = 0$ in D . Denote by S the zero set of $\omega_1 \wedge \dots \wedge \omega_r = 0$. A point of S is called a singular point of (10). If the codimension of S is ≥ 1 , then system (10) is completely integrable in $D - S$. The following theorem was proved by B. Malgrange [21]: Let $x^0 \in S$, and suppose that the codimension of S is ≥ 3 around x^0 ; then there exist functions $f_j, j = 1, \dots, r$, and $g_{jk}, j, k = 1, \dots, r$, that are holomorphic at x^0 and satisfy $\omega_j = \sum_{k=1}^r g_{jk} df_k$ and $\det(g_{jk}(x^0)) \neq 0$.

References

- [1] P. Appell and J. Kampé de Fériet, *Fonctions hypergéométriques et hypersphériques*, Gauthier-Villars, 1926.
- [2] E. Cartan, *Sur l'intégration des systèmes d'équations aux différentielles totales*, Ann. Sci. Ecole Norm. Sup., 18 (1901), 241–311.
- [3] E. Cartan, *Leçons sur les invariants intégraux*, Hermann, 1922.
- [4] E. Cartan, *Les systèmes différentielles extérieures et leurs applications géométriques*, Hermann, Actualités Sci. Ind., 1945.
- [5] P. Deligne, *Equations différentielles à points singuliers réguliers*, Lecture notes in math. 163, Springer, 1970.
- [6] A. R. Forsyth, *Theory of differential equations*, pt. I. Exact equations and Pfaff's problem, Cambridge Univ. Press, 1890.
- [7] A. R. Forsyth, *Theory of differential equations*, pt. IV. Partial differential equations, Cambridge Univ. Press, 1906.
- [8] R. Gérard, *Théorie de Fuchs sur une variété analytique complexe*, J. Math. Pures Appl., 47 (1968), 321–404.
- [9] R. Gérard and A. H. M. Levelt, *Sur les connexions à singularités régulières dans le cas de plusieurs variables*, Funkcial. Ekvac., 19 (1976), 149–173.
- [10] R. Gérard and Y. Sibuya, *Etude de certains systèmes de Pfaff avec singularités*, Lecture notes in math. 712, Springer, 1979.
- [11] E. Goursat, *Leçons sur l'intégration des équations aux dérivées partielles du premier ordre*, Hermann, second edition, 1920.
- [12] E. Goursat, *Leçons sur le problème de Pfaff*, Hermann, 1922.
- [13] V. W. Guillemin and S. Sternberg, *An algebraic model of transitive differential geometry*, Bull. Amer. Math. Soc., 70 (1964), 16–47.
- [14] E. Kähler, *Einführung in die Theorie der Systeme von Differentialgleichungen*, Teubner, 1934.
- [15] M. Kashiwara and T. Kawai, *On holonomic systems of microdifferential equations IV, systems with regular singularities*, Publ. Res. Inst. Math. Sci., 17 (1981), 813–979.
- [16] M. Kuranishi, *On E. Cartan's prolongation theorem of exterior differential systems*, Amer. J. Math., 79 (1957), 1–47.
- [17] M. Kuranishi, *Lectures on exterior differential systems*, Lecture notes, Tata Inst., 1962.
- [18] M. Kuranishi, *On the local theory of continuous infinite pseudo groups I, II*, Nagoya Math. J., 15 (1959), 225–260; 19 (1961), 55–91.
- [19] M. Kuranishi, *Lectures on involutive systems of partial differential equations*, Publ. Soc. Math. São Paulo, 1967.
- [20] H. Majima, *Asymptotic analysis for integrable connections with irregular singular points*, Lecture notes in math. 1075, Springer, 1984.
- [21] B. Malgrange, *Frobenius avec singularités II, Le cas général*, Inventiones Math., 39 (1976), 67–89.
- [22] M. Matsuda, *Cartan-Kuranishi's prolongation of differential systems combined with that of Lagrange and Jacobi*, Publ. Res. Inst. Math. Sci., 3 (1967/68), 69–84.
- [23] C. Riquier, *Les systèmes d'équations aux dérivées partielles*, Gauthier-Villars, 1910.
- [24] J. F. Ritt, *Differential algebra*, Amer. Math. Soc. Colloq. Publ., 1950.
- [25] I. M. Singer and S. Sternberg, *The infinite groups of Lie and Cartan I. The transitive groups*, J. Analyse Math., 15 (1965), 1–114.
- [26] J. A. Schouten and W. v. d. Kulk, *Pfaff's problem and its generalizations*, Clarendon Press, 1949.
- [27] D. C. Spencer, *Overdetermined systems of linear partial differential equations*, Bull. Amer. Math. Soc., 75 (1969), 179–239.
- [28] T. Terada, *Problème de Riemann et fonctions automorphes provenant des fonctions hypergéométriques de plusieurs variables*, J. Math. Kyoto Univ., 13 (1973), 557–578.
- [29] M. Yoshida, *Local theory of Fuchsian systems with certain discrete monodromy groups I, II*, Funkcial. Ekvac., 21 (1978), 105–137, 203–221.
- [30] M. Yoshida and K. Takano, *On a linear system of Pfaffian equations with regular singular point*, Funkcial. Ekvac., 19 (1976), 175–189.

429 (XI.6) Transcendental Entire Functions

A. General Remarks

An **entire function** (or **integral function**) $f(z)$ is a complex-valued function of a complex variable

z that is holomorphic in the finite z -plane, $z \neq \infty$. If $f(z)$ has a pole at ∞ , then $f(z)$ is a polynomial in z . A polynomial is called a **rational entire function**. If an entire function is bounded, it is constant († Liouville's theorem). A **transcendental entire function** is an entire function that is not a polynomial, for example, $\exp z$, $\sin z$, $\cos z$. An entire function can be developed in a power series $\sum_{n=0}^{\infty} a_n z^n$ with infinite radius of convergence. If $f(z)$ is a transcendental entire function, this is actually an infinite series.

B. The Order of an Entire Function

If a transcendental entire function $f(z)$ has a zero of order m ($m \geq 0$) at $z=0$ and other zeros at $\alpha_1, \alpha_2, \dots, \alpha_n, \dots$ ($0 < |\alpha_1| \leq |\alpha_2| \leq |\alpha_3| \leq \dots \rightarrow \infty$), multiple zeros being repeated, then $f(z)$ can be written in the form

$$f(z) = e^{g(z)} z^m \prod_{k=1}^{\infty} \left(1 - \frac{z}{\alpha_k}\right) e^{g_k(z)},$$

where $g(z)$ is an entire function, $g_k(z) = (z/\alpha_k) + (1/2)(z/\alpha_k)^2 + (1/3)(z/\alpha_k)^3 + \dots + (1/p_k)(z/\alpha_k)^{p_k}$, and p_1, p_2, \dots are integers with the property that $\sum_{k=1}^{\infty} |z/\alpha_k|^{p_k+1}$ converges for all z (**Weierstrass's canonical product**).

E. N. Laguerre introduced the concept of the genus of a transcendental entire function $f(z)$. Assume that there exists an integer p for which $\sum_{k=0}^{\infty} |\alpha_k|^{-(p+1)}$ converges, and take the smallest such p . Assume further that in the representation for $f(z)$ in the previous paragraph, when $p_1 = p_2 = \dots = p$, the function $g(z)$ reduces to a polynomial of degree q ; then $\max(p, q)$ is called the **genus** of $f(z)$. For transcendental entire functions, however, the order is more essential than the genus. The **order** ρ of a transcendental entire function $f(z)$ is defined by

$$\rho = \limsup_{r \rightarrow \infty} \log \log M(r) / \log r,$$

where $M(r)$ is the maximum value of $|f(z)|$ on $|z|=r$. By using the coefficients of $f(z) = \sum a_n z^n$, we can write

$$\rho = \limsup_{n \rightarrow \infty} n \log n / \log(1/|a_n|).$$

The entire functions of order 0, which were studied by Valiron and others, have properties similar to polynomials, and the entire functions of order less than $1/2$ satisfy $\lim_{r_n \rightarrow \infty} \min_{|z|=r_n} |f(z)| = \infty$ for some increasing sequence $r_n \uparrow \infty$ (**Wiman's theorem**). Hence entire functions of order less than $1/2$ cannot be bounded in any domain extending to infinity. Among the functions of order greater than $1/2$ there exist functions bounded in a given angular domain $D: \alpha < \arg z < \alpha + \pi/\mu$. If $|f(z)|$

$< \exp r^{\rho}$ ($\rho < \mu$) and $f(z)$ is bounded on the boundary of D , then $f(z)$ is bounded in the angular domain (\rightarrow 272 Meromorphic Functions). In particular, if the order ρ of $f(z)$ is an integer p , then it is equal to the genus, and $g(z)$ reduces to a polynomial of degree $\leq p$ (J. Hadamard). These theorems originated in the study of the zeros of the † Riemann zeta function and constitute the beginning of the theory of entire functions.

There is some difference between the properties of functions of integral order and those of others. Generally, the point z at which $f(z) = w$ is called a **w-point** of $f(z)$. If $\{z_n\}$ consists of w -points different from the origin, the infimum $\rho_1(w)$ of k for which $\sum 1/|z_n|^k$ converges is called the **exponent of convergence** of $f-w$. If the order ρ of an entire function is integral, then $\rho_1(w) = \rho$ for each value w with one possible exception, and if ρ is not integral, then $\rho_1(w) = \rho$ for all w (É. Borel). Therefore any transcendental entire function has an infinite number of w -points for each value w except for at most one value, called an **exceptional value** of $f(z)$ (**Picard's theorem**). In particular, $f(z)$ has no exceptional values if ρ is not integral. For instance, $\sin z$ and $\cos z$ have no exceptional values, while e^z has 0 as an exceptional value. Since transcendental entire functions have no poles, ∞ can be counted as an exceptional value. Then we must change the statement in Picard's theorem to "except for at most two values." Since the theorem was obtained by E. Picard in 1879, problems of this type have been studied intensively (\rightarrow 62 Cluster Sets, 272 Meromorphic Functions).

After Picard proved the theorem by using the inverse of a † modular function, several alternative proofs were given. For instance, there is a proof using the Landau-Schottky theorem and † Bloch's theorem and one using † normal families. Picard's theorem was extended to meromorphic functions and has also been studied for analytic functions defined in more general domains. There are many fully quantitative results, too. For instance, Valiron [3] gave such results by performing some calculations on neighborhoods of points where entire functions attain their maximum absolute values.

Thereafter, the distribution of w -points in a neighborhood of an essential singularity was studied by many people, and in 1925 the Nevanlinna theory of meromorphic functions was established. The core of the theory consists of two fundamental theorems, † Nevanlinna's first and second fundamental theorems (\rightarrow 272 Meromorphic Functions). Concerning composite entire functions $F(z) = f(g(z))$, Pólya proved the following fact: The finiteness of the order of F implies that the order of f should

Transcendental Entire Functions

be zero unless g is a polynomial. This gives the starting point of the factorization theory, on which several people have been working recently. Several theorems in the theory of meromorphic functions can be applied to the theory. One of the fundamental theorems is the following: Let $F(z)$ be an entire function, which admits the factorizations $F(z) = P_m(f_m(z))$ with a polynomial P_m of degree m and an entire function f_m for all integers m . Then $F(z) = A \cos \sqrt{H(z)} + B$ unless $F(z) = A \exp H(z) + B$. Here, H is a nonconstant entire function and A, B are constant, $A \neq 0$.

C. Julia Directions

Applying the theory of †normal families of holomorphic functions, G. Julia proved the existence of Julia directions as a precise form of Picard's theorem [5]. A transcendental entire function $f(z)$ has at least one direction $\arg z = \theta$ such that for any $\varepsilon > 0$, $f(z)$ takes on every (finite) value with one possible exception infinitely often in the angular domain $\theta - \varepsilon < \arg z < \theta + \varepsilon$. This direction $\arg z = \theta$ is called a **Julia direction** of $f(z)$.

D. Asymptotic Values

†Asymptotic values, †asymptotic paths, etc., are defined for entire functions as for meromorphic functions. In relation to †Iversen's theorem and †Gross's theorem for inverse functions and results on †cluster sets, †ordinary singularities of inverse functions hold for entire functions in the same way as for meromorphic functions. Also, as for meromorphic functions, †transcendental singularities of inverse functions are divided into two classes, the †direct and the †indirect transcendental singularities.

The exceptional values in Picard's theorem are asymptotic values of the functions, and ∞ is an asymptotic value of any transcendental entire function. Therefore $f(z) \rightarrow \infty$ along some curve extending to infinity. Between the asymptotic paths corresponding to two distinct asymptotic values, there is always an asymptotic path with asymptotic value ∞ . By †Bloch's theorem, A. Bloch showed that the †Riemann surface of the inverse function of a transcendental entire function contains a disk with arbitrarily large radius. Denjoy conjectured in 1907 that $\mu \leq 2\rho$, where ρ is the order of an entire function and μ is the number of distinct finite asymptotic values of the function, and L. V. Ahlfors gave the first proof (1929). This result contains Wiman's theorem. There are transcendental entire functions with $\mu = 2\rho$. It was shown by W. Gross that among entire functions of infinite order there exists

an entire function having every value as its asymptotic value.

References

- [1] E. C. Titchmarsh, *The theory of functions*, Clarendon Press, second edition, 1939.
- [2] R. P. Boas, *Entire functions*, Academic Press, 1954.
- [3] G. Valiron, *Lectures on the general theory of integral functions*, Librairie de l'Université, Deighton, Bell and Co., 1923 (Chelsea, 1949).
- [4] L. Bieberbach, *Lehrbuch der Funktionentheorie II*, Teubner, 1931 (Johnson Reprint Co., 1969).
- [5] G. Julia, *Sur quelques propriétés nouvelles des fonctions entières ou méromorphes*, Ann. Sci. Ecole Norm. Sup., (3) 36 (1919), 93–125.

430 (V.11) Transcendental Numbers

A. History

A complex number α is called a **transcendental number** if α is not †algebraic over the field of rational numbers \mathbb{Q} . C. Hermite showed in 1873 that e is a transcendental number. Following a similar line of thought as that taken by Hermite, C. L. F. Lindemann showed that π is also transcendental (1882). Among the 23 problems posed by D. Hilbert in 1900 (→ 196 Hilbert), the seventh was the problem of establishing the transcendence of certain numbers (e.g., $2^{\sqrt{2}}$). This stimulated fruitful investigations by A. O. Gel'fond, T. Schneider, C. L. Siegel, and others. The theory of transcendental numbers is, however, far from complete. There is no general criterion that can be utilized to characterize transcendental numbers. For example, neither the transcendence nor even the irrationality of the †Euler constant $C = \lim_{n \rightarrow \infty} (1 + 1/2 + \dots + 1/n - \log n)$ has been established. A survey of the development of the theory of transcendental numbers can be found in [18], in which an extensive list of relevant publications up to 1966 is given.

B. Construction of Transcendental Numbers

Let $\bar{\mathbb{Q}}$ be the field of †algebraic numbers. Suppose that α is an element of $\bar{\mathbb{Q}}$ that satisfies the irreducible equation $f(x) = a_0 x^n + a_1 x^{n-1} + \dots + a_n = 0$, where the a_i are rational integers, $a_0 \neq 0$, and a_0, a_1, \dots, a_n have no common factors. Then we define $H(\alpha)$ to be the maxi-

mum of $|a_i|$ ($i=0, \dots, n$) and call it the **height** of α . J. Liouville proved the following theorem (1844): Let ξ be a real number ($\xi \notin \mathbf{Q}$). If $\inf\{q^n |\xi - p/q| \mid p/q \in \mathbf{Q}\} = 0$ for any positive integer n , then ξ is transcendental.

Transcendental numbers having this property are called **Liouville numbers**. Examples are: (i) $\xi = \sum_{v=1}^{\infty} g^{-v!}$, where g is an integer not smaller than 2. (ii) Suppose that we are given a sequence $\{n_k\}$ of positive integers such that $n_k \rightarrow \infty$ ($k \rightarrow \infty$). Let ξ be the real number expressed as an infinite simple continued fraction $b_0 + 1/b_1 + 1/b_2 + \dots$. Let B_l be the denominator of the l th convergent of the continued fraction. If $b_{n_k+1} \geq B_{n_k}^{n_k-2}$ for $k \geq 1$, then ξ is a Liouville number.

On the other hand, K. Mahler [8, 9] proved the existence of transcendental numbers that are not Liouville numbers. For example, he showed that if $f(x)$ is a nonconstant integral polynomial function mapping the set of positive integers into itself, then a number ξ expressed, e.g., in the decimal system as $0.y_1 y_2 y_3 \dots$ is such a number if we put $y_n = f(n)$, $n = 1, 2, 3, \dots$ (In particular, from $f(x) = x$ we get the non-Liouville transcendental number $\xi = 0.123456789101112\dots$) Mahler proved this result by using Roth's theorem (1955) (\rightarrow 182 Geometry of Numbers). Both Liouville and Mahler utilized the theory of Diophantine approximation to construct transcendental numbers.

On the other hand, Schneider [10–12] and Siegel [3] constructed transcendental numbers using certain functions. Examples are: $\exp \alpha$ ($\alpha \in \mathbf{Q}$, $\alpha \neq 0$); α^β ($\alpha \in \mathbf{Q}$, $\alpha \neq 0, 1$; $\beta \in \mathbf{Q} - \mathbf{Q}$); $J(\tau)$, where J is the modular function and τ is an algebraic number that is not contained in any imaginary quadratic number field; $\wp(2\pi i/\alpha)$, where \wp is the Weierstrass \wp -function, $\alpha \in \overline{\mathbf{Q}}$, and $\alpha \neq 0$; and $B(p, q)$, where B is the Beta function and $p, q \in \mathbf{Q} - \mathbf{Z}$.

Since $e = \exp 1$ and $1 = \exp 2\pi i$, the transcendence of e and π is directly implied by the transcendence of $\exp \alpha$ ($\alpha \in \overline{\mathbf{Q}}$, $\alpha \neq 0$).

C. Classification of Transcendental Numbers

(1) Mahler's classification: Given a complex number ξ and positive integers n and H , we consider the following:

$$w_n(H, \xi) = \min \left\{ \left| \sum_{v=0}^n a_v \xi^v \right| \mid a_v \in \mathbf{Z}, |a_v| \leq H, \sum_{v=0}^n a_v \xi^v \neq 0 \right\},$$

$$w_n(\xi) = w_n = \limsup_{H \rightarrow \infty} (-\log w_n(H, \xi) / \log H),$$

$$w(\xi) = w = \limsup_{n \rightarrow \infty} w_n(\xi) / n,$$

and let μ = the first number n for which w_n is ∞ . Then we have the following four cases:

(i) $w = 0$, $\mu = \infty$; (ii) $0 < w < \infty$, $\mu = \infty$; (iii) $w = \mu = \infty$; (iv) $w = \infty$, $\mu < \infty$, corresponding to which we call ξ an **A-number**, **S-number**, **T-number**, or **U-number**. The set of A-numbers is denoted by **A**, and similarly we have the classes **S**, **T**, and **U**. It is known that **A** = $\overline{\mathbf{Q}}$. If two numbers ξ and η are algebraically dependent over \mathbf{Q} , then they belong to the same class. If ξ belongs to **S**, the quantity $\theta(\xi) = \sup\{w_n(\xi)/n \mid n = 1, 2, \dots\}$ is called the **type** of ξ (in the sense of Mahler). Mahler conjectured that almost all transcendental numbers (except a set of Lebesgue measure zero) are S-numbers of the type 1 or 1/2 according as they belong to **R** or not. Various results were obtained concerning this conjecture (W. J. LeVeque, J. F. Koksma, B. Volkmann) until it was proved by V. G. Sprindzhuk in 1965 [14, 15]. The existence of T-numbers was proved by W. M. Schmidt (1968) [16]. All Liouville numbers are U-numbers [7]. On the other hand, $\log \alpha$ ($\alpha \in \mathbf{Q}$, $\alpha > 0$, $\alpha \neq 1$) and π are transcendental numbers that do not belong to **U**.

(2) Koksma's classification: For a given transcendental number ξ and positive numbers n and H , we consider the following:

$$w_n^*(H, \xi) = \min\{|\xi - \alpha| \mid \alpha \in \overline{\mathbf{Q}},$$

$$H(\alpha) \leq H, [Q(\alpha):\mathbf{Q}] \leq n\},$$

$$w_n^*(\xi) = w_n^* = \limsup_{H \rightarrow \infty} (-\log H w_n^*(H, \xi) / \log H),$$

$$w^*(\xi) = w^* = \limsup_{n \rightarrow \infty} w_n^*(\xi) / n,$$

and let μ^* = the first number n for which w_n^* is ∞ . Then we have the following three cases: (i) $w^* < \infty$, $\mu^* = \infty$; (ii) $w^* = \mu^* = \infty$; (iii) $w^* = \infty$, $\mu^* < \infty$. We call ξ an **S*-number**, **T*-number**, or **U*-number** according as (i), (ii), or (iii) holds and denote the set of S*-numbers by **S***, etc. If ξ belongs to **S***, we call $\theta^*(\xi) = \sup\{w_n^*(\xi)/n \mid n = 1, 2, \dots\}$ the **type** of ξ (in the sense of Koksma). It can be shown that **S** = **S***, **T** = **T***, and **U** = **U***, and that if $\xi \in \mathbf{S}$, then $\theta^*(\xi) \leq \theta(\xi) \leq \theta^*(\xi) + 1$.

D. Algebraic Independence

Concerning the algebraic relations of transcendental numbers, we have the following three principal theorems:

(1) Let $\alpha_1, \dots, \alpha_m$ be elements of $\overline{\mathbf{Q}}$ that are linearly independent over \mathbf{Q} . Then $\exp \alpha_1, \dots, \exp \alpha_m$ are transcendental and algebraically independent over $\overline{\mathbf{Q}}$ (**Lindemann-Weierstrass theorem**).

(2) Let $J_0(x)$ be the Bessel function and α a nonzero algebraic number. Then $J_0(x)$ and $J'_0(x)$ are transcendental and algebraically independent over \mathbf{Q} (Siegel).

(3) Let $\alpha_1, \dots, \alpha_n$ be nonzero elements of $\bar{\mathbf{Q}}$ such that $\log \alpha_1, \dots, \log \alpha_n$ are linearly independent over \mathbf{Q} . Then $1, \log \alpha_1, \dots, \log \alpha_n$ are linearly independent over $\bar{\mathbf{Q}}$ (A. Baker).

Besides these theorems, various related results have been obtained by A. B. Shidlovskii, Gel'fond, N. I. Fel'dman, and others. A quantitative extension of theorem (3), also by Baker, will be discussed later.

First we give more detailed descriptions of theorems (1) and (2). Let $\alpha_1, \dots, \alpha_m$ be as in theorem (1), $s = [\mathbf{Q}(\alpha_1, \dots, \alpha_m) : \mathbf{Q}]$, $P(X_1, \dots, X_m)$ be an arbitrary polynomial in $\bar{\mathbf{Q}}[X_1, \dots, X_m]$ of degree n , and $H(P)$ be the maximum of the absolute values of the coefficients of the polynomial P . Then there exists a positive number C determined only by the numbers $\alpha_1, \dots, \alpha_m$ and $n (= \deg P)$ such that

$$|P(e^{\alpha_1}, \dots, e^{\alpha_m})| > CH(P)^{-2s(2^{2smn+m+n}-1)}.$$

In particular, if α is a nonzero algebraic number, then $\exp \alpha$ belongs to \mathbf{S} and $\theta(\exp \alpha) \leq 8s^2 + 6s$.

(2') Let α be a nonzero algebraic number, $s = [\mathbf{Q}(\alpha) : \mathbf{Q}]$, $P \in \mathbf{Q}[X_1, X_2]$, $\deg P = n$. Then there exists a positive number C determined only by α and n such that $|P(J_0(\alpha), J'_0(\alpha))| > CH(P)^{-82s^3n^3}$.

Theorems (1) and (2) are actually special cases of a theorem obtained by Siegel. To state this theorem, the following terminology is used: An entire function $f(z) = \sum_{n=0}^{\infty} C_n \cdot z^n/n!$ is called an *E-function* defined over an algebraic number field K of finite degree if the following three conditions are satisfied: (i) $C_n \in K$ ($n = 0, 1, 2, \dots$). (ii) For any positive number ε , $C_n = O(n^{\varepsilon n})$. (iii) Let q_n be the least positive integer such that $C_k q_n$ belongs to the ring \mathfrak{O} of algebraic integers in K ($0 \leq n, 0 \leq k \leq n$). Then for an arbitrary positive number ε , $q_n = O(n^{\varepsilon n})$.

A system $\{f_1(z), \dots, f_m(z)\}$ of *E-functions* defined over K is said to be **normal** if it satisfies the following two conditions: (i) None of the functions $f_i(z)$ is identically zero. (ii) If the functions $w_k = f_k(z)$ ($k = 1, \dots, m$) satisfy a system of homogeneous linear differential equations of the first order, then $w'_k = \sum_{i=1}^m Q_{ki}(z)w_i$, where the $Q_{ki}(z)$ are rational functions of z , with coefficients in the ring \mathfrak{O} . The matrix (Q_{ki}) can be decomposed by rearranging the order of the indices k, l if necessary into the form

$$\begin{pmatrix} W_1 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & W_r \end{pmatrix},$$

where

$$W_t = \begin{bmatrix} Q_{11,t} & \dots & Q_{1m_t,t} \\ \vdots & \ddots & \vdots \\ Q_{m_t 1,t} & \dots & Q_{m_t m_t,t} \end{bmatrix}, \quad 1 \leq t \leq r, \quad \sum_{t=1}^r m_t = m.$$

The decomposition is unique if we choose r as large as possible, in which case we call W_1, \dots, W_r the primitive parts of (Q_{ki}) . The requirement is that the primitive parts W_i are independent in the following sense: If there are numbers $C_{st} \in K$ and polynomial functions $P_{kt}(z) \in K[z]$ such that

$$\sum_{t=1}^r (C_{1t} \dots C_{m_t t}) W_t \begin{bmatrix} P_{1t}(z) \\ \vdots \\ P_{m_t t}(z) \end{bmatrix} = 0,$$

then $C_{st} = 0$, $P_{kt}(z) = 0$.

Let N be a positive integer. A normal system $\{f_1(z), \dots, f_m(z)\}$ of *E-functions* is said to be of degree N if the system $\{F_{n_1, \dots, n_m}(z) = f_1(z)^{n_1} \dots f_m(z)^{n_m} \mid n_i \geq 0, \sum_{i=1}^m n_i \leq N\}$ is also a normal system of *E-functions*. Then the theorem obtained by Siegel [4] is: Let N be an arbitrary positive integer and $\{f_1(z), \dots, f_m(z)\}$ be a normal system of *E-functions* of degree N defined over an algebraic number field of finite degree K satisfying the system of differential equations $f'_k(z) = \sum_{i=1}^m Q_{ki}(z)f_i(z)$, where $Q_{ki}(z) \in \mathfrak{O}(z)$, $1 \leq k \leq m$. If α is a nonzero algebraic number that is not a pole of any one of the functions $Q_{ki}(z)$, then $f_1(\alpha), \dots, f_m(\alpha)$ are transcendental numbers that are algebraically independent over the field $\bar{\mathbf{Q}}$.

Theorem (3) at the beginning of this section implies, for example, the following: (i) If $\alpha_1, \dots, \alpha_n$ and β_1, \dots, β_n all belong to $\bar{\mathbf{Q}}$ and $\gamma = \alpha_1 \log \beta_1 + \dots + \alpha_n \log \beta_n \neq 0$, then γ is transcendental. (ii) If $\alpha_1, \dots, \alpha_n, \beta_0, \beta_1, \dots, \beta_n$ are nonzero algebraic numbers, then $e^{\beta_0 \alpha_1^{\beta_1}} \dots \alpha_n^{\beta_n}$ is transcendental. (iii) If $\alpha_1, \dots, \alpha_n$ are algebraic numbers other than 0 and 1, and β_1, \dots, β_n also belong to $\bar{\mathbf{Q}}$, with $1, \beta_1, \dots, \beta_n$ linearly independent over \mathbf{Q} , then $\alpha_1^{\beta_1} \dots \alpha_n^{\beta_n}$ is transcendental.

Baker [17] also obtained a quantitative extension of theorem (3): Suppose that we are given integers $A \geq 4$, $d \geq 4$ and nonzero algebraic numbers $\alpha_1, \dots, \alpha_n$ ($n \geq 2$) whose heights and degrees do not exceed A and d , respectively. Suppose further that $0 < \delta \leq 1$, and let $\log \alpha_1, \dots, \log \alpha_n$ be the principal values of the logarithms. If there exist rational integers b_1, \dots, b_n with absolute value at most H such that

$$0 < |b_1 \log \alpha_1 + \dots + b_n \log \alpha_n| < e^{-\delta H},$$

then

$$H < (4n^2 \delta^{-1} d^{2n} \log A)^{(2n+1)^2}.$$

This theorem has extensive applications in various problems of number theory, including a wide class of Diophantine problems [19].

A number of new, interesting results on the algebraic independence of values of exponential functions, elliptic functions, and some other special functions have been obtained

recently by D. Masser, G. V. Chudnovskii, M. Waldschmidt, and other writers. In particular, Chudnovskii (1975) obtained the remarkable result that $\Gamma(1/3)$ and $\Gamma(1/4)$ are transcendental numbers. See [20–24].

References

- [1] A. O. Gel'fond, *Transcendental and algebraic numbers*, Dover, 1960. (Original in Russian, 1952.)
- [2] T. Schneider, *Einführung in die transzendenten Zahlen*, Springer, 1957.
- [3] C. L. Siegel, *Über einige Anwendungen diophantischer Approximationen*, Abh. Preuss. Akad. Wiss., 1929 (Gesammelte Abhandlungen, Springer, 1966, vol. 1, 209–266).
- [4] C. L. Siegel, *Transcendental numbers*, Ann. Math. Studies, Princeton Univ. Press, 1949.
- [5] N. I. Fel'dman, *Approximation of certain transcendental numbers I*, Amer. Math. Soc. Transl., 59 (1966), 224–245. (Original in Russian, 1951.)
- [6] W. J. LeVeque, *Note on S-numbers*, Proc. Amer. Math. Soc., 4 (1953), 189–190.
- [7] W. J. LeVeque, *On Mahler's U-numbers*, J. London Math. Soc., 28 (1953), 220–229.
- [8] K. Mahler, *Arithmetische Eigenschaften einer Klasse von Dezimalbrüchen*, Proc. Acad. Amsterdam, 40 (1937), 421–428.
- [9] K. Mahler, *Über die Dezimalbruchentwicklung gewisser Irrationalzahlen*, Mathematika B, Zutphen, 6 (1937), 22–36.
- [10] T. Schneider, *Transzendenzuntersuchungen periodischer Funktionen I*, J. Reine Angew. Math., 172 (1935), 65–69.
- [11] T. Schneider, *Arithmetische Untersuchungen elliptischer Integrale*, Math. Ann., 113 (1936), 1–13.
- [12] T. Schneider, *Zur Theorie der Abelschen Funktionen und Integrale*, J. Reine Angew. Math., 183 (1941), 110–128.
- [13] S. Lang, *Introduction to transcendental numbers*, Addison-Wesley, 1966.
- [14] V. G. Sprindzhuk, *A proof of Mahler's conjecture on the measure of the set of S-numbers*, Amer. Math. Soc. Transl., 51 (1966), 215–272. (Original in Russian, 1965.)
- [15] V. G. Sprindzhuk, *Mahler's problem in metric number theory*, Amer. Math. Soc. Transl. Math. Monographs, 1969. (Original in Russian, 1967.)
- [16] W. M. Schmidt, *T-numbers do exist*, Rend. Convegno di Teoria dei Numeri, Roma, 1968.
- [17] A. Baker, *Linear forms in the logarithms of algebraic numbers I, II, III, IV*, Mathematika, 13 (1966), 204–216; 14 (1967), 102–107; 14 (1967), 220–228; 15 (1968), 204–216.
- [18] N. I. Fel'dman and A. B. Shidlovskii, *The development and present state of the theory of transcendental numbers*, Russian Math. Surveys, 22 (1967), 1–79. (Original in Russian, 1967.)
- [19] A. Baker, *Transcendental number theory*, Cambridge Univ. Press, 1975.
- [20] D. Masser, *Elliptic functions and transcendence*, Lecture notes in math. 437, Springer, 1975.
- [21] G. V. Chudnovskii, *Algebraic independence of values of exponential and elliptic functions*, Proc. Int. Congr. Math., Helsinki, 1978.
- [22] A. Baker and D. Masser (eds.), *Transcendence theory: Advances and applications*, Academic Press, 1977.
- [23] M. Waldschmidt, *Transcendence methods*, Queen's papers in pure and appl. math. 52, Queen's Univ., 1979.
- [24] M. Waldschmidt, *Nombres transcendants et groupes algébriques*, Astérisque, 69–70 (1980).

431 (IX.19) Transformation Groups

A. Topological Transformation Groups

Let G be a group, M a set, and f a mapping from $G \times M$ into M . Put $f(g, x) = g(x)$ ($g \in G$, $x \in M$). Then the group G is said to be a **transformation group** of the set M if the following two conditions are satisfied: (i) $e(x) = x$ ($x \in M$), where e is the identity element of G ; and (ii) $(gh)(x) = g(h(x))$ ($x \in M$) for any $g, h \in G$. In this case the mapping $x \rightarrow g(x)$ is a one-to-one mapping of M onto itself.

Let G be a transformation group of M . If G is a topological group, M a topological space, and the mapping $(g, x) \rightarrow g(x)$ a continuous mapping from $G \times M$ into M , then G is called a **topological transformation group** of M . In this case $x \rightarrow g(x)$ is a homeomorphism of M onto itself. The mapping $(g, x) \rightarrow g(x)$ is called an **action** of G on M . The space M , together with a given action of G , is called a **G -space**.

For a point x of M , the set $G(x) = \{g(x) \mid g \in G\}$ is called the **orbit** of G passing through the point x . Defining as equivalent two points x and y of M belonging to the same orbit, we get an equivalence relation in M . The quotient space of M by this equivalence relation, denoted by M/G , is called the **orbit space** of G -space M .

If $G(x) = \{x\}$, then x is called a **fixed point**. The set of all fixed points is denoted by M^G . For a point x of M , the set $G_x = \{g \in G \mid g(x) =$

Transformation Groups

$x\}$ is a subgroup of G called the **isotropy subgroup (stabilizer, stability subgroup)** of G at the point x . A conjugacy class of the subgroup G_x is called an **isotropy type** of the transformation group G on M .

The group G is said to act **nontrivially** (resp. **trivially**) on M if $M \neq M^G$ (resp. $M = M^G$). The group G is said to act **freely** on M if the isotropy subgroup G_x consists only of the identity element for any point x of M .

The group G is said to act **transitively** on M if for any two points x and y of M , there exists an element $g \in G$ such that $g(x) = y$.

Let N be the set of all elements $g \in G$ such that $g(x) = x$ for all points x of M . Then N is a normal subgroup of G . If N consists only of the identity element e , we say that G acts **effectively** on M , and if N is a discrete subgroup of G , we say that G acts **almost effectively** on M . When $N \neq \{e\}$, the quotient topological group G/N acts effectively on M in a natural fashion.

An **equivariant mapping (equivariant map)** (or a **G-mapping, G-map**) $h: X \rightarrow Y$ between G -spaces is a continuous mapping which commutes with the group actions, that is, $h(g(x)) = g(h(x))$ for all $g \in G$ and $x \in X$. An equivariant mapping which is also a homeomorphism is called an equivalence of G -spaces.

For a G -space M , an equivalence class of the G -spaces $G(x)$, $x \in M$, is called an **orbit type** of the G -space M .

B. Cohomological Properties

We consider only * paracompact G -spaces and * Čech cohomology theory in this section. We shall say that a topological space X is **finitistic** if every open covering has a finite-dimensional refinement. The following theorems are useful [1–3].

(1) If G is finite, X a finitistic paracompact G -space, and K a field of characteristic zero or prime to the order of G , then the induced homomorphism $\pi^*: H^*(X/G; K) \rightarrow H^*(X; K)^G$ is an isomorphism. Here, π is a natural projection of X onto X/G . The group G acts naturally on $H^*(X; K)$, and $H^*(X; K)^G$ denotes the fixed-point set of this G -action.

(2) Let X be a finitistic G -space and G cyclic of prime order p . Then, with coefficients in $\mathbb{Z}/p\mathbb{Z}$, we have

$$(a) \text{ for each } n \quad \sum_{i=n}^{\infty} \text{rank } H^i(X^G) \leq \sum_{i=n}^{\infty} \text{rank } H^i(X),$$

$$(b) \quad \chi(X) + (p-1)\chi(X^G) = p\chi(X/G).$$

Here the * Euler-Poincaré characteristics $\chi(\quad)$ are defined in terms of mod p cohomology.

(3) **Smith's theorem:** If G is a p -group (p prime) and if x is a finitistic G -space whose mod p cohomology is isomorphic to the n -

sphere, then the mod p cohomology of the fixed-point set X^G is isomorphic to that of the r -sphere for some $-1 \leq r \leq n$, where (-1) -sphere means the empty set.

(4) Let T^k denote the k -dimensional toral group. Let X be a T^k -space whose rational cohomology is isomorphic to the n -sphere, and assume that there are only a finite number of orbit types and that the orbit spaces of all subtori are finitistic. Let H be a subtorus of T^k . Then by the above theorem the rational cohomology of X^H is isomorphic to that of the $r(H)$ -sphere for some $-1 \leq r(H) \leq n$. Assume further that there is no fixed point of the T^k -action. Then, with H ranging over all subtori of dimension $k-1$, we have

$$n+1 = \sum_H (r(H) + 1).$$

C. Differentiable Transformation Groups

Suppose that the group G is a transformation group of a * differentiable manifold M , G is a * Lie group, and the mapping $(g, x) \rightarrow g(x)$ of $G \times M$ into M is a differentiable mapping. Then G is called a **differentiable transformation group** (or **Lie transformation group**) of M , and M is called a **differentiable G -manifold**.

The following are basic facts about compact differentiable transformation groups [3, 4]:

(5) **Differentiable slice theorem:** Let G be a compact Lie group acting differentiably on a manifold M . Then, by averaging an arbitrary * Riemannian metric on M , we may have a G -invariant Riemannian metric on M . That is, the mapping $x \rightarrow g(x)$ is an * isometry of this Riemannian manifold M for each $g \in G$. For each point $x \in M$, the orbit $G(x)$ through x is a compact submanifold of M and the mapping $g \mapsto g(x)$ defines a G -equivariant diffeomorphism $G/G_x \cong G(x)$, where G/G_x is the left quotient space by the isotropy subgroup G_x . G_x acts orthogonally on the * tangent space $T_x M$ at x (resp. the * normal vector space N_x of the orbit $G(x)$); we call it the **isotropy representation** (resp. **slice representation**) of G_x at x . Let E be the * normal vector bundle of the orbit $G(x)$. Since G acts naturally on E as a bundle mapping, the bundle E is equivalent to the bundle $(G \times N_x)/G_x$ over G/G_x as a * G -vector bundle, where G_x acts on N_x by means of the slice representation and G_x acts on G by the right translation. We can choose a small positive real number ε such that the * exponential mapping gives an equivariant * diffeomorphism of the ε -disk bundle of E onto an invariant * tubular neighborhood of $G(x)$.

(6) Assume that a compact Lie group G acts differentiably on M with the orbit space $M^* = M/G$ connected. Then there exists a maximum

orbit type G/H for G on M (i.e., H is an isotropy subgroup and H is conjugate to a subgroup of each isotropy group). The union $M_{(H)}$ of the orbits of type G/H is open and dense in M , and its image $M_{(H)}^*$ in M^* is connected.

The maximum orbit type for orbits in M guaranteed by the above theorem is called the **principal orbit type**, and orbits of this type are called **principal orbits**. The corresponding isotropy groups are called **principal isotropy groups**. Let P be a principal orbit and Q any orbit. If $\dim P > \dim Q$, then Q is called a **singular orbit**. If $\dim P = \dim Q$ but P and Q are not equivalent, then Q is called an **exceptional orbit**.

(7) Let G be a compact Lie group and M a compact G -manifold. Then the orbit types are finite in number.

By applying (5) and (6) we have that an isotropy group is principal if and only if its slice representation is trivial.

The situation is quite different in the case of noncompact transformation groups. For example, there exists an analytic action of $G = SL(4, \mathbf{R})$ on an analytic manifold M such that each orbit of G on M is closed and of codimension one and such that, for $x, y \in M$, G_x is not isomorphic to G_y unless x and y lie on the same G -orbit [5].

D. Compact Differentiable Transformation Groups

Many powerful techniques in \dagger differential topology have been applied to the study of differentiable transformation groups. For example, using the techniques of \dagger surgery, we can show that there are infinitely many free differentiable circle actions on \dagger homotopy $(2n+1)$ -spheres ($n \geq 3$) that are differentially inequivalent and distinguished by the rational \dagger Pontryagin classes of the orbit manifolds (W. C. Hsiang [6]). Also, using \dagger Brieskorn varieties, we can construct many examples of differentiable transformation groups on homotopy spheres [3, 4, 7]. Differentiable actions of compact connected Lie groups on homology spheres have been studied systematically (Hsiang and W. Y. Hsiang [4]).

The Atiyah-Singer \dagger index theorem has many applications in the study of transformation groups. The following are notable applications:

(8) Let M be a compact connected \dagger oriented differentiable manifold of dimension $4k$ with a \dagger spin-structure. If a compact connected Lie group G acts differentiably and nontrivially on M , then the \hat{A} -genus $\langle \hat{\mathcal{A}}(M), [M] \rangle$ of M vanishes (where $\hat{\mathcal{A}}(M)$ denotes the $\dagger\hat{A}$ -characteristic class of M) (M. F. Atiyah and F.

Hirzebruch [8], K. Kawakubo [9]). For further developments, see A. Hattori [10].

(9) Let M be a closed oriented manifold with a differentiable circle action. Then each connected component F_k of the fixed point set can be oriented canonically, and we have

$$I(M) = \sum_k I(F_k),$$

where $I(\)$ denotes the \dagger Thom-Hirzebruch index [8, 9].

Let G be a compact Lie group and $G \rightarrow EG \rightarrow BG$ the \dagger universal G -bundle. Then the \dagger singular cohomology $H^*(EG \times_G X)$ is called **equivariant cohomology** for a G -space X and is an $H^*(BG)$ -module. Let $G = U(1)$, M a differentiable $U(1)$ -manifold, $F = M^G$, and $i: F \rightarrow M$ the inclusion mapping. Then the \dagger localization of the induced homomorphism

$$S^{-1}i^*: S^{-1}H^*(EG \times_G M) \rightarrow S^{-1}H^*(BG \times F)$$

is an isomorphism, where S^{-1} denotes the localization with respect to the multiplicative set $S = \{at^k\}$ with a, k ranging over all positive integers and t the generator of $H^2(BG)$. Theorems (8) and (9) can be proved by the above localization isomorphism.

Let M be a differentiable manifold. The upper bound $N(M)$ of the dimension of all the compact Lie groups that acts effectively and differentiably on M is called the **degree of symmetry** of M . It measures, in some crude sense, the symmetry of the differentiable manifold M . The number $N(M)$ depends heavily on the differentiable structure. For example, $N(S^m) = m(m+1)/2$ for the standard m -sphere, but $N(\Sigma^m) < (m+1)^2/16 + 5$ for a \dagger homotopy m -sphere ($m \geq 300$) that does not bound a $\dagger\pi$ -manifold [11]. Also, $N(P_n(\mathbf{C})) = n(n+2)$ for the complex projective n -space $P_n(\mathbf{C})$, but $N(hP_n(\mathbf{C})) < (n+1)(n+2)/2$ for any homotopy complex projective n -space $hP_n(\mathbf{C})$ ($n \geq 13$) other than $P_n(\mathbf{C})$ (T. Watabe [12]).

Let X be a differentiable closed manifold and $h: X \rightarrow P_n(\mathbf{C})$ be an orientation-preserving \dagger homotopy equivalence. There is a conjecture about the total \hat{A} -classes that states: If X admits a nontrivial differentiable circle action, then $\hat{\mathcal{A}}(X) = h^*\hat{\mathcal{A}}(P_n(\mathbf{C}))$ (T. Petrie [13]). It is known that if the action is free outside the fixed-point set, then the conjecture is true (T. Yoshida [14]).

E. Equivariant Bordism

Fix a compact Lie group G ; a compact **oriented G -manifold** (ψ, M) consists of a compact \dagger oriented differentiable manifold M and an orientation-preserving differentiable G -action $\psi: G \times M \rightarrow M$ on M .

Given families $F \supset F'$ of subgroups of G , a compact oriented G -manifold (ψ, M) is (F, F') -free if the following conditions are satisfied: (i) if $x \in M$, then the isotropy group G_x is conjugate to a member of F ; (ii) if $x \in \partial M$, then G_x is conjugate to a member of F' .

If F' is the empty family, then necessarily ∂M is empty and M is closed. In this case we say that (ψ, M) is F -free.

Given (ψ, M) , define $-(\psi, M) = (\psi, -M)$ with the structure precisely the same as (ψ, M) except for \dagger orientation. Also define $\partial(\psi, M) = (\psi, \partial M)$. Note that if (ψ, M) is (F, F') -free, then $(\psi, \partial M)$ is F' -free. Define (ψ, M) and (ψ', M') to be isomorphic if there exists an equivariant orientation-preserving diffeomorphism of M onto M' .

An (F, F') -free compact oriented n -dimensional G -manifold (ψ, M) is said to **bord** if there exists an (F, F') -free compact oriented $(n+1)$ -dimensional G -manifold (Φ, W) together with a regularly embedded compact n -dimensional manifold M_1 in ∂W with M_1 invariant under the G -action Φ such that (Φ, M_1) is isomorphic to (ψ, M) and G_x is conjugate to a member of F' for $x \in \partial W - M_1$. Also, M_1 is required to have its orientation induced by that of W .

We say that (ψ_1, M_1) is **bordant** to (ψ_2, M_2) if the disjoint union $(\psi_1, M_1) + (\psi_2, -M_2)$ bords. Bordism is an equivalence relation on the class of (F, F') -free compact oriented n -dimensional G -manifolds. The bordism classes constitute an Abelian group $\mathbf{O}_n^G(F, F')$ under the operation of disjoint union. If F' is empty, denote the above group by $\mathbf{O}_n^G(F)$. The direct sum

$$\mathbf{O}_*^G(F, F') = \bigoplus_n \mathbf{O}_n^G(F, F')$$

is naturally an Ω -module, where Ω is the \dagger oriented cobordism ring. If F consists of all subgroups of G , then $\mathbf{O}_*^G(F)$ is denoted by \mathbf{O}_*^G .

Suppose now that $F \supset F'$ are fixed families of subgroups of G . Every F' -free G -manifold is also F -free, and so this inclusion induces a homomorphism $\alpha: \mathbf{O}_n^G(F') \rightarrow \mathbf{O}_n^G(F)$. Similarly every F -free G -manifold is also (F, F') -free, inducing a homomorphism $\beta: \mathbf{O}_n^G(F) \rightarrow \mathbf{O}_n^G(F, F')$. Finally, there is a homomorphism $\partial: \mathbf{O}_n^G(F, F') \rightarrow \mathbf{O}_{n-1}^G(F')$ given by $\partial(\psi, M) = (\psi, \partial M)$. Then the following sequence is exact [15]:

$$\dots \xrightarrow{\hat{\alpha}} \mathbf{O}_n^G(F') \xrightarrow{\alpha} \mathbf{O}_n^G(F) \xrightarrow{\beta} \mathbf{O}_n^G(F, F') \xrightarrow{\partial} \mathbf{O}_{n-1}^G(F') \xrightarrow{\hat{\alpha}} \dots$$

A weakly almost complex compact G -manifold (ψ, M) consists of a \dagger weakly almost complex compact manifold M and a differentiable G -action $\psi: G \times M \rightarrow M$ that preserves the weakly almost complex structure on M . $\mathbf{U}_*^G(F, F')$, \mathbf{U}_*^G are defined similarly, and they are \mathbf{U}_* -modules, where \mathbf{U}_* is the \dagger complex

cobordism ring of compact weakly almost complex manifolds.

To study \mathbf{O}_*^G and \mathbf{U}_*^G , (co)bordism theory is introduced (P. E. Conner and E. E. Floyd [16]), which is one of the \dagger generalized (co)-homology theories. Miscellaneous results are known, in particular, for G a cyclic group of prime period. By means of the equivariant \dagger Thom spectrum, equivariant cobordism theory can be developed (T. tom Dieck [17]); this is a multiplicative generalized cohomology theory with Thom classes (\rightarrow 114 Differential Topology; also \rightarrow 201 Homology Theory, 56 Characteristic Classes).

F. Equivariant Homotopy

Let G be a compact Lie group. On the category of closed G -manifolds, we say that two objects M, N are χ -equivalent if $\chi(M^H) = \chi(N^H)$ for all closed subgroups H of G , where $\chi(\)$ is the \dagger Euler-Poincaré characteristic. On the set of equivalence classes $\mathbf{A}(G)$, a ring structure is imposed by disjoint union and the Cartesian product. We call $\mathbf{A}(G)$ the **Burnside ring** of G . If G is finite, $\mathbf{A}(G)$ is naturally isomorphic to the classical Burnside ring of G [18].

Denote by $S(V)$ the unit sphere of an orthogonal G -representation space V . Let V, W be orthogonal G -representation spaces. The equivariant stable homotopy group $[[S(V), S(W)]]$, which is defined as the direct limit of the equivariant homotopy sets $[S(V+U), S(W+U)]_G$ taken over orthogonal G -representation spaces U and suspensions, is denoted by ω_α for $\alpha = V - W \in RO(G)$. The \dagger smash product of representatives induces a bilinear pairing $\omega_\alpha \times \omega_\beta \rightarrow \omega_{\alpha+\beta}$. Then ω_0 is a ring, and ω_α is an ω_0 -module. The ring ω_0 is isomorphic to the Burnside ring of G , and ω_α is a \dagger projective ω_0 -module of rank one. The ω_0 -module ω_α is free if and only if $S(V)$ and $S(W)$ are stably G -homotopy equivalent [18].

Let E be an orthogonal G -vector bundle over a compact G -space X . Denote by $S(E)$ the sphere bundle associated with E . Let E, F be orthogonal G -vector bundles over X . Then E and F have the **same spherical G -fiber homotopy type** if there exist fiber-preserving G -mappings $f: S(E) \rightarrow S(F)$, $f': S(F) \rightarrow S(E)$ and fiber-preserving G -homotopies $h_t: S(E) \rightarrow S(E)$, $h'_t: S(F) \rightarrow S(F)$ such that $h_0 = f' \circ f$, $h_1 = \text{identity}$, $h'_0 = f \circ f'$, $h'_1 = \text{identity}$. Let $KO_G(X)$ be the \dagger equivariant K -group of real G -vector bundles over X . Let $T_G(X)$ be the additive subgroup of $KO_G(X)$ generated by elements of the form $[E] - [F]$, where E and F are orthogonal G -vector bundles having the same spherical G -fiber homotopy type. The factor group $J_G(X) = KO_G(X)/T_G(X)$ and the natural projection

$J_G: KO_G(X) \rightarrow J_G(X)$ are called an **equivariant J -group** and an **equivariant J -homomorphism**, respectively (\rightarrow 237 K -Theory).

In particular, $J_G(\{x_0\})$ is a factor group of the real representation ring $RO(G)$. ^+Adams operations on representation rings are the main tools for studying the group $J_G(\{x_0\})$ [18].

G. Infinitesimal Transformations

Let $f: G \times M \rightarrow M$ be a differentiable action of a Lie group G on a differentiable manifold M . Let X be a ^+left invariant vector field on G . Then we can define a differentiable vector field $f^+(X)$ on M as

$$f^+(X)_q h = \lim_{t \rightarrow 0} (h(f(\exp(-tX), q)) - h(q))/t$$

for each $q \in M$ and any differentiable function h defined on a neighborhood of q . It is easy to see that $f^+(X)_q = 0$ if and only if q is a fixed point of the one-parameter subgroup $\{\exp(tX)\}$. A vector field $f^+(X)$ is called an **infinitesimal transformation** of the differentiable transformation group G .

The set \mathfrak{g} of all infinitesimal transformations of G forms a finite-dimensional ^+Lie algebra (the laws of addition and $^+bracket$ product are defined from those for the vector fields on M). If G acts effectively on M , \mathfrak{g} is isomorphic to the Lie algebra of the Lie group G (\rightarrow 249 Lie Groups). In fact, the correspondence $X \rightarrow f^+(X)$ defines a Lie algebra homomorphism f^+ from the Lie algebra of all left invariant vector fields on G into the Lie algebra of all differentiable vector fields on M [19].

The following fact [20] is useful for the study of noncompact real analytic transformation groups. Let \mathfrak{g} be a real $^+semisimple$ Lie algebra and $\rho: \mathfrak{g} \rightarrow L(M)$ be a Lie algebra homomorphism of \mathfrak{g} into a Lie algebra of real analytic vector fields on a ^+real analytic manifold M . Let p be a point at which the vector fields in the image $\rho(\mathfrak{g})$ have common zero. Then there exists an analytic system of coordinates $(U; u_1, \dots, u_m)$ with origin at p in which all the vector fields in $\rho(\mathfrak{g})$ are linear. Namely, there exists $a_{ij} \in \mathfrak{g}^* = \text{Hom}_{\mathbf{R}}(\mathfrak{g}, \mathbf{R})$ such that

$$\rho(X)_q = - \sum_{i,j} a_{ij}(X) u_j(q) \frac{\partial}{\partial u_i}; \quad X \in \mathfrak{g}, \quad q \in U.$$

The correspondence $X \rightarrow (a_{ij}(X))$ defines a Lie algebra homomorphism of \mathfrak{g} into $\mathfrak{sl}(m, \mathbf{R})$.

For example, we can show that a real analytic $SL(n, \mathbf{R})$ action on the m -sphere is characterized by a certain real analytic vector field on $(m-n+1)$ -sphere ($5 \leq n \leq m \leq 2n-2$) [21]. In particular, there are infinitely many (at least the cardinality of the real numbers) inequivalent

real analytic $SL(n, \mathbf{R})$ actions on the m -sphere ($3 \leq n \leq m$).

Conversely, let \mathfrak{g} be a finite-dimensional Lie algebra of vector fields on M . Although there is not always a differentiable transformation group G that admits \mathfrak{g} as its Lie algebra of infinitesimal transformations, the following local result holds. Let \tilde{G} be the ^+simply connected Lie group corresponding to the Lie algebra \mathfrak{g} . Then for each point x of M , there exist a neighborhood \tilde{U} of the identity element e of \tilde{G} , neighborhoods V, W ($V \subset W$) of x , and a differentiable mapping f of $\tilde{U} \times V$ into W with the following properties. Putting $f(g, y) = g(y)$ ($g \in \tilde{U}, y \in V$), we have: (i) For all $y \in V$, $e(y) = y$. (ii) If $g, h \in \tilde{U}$, $y \in V$, then $(gh)(y) = g(h(y))$, provided that $gh \in \tilde{U}$, $h(y) \in V$. (iii) Let X be an arbitrary element of \mathfrak{g} . Put $g_t = \exp(-tX)$, the corresponding one-parameter subgroup of \tilde{G} . If $\varepsilon > 0$ is taken small enough, then we have $g_t \in \tilde{U}$ for $|t| < \varepsilon$ so that $g_t(y)$ ($|t| < \varepsilon, y \in V$) is well defined. Therefore g_t determines a vector field \tilde{X} on V by the formula

$$\tilde{X}_y h = \lim_{t \rightarrow 0} (h(g_t(y)) - h(y))/t.$$

The vector field \tilde{X} coincides with the restriction of X to V . This local proposition is often expressed by the statement that \mathfrak{g} generates a **local Lie group of local transformations**, which is called **Lie's fundamental theorem** on local Lie groups of local transformations.

H. Criteria

It is important to know whether a given transformation group is a topological or a Lie transformation group. The following theorems are useful for this purpose [22, 23]:

(10) Let G be a transformation group of a $^+locally$ compact Hausdorff space M . If we introduce the $^+compact$ -open topology in G , then G is a topological transformation group of M when M is locally connected or M is a $^+uniform$ topological space and G acts $^+equicontinuously$ on M .

(11) Suppose that M is a $^+C^1$ -manifold and G is a topological transformation group of M acting effectively on M . If G is locally compact and the mapping $x \rightarrow g(x)$ of M is of class C^1 for each element g of G , then G is a Lie transformation group of M .

(12) Assume that G is a transformation group of a differentiable manifold M and G acts effectively on M . Let \mathfrak{g} be the set of all vector fields on M defined by one-parameter groups of transformations of M contained in G as subgroups. If \mathfrak{g} is a finite-dimensional Lie algebra, then G has a Lie group structure with respect to which G is a Lie transformation group of M , and then \mathfrak{g} coincides with the Lie

algebra formed by the infinitesimal transformations of G .

By applying theorems (10), (11), and (12) we can show that the following groups are Lie transformation groups: the group of all isometries of a Riemannian manifold; the group of all affine transformations of a differentiable manifold with a linear connection (generally, the group of all transformations of a differentiable manifold that leave invariant a given Cartan connection); the group of all analytic transformations of a compact complex manifold (this group is actually a complex Lie group); and the group of all analytic (holomorphic) transformations of a bounded domain in \mathbb{C}^n .

For related topics — 105 Differentiable Manifolds, 114 Differential Topology, 122 Discontinuous Groups, 153 Fixed-Point Theorems, 427 Topology of Lie Groups and Homogeneous Spaces, etc.

References

- [1] A. Borel et al., Seminar on transformation groups, Ann. Math. Studies, Princeton Univ. Press, 1960.
- [2] W. Y. Hsiang, Cohomology theory of topological transformation groups, Erg. math. 85, Springer, 1975.
- [3] G. E. Bredon, Introduction to compact transformation groups, Academic Press, 1972.
- [4] W. C. Hsiang and W. Y. Hsiang, Differentiable actions of compact connected classical groups I, II, III, Amer. J. Math. 89 (1967); Ann. Math., (2) 92 (1970); Ann. Math., (2) 99 (1974).
- [5] R. W. Richardson, Deformations of Lie subgroups and the variation of isotropy subgroups, Acta Math., 129 (1972).
- [6] W. C. Hsiang, A note on free differentiable actions of S^1 and S^3 on homotopy spheres, Ann. Math., (2) 83 (1966).
- [7] F. Hirzebruch and K. H. Mayer, $O(n)$ -Mannigfaltigkeiten, Exotische Sphären und Singularitäten, Lecture notes in math. 57, Springer, 1968.
- [8] M. F. Atiyah and F. Hirzebruch, Spin-manifolds and group actions, Essays on Topology and Related Topics, Memoires dédiés à Georges de Rham, Springer, 1970.
- [9] K. Kawakubo, Equivariant Riemann-Roch theorems, localization and formal group law, Osaka J. Math., 17 (1980).
- [10] A. Hattori, Spin^c-structures and S^1 -actions, Inventiones Math., 48 (1978).
- [11] W. C. Hsiang and W. Y. Hsiang, The degree of symmetry of homotopy spheres, Ann. Math., (2) 89 (1969).
- [12] T. Watabe, On the degree of symmetry of complex quadric and homotopy complex

projective space, Sci. Rep. Niigata Univ., 11 (1974).

- [13] T. Petrie, Smooth S^1 actions on homotopy complex projective spaces and related topics, Bull. Amer. Math. Soc., 78 (1972).
- [14] T. Yoshida, On smooth semifree S^1 actions on cohomology complex projective spaces, Publ. Res. Inst. Math. Sci., 11 (1976).
- [15] P. E. Conner and E. E. Floyd, Maps of odd period, Ann. Math., (2) 84 (1966).
- [16] P. E. Conner and E. E. Floyd, Differentiable periodic maps, Erg. Math., 33 (1964).
- [17] T. tom Dieck, Bordism of G -manifolds and integrality theorems, Topology, 9 (1970).
- [18] T. tom Dieck, Transformation groups and representation theory, Lecture notes in math. 766, Springer, 1979.
- [19] R. S. Palais, A global formulation of the Lie theory of transformation groups, Mem. Amer. Math. Soc., 22 (1957).
- [20] V. Guillemin and S. Sternberg, Remarks on a paper of Hermann, Trans. Amer. Math. Soc., 130 (1968).
- [21] F. Uchida, Real analytic $SL(n, \mathbb{R})$ actions on spheres, Tôhoku Math. J., 33 (1981).
- [22] D. Montgomery and L. Zippin, Topological transformation groups, Interscience, 1955.
- [23] S. Kobayashi, Transformation groups in differential geometry, Erg. math. 70, Springer, 1972.

432 (VI.8) Trigonometry

A. Plane Trigonometry

Fix an orthogonal frame $O-XY$ in a plane, and take a point P on the plane such that the angle POX is α . Denote by (x, y) the coordinates of P , and put $OP = r$ (Fig. 1). We call the six ratios $\sin \alpha = y/r$, $\cos \alpha = x/r$, $\tan \alpha = y/x$, $\cot \alpha = x/y$, $\sec \alpha = r/x$, $\csc \alpha = r/y$ the **sine**, **cosine**, **tangent**, **cotangent**, **secant**, and **cosecant** of α , respectively. These functions of the angle α are called **trigonometric functions** or **circular functions** (\rightarrow 131 Elementary Functions). They are periodic functions with the fundamental period π for the tangent and cotangent, and 2π for the others. The relation $\sin^2 \alpha + \cos^2 \alpha = 1$ and the **addition formulas** $\sin(\alpha \pm \beta) = \sin \alpha \cos \beta \pm \cos \alpha \sin \beta$, $\cos(\alpha \pm \beta) = \cos \alpha \cos \beta \mp \sin \alpha \sin \beta$ follow from the definitions (\rightarrow Appendix A, Table 2). Given a plane triangle ABC (Fig. 2), we have the following three properties: (i) $a = b \cos C + c \cos B$ (**the first law of cosines**); (ii) $a^2 = b^2 + c^2 - 2bc \cos A$ (**the second law of cosines**); (iii) $a/\sin A = b/\sin B = c/\sin C = 2R$, where R is the radius of the circle circum-

scribed about $\triangle ABC$ (**laws of sines**) (\rightarrow Appendix A, Table 2). Thus we obtain relations among the six quantities $a, b, c, \angle A, \angle B$, and $\angle C$ associated with the triangle ABC . The study of plane figures by means of trigonometric functions is called **plane trigonometry**. For example, if a suitable combination of three of these six quantities (including a side) associated with a triangle is given, then the other three quantities are uniquely determined. The determination of unknown quantities associated with a triangle by means of these laws is called **solving a triangle**.

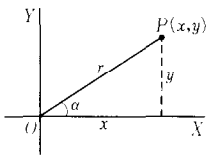


Fig. 1

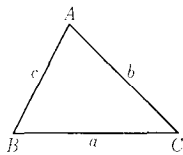


Fig. 2

B. Spherical Trigonometry

The part ABC of a spherical surface bounded by three arcs of great circles is called a **spherical triangle**. Points A, B, C are called the **vertices**; the three arcs a, b, c are called the **sides**; and the angles formed by lines tangent to the sides and intersecting at the vertices are called the **angles** of the spherical triangle (Fig. 3). If we denote the angles by A, B, C , we have the relation $A + B + C - \pi = E > 0$, and E is called the **spherical excess**. Spherical triangles have properties similar to those of plane triangles: $\sin a/\sin A = \sin b/\sin B = \sin c/\sin C$ (**laws of sines**), and $\cos a = \cos b \cos c + \sin b \sin c \cos A$ (**law of cosines**). The study of spherical figures by means of trigonometric functions, called **spherical trigonometry**, is widely used in astronomy, geodesy, and navigation (\rightarrow Appendix A, Table 2).

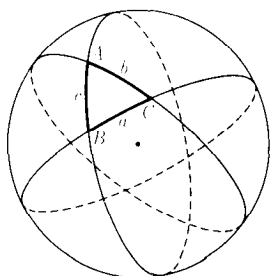


Fig. 3

C. History

Trigonometry originated from practical problems of determining a triangle from three of its elements. The development of spherical trigonometry, which was spurred on by its applications to astronomy, preceded the development of plane trigonometry. In Egypt, Babylon, and China, people had some knowledge of trigonometry, and the founder of trigonometry is believed to have been Hipparchus of Greece (fl. 150 B.C.). In the *Almagest* of Ptolemy (c. 150 A.D.) we find a table for $2 \sin \alpha$ for $\alpha = 0, 30', 1^\circ, 1^\circ 30', \dots$ that is exact to five decimal places, and the addition formulas. The Greeks calculated $2 \sin \alpha$, which is the length of the chord corresponding to the double arc. Indian mathematicians, on the other hand, calculated half of the above quantities, that is, $\sin \alpha$ and $1 - \cos \alpha$ for the arc α . In the book by Aryabhatta (c. 500 A.D.) we find laws of cosines. The Arabs, influenced by Indian mathematicians, expressed geometric computations algebraically, a technique also known to the Greeks. Abûl Wafâ (in the latter half of the 10th century A.D.) gave the correct sines of angles for every $30'$ to 9 decimal places and studied with Al Battani the projection triangle of the sundial, thereby obtaining the concepts of sine, cosine, secant, and cosecant. Later, a table of sines and cosines for every minute was established by the Arabs. Regiomontanus (d. 1476), a German, elaborated on this table. The form he gave to trigonometry has been maintained nearly intact to the present day. Various theorems in trigonometry were established by G. J. Rheticus, J. Napier, J. Kepler, and L. Euler (1748). Euler treated trigonometry as a branch of analysis, generalized it to functions of complex variables, and introduced the abbreviated notations that are still in use (\rightarrow 131 Elementary Functions).

References

- [1] H. Flanders and J. Price, *Trigonometry*, Academic Press, 1975.
- [2] A. Washington and C. Edmond, *Plane trigonometry*, Addison-Wesley, 1977.

433 (XX.12) Turbulence and Chaos

Turbulent flow is the irregular motion of fluids, whereas relatively simple types of flows that are either stationary, slowly varying, or periodic in time are called **laminar flow**. When

a laminar flow is stable against external disturbances, it remains laminar, but if the flow is unstable, it usually changes into either another type of laminar flow or a turbulent flow.

A. Stability and Bifurcation of Flows

The velocity field $\mathbf{u}(\mathbf{x}, t)$, \mathbf{x} being the space coordinates and t the time, of a flow of an incompressible viscous fluid in a bounded domain G is determined by the [†]Navier-Stokes equation of motion,

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \text{grad})\mathbf{u} - \nu \Delta \mathbf{u} + \frac{1}{\rho} \text{grad } p = 0, \quad (1)$$

and the equation of continuity,

$$\text{div } \mathbf{u} = 0, \quad (2)$$

with the prescribed initial and boundary conditions, where Δ denotes the Laplacian, p the pressure, ρ the density, and ν the kinematic viscosity of the fluid. Suitable extensions must be made in the foregoing system of equations if other field variables, such as the temperature in thermal-convection problems, are to be considered.

The stability of a fluid flow is studied by examining the behavior of the solution of equations (1) and (2) against external disturbances, and, in particular, stability against infinitesimal disturbances constitutes the linear stability problem. The stability characteristics of the solution of equations (1) and (2) depend largely upon the value of the [†]Reynolds number $R = UL/\nu$, U and L being the representative velocity and length of the flow, respectively. Let a stationary solution of equations (1) and (2) be $\mathbf{u}_0(\mathbf{x}, R)$. If the perturbed flow is given by $\mathbf{u}_0(\mathbf{x}, R) + \mathbf{v}(\mathbf{x}, R)\exp(\sigma t)$, \mathbf{v} being the perturbation velocity, and equation (1) is linearized with respect to \mathbf{v} , we obtain a [†]linear eigenvalue problem for σ . The flow is called linearly stable if $\max(\text{Re } \sigma)$ is negative, and linearly unstable if it is positive. For small values of R , a flow is generally stable, but it becomes unstable if R exceeds a critical value R_c , which is called the critical Reynolds number [1].

The instability of a stationary solution gives rise to the [†]bifurcation to another solution at a [†]bifurcation point R_c of the parameter R . If $\text{Im } \sigma = 0$ for an eigenvalue σ at $R = R_c$, a stationary solution bifurcates from the solution \mathbf{u}_0 at R_c , and if $\text{Im } \sigma \neq 0$, a time-periodic solution bifurcates at R_c . The latter bifurcation is called the Hopf bifurcation. A typical example of stationary bifurcation is the generation of an axially periodic row of Taylor vortices in Couette flow between two rotating coaxial cylinders, which was studied by G. I.

Taylor (1923), with excellent agreement between theory and experiment [2]. Hopf bifurcation is exemplified by the generation of Tollmien-Schlichting waves in the laminar [†]boundary layer along a flat plate, which was predicted theoretically by W. Tollmien (1929) and H. Schlichting (1933) and later confirmed experimentally by G. B. Schubauer and H. K. Skramstad (1947) [3].

In either type of bifurcation ($\text{Im } \sigma = 0$ or $\neq 0$) the bifurcation is called supercritical if the bifurcating solution exists only for $R > R_c$, subcritical if it exists only for $R < R_c$, and transcritical if it happens to exist on both sides of R_c . The amplitude of the departure of the bifurcating solution from the unperturbed solution \mathbf{u}_0 tends to zero as $R \rightarrow R_c$. The behavior of the bifurcating solution around the bifurcation point R_c is dealt with systematically by means of bifurcation analysis. In supercritical bifurcation, the bifurcating solution is stable and represents an equilibrium state to which the perturbed flow approaches just as in the cases of Taylor vortices and Tollmien-Schlichting waves. On the other hand, for subcritical bifurcation the bifurcating solution is unstable and gives a critical amplitude of the disturbance above which the linearly stable basic flow ($R < R_c$) becomes unstable. In this case, the instability of the basic flow gives rise to a sudden change of the flow pattern resulting in either a stationary (or time-periodic) or even turbulent flow. The transition to turbulent flow that takes place in Hagen-Poiseuille flow through a circular tube and is linearly stable at all values of R ($R_c = \infty$) may be attributed to this type of bifurcation.

The concept of bifurcation can be extended to the case where the flow \mathbf{u}_0 is nonstationary, but the bifurcation analysis then becomes much more difficult.

B. Onset of Turbulence

The fluctuating flow resulting from an instability does not itself necessarily constitute a turbulent flow. In order that a flow be turbulent, the fluctuations must take on some irregularity. The turbulent flow is usually defined in terms of the long-time behavior of the flow velocity $\mathbf{u}(\mathbf{x}, t)$ at a fixed point \mathbf{x} in space. The flow is expected to be turbulent if the fluctuating velocity

$$\delta \mathbf{u}(\mathbf{x}, t) = \mathbf{u}(\mathbf{x}, t) - \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T \mathbf{u}(\mathbf{x}, t) dt \quad (3)$$

satisfies the condition

$$\lim_{\tau \rightarrow \infty} \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T \delta u_i(\mathbf{x}, t) \delta u_i(\mathbf{x}, t + \tau) dt = 0, \quad (4)$$

where the subscripts label the components. Condition (4) implies that the dynamical system of a fluid has the mixing property. This condition also states that the velocity fluctuation δu_i has a continuous frequency spectrum. In practical situations the frequency spectrum of a turbulent flow may contain both the line and continuous spectra, in which case the flow is said to be partially turbulent.

L. D. Landau (1959) and E. Hopf (1948) proposed a picture of turbulent flow as one composed of a quasiperiodic motion, $\mathbf{u}(t) = \mathbf{f}(\omega_1 t, \omega_2 t, \dots, \omega_n t)$, with a large number of rationally independent frequencies $\omega_1, \dots, \omega_n$ produced by successive supercritical bifurcations of Hopf type. This picture of turbulence is not compatible with the foregoing definition of turbulence, since it does not satisfy the mixing property (4). The fact that the generation of real turbulence is not necessarily preceded by successive supercritical bifurcations casts another limitation on the validity of this picture.

The concept of turbulence is more clearly exhibited with respect to a dynamical system of finite dimension. Although we are without a general proof, it is expected that the Navier-Stokes equation with nonzero viscosity ν can be approximated within any degree of accuracy by a system of finite-dimensional first-order ordinary differential equations

$$\frac{dX}{dt} = F(X). \quad (5)$$

Thus the onset and some general properties of turbulence are understood in the context of the theory of dynamical systems. Turbulence is related to those solutions of equation (5) that tend to a set in the phase space that is neither a fixed point, a closed orbit, nor a torus. A set of such complicated structure is called a nonperiodic attractor or a strange attractor. Historically, the strange attractor originates from the strange Axiom A attractor that was found in a certain class of dynamical systems called the Axiom A systems. However, this term has come to be used in a broader sense, and it now represents a variety of nonperiodic motions exhibited by a system that is not necessarily of Axiom A type. The above-mentioned Landau-Hopf picture of turbulence was criticized by D. Ruelle and F. Takens (1971), who proved for the dynamical system (5) that an arbitrary small perturbation on a quasiperiodic flow on a k -dimensional torus ($k \geq 4$) generically (in the sense of residual sets) produces a flow with a strange Axiom A attractor [4].

There exist a number of examples of first-order ordinary differential equations of relatively low dimension whose solutions exhibit

nonperiodic behavior. An important model system related to turbulence is the Lorenz model (1963) of thermal convection in a horizontal fluid layer. This model is obtained by taking only three components out of an infinite number of spatial Fourier components of the velocity and temperature fields. The model is written as

$$\begin{aligned} \frac{dX}{dt} &= -\sigma X + \sigma Y, \\ \frac{dY}{dt} &= -XZ + rX - Y, \\ \frac{dZ}{dt} &= XY - bZ, \end{aligned} \quad (6)$$

where σ ($> b + 1$) and b are positive constants and r is a parameter proportional to the Rayleigh number. Obviously, equations (6) have a fixed point $X = Y = Z = 0$ representing the state of thermal convection without fluid flow. For $r < 1$, this fixed point is stable, but it becomes unstable for $r > 1$, and a pair of new fixed points $X = Y = \pm \sqrt{b(r-1)}$, $Z = r-1$ emerges supercritically. This corresponds to the onset of stationary convection at $r = 1$. At a still higher value of $r = \sigma(\sigma + b + 3)/(\sigma - b - 1)$, a subcritical Hopf bifurcation occurs with respect to this pair of fixed points, and for a certain range of r above this threshold the solutions with almost any initial conditions exhibit nonperiodic behavior. This corresponds to the generation of turbulence. The property

$$\frac{\partial \dot{X}}{\partial X} + \frac{\partial \dot{Y}}{\partial Y} + \frac{\partial \dot{Z}}{\partial Z} = -(\sigma + b + 1) < 0, \quad (7)$$

where the dots denote time derivatives, shows that each volume element of the phase space shrinks asymptotically to zero as the time increases indefinitely. This property is characteristic of dynamical systems with energy dissipation, in sharp contrast to the measure-preserving character of Hamiltonian systems [5].

For a certain class of ordinary differential equations, the bifurcation to nonperiodic motion corresponds neither to the bifurcation of tori, just as in the Ruelle-Takens theory, nor to subcritical bifurcation, as in the Lorenz model. Such a bifurcation takes place when nonperiodic motion emerges as the consequence of an infinite sequence of supercritical bifurcations at each of which a periodic orbit of period T bifurcates into one of period $2T$. If we denote the n th bifurcation point by r_n , the distance $r_{n+1} - r_n$ between two successive bifurcation points decreases exponentially with increasing n , and eventually the bifurcation points accumulate at a point r_c , beyond which nonperiodic motion is expected to emerge. It is

not yet clear if any of the above three types of bifurcation leading to nonperiodic behavior is actually responsible for the generation of real turbulence.

Some important properties of a dynamical system with a nonperiodic attractor, which may be either a flow or a [†]diffeomorphism, can be stated as follows:

- (i) The distance between two points in the phase space that are initially close to each other grows exponentially in time, so that the solutions exhibit a sensitive dependence on the initial conditions.
- (ii) The nonperiodic attractor has [†]Lebesgue measure zero, and such a system is expected to have many other [†]ergodic [†]invariant measures.

The irregular behavior of a deterministic dynamical system is also called **chaos**, but this concept is more abstract and general than that of turbulence, and covers phenomena exhibited by systems such as nonlinear electric circuits, chemical reactions, and ecological systems.

C. Statistical Theory of Turbulence

The statistical theory of turbulence deals with the statistical behavior of fully developed turbulence. The turbulent field is sometimes idealized for mathematical simplicity to be homogeneous or isotropic. In **homogeneous turbulence** the statistical laws are invariant under all parallel displacements of the coordinates, whereas in **isotropic turbulence** invariance under rotations and reflections of the coordinates is required in addition.

The instantaneous state of the fluid motion is completely determined by specifying the fluid velocity **u** at all space points **x** and can be expressed as a phase point in the infinite-dimensional [†]phase space spanned by these velocities. The phase point moves with time along a path uniquely determined by the solution of the Navier-Stokes equation. In the turbulent state the path is unstable to the initial disturbance and describes an irregular line in the phase space. In this situation the deterministic description is no longer useful and should be replaced by a statistical treatment. Abstractly speaking, turbulence is just a view of fluid motion as the random motion of the phase point **u(x)** (→ 407 Stochastic Processes). The equation for the [†]characteristic functional of the random velocity **u(x)** was first given by E. Hopf (1952). An exact solution obtained by Hopf represents a [†]normal distribution associated with a white energy spectrum, but so far no general solution has been obtained [6].

Besides the formulation in terms of the

[†]probability distribution or the characteristic functional, there is another way of describing turbulence by [†]moments of lower orders. This is the conventional statistical theory originated by G. I. Taylor (1935) and T. von Kármán (1938), which made remarkable progress after World War II. The principal moments in this theory are the **correlation tensor**, whose (i, j) -component is the mean of the product of two velocity components u_i at a point **x** and u_j at another point **x + r**,

$$B_{ij}(\mathbf{r}) = \langle u_i(\mathbf{x})u_j(\mathbf{x} + \mathbf{r}) \rangle, \quad (8)$$

and its [†]Fourier transform, or the **energy spectrum tensor**,

$$\Phi_{ij}(\mathbf{k}) = \frac{1}{(2\pi)^3} \int B_{ij}(\mathbf{r}) \exp(-\sqrt{-1} \mathbf{k} \cdot \mathbf{r}) d\mathbf{r}. \quad (9)$$

In isotropic turbulence Φ_{ij} is expressed as

$$\Phi_{ij}(\mathbf{k}) = \frac{1}{4\pi k^2} E(k) \left(\delta_{ij} - \frac{k_i k_j}{k^2} \right), \quad k = |\mathbf{k}|, \quad (10)$$

where $E(k)$ is the **energy spectrum function**, representing the amount of energy included in a spherical shell of radius k in the wave number space. The energy of turbulence \mathcal{E} per unit mass is expressed as

$$\begin{aligned} \mathcal{E} &= \frac{1}{2} \langle |\mathbf{u}|^2 \rangle = \frac{1}{2} B_{ii}(0) = \frac{1}{2} \int \Phi_{ii}(\mathbf{k}) d\mathbf{k} \\ &= \int_0^\infty E(k) dk. \end{aligned} \quad (11)$$

The state of turbulence is characterized by the Reynolds number $R = E_0^{1/2}/(\nu k_0^{1/2})$, where E_0 and k_0 are representative values of $E(k)$ and k , respectively. For weak turbulence of small R , $E(k)$ is governed by a linear equation with the general solution

$$E(k, t) = E(k, 0) \exp(-2\nu k^2 t), \quad (12)$$

$E(k, 0)$ being an arbitrary function. Thus $E(k)$ decays in time due to the viscous dissipation. For strong turbulence of large R , it is difficult to obtain the precise form of $E(k)$, and this is usually done by way of some assumption that allows us to approximate the nonlinear effects [7].

Some of the similarity laws governing the energy spectrum and other statistical functions can be determined rigorously but not necessarily uniquely. For 3-dimensional incompressible turbulence, the energy spectrum satisfies an inviscid similarity law

$$E(k)/E_0 = F_e(k/k_0) \quad (13)$$

in the energy-containing region $k = O(k_0)$ characterized by a wave number k_0 , and a viscous similarity law

$$E(k)/E_0 = R^{-5/4} F_d(k/(R^{3/4} k_0)), \quad (14)$$

in the energy dissipation region $k = O(R^{3/4}k_0)$, where F_e and F_d denote dimensionless functions generally dependent on the initial condition and the time [6].

If an assumption is made to the effect that the statistical state in the energy-dissipation region depends only upon the energy-dissipation rate $\varepsilon = -d\mathcal{E}/dt$ besides the viscosity ν (or R), then (14) becomes Kolmogorov's equilibrium similarity law (1941):

$$E(k) = \varepsilon^{1/4} \nu^{5/4} F(k/(\varepsilon^{1/4} \nu^{-3/4})), \quad (15)$$

where F is a dimensionless function. For extremely large R (or small ν) there exists an inertial subregion between the energy-containing and energy-dissipation regions such that the viscous effect vanishes and (15) takes the form

$$E(k) = K \varepsilon^{2/3} k^{-5/3}, \quad (16)$$

where K is an absolute constant. **Kolmogorov's spectrum** (16) has been observed experimentally several times, and now its consistency with experimental results at large Reynolds numbers is well established [8].

Kolmogorov (1962) and others modified (16) by taking account of the fluctuation of ε due to the **intermittent structure** of the energy-dissipation region as

$$E(k) = K' \varepsilon^{2/3} k^{-5/3} (Lk)^{-\mu/9}, \quad (17)$$

where ε is now the average of the fluctuating ε , μ is the covariance of the log-normal distribution of ε , and L is the length scale of the spatial domain in which the average of ε is taken [8]. A similar modification, with the exponent $-\mu/3$ in place of $-\mu/9$, is obtained using a fractal model of the energy-cascade process. These corrections to $E(k)$, based upon the experimentally estimated μ of 0.3–0.5, are too small to be detected experimentally, but the deviation is expected to appear more clearly in the higher-order moments [8–10]. It should be noted that Kolmogorov's spectrum (16) itself does not contradict the notion of intermittent turbulence and gives one of the possible asymptotic forms in the limit $R \rightarrow \infty$.

The 1-dimensional Burgers model of turbulence satisfies the same similarity laws as (13) and (14), but it has an inviscid spectrum $E(k) \propto k^{-2}$ instead of (16). Two-dimensional incompressible turbulence has no energy-dissipation region, and hence Kolmogorov's theory is not valid for this turbulence. It has an inviscid spectrum $E(k) \propto k^{-3}$, first derived by R. H. Kraichnan (1967), C. E. Leith (1968), and G. K. Batchelor (1969). These inviscid spectra for 1- and 2-dimensional turbulence have been confirmed by numerical simulation [11].

References

- [1] C. C. Lin, The theory of hydrodynamical stability, Cambridge Univ. Press, 1955.
- [2] S. Chandrasekhar, Hydrodynamic and hydromagnetic stability, Clarendon Press, 1961.
- [3] H. L. Dryden, Recent advances in the mechanics of boundary layer flow, Adv. Appl. Mech., 1 (1948), 1–40.
- [4] D. Ruelle and F. Takens, On the nature of turbulence, Comm. Math. Phys., 20 (1971), 167–192.
- [5] E. N. Lorenz, Deterministic nonperiodic flow, J. Atmospheric Sci., 20 (1963), 130–141.
- [6] T. Tatsumi, Theory of homogeneous turbulence, Adv. Appl. Mech., 20 (1980), 39–133.
- [7] G. K. Batchelor, The theory of homogeneous turbulence, Cambridge Univ. Press, 1953.
- [8] A. S. Monin and A. M. Yaglom, Statistical fluid mechanics II, MIT Press, 1975.
- [9] B. Mandelbrot, Fractals and turbulence: Attractors and dispersion, Lecture notes in math. 615, Springer, 1977, 83–93.
- [10] B. Mandelbrot, Intermittent turbulence and fractal dimension: Kurtosis and the spectral exponent $5/3 + B$, Lecture notes in math. 565, Springer, 1977, 121–145.
- [11] T. Tatsumi, Analytical theories and numerical experiments on two-dimensional turbulence, Theor. Appl. Mech., 29 (1979), 375–393.

U

434 (XX.22) Unified Field Theory

A. History

Unified field theory is a branch of theoretical physics that arose from the success of [†]general relativity theory. Its purpose is to discuss in a unified way the fields of gravitation, electromagnetism, and nuclear force from the standpoint of the geometric structure of space and time. Studies have continued since 1918, and many theories of mathematical interest have been published without attaining, however, any conclusive physical theory.

A characteristic feature of relativity theory is that it is based on a completely new concept of space and time. That is, in general relativity theory it is considered that when a gravitational field is generated by matter, the structure of space and time changes, and the flat [†]Minkowski world becomes a 4-dimensional [†]Riemannian manifold (with signature (1, 3)) having nonvanishing curvature. The [†]fundamental tensor g_{ij} of the manifold is interpreted as the gravitational potential, and the basic gravitational equation can be described as a geometric law of the manifold. It is characteristic of general relativity theory that gravitational phenomena are reduced to space-time structure (\rightarrow 359 Relativity). The introduction of the Minkowski world in [†]special relativity theory was a revolutionary advance over the 3-dimensional space of Newtonian mechanics. But the inner structure of the Minkowski world does not reflect gravitational phenomena. The latter shortcoming is overcome by introducing the concept of space-time represented by a Riemannian manifold into general relativity theory.

When a coexisting system of gravitational and electromagnetic fields is discussed in general relativity theory, simultaneous equations (Einstein-Maxwell equations) must be solved for the gravitational potential g_{ij} and the electromagnetic field tensor F_{ij} . Thus the gravitational potential g_{ij} is affected by the existence of an electromagnetic field. As the validity of general relativity began to be accepted, it came to be expected that all physical actions might be attributed to the gravitational and electromagnetic fields. Thus various extensions of general relativity theory have been proposed in order to devise a geometry in which the electromagnetic as well as the gravitational field directly contributes to the space-time structure, and to establish a unified theory of both fields on the basis of the geometry thus obtained. These attempts are illustrated in Fig. 1.

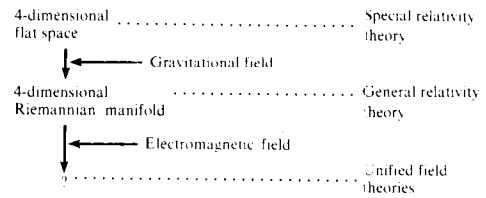


Fig. 1

B. Weyl's Theory

The first unified field theory was proposed by H. Weyl in 1918. In Riemannian geometry, which is the mathematical framework of general relativity theory, the [†]covariant derivative of the [†]fundamental tensor g_{ij} vanishes, i.e.,

$$\nabla_i g_{jk} \equiv \partial g_{jk} / \partial x^i - g_{ja} \Gamma_{ik}^a - g_{ak} \Gamma_{ij}^a = 0, \quad (1)$$

where Γ_{jk}^i is the [†]Christoffel symbol derived from g_{ij} . Conversely, if Γ_{jk}^i is considered as the coefficient of a general [†]affine connection and (1) is solved with respect to Γ_{jk}^i under the condition $\Gamma_{jk}^i = \Gamma_{kj}^i$, then the Christoffel symbol derived from g_{ij} coincides with Γ_{jk}^i . In this sense, (1) means that the space-time manifold has Riemannian structure. On the other hand, Weyl considered a space whose structure is given by an extension of (1),

$$\nabla_i g_{jk} = 2A_i g_{jk}, \quad (2)$$

and developed a unified field theory by regarding A_i as the electromagnetic potential. This theory has mathematical significance in that it motivated the discovery of Cartan's geometry of connection, but it has some unsatisfactory points concerning the derivation of the field equation and the equation of motion for a charged particle.

The scale transformation given by $\bar{g}_{ij} = \rho^2 g_{ij}$ is important in Weyl's theory. If in addition to this transformation, A_i is changed to

$$\bar{A}_i = A_i - \partial \log \rho / \partial x^i, \quad (3)$$

then (2) is left invariant and the space-time structure in Weyl's theory remains unchanged. We call (3) the **gauge transformation**, corresponding to the fact that the electromagnetic potential A_i is determined by the electromagnetic field tensor F_{ij} up to a gradient vector. In the [†]field theories known at present, the gauge transformation is generalized to various fields, and the law of charge conservation is derived from the invariance of field equations under generalized gauge transformation.

C. Further Developments

A unified field theory that appeared after Weyl's is **Kaluza's 5-dimensional theory** (Th.

Kaluza, 1921). This theory has been criticized as being artificial, but it is logically consistent, and therefore many of the later unified field theories are improved or generalized versions of it. The underlying space of Kaluza's theory is a 5-dimensional Riemannian manifold with the fundamental form

$$ds^2 = (dx^4 + A_a dx^a)^2 + g_{ab} dx^a dx^b,$$

where A_i and g_{ij} are functions of x^i alone ($a, b, \dots, i, j = 0, 1, 2, 3$). The field equation and the equation of motion of a particle are derived from the variational principle in general relativity theory. The field equation is equivalent to the Einstein-Maxwell equations. The trajectory of a charged particle is given by a geodesic in the manifold, and its equation is reducible to the Lorentz equation in general relativity.

After the introduction of Kaluza's theory, various unified field theories were proposed, and we give here the underlying manifolds or geometries of some mathematically interesting theories: a manifold with \dagger affine connection admitting absolute parallelism (A. Einstein, 1928); a manifold with \dagger projective connection (O. Veblen, B. Hoffman, 1930 [4]; J. A. Schouten, D. van Dantzig, 1932); **wave geometry** (a theory based on the linearization of the fundamental form; Y. Mimura, 1934 [3]); a nonholonomic geometry (G. Vranceanu, 1936); a manifold with \dagger conformal connection (Hoffman, 1948).

The investigations since 1945 have been motivated by the problem of the representation of matter in general relativity theory. Einstein first represented matter by an energy-momentum tensor T_{ij} of class C^0 , which must be determined by information obtained from outside relativity. Afterward he felt that this point was unsatisfactory and tried to develop a theory on the basis of field variables alone, without introducing such a quantity as T_{ij} . This theory is the so-called **unitary field theory**, and a solution without singularities is required from a physical point of view. His first attempt was to remove singularities from an exterior solution in general relativity by changing the topological structure of the space-time manifold. This idea was then extended to a unified field theory by J. A. Wheeler, and an interpretation was given to mass and charge by applying the theory of \dagger harmonic integrals (1957) [2].

Einstein's second attempt was to propose a **nonsymmetric unified field theory** (1945) [1, Appendix II; 6]. The fundamental quantities in this theory are a nonsymmetric tensor g_{ij} and a nonsymmetric affine connection Γ_{jk}^i . The underlying space of the theory can be considered a direct extension of the Riemannian

manifold, since (1) is contained in the field equations (notice the order of indices in this equation). E. Schrödinger obtained field equations of almost the same form by taking only Γ_{jk}^i as a fundamental quantity (1947) [5].

References

- [1] A. Einstein, The meaning of relativity, Princeton Univ. Press, fifth edition, 1956.
- [2] J. A. Wheeler, Geometrodynamics, Academic Press, 1962.
- [3] Y. Mimura and H. Takeno, Wave geometry, Sci. Rep. Res. Inst. Theoret. Phys. Hiroshima Univ., no. 2 (1962).
- [4] O. Veblen, Projektive Relativitätstheorie, Springer, 1933.
- [5] E. Schrödinger, Space-time structure, Cambridge Univ. Press, 1950.
- [6] A. Einstein, A generalization of the relativistic theory of gravitation, Ann. Math., (2) 46 (1945), 578–584.
- [7] V. Hlavatý, Geometry of Einstein's unified field theory, Noordhoff, 1957.
- [8] M. A. Tonnelat, Einstein's unified field theory, Gordon & Breach, 1966. (Original in French, 1955.)

435 (II.23) Uniform Convergence

A. Uniform Convergence of a Sequence of Real-Valued Functions

A sequence of real-valued functions $\{f_n(x)\}$ defined on a set B is said to be **uniformly convergent** (or to **converge uniformly**) to a function $f(x)$ on the set B if it converges with respect to the \dagger norm $\|\varphi\| = \sup\{|\varphi(x)| \mid x \in B\}$, i.e., $\lim_{n \rightarrow \infty} \|f_n - f\| = 0$ (\rightarrow 87 Convergence). In other words, $\{f_n(x)\}$ converges uniformly to $f(x)$ on B if for every positive constant ε we can select a number N independent of the point x such that $|f_n(x) - f(x)| < \varepsilon$ holds for all $n > N$ and $x \in B$. By the \dagger completeness of the real numbers, a sequence of functions $\{f_n(x)\}$ converges uniformly on B if and only if we can select for every positive constant ε a number N independent of the point x such that $|f_m(x) - f_n(x)| < \varepsilon$ holds for all $m, n > N$ and $x \in B$.

The **uniform convergence** of a series $\sum_n f_n(x)$ or of an infinite product $\prod_n f_n(x)$ is defined by the uniform convergence of the sequence of its partial sums or products. If the series of the absolute values $\sum_n |f_n(x)|$ converges uniformly, then the series $\sum_n f_n(x)$ also converges uniformly. In this case the series $\sum_n f_n(x)$ is said to

be **uniformly absolutely convergent**. A sequence of (nonnegative) constants M_n satisfying $|f_n(x)| \leq M_n$ is called a **dominant** (or **majorant**) of the sequence of functions $\{f_n(x)\}$. A series of functions $\sum_n f_n(x)$ with converging **majorant series** $\sum_n M_n$ is uniformly absolutely convergent (**Weierstrass's criterion for uniform convergence**).

Let $\{\lambda_n(x)\}$ be another sequence of functions on B . The series $\sum_n \lambda_n(x) f_n(x)$ is uniformly convergent if either of the following conditions holds: (i) The series $\sum_n f_n(x)$ converges uniformly and the partial sums of the series $\sum_n |\lambda_n(x) - \lambda_{n+1}(x)|$ are uniformly bounded, i.e., bounded by a constant independent of $x \in B$ and of the number of terms; or (ii) the series $\sum_n |\lambda_n(x) - \lambda_{n+1}(x)|$ converges uniformly, the sequence $\{\lambda_n(x)\}$ converges uniformly to 0, and the partial sums of $\sum_n |f_n(x)|$ are uniformly bounded.

B. Uniform Convergence and Pointwise Convergence

Let $\{f_n(x)\}$ be a sequence of real-valued functions on B , and let $f(x)$ be a real-valued function also defined on B . If the sequence of numbers $\{f_n(x_0)\}$ converges to $f(x_0)$ for every point $x_0 \in B$, we say that $\{f_n(x)\}$ is **pointwise convergent** (or **simply convergent**) to the function $f(x)$. Pointwise convergence is, of course, weaker than uniform convergence. If we represent the function $f(x)$ by the point $\prod_{x \in B} f(x) = [f]$ of the t Cartesian product $\mathbf{R}^B = \prod_{x \in B} \mathbf{R}$, then the pointwise convergence of $\{f_n(x)\}$ to $f(x)$ is equivalent to the convergence of the sequence of points $\{[f_n]\}$ to $[f]$ in the t product topology of \mathbf{R}^B .

When B is a t topological space and every $f_n(x)$ is continuous, the pointwise limit $f(x)$ of the sequence $\{f_n(x)\}$ is not necessarily continuous. However, if the sequence of continuous functions $\{f_n(x)\}$ converges uniformly to $f(x)$, then the limit function $f(x)$ is continuous. On the other hand, the continuity of the limit does not imply that the convergence is uniform. If the set B is t compact and the sequence of continuous functions $\{f_n(x)\}$ is monotone (i.e., $f_n(x) \leq f_{n+1}(x)$ for all n or $f_n(x) \geq f_{n+1}(x)$ for all n) and pointwise convergent to a continuous function $f(x)$, then the convergence is uniform (**Dini's theorem**).

C. Uniform Convergence on a Family of Sets

Let B be a topological space. We say that a sequence of functions $\{f_n(x)\}$ is **uniformly convergent in the wider sense** to the function

$f(x)$, depending on circumstances, in either of the following two cases: (i) Every point $x_0 \in B$ has a neighborhood U on which the sequence $\{f_n(x)\}$ converges uniformly to $f(x)$; or (ii) $\{f_n(x)\}$ converges uniformly to $f(x)$ on every compact subset K in B . If B is t locally compact, the two definitions coincide. The term **uniform convergence on compact sets** is also used for (ii).

In general, given a family \mathcal{P} of subsets in B , we may introduce in the space \mathcal{F} of real-valued functions on B a family of t seminorms $\|f\|_K = \sup\{|f(x)| \mid x \in K\}$ for every set $K \in \mathcal{P}$. Let T be the topology of \mathcal{F} defined by this family of seminorms (\rightarrow 424 Topological Linear Spaces). A sequence $\{f_n(x)\}$ is called **uniformly convergent on \mathcal{P}** if it is convergent with respect to T . In particular, when \mathcal{P} coincides with $\{B\}$, $\{\{x\} \mid x \in B\}$, or the family of all compact sets in B , then uniform convergence on \mathcal{P} coincides with the usual uniform convergence, pointwise convergence, or uniform convergence on compact sets, respectively. If \mathcal{P} is a countable set, the topology T is t metrizable. Most of these definitions and results may be extended to the case of functions whose values are in the complex number field, in a t normed space, or in any t uniform space.

D. Topology of the Space of Mappings

Let X, Y be two topological spaces. Denote by $C(X, Y)$ the space of all continuous mappings $f: X \rightarrow Y$. This space $C(X, Y)$, or a subspace \mathcal{F} of $C(X, Y)$, is called a **mapping space** (or **function space** or **space of continuous mappings**) from X to Y . A natural mapping $\Phi: \mathcal{F} \times X \rightarrow Y$ is defined by $\Phi(f, x) = f(x)$ ($f \in \mathcal{F}, x \in X$). We define a topology in \mathcal{F} as follows: for a compact set K in X and an open set U in Y , put $W(K, U) = \{f \in \mathcal{F} \mid f(K) \in U\}$, and introduce a topology in \mathcal{F} such that the base for the topology consists of intersections of finite numbers of $W(K_i, U_i)$. This topology is called the **compact-open topology** (R. H. Fox, *Bull. Amer. Math. Soc.*, 51 (1945)). When X is a t locally compact Hausdorff space and Y is a t Hausdorff space, the compact-open topology is the t weakest topology on \mathcal{F} for which the function Φ is continuous. If, in this case, \mathcal{F} is compact with respect to the compact-open topology, then the compact-open topology coincides with the topology of pointwise convergence.

In particular, when Y is a t metric space (or, in general, a t uniform space with the uniformity \mathcal{U}), the compact-open topology in \mathcal{F} coincides with the topology of uniform convergence on compact sets. A family \mathcal{F} is called

equicontinuous at a point $x \in X$ if for every positive number ε (in the case of uniform space, for every $V \in \mathcal{U}$) there exists a neighborhood U of x such that $\rho(f(x), f(p)) < \varepsilon$ ($f(x), f(p) \in V$) for every point $p \in U$ and for every function $f \in \mathcal{F}$ (G. Ascoli, 1883–1884). If X is a * locally compact Hausdorff space, a necessary and sufficient condition for \mathcal{F} to be relatively compact (i.e., for the closure of \mathcal{F} to be compact) with respect to the compact-open topology (i.e., to the topology of uniform convergence on compact sets) is that \mathcal{F} be equicontinuous at every point $x \in X$ and that the set $\{f(x) | f \in \mathcal{F}\}$ be relatively compact in Y for every point $x \in X$ (**Ascoli's theorem**). In particular, when X is a σ -compact locally compact Hausdorff space and Y is the space of real numbers, a family of functions \mathcal{F} that are equicontinuous (at every point $x \in X$) and uniformly bounded is relatively compact. Hence, for any sequence of functions $\{f_n\}$ in \mathcal{F} , we can select a subsequence $\{f_{n(v)}\}$ which converges uniformly on compact sets (**Ascoli-Arzelà theorem**).

E. Normal Families

P. Montel (1912) gave the name **normal family** to the family of functions that is relatively compact with respect to the topology of uniform convergence on compact sets. This terminology is used mainly for the family of complex analytic functions. In that case, it is customary to compactify the range space and consider Y to be the * Riemann sphere. Using this notion, Montel succeeded in giving a unified treatment of various results in the theory of complex functions.

A family of analytic functions \mathcal{F} on a finite-dimensional * complex manifold X is a normal family if it is uniformly bounded on each compact set (**Montel's theorem**). Another criterion is that there are three values on the Riemann sphere which no function $f \in \mathcal{F}$ takes. More generally, three exceptional values not taken by $f \in \mathcal{F}$ may depend on f , if there is a positive lower bound for the distances between these three values on the Riemann sphere. This gives an easy proof of the * Picard theorem stating that every * transcendental meromorphic function $f(z)$ in $|z| < \infty$ must take all values except possibly two values. In fact the family of functions $f_n(z) = f(z/2^n)$, $n = 1, 2, 3, \dots$, in $\{1 < |z| < 2\}$ cannot be normal. Using a similar procedure, G. Julia obtained the results on * Julia's direction.

F. Marty introduced the notion of **spherical derivative** $|f'(z)|/(1 + |f(z)|^2)$ for the analytic or meromorphic function $f(z)$ and proved that

for a family $\mathcal{F} = \{f(z)\}$ of analytic functions to be normal, it is necessary and sufficient that the spherical derivatives of $f \in \mathcal{F}$ be uniformly bounded. This theorem implies Montel's theorem and its various extensions, including, for example, quantitative results concerning * Borel's direction.

A family \mathcal{F} of analytic functions of one variable defined on X is said to form a **quasi-normal family** if there exists a subset P of X consisting only of isolated points such that from any sequence $\{f_n\}$ ($f_n \in \mathcal{F}$) we can select a subsequence $\{f_{n(v)}\}$ converging uniformly on $X - P$. If P is finite and consists of p points, the family \mathcal{F} is called a quasinormal family of order p . For example, the family of at most *p -valent functions is quasinormal of order p .

The theory of normal families of complex analytic functions is not only applied to * value distribution theory, as above, but also used to show the existence of a function that gives the extremal of functionals. The extremal function is usually obtained as a limit of a subsequence of a sequence in a normal family. A typical example of this method is seen in the proof of the * Riemann mapping theorem. This is perhaps the only general method known today in the study of the iteration of * holomorphic functions. By this method, Julia (1919) made an exhaustive study of the iteration of meromorphic functions; there are several other investigations on the iteration of elementary transcendental functions. On the other hand, A. Wintner (*Comm. Math. Helv.*, 23 (1949)) gave the implicit function theorem for analytic functions in a precise form using the theory of normal families of analytic functions of several variables.

References

- [1] N. Bourbaki, *Eléments de mathématique*, III., Topologie générale, ch. 10, Espaces fonctionnels, Actualités Sci. Ind., 1084b, Hermann, second edition, 1967; English translation, *Theory of sets*, Addison-Wesley, 1966.
- [2] J. L. Kelley, *General topology*, Van Nostrand, 1955, ch. 7.
- [3] J. Dieudonné, *Foundations of modern analysis*, Academic Press, 1960, enlarged and corrected printing, 1969.
- For normal families of complex functions,
- [4] P. Montel, *Leçons sur les familles normales de fonctions analytiques et leurs applications*, Gauthier-Villars, 1927.
- [5] G. Valiron, *Familles normales et quasinormales de fonctions méromorphes*, Gauthier-Villars, 1929.

436 (II.22) Uniform Spaces

A. Introduction

There are certain properties defined on * metric spaces but not on general * topological spaces, for example, * completeness or * uniform continuity of functions. Generalizing metric spaces, A. Weil introduced the notion of uniform spaces. This notion can be defined in several ways [3, 4]. The definition in Section B is that of Weil [1] without the * separation axiom for topology.

We denote by Δ_X the **diagonal** $\{(x, x) | x \in X\}$ of the Cartesian product $X \times X$ of a set X with itself. If U and V are subsets of $X \times X$, then the **composite** $V \circ U$ is defined to be the set of all pairs (x, y) such that for some element z of X , the pair (x, z) is in U and the pair (z, y) is in V . The inverse U^{-1} of U is defined to be the set of all pairs (x, y) such that $(y, x) \in U$.

B. Definitions

Let \mathcal{U} be a nonempty family of subsets of $X \times X$ such that (i) if $U \in \mathcal{U}$ and $U \subset V$, then $V \in \mathcal{U}$; (ii) if $U, V \in \mathcal{U}$, then $U \cap V \in \mathcal{U}$; (iii) if $U \in \mathcal{U}$, then $\Delta_X \subset U$; (iv) if $U \in \mathcal{U}$, then $U^{-1} \in \mathcal{U}$; and (v) if $U \in \mathcal{U}$, then $V \circ V \subset U$ for some $V \in \mathcal{U}$. Then we say that a **uniform structure** (or simply a **uniformity**) is defined on X by \mathcal{U} . If a uniformity is defined on X by \mathcal{U} , then the pair (X, \mathcal{U}) or simply the set X itself is called a **uniform space**, and \mathcal{U} is usually called a **uniformity** for X .

A subfamily \mathcal{B} of the uniformity \mathcal{U} is called a **base for the uniformity** \mathcal{U} if every member of \mathcal{U} contains a member of \mathcal{B} . If a family \mathcal{B} of subsets of $X \times X$ is a base for a uniformity \mathcal{U} , then the following propositions hold: (ii') if $U, V \in \mathcal{B}$, then there exists a $W \in \mathcal{B}$ such that $W \subset U \cap V$; (iii') if $U \in \mathcal{B}$, then $\Delta_X \subset U$; (iv') if $U \in \mathcal{B}$, then there exists a $V \in \mathcal{B}$ such that $V \subset U^{-1}$; (v') if $U \in \mathcal{B}$, then there exists a $V \in \mathcal{B}$ such that $V \circ V \subset U$. Conversely, if a family \mathcal{B} of subsets of a Cartesian product $X \times X$ satisfies (ii')–(v'), then the family $\mathcal{U} = \{U | U \subset X \times X, V \subset U \text{ for some } V \in \mathcal{B}\}$ defines a uniformity on X and \mathcal{B} is a base for \mathcal{U} . Given a uniform space (X, \mathcal{U}) , a member V of \mathcal{U} is said to be **symmetric** if $V = V^{-1}$. The family of all symmetric members of \mathcal{U} is a base for \mathcal{U} .

C. Topology of Uniform Spaces

Given a uniform space (X, \mathcal{U}) , an element $x \in X$, and $U \in \mathcal{U}$, we put $U(x) = \{y | (x, y) \in U\}$.

Then the family $\mathcal{U}(x) = \{U(x) | U \in \mathcal{U}\}$ forms a neighborhood system of $x \in X$, which gives rise to a topology of X (\rightarrow 425 Topological Spaces). This topology is called the **uniform topology** (or **topology of the uniformity**). When we refer to a topology of a uniform space (X, \mathcal{U}) , it is understood to be the uniform topology; thus a uniform space is also called a **uniform topological space**. If \mathcal{B} is a base for the uniformity of a uniform space (X, \mathcal{U}) , then $\mathcal{B}(x) = \{U(x) | U \in \mathcal{B}\}$ is a base for the neighborhood system at each point $x \in X$. Each member of \mathcal{U} is a subset of the topological space $X \times X$, which is supplied with the product topology. The family of all open (closed) symmetric members of \mathcal{U} forms a base for \mathcal{U} . A uniform space (X, \mathcal{U}) is a T_1 -topological space if and only if the intersection of all members of \mathcal{U} is the diagonal Δ_X . In this case, the uniformity of (X, \mathcal{U}) is called a **T_1 -uniformity**, and (X, \mathcal{U}) is called a **T_1 -uniform space**. A T_1 -uniform space is always * regular; a fortiori, it is a T_2 -topological space. Hence a T_1 -uniform space is also said to be a **Hausdorff uniform space** (or **separated uniform space**). Moreover, a uniform topology satisfies * Tikhonov's separation axiom; in particular, a T_1 -uniform space is * completely regular.

D. Examples

(1) **Discrete Uniformity.** Let X be a nonempty set, and let $\mathcal{U} = \{U | \Delta_X \subset U \subset X \times X\}$. Then (X, \mathcal{U}) is a T_1 -uniform space and $\mathcal{B} = \{\Delta_X\}$ is a base for \mathcal{U} . This uniformity is called the **discrete uniformity** for X .

(2) **Uniform Family of Neighborhood System.**

A family $\{U_\alpha(x)\}_{x \in A}$ ($x \in X$) of subsets of a set X is called a **uniform neighborhood system** in X if it satisfies the following four requirements: (i) $x \in U_\alpha(x)$ for each $\alpha \in A$ and each $x \in X$; (ii) if x and y are distinct elements of X , then $y \notin U_\alpha(x)$ for some $\alpha \in A$; (iii) if α and β are two elements of A , then there is another element $\gamma \in A$ such that $U_\gamma(x) \subset U_\alpha(x) \cap U_\beta(x)$ for all $x \in X$; (iv) if α is an arbitrary element in A , then there is an element β in A such that $y \in U_\alpha(x)$ whenever $x, y \in U_\beta(z)$ for some z in X . If we denote by $U_\alpha(\alpha \in A)$ the subset of $X \times X$ consisting of all elements (x, y) such that $x \in X$ and $y \in U_\alpha(x)$, then the family $\{U_\alpha | \alpha \in A\}$ satisfies all the conditions for a base for a uniformity. In particular, it follows from (ii) that $\bigcap_{\alpha \in A} U_\alpha = \Delta_X$, which is a stronger condition than (iii') in Section B. For instance, if $\{U_\alpha | \alpha \in A\}$ is a base for the neighborhood system at the identity element of a T_1 -topological group G , then we have two uniform neighborhood systems $\{U_\alpha^l(x)\}$ and $\{U_\alpha^r(x)\}$, where $U_\alpha^l(x) = xU_\alpha$ and

$U'_\alpha(x) = U_\alpha x$. Two uniformities derived from these uniform neighborhood systems are called a \dagger left uniformity and a \dagger right uniformity, respectively. Generally, these two uniformities do not coincide (\rightarrow 423 Topological Groups).

(3) Uniform Covering System [4]. A family $\{U_\alpha\}_{\alpha \in A}$ of \dagger coverings of a set X is called a **uniform covering system** if the following three conditions are satisfied: (i) if U is a covering of X such that $U < U_\alpha$ for all $\alpha \in A$, then U coincides with the covering $\Delta = \{\{x\}\}_{x \in X}$; (ii) if $\alpha, \beta \in A$, then there is a $\gamma \in A$ such that $U_\gamma < U_\alpha$ and $U_\gamma < U_\beta$; (iii) if $\alpha \in A$, then there is a $\beta \in A$ such that U_β is a $\dagger\Delta$ -refinement of U_α ($(U_\beta)^\Delta < U_\alpha$). For an example of a uniform covering system of X , suppose that we are given a uniform neighborhood system $\{U_\alpha(x)\}_{\alpha \in A}$ ($x \in X$). Let $U_\alpha = \{U_\alpha(x)\}_{x \in X}$ ($\alpha \in A$). Then $\{U_\alpha\}_{\alpha \in A}$ is a uniform covering system. On the other hand, for a covering $U = \{U_\lambda\}_{\lambda \in \Lambda}$, let $S(x, U)$ be the union of all members of U that contain x . If $\{U_\alpha\}_{\alpha \in A}$ is a uniform covering system and $U_\alpha(x) = S(x, U_\alpha)$, then $\{U_\alpha(x)\}_{\alpha \in A}$ ($x \in X$) is a uniform neighborhood system. Hence defining a uniform covering system of X is equivalent to defining a T_1 -uniformity on X .

(4). In a metric space (x, d) the subsets $U_r = \{(x, y) | d(x, y) < r\}$, $r > 0$, form a base of uniformity. The uniform topology defined by this coincides with the topology defined by the metric.

E. Some Notions on Uniform Spaces

Some of the terminology concerning topological spaces can be restated in the language of uniform structures. A mapping f from a uniform space (X, \mathcal{U}) into another (X', \mathcal{U}') is said to be **uniformly continuous** if for each member U' in \mathcal{U}' there is a member U in \mathcal{U} such that $(f(x), f(y)) \in U'$ for every $(x, y) \in U$. This condition implies that f is continuous with respect to the uniform topologies of the uniform spaces. Equivalently, the mapping is uniformly continuous with respect to the uniform neighborhood system $\{U_\alpha(x)\}_{\alpha \in A}$ if for any index β there is an index α such that $y \in U_\alpha(x)$ implies $f(y) \in U_\beta(f(x))$. If $f: X \rightarrow X'$ and $g: X' \rightarrow X''$ are uniformly continuous, then the composite $g \circ f: X \rightarrow X''$ is also uniformly continuous. A bijection f of a uniform space (X, \mathcal{U}) to another (X', \mathcal{U}') is said to be a **uniform isomorphism** if both f and f^{-1} are uniformly continuous; in this case (X, \mathcal{U}) and (X', \mathcal{U}') are said to be **uniformly equivalent**. A uniform isomorphism is a homeomorphism with respect to the uniform topologies, and a uniform equivalence

defines an equivalence relation between uniform spaces.

If \mathcal{U}_1 and \mathcal{U}_2 are uniformities for a set X , we say that the uniformity \mathcal{U}_1 is **stronger** than the uniformity \mathcal{U}_2 and \mathcal{U}_2 is **weaker** than \mathcal{U}_1 if the identity mapping of (X, \mathcal{U}_1) to (X, \mathcal{U}_2) is uniformly continuous. The discrete uniformity is the strongest among the uniformities for a set X . The weakest uniformity for X is defined by the single member $X \times X$; this uniformity is not a T_1 -uniformity unless X is a singleton. Generally, there is no weakest T_1 -uniformity. A uniformity \mathcal{U}_1 for X is stronger than another \mathcal{U}_2 if and only if every member of \mathcal{U}_2 is also a member of \mathcal{U}_1 .

If f is a mapping from a set X into a uniform space (Y, \mathcal{V}) and g is the mapping of $X \times X$ into $Y \times Y$ defined by $g(x, y) = (f(x), f(y))$, then $\mathcal{B} = \{g^{-1}(V) | V \in \mathcal{V}\}$ satisfies conditions (ii')–(v') in Section B for a base for a uniformity. The uniformity \mathcal{U} for X determined by \mathcal{B} is called the **inverse image** of the uniformity \mathcal{V} for Y by f ; \mathcal{U} is the weakest uniformity for X such that f is uniformly continuous. Hence a mapping f from a uniform space (X, \mathcal{U}) into another (Y, \mathcal{V}) is uniformly continuous if and only if the inverse image of the uniformity \mathcal{V} under f is weaker than the uniformity \mathcal{U} . If A is a subset of a uniform space (X, \mathcal{U}) , then there is a uniformity \mathcal{V} for A determined as the inverse image of \mathcal{U} by the inclusion mapping of A into X . This uniformity \mathcal{V} for A is called the **relative uniformity** for A induced by \mathcal{U} , or the **relativization** of \mathcal{U} to A , and the uniform space (A, \mathcal{V}) is called a **uniform subspace** of (X, \mathcal{U}) . The uniform topology for (A, \mathcal{V}) is the relative topology for A induced by the uniform topology for (X, \mathcal{U}) .

If $\{(X_\lambda, \mathcal{U}_\lambda)\}_{\lambda \in \Lambda}$ is a family of uniform spaces, then the **product uniformity** for $X = \prod_{\lambda \in \Lambda} X_\lambda$ is defined to be the weakest uniformity \mathcal{U} such that the projection of X onto each X_λ is uniformly continuous, and (X, \mathcal{U}) is called the **product uniform space** of $\{(X_\lambda, \mathcal{U}_\lambda)\}_{\lambda \in \Lambda}$. The topology for (X, \mathcal{U}) is the product of the topologies for $(X_\lambda, \mathcal{U}_\lambda)$ ($\lambda \in \Lambda$).

F. Metrization

Each \dagger pseudometric d for a set X generates a uniformity in the following way. For each positive number r , let $V_{d,r} = \{(x, y) \in X \times X | d(x, y) < r\}$. Then the family $\{V_{d,r} | r > 0\}$ satisfies conditions (ii')–(v') in Section B for a base for a uniformity \mathcal{U} . This uniformity is called the **pseudometric uniformity** or **uniformity generated by d** . The uniform topology for (X, \mathcal{U}) is the pseudometric topology. A uniform space (X, \mathcal{U}) is said to be **pseudometrizable** (**metrizable**) if there is a pseudometric (metric)

d such that the uniformity \mathcal{U} is identical with the uniformity generated by d . A uniform space is pseudometrizable if and only if its uniformity has a countable base. Consequently, a uniform space is metrizable if and only if its uniformity is a T_1 -uniformity and has a countable base. For a family P of pseudometrics on a set X , let $V_{d,r} = \{(x, y) \in X \times X \mid d(x, y) < r\}$ for $d \in P$ and positive r . The weakest uniformity containing every $V_{d,r}$ ($d \in P, r > 0$) is called the **uniformity generated by P** . This uniformity may also be described as the weakest one such that each pseudometric in P is uniformly continuous on $X \times X$ with respect to the product uniformity.

Each uniformity \mathcal{U} on a set X coincides with the uniformity generated by the family P_X of all pseudometrics that are uniformly continuous on $X \times X$ with respect to the product uniformity of \mathcal{U} with itself. It follows that each uniform space is uniformly isomorphic to a subspace of a product of pseudometric spaces (in which the number of components is equal to the cardinal number of P_X) and that each T_1 -uniform space is uniformly isomorphic to a subspace of a product of metric spaces. A topology τ for a set X is the uniform topology for some uniformity for X if and only if the topological space (X, τ) satisfies Tikhonov's separation axiom; in particular, the uniformity is a T_1 -uniformity if and only if (X, τ) is \ast completely regular.

G. Completeness

If (X, \mathcal{U}) is a uniform space, a subset A of X is called a **small set of order U** ($U \in \mathcal{U}$) if $A \times A \subset U$. A \ast filter on X is called a **Cauchy filter** (with respect to the uniformity \mathcal{U}) if it contains a small set of order U for each U in \mathcal{U} . If a filter on X converges to some point in X , then it is a Cauchy filter. If f is a uniformly continuous mapping from a uniform space X into another X' , then the image of a base for a Cauchy filter on X under f is a base for a Cauchy filter on X' . A point contained in the closure of every set in a Cauchy filter \mathfrak{F} is the limit point of \mathfrak{F} . Hence if a filter converges to x , a Cauchy filter contained in the filter also converges to x .

A \ast net $x(\mathfrak{A}) = \{x_\alpha\}_{\alpha \in \mathfrak{A}}$ (where \mathfrak{A} is a directed set with a preordering \leq) in a uniform space (X, \mathcal{U}) is called a **Cauchy net** if for each U in \mathcal{U} there is a γ in \mathfrak{A} such that $(x_\alpha, x_\beta) \in U$ for every α and β such that $\gamma \leq \alpha, \gamma \leq \beta$. If \mathfrak{A} is the set \mathbb{N} of all natural numbers, a Cauchy net $\{x_n\}_{n \in \mathbb{N}}$ is called a **Cauchy sequence** (or **fundamental sequence**). Given a Cauchy net $\{x_\alpha\}_{\alpha \in \mathfrak{A}}$, let $A_x = \{x_\beta \mid \beta \geq \alpha\}$. Then $\mathfrak{B} = \{A_\alpha \mid \alpha \in \mathfrak{A}\}$ is a base for a filter, and the filter is a Cauchy filter. On the

other hand, let \mathfrak{B} be a base for a Cauchy filter \mathfrak{F} . For $U, V \in \mathfrak{B}$, we put $U \leq V$ if and only if $U \supset V$. Then \mathfrak{B} is a directed set with respect to \leq . The net $\{x_U\}_{U \in \mathfrak{B}}$, where x_U is an arbitrary point in U , is a Cauchy net. A proposition concerning convergence of a Cauchy filter is always equivalent to a proposition concerning convergence of the corresponding Cauchy net.

A Cauchy filter (or Cauchy net) in a uniform space X does not always converge to a point of X . A uniform space is said to be **complete** (with respect to the uniformity) if every Cauchy filter (or Cauchy net) converges to a point of that space. A complete uniform space is called for brevity a **complete space**. A closed subspace of a complete space is complete with respect to the relative uniformity. A pseudometrizable uniform space is complete if and only if every Cauchy sequence in the space converges to a point. Hence in the case of a metric space, our definition of completeness coincides with the usual one (\rightarrow 273 Metric Spaces).

A mapping f from a uniform space X to another X' is said to be **uniformly continuous on a subset A of X** if the restriction of f to A is uniformly continuous with respect to the relative uniformity for A . If f is a uniformly continuous mapping from a subset A of a uniform space into a complete T_1 -uniform space, then there is a unique uniformly continuous extension \bar{f} of f on the closure \bar{A} .

Each T_1 -uniform space is uniformly equivalent to a dense subspace of a complete T_1 -uniform space; this property is a generalization of the fact that each metric space can be mapped by an isometry onto a dense subset of a complete metric space. A **completion** of a uniform space (X, \mathcal{U}) is a pair $(f, (X^*, \mathcal{U}^*))$, where (X^*, \mathcal{U}^*) is a complete space and f is a uniform isomorphism of X onto a dense subspace of X^* . The T_1 -completion of a T_1 -uniform space is unique up to uniform equivalence.

H. Compact Spaces

A uniformity \mathcal{U} for a topological space (X, τ) is said to be **compatible** with the topology τ if the uniform topology for (X, \mathcal{U}) coincides with τ . A topological space (X, τ) is said to be **uniformizable** if there is a uniformity compatible with τ . If (X, τ) is a compact Hausdorff space, then there is a unique uniformity \mathcal{U} compatible with τ ; in fact, \mathcal{U} consists of all neighborhoods of the diagonal Δ_X in $X \times X$; and the compact Hausdorff space is complete with this uniformity. Hence every subspace of a compact Hausdorff space is uniformizable, and every \ast locally compact Hausdorff space is

uniformizable. Any continuous mapping from a compact Hausdorff space to a uniform space is uniformly continuous. A uniform space (X, \mathcal{U}) is said to be **totally bounded** (or **precompact**) if for each $U \in \mathcal{U}$ there is a finite covering consisting of small sets of order U ; a subset of a uniform space is called **totally bounded** if it is totally bounded with respect to the relative uniformity. A uniform space X is said to be **locally totally bounded** if for each point of X there is a base for a neighborhood system consisting of totally bounded open subsets. A uniform space is compact if and only if it is totally bounded and complete. If f is a uniformly continuous mapping from a uniform space X to another, then the image $f(A)$ of a totally bounded subset A of X is totally bounded.

I. Topologically Complete Spaces

A topological space (X, τ) is said to be **topologically complete** (or **Dieudonné complete**) if it admits a uniformity compatible with τ with respect to which X is complete. Each \ast paracompact Hausdorff space is topologically complete. Actually such a space is complete with respect to its strongest uniformity. A Hausdorff space which is homeomorphic to a $\ast G_\delta$ -set in a compact Hausdorff space is said to be **Čech-complete**; A metric space is homeomorphic to a complete metric space if and only if it is Čech-complete. A Hausdorff space X is paracompact and Čech-complete if and only if there is a \ast perfect mapping from X onto a complete metric space.

References

- [1] A. Weil, Sur les espaces à structure uniforme et sur la topologie générale, Actualités Sci. Ind., Hermann, 1938.
- [2] J. W. Tukey, Convergence and uniformity in topology, Ann. Math. Studies, Princeton Univ. Press, 1940.
- [3] J. L. Kelley, General topology, Van Nostrand, 1955.
- [4] N. Bourbaki, Éléments de mathématique, III. Topologie générale, ch. 2, Actualités Sci. Ind., 1142d, Hermann, fourth edition 1965; English translation, General topology, Addison-Wesley, 1966.
- [5] J. R. Isbell, Uniform spaces, Amer. Math. Soc. Math. Surveys, 1964.
- [6] H. Nakano, Uniform spaces and transformation groups, Wayne State Univ. Press, 1968.
- [7] R. Engelking, General topology, Polish Scientific Publishers, 1977.

- [8] Z. Frolik, Generalization of the G_δ -property of complete metric spaces, Czech. Math. J., 10 (1960), 359–379.

437 (IV.17) Unitary Representations

A. Definitions

A homomorphism U of a \ast topological group G into the group of \ast unitary operators on a \ast Hilbert space \mathfrak{H} ($\neq \{0\}$) is called a **unitary representation** of G if U is **strongly continuous** in the following sense: For any element $x \in \mathfrak{H}$, the mapping $g \rightarrow U_g x$ is a continuous mapping from G into \mathfrak{H} . The Hilbert space \mathfrak{H} is called the **representation space** of U and is denoted by $\mathfrak{H}(U)$. Two unitary representations U and U' are said to be **equivalent** (**similar** or **isomorphic**), denoted by $U \cong U'$, if there exists a \ast isometry T from $\mathfrak{H}(U)$ onto $\mathfrak{H}(U')$ that satisfies the equality $T \circ U_g = U'_g \circ T$ for every g in G . If the representation space $\mathfrak{H}(U)$ contains no closed subspace other than \mathfrak{H} and $\{0\}$ that is invariant under every U_g , the unitary representation U is said to be **irreducible**. An element x in $\mathfrak{H}(U)$ is called a **cyclic vector** if the set of all finite linear combinations of the elements $U_g x (g \in G)$ is dense in $\mathfrak{H}(U)$. A representation U having a cyclic vector is called a **cyclic representation**. Every nonzero element of the representation space of an irreducible representation is a cyclic vector.

Examples. Let G be a \ast topological transformation group acting on a \ast locally compact Hausdorff space X from the right. Suppose that there exists a \ast Radon measure μ that is invariant under the group G . Then a unitary representation R^μ is defined on the Hilbert space $\mathfrak{H} = L^2(X, \mu)$ by the formula $(R_g^\mu f)(x) = f(xg)$ ($f \in \mathfrak{H}, x \in X, g \in G$). The representation R^μ is called the **regular representation** of G on (X, μ) . If G acts on X from the left, then the regular representation L^μ is defined by $(L_g^\mu f)(x) = f(g^{-1}x)$. In particular, when X is the \ast quotient space $H \backslash G$ of a \ast locally compact group G by a closed subgroup H , any two invariant measures μ, μ' (if they exist) coincide up to a constant factor. Hence the regular representation R^μ on (X, μ) and the regular representation $R^{\mu'}$ on (X, μ') are equivalent. In this case, the representation R^μ is called the regular representation on X . When $H = \{e\}$, a locally compact group G has a Radon measure $\mu \neq 0$ that is invariant under every right (left) translation $h \rightarrow hg$ ($h \rightarrow gh$) and is called a right (left) \ast Haar measure on G . So G has the regu-

Unitary Representations

lar representation $R(L)$ on G . $R(L)$ is called the right (left) regular representation of G .

B. Positive Definite Functions and Existence of Representations

A complex-valued continuous function φ on a topological group G is called **positive definite** if the matrix having $\varphi(g_i^{-1}g_j)$ as the (i,j) -component is a † positive semidefinite Hermitian matrix for any finite number of elements g_1, \dots, g_n in G . If U is a unitary representation of G , then the function $\varphi(g) = (U_g x, x)$ is positive definite for every element x in $\mathfrak{H}(U)$. Conversely, any positive definite function $\varphi(g)$ on a topological group G can be expressed as $\varphi(g) = (U_g x, x)$ for some unitary representation U and x in $\mathfrak{H}(U)$. Using this fact and the † Kreĭn-Milman theorem, it can be proved that every locally compact group G has **sufficiently many irreducible unitary representations** in the following sense: For every element g in G other than the identity element e , there exists an irreducible unitary representation U , generally depending on g , that satisfies the inequality $U_g \neq 1$. The groups having sufficiently many finite-dimensional (irreducible) unitary representations are called † maximally almost periodic. If a connected locally compact group G is maximally almost periodic, then G is the direct product of a compact group and a vector group \mathbb{R}^m . On the other hand, any non-compact connected † simple Lie group has no finite-dimensional irreducible unitary representation other than the unit representation $g \rightarrow 1$ (\rightarrow 18 Almost Periodic Functions).

C. Subrepresentations

Let U be a unitary representation of a topological group G . A closed subspace \mathfrak{N} of $\mathfrak{H}(U)$ is called **U -invariant** if \mathfrak{N} is invariant under every U_g ($g \in G$). Let $\mathfrak{N} \neq \{0\}$ be a closed invariant subspace of $\mathfrak{H}(U)$ and V_g be the restriction of U_g on \mathfrak{N} . Then V is a unitary representation of G on the representation space \mathfrak{N} and is called a **subrepresentation** of U . Two unitary representations L and M are called **disjoint** if no subrepresentation of L is equivalent to a subrepresentation of M ; they are called **quasi-equivalent** if no subrepresentation of L is disjoint from M and no subrepresentation of M is disjoint from L .

D. Irreducible Representations

Let U be a unitary representation of G , \mathbf{M} be the † von Neumann algebra generated by $\{U_g | g \in G\}$, and \mathbf{M}' be the † commutant of \mathbf{M} .

Then a closed subspace \mathfrak{N} of $\mathfrak{H}(U)$ is invariant under U if and only if the † projection operator P corresponding to \mathfrak{N} belongs to \mathbf{M}' . Therefore U is irreducible if and only if \mathbf{M}' consists of scalar operators $\{\alpha 1 | \alpha \in \mathbb{C}\}$ (**Schur's lemma**). A representation space of a cyclic or irreducible representation of a † separable topological group is † separable.

E. Factor Representations

A unitary representation U of G is called a **factor representation** if the von Neumann algebra $\mathbf{M} = \{U_g | g \in G\}$ is a † factor, that is, $\mathbf{M} \cap \mathbf{M}' = \{\alpha 1 | \alpha \in \mathbb{C}\}$. Two factor representations are quasi-equivalent if and only if they are not disjoint. U is called a **factor representation of type I, II, or III** if the von Neumann algebra \mathbf{M} is a factor of † type I, II, or III, respectively (\rightarrow 308 Operator Algebras). A topological group G is called a **group of type I** (or **type I group**) if every factor representation of G is of type I. Compact groups, locally compact Abelian groups, connected † nilpotent Lie groups, connected † semisimple Lie groups, and real or complex † linear algebraic groups are examples of groups of type I. There exists a connected solvable Lie group that is not of type I (\rightarrow Section U), but a connected solvable Lie group is of type I if the exponential mapping is surjective (O. Takenouchi). A discrete group G with countably many elements is a type I group if and only if G has an Abelian normal subgroup with finite index (E. Thoma).

F. Representation of Direct Products

Let G_1 and G_2 be topological groups, G the † direct product of G_1 and G_2 ($G = G_1 \times G_2$), and U_i an irreducible unitary representation of G_i ($i = 1, 2$). Then the † tensor product representation $U_1 \otimes U_2: (g_1, g_2) \rightarrow U_{g_1} \otimes U_{g_2}$ is an irreducible unitary representation of G . Conversely, if one of the groups G_1 and G_2 is of type I, then every irreducible unitary representation of G is equivalent to the tensor product $U_1 \otimes U_2$ of some irreducible representations U_i of G_i ($i = 1, 2$).

G. Direct Sums

If the representation space \mathfrak{H} of a unitary representation U is the † direct sum $\bigoplus_{\alpha \in I} \mathfrak{H}(\alpha)$ of mutually orthogonal closed invariant subspaces $\{\mathfrak{H}(\alpha)\}_{\alpha \in I}$, then U is called the **direct sum** of the subrepresentations $U(\alpha)$ induced on $\mathfrak{H}(\alpha)$ by U , and is denoted by $U = \bigoplus_{\alpha \in I} U(\alpha)$. Any unitary representation is the direct sum of cyclic representations. A unitary representa-

tion U is called a **representation without multiplicity** if U cannot be decomposed as a direct sum $U_1 \oplus U_2$ unless U_1 and U_2 are disjoint. If U is the direct sum of $\{U(\alpha)\}_{\alpha \in I}$ and every $U(\alpha)$ is irreducible, then U is said to be **decomposed into the direct sum of irreducible representations**. Decomposition into direct sums of irreducible representations is essentially unique if it exists; that is, if $U = \bigoplus_{\alpha \in I} U(\alpha) = \bigoplus_{\beta \in J} V(\beta)$ are two decompositions of U into direct sums of irreducible representations, then there exists a bijection φ from I onto J such that $U(\alpha)$ is equivalent to $V(\varphi(\alpha))$ for every α in I . A factor representation U of type I can be decomposed as the direct sum $U = \bigoplus_{\alpha \in I} U(\alpha)$ of equivalent irreducible representations $U(\alpha)$. In general, a unitary representation U cannot be decomposed as the direct sum of irreducible representations even if U is not irreducible. Thus it becomes necessary to use direct integrals to obtain an irreducible decomposition.

H. Direct Integrals

Let U be a unitary representation of a group G and (X, μ) be a \ast measure space. Assume that the following two conditions are satisfied by U : (i) There exists a unitary representation $U(x)$ of G corresponding to every element x of X , and $\mathfrak{H}(U)$ is a \ast direct integral (\rightarrow 308 Operator Algebras) of $\mathfrak{H}(U(x))$ ($x \in X$) (written $\mathfrak{H}(U) = \int_X \mathfrak{H}(U(x)) d\mu(x)$); (ii) for every g in G , the operator U_g is a decomposable operator and can be written as $U_g = \int_X U_g(x) d\mu(x)$. Then the unitary representation U is called the **direct integral** of the family $\{U(x)\}_{x \in X}$ of unitary representations and is denoted by $U = \int_X U(x) d\mu(x)$. If every point of X has measure 1, then a direct integral is reduced to a direct sum.

I. Decomposition into Factor Representations

We assume that G is a locally compact group satisfying the \ast second countability axiom, and also that a Hilbert space is separable. Every unitary representation U of G can be decomposed as a direct integral $U = \int_X U(x) d\mu(x)$ in such a way that the center \mathbf{A} of the von Neumann algebra $\mathbf{M}'' = \{U_g | g \in G\}''$ is the set of all \ast diagonalizable operators. In this case almost all the $U(x)$ are factor representations. Such a decomposition of U is essentially unique. There exists a \ast null set N in X such that for every x and x' in $X - N$ ($x \neq x'$), $U(x)$ and $U(x')$ are mutually disjoint factor representations. Hence the space X can be identified with the set G^\ast of all quasi-equivalence classes

of factor representations of G endowed with a suitable structure of a measure space. The space G^\ast is called the **quasidual** of G . The measure μ is determined by U up to \ast equivalence of measures.

J. Duals

A topology is introduced on the set \hat{G} of all equivalence classes of irreducible unitary representations of a locally compact group G in the following way. Let H_n be the n -dimensional Hilbert space $l_2(n)$ and I_n the set of all irreducible unitary representations of G realized on H_n ($1 \leq n \leq \infty$). We topologize I_n in such a way that a \ast net $\{U^\lambda\}_{\lambda \in L}$ in I_n converges to U if and only if $(U_g^\lambda x, y)$ converges uniformly to $(U_g x, y)$ on every compact subset of G for any x and y in H_n . Equivalence between representations in I_n is an open relation. Let \hat{G}_n be the set of all equivalence classes of n -dimensional irreducible unitary representations of G with the topology of a quotient space of I_n and $\hat{G} = \bigcup_n \hat{G}_n$ be the direct sum of topological spaces \hat{G}_n . Then the topological space \hat{G} is called the **dual** of G . \hat{G} is a locally compact \ast Baire space with countable open base, but it does not satisfy the \ast Hausdorff separation axiom in general. If G is a compact Hausdorff topological group, then \hat{G} is discrete. If G is a locally compact Abelian group, then \hat{G} coincides with the \ast character group of G in the sense of Pontryagin. If G is a type I group, then there exists a dense open subset of \hat{G} that is a locally compact Hausdorff space. The $\ast\sigma$ -additive family generated by closed sets in \hat{G} is denoted by \mathfrak{B} . In the following sections, a measure on \hat{G} means a measure defined on \mathfrak{B} .

K. Irreducible Decompositions

In this section G is assumed to be a locally compact group of type I with countable open base. For any equivalence class x in \hat{G} , we choose a representative $U(x) \in x$ with the representation space $H(U(x)) = l_2(n)$ if x is n -dimensional. For any measure μ on \hat{G} , the representation $U^\mu = \int_{\hat{G}} U(x) d\mu(x)$ is a unitary representation without multiplicity. Conversely, any unitary representation of G without multiplicity is equivalent to a U^μ for some measure μ on \hat{G} . Moreover, U^μ is equivalent to U^ν if and only if the two measures μ and ν are equivalent (that is, μ is absolutely continuous with respect to ν , and vice versa). A unitary representation U with multiplicity on a separable Hilbert space \mathfrak{H} can be decomposed as follows: There exists a countable set of measures $\mu_1, \mu_2, \dots, \mu_\infty$ whose supports are mutu-

ally disjoint such that $U \cong \int_G U(x) d\mu_1(x) \oplus 2 \int_G U(x) d\mu_2(x) \oplus \dots \oplus \infty \int_G U(x) d\mu_\infty(x)$. The measures $\mu_1, \mu_2, \dots, \mu_\infty$ are uniquely determined by U up to equivalence of measures. Any unitary representation U on a separable Hilbert space \mathfrak{H} of an arbitrary locally compact group with countable open base (even if not of type I) can be decomposed as a direct integral of irreducible representations. In order to obtain such a decomposition, it is sufficient to decompose \mathfrak{H} as a direct integral in such a way that a maximal Abelian von Neumann subalgebra \mathbf{A} of $\mathbf{M}' = \{U_g | g \in G\}'$ is the set of all diagonalizable operators. In this case, however, a different choice of \mathbf{A} induces in general an essentially different decomposition, and uniqueness of the decomposition does not hold. For a group of type I, the irreducible representations are the "atoms" of representations, as in the case of compact groups. For a group not of type I, it is more natural to take the factor representations for the irreducible representations, quasi-equivalence for the equivalence, and the quasidual for the dual of G . Therefore the theory of unitary representations for a group not of type I has different features from the one for a type I group. The theory of unitary representation for groups not of type I has not yet been successfully developed, but some important results have been obtained (e.g., L. Pukanszky, *Ann. Sci. Ecole Norm. Sup.*, 4 (1971)).

Tatsuuma [1] proved a duality theorem for general locally compact groups which is an extension of both Pontryagin's and Tannaka's duality theorems considering the direct integral decomposition of tensor product representations.

L. The Plancherel Formula

Let G be a unimodular locally compact group with countable open base, $R(L)$ be the right (left) regular representation of G , and \mathbf{M}, \mathbf{N} , and \mathbf{P} be the von Neumann algebras generated by $\{R_g\}$, $\{L_g\}$, and $\{R_g, L_g\}$, respectively. Then $\mathbf{M}' = \mathbf{N}$, $\mathbf{N}' = \mathbf{M}$, and $\mathbf{P}' = \mathbf{M} \cap \mathbf{N}$. If we decompose \mathfrak{H} into a direct integral in such a way that \mathbf{P}' is the algebra of all diagonalizable operators, then $\mathbf{M}(x)$ and $\mathbf{N}(x)$ are factors for almost all x . This decomposition of \mathfrak{H} produces a decomposition of the two-sided regular representation $\{R_g, L_g\}$ into irreducible representations and a decomposition of the regular representation $R(L)$ into factor representations. Hence the decomposition is realized as the direct integral over the quasidual G^* of G . Moreover, the factors $\mathbf{M}(x)$ and $\mathbf{N}(x)$ are of type I or II for almost all x in G^* , and there

exists a \dagger trace t in the factor $\mathbf{M}(x)$. For any f and g in $L_1(G) \cap L_2(G)$, the **Plancherel formula**

$$\int_G f(s) \overline{g(s)} ds = \int_{G^*} t(U_g^*(x) U_f(x)) d\mu(x) \quad (1)$$

holds, where $U_f(x) = \int_G f(s) U_s(x) ds$ and U^* is the \dagger adjoint of U . The **inversion formula**

$$h(s) = \int_{G^*} t(U_h(x) U_s^*(x)) d\mu(x) \quad (2)$$

is derived from (1) for a function $h = f * g$ ($f, g \in L_1(G) \cap L_2(G)$). In (1) and (2), because of the impossibility of normalization of the trace t in a factor of type II_∞ , the measure μ cannot in general be determined uniquely. However, if G is a type I group, then (1) and (2) can be rewritten as similar formulas, where the representation $U(x)$ in (1) and (2) is irreducible, the trace t is the usual trace, and the domain of integration is not the quasidual G^* but the dual \hat{G} of G . The revised formula (1) is also called the **Plancherel formula**. In this case the measure μ on \hat{G} in formulas (1) and (2) is uniquely determined by the given Haar measure on G . The measure μ is called the **Plancherel measure** of G . The support \hat{G}_r of the Plancherel measure μ is called the **reduced dual** of G . The Plancherel formula gives the direct integral decomposition of the regular representation into the irreducible representations belonging to \hat{G}_r . Each U in \hat{G}_r is contained in this decomposition, with the multiplicity equal to $\dim \mathfrak{H}(U)$.

M. Square Integrable Representations

An irreducible unitary representation U of a unimodular locally compact group G is said to be **square integrable** when for some element $x \neq 0$, in $\mathfrak{H}(U)$, the function $\varphi(g) = (U_g x, x)$ belongs to $L^2(G, dg)$, where dg is the Haar measure of G . If U is square integrable, then $\varphi_{x,y}(g) = (U_g x, y)$ belongs to $L^2(G, dg)$ for any x and y in $\mathfrak{H}(U)$. Let U and U' be the two square integrable representations of G . Then the following **orthogonality relations** hold:

$$\int_G (U_g x, y) \overline{(U'_g u, v)} dg = \begin{cases} 0 & \text{if } U \text{ is not} \\ & \text{equivalent to } U', \\ d_v^{-1}(x, u) \overline{(v, y)} & \text{if } U = U'. \end{cases} \quad (3)$$

When G is compact, every irreducible unitary representation U is square integrable and finite-dimensional. Moreover, the scalar d_v in (3) is the degree of U if the total measure of G is normalized to 1. In the general case, the scalar d_v in (3) is called the **formal degree** of U and is determined uniquely by the given Haar

measure dg . Let y be an element in $\mathfrak{H}(U)$ with norm 1 and V be the subspace $\{\varphi_{x,y} \mid x \in \mathfrak{H}(U)\}$ of $L^2(G)$. Then the linear mapping $T: x \rightarrow \sqrt{d_U} \varphi_{x,y}$ is an isometry of $\mathfrak{H}(U)$ onto V . Hence U is equivalent to a subrepresentation of the right regular representation R of G . Conversely, every irreducible subrepresentation of R is square integrable. Thus a square integrable representation is an irreducible subrepresentation of R ($\cong L$). Therefore, in the irreducible decomposition of R , the square integrable representations appear as discrete direct summands. Hence every square integrable representation U has a positive Plancherel measure $\mu(U)$ that is equal to the formal degree d_U . There exist noncompact groups that have square integrable representations. An example of such a group is $SL(2, \mathbf{R})$ (\rightarrow Section X).

N. Representations of $L_1(G)$

Let G be a locally compact group and $L_1(G)$ be the space of all complex-valued integrable functions on G . Then $L_1(G)$ is an algebra over \mathbf{C} , where the convolution

$$(f * g)(s) = \int_G f(st^{-1})g(t)dt$$

is defined to be the product of f and g . Let Δ be the \ast modular function of G . Then the mapping $f(s) \rightarrow f^\ast(s) = \Delta(s^{-1})f(s^{-1})$ is an \ast involution of the algebra $L_1(G)$. Let U be a unitary representation of G , and put $U_f = \int_G U_s f(s)ds$. Then the mapping $f \rightarrow U_f$ gives a **nondegenerate representation** of the Banach algebra $L_1(G)$ with an involution, where nondegenerate means that $\{U_f x \mid f \in L_1(G), x \in \mathfrak{H}(U)\}^\perp$ reduces to $\{0\}$. The mapping $U \rightarrow U'$ gives a bijection between the set of equivalence classes of unitary representations of G and the set of equivalence classes of nondegenerate representations of the Banach algebra $L_1(G)$ with an involution on Hilbert spaces. U is an irreducible (factor) representation if and only if U' is an irreducible (factor) representation. Therefore the study of unitary representations of G reduces to that of representations of $L_1(G)$. If U_f is a \ast compact operator for every f in $L_1(G)$, then U is the discrete direct sum of irreducible representations, and the multiplicity of every irreducible component is finite. (See [2] for Sections A–N.)

O. Induced Representations

Induced representation is the method of constructing a representation of a group G in a canonical way from a representation of a subgroup H of G . It is a fundamental method

of obtaining a unitary representation of G . Let G be a locally compact group satisfying the second countability axiom, L be a unitary representation on a separable Hilbert space $\mathfrak{H}(L)$ of a closed subgroup H of G , and m, n, Δ , and δ be the right Haar measures and the modular functions of the groups G and H , respectively. Then there exists a continuous positive function ρ on G satisfying $\rho(hg) = \delta(h)\Delta(h)^{-1}\rho(g)$ for every h in H and g in G . The \ast quotient measure $\mu = (\rho m)/n$ is a quasi-invariant measure on the coset space $H \backslash G$ (\rightarrow 225 Invariant Measures). Let \mathfrak{H} be the vector space of weakly measurable functions f on G with values in $\mathfrak{H}(L)$ satisfying the following two conditions: (i) $f(hg) = L_h f(g)$ for every h in H and g in G ; and (ii) $\|f\|^2 = \int_{H \backslash G} \|f(g)\|^2 d\mu(\dot{g}) < +\infty$, where \dot{g} represents the coset Hg . By condition (i), the norm $\|f(g)\|$ is constant on a coset $Hg = \dot{g}$ and is a function on $H \backslash G$, so the integral in condition (ii) is well defined. Then \mathfrak{H} is a Hilbert space with the norm defined in (ii). A unitary representation U of G on the Hilbert space \mathfrak{H} is defined by the formula

$$(U_s f)(g) = \sqrt{\rho(gs)/\rho(g)} f(gs).$$

U is called the **unitary representation induced by the representation L of a subgroup H** and is denoted by $U = U^L$ or $\text{Ind}_H^G L$. **Induced representations** have the following properties.

(1) $U^{L_1 \oplus L_2} \cong U^{L_1} \oplus U^{L_2}$ or more generally, $U^{\int U(x)d\mu(x)} \cong \int U^L(x)d\mu(x)$. Therefore if U^L is irreducible, L is also irreducible (the converse does not hold in general).

(2) Let H, K be two subgroups of G such that $H \subset K$, L be a unitary representation of H , and M be the representation of K induced by L . Then two unitary representations U^M and U^L of G are equivalent.

An induced representation U^L is the representation on the space of square integrable sections of the \ast vector bundle with fiber $\mathfrak{H}(L)$ \ast associated with the principal bundle $(G, H \backslash G, H)$ (\rightarrow G. W. Mackey [3], F. Bruhat [4]).

P. Unitary Representations of Special Groups

In the following sections we describe the fundamental results on the unitary representations of certain special groups.

Q. Compact Groups

Irreducible unitary representations of a compact group are always finite-dimensional. Every unitary representation of a compact group is decomposed into the direct sum of irreducible representations. Irreducible unitary representations of a compact connected Lie

Unitary Representations

group are completely classified. The characters of irreducible representations are calculated in an explicit form (\rightarrow 69 Compact Groups; 249 Lie Groups). Every irreducible unitary representation U of a connected compact Lie group G can be extended uniquely to an irreducible holomorphic representation U^C of the complexification G^C of G . U^C is holomorphically induced from a 1-dimensional representation of a Borel subgroup B of G^C (**Borel-Weil theorem**; \rightarrow R. Bott [5]).

R. Abelian Groups

Every irreducible unitary representation of an Abelian group G is 1-dimensional. *Stone's theorem concerning one-parameter groups of unitary operators, $U_t = \int_{-\infty}^{\infty} e^{i\lambda t} dE_\lambda$, gives irreducible decompositions of unitary representations of the additive group \mathbf{R} of real numbers. *Bochner's theorem on *positive definite functions on \mathbf{R} is a restatement of Stone's theorem in terms of positive definite functions. The theory of the *Fourier transform on \mathbf{R} , in particular *Plancherel's theorem, gives the irreducible decomposition of the regular representation of \mathbf{R} . The theorems of Stone, Bochner, and Plancherel have been extended to an arbitrary locally compact Abelian group (\rightarrow 192 Harmonic Analysis).

S. Representations of Lie Groups and Lie Algebras

Let U be a unitary representation of a Lie group G with the Lie algebra \mathfrak{g} . An element x in $\mathfrak{H}(U)$ is called an **analytic vector** with respect to U if the mapping $g \rightarrow U_g x$ is a real analytic function on G with values in $\mathfrak{H}(U)$. The set of all analytic vectors with respect to U forms a dense subspace $\mathfrak{A} = \mathfrak{A}(U)$ of $\mathfrak{H}(U)$. For any elements X in \mathfrak{g} and x in $\mathfrak{A}(U)$, the derivative at $t=0$ of a real analytic function $U_{\exp tX} x$ is denoted by $V(X)x$. Then $V(X)$ is a linear transformation on \mathfrak{A} , and the mapping $V: X \rightarrow V(X)$ is a representation of \mathfrak{g} on \mathfrak{A} . We call V the **differential representation** of U . The representation V of \mathfrak{g} can be extended uniquely to a representation of the *universal enveloping algebra \mathfrak{B} of \mathfrak{g} . Two unitary representations $U^{(1)}$ and $U^{(2)}$ of a connected Lie group G are equivalent if and only if there exists a bijective bounded linear mapping T from $\mathfrak{H}(U^{(1)})$ onto $\mathfrak{H}(U^{(2)})$ such that T maps $\mathfrak{A}(U^{(1)})$ onto $\mathfrak{A}(U^{(2)})$ and satisfies the equality

$$(T \circ V^{(1)}(X))x = (V^{(2)}(X) \circ T)x$$

for all X in \mathfrak{g} and x in $\mathfrak{A}(U^{(1)})$. Let X_1, \dots, X_n be a basis of \mathfrak{g} and U be a unitary representation of G . Then the element $\Delta = X_1^2 + \dots + X_n^2$

in the universal enveloping algebra \mathfrak{B} of \mathfrak{g} is represented in the differential representation V of U by an *essentially self-adjoint operator $V(\Delta)$. Conversely, if to each element X in \mathfrak{g} there corresponds a (not necessarily bounded) *skew-Hermitian operator $\rho(X)$ that satisfies the following three conditions, then there exists a unique unitary representation U of the simply connected Lie group G with the Lie algebra \mathfrak{g} such that the *closure of $V(X)$ coincides with the closure of $\rho(X)$ for every X in \mathfrak{g} : (i) There exists a dense subspace \mathfrak{D} contained in the domain of $\rho(X)\rho(Y)$ for every X and Y in \mathfrak{g} ; (ii) for each X and Y in \mathfrak{g} , a and b in \mathbf{R} , and x in \mathfrak{D} , $\rho(aX + bY)x = a\rho(X)x + b\rho(Y)x$, $\rho([X, Y])x = (\rho(X)\rho(Y) - \rho(Y)\rho(X))x$; (iii) the restriction of $\rho(X_1)^2 + \dots + \rho(X_n)^2$ to \mathfrak{D} is an essentially self-adjoint operator if X_1, \dots, X_n is a basis of \mathfrak{g} (E. Nelson [6]).

T. Nilpotent Lie Groups

For every irreducible unitary representation of a connected nilpotent Lie group G , there is some 1-dimensional unitary representation of some subgroup of G that induces it. Let G be a simply connected nilpotent Lie group, \mathfrak{g} be the Lie algebra of G , and ρ be the contragredient representation of the adjoint representation of G . The representation space of ρ is the dual space \mathfrak{g}^* of \mathfrak{g} . A subalgebra \mathfrak{h} of \mathfrak{g} is called **subordinate** to an element f in \mathfrak{g}^* if f annihilates each bracket $[X, Y]$ for every X and Y in \mathfrak{h} : $(f, [X, Y]) = 0$. When \mathfrak{h} is subordinate to f , a 1-dimensional unitary representation L of the analytic subgroup H of G with the Lie algebra \mathfrak{h} is defined by the formula $\lambda_f(\exp X) = e^{2\pi i(f, X)}$ ($X \in \mathfrak{h}$). Every 1-dimensional unitary representation λ_f of H is defined as in this formula by an element f in \mathfrak{g}^* to which \mathfrak{h} is subordinate. The unitary representation of G induced by such a λ_f is denoted by $U(f, \mathfrak{h})$. The representation $U(f, \mathfrak{h})$ is irreducible if and only if \mathfrak{h} has maximal dimension among the subalgebras subordinate to f . Two irreducible representations $U(f, \mathfrak{h})$ and $U(f', \mathfrak{h}')$ are equivalent if and only if f and f' are conjugate under the group $\rho(G)$. Therefore there exists a bijection between the set of equivalence classes of the irreducible unitary representations of a simply connected nilpotent Lie group G and the set of orbits of $\rho(G)$ on \mathfrak{g}^* (A. A. Kirillov [7]).

U. Solvable Lie Groups

Let G be a simply connected solvable Lie group. If the exponential mapping is bijective, G is called an **exponential group**. All results stated above for nilpotent Lie groups hold for exponential groups except the irreducibility

criterion. In this case the representation $U(f, \mathfrak{h})$ is irreducible if and only if \mathfrak{h} is of maximal dimension among subordinate subalgebras and the orbit $O = \rho(G)f$ contains the affine subspace $f + \mathfrak{h}^\perp = f + \{g | g(\mathfrak{h}) = 0\}$ (Pukanszky condition).

The situation is more complicated for general solvable Lie groups. The isotropy subgroup $G_f = \{g \in G | \rho(g)f = f\}$ at $f \in \mathfrak{g}^*$ is, in general, not connected. A linear form f is called integral if there exists a unitary character η_f of G_f whose differential is the restriction of $2\pi i f$ to \mathfrak{g}_f (the Lie algebra of G_f). Using the notion of "polarization," an irreducible unitary representation of G is constructed from a pair (f, η_f) of an integral form $f \in \mathfrak{g}^*$ and a character η_f . If G is of type I, then every irreducible unitary representation of G is obtained in this way. A simply connected solvable Lie group G is of type I if and only if (i) every $f \in \mathfrak{g}^*$ is integral and (ii) every G -orbit $\rho(G)f$ in \mathfrak{g}^* is locally closed (Auslander and Kostant [8]).

As an example, let α be an irrational real number. Then the following Lie group G is not

$$\text{of type I: } G = \left\{ \begin{pmatrix} e^{it} & 0 & z \\ 0 & e^{i\alpha t} & w \\ 0 & 0 & 1 \end{pmatrix} \mid t \in \mathbf{R}, z, w \in \mathbf{C} \right\}.$$

V. Semisimple Lie Groups

A connected semisimple Lie group is of type I. The character $\chi = \chi_U$ of an irreducible unitary representation U of G is defined as follows: Let $C_0^\infty(G)$ be the set of all complex-valued C^∞ -functions with compact support on G . Then for any function f in $C_0^\infty(G)$, the operator $U_f = \int_G U_g f(g) dg$ belongs to the \dagger trace class, and the linear form $\chi: f \rightarrow \text{tr } U_f$ is a \dagger distribution in the sense of Schwartz. The distribution χ is called the **character** of an irreducible unitary representation U . A character χ is invariant under any inner automorphism of G and is a simultaneous eigendistribution of the algebra of all two-sided invariant linear differential operators on G . Two irreducible unitary representations of G are equivalent if and only if their characters coincide. The distribution χ is a \dagger locally summable function on G and coincides with a real analytic function on each connected component of the dense open submanifold G' consisting of regular elements in G . In general, χ is not real analytic on all of G (Harish-Chandra [9, III; 10]).

W. Complex Semisimple Lie Groups

There are four series of irreducible representations of a complex semisimple Lie group G .

(1) A **principal series** consists of unitary representations of G induced from 1-

dimensional unitary representations L of a \dagger Borel subgroup B of G . L is uniquely determined by a unitary character $v \in \text{Hom}(A, U(1)) = A^*$ of the \dagger Cartan subgroup A of G contained in B . Hence the representations in the principal series are parametrized by the elements in the character group A^* of the Cartan subgroup A . If we denote U^L by U^v , two representations U^v and $U^{v'}$ ($v, v' \in A^*$) are equivalent if and only if v and v' are conjugate under the \dagger Weyl group W of G with respect to A .

(2) A **degenerate series** consists of unitary representations induced by 1-dimensional unitary representations of a \dagger parabolic subgroup P of G other than B . (A parabolic subgroup P is any subgroup of G containing a Borel subgroup B .)

(3) A **complementary series** consists of irreducible unitary representations U^L induced by nonunitary 1-dimensional representations L of a Borel subgroup B . In this case, condition (ii) in the definition of U^L (\rightarrow Section O) must be changed. When L is a nonunitary representation, then the operator U_g^L is not a unitary operator with respect to the usual L_2 -inner product (ii). However, if L satisfies a certain condition, then U_g^L leaves invariant some positive definite Hermitian form on the space of sufficiently nice functions. Completing this space, we get a unitary representation U^L . The representations thus obtained form the complementary series.

(4) A **complementary degenerate series** consists of irreducible unitary representations induced by nonunitary 1-dimensional representations of a parabolic subgroup $P \neq B$.

Representations belonging to different series are never equivalent. It seems certain that any irreducible unitary representation of a connected complex semisimple Lie group is equivalent to a representation belonging to one of the above four series, but this conjecture has not yet been proved. Moreover, E. M. Stein [11] constructed irreducible unitary representations different from any in the list obtained by I. M. Gel'fand and M. A. Naimark (Neumark) [12]. These representations belong to the complementary degenerate series. The characters of the representations in these four series are computed in explicit form. For example, the character χ_v of the representation U^v in the principal series can be calculated as follows: Let λ be a linear form on a Cartan subalgebra \mathfrak{a} such that $v(\exp H) = e^{\lambda(H)}$ for every H in \mathfrak{a} , let D be the function on A defined by $D(\exp H) = \prod_{\alpha} |e^{\alpha(H)/2} - e^{-\alpha(H)/2}|^2$, where α runs over all positive roots. Then the character χ_v of a representation U^v in the principal series is given by the formula

$$\chi_v(\exp H) = D(\exp H)^{-1} \sum_{s \in W} e^{s\lambda(H)}.$$

In the irreducible decomposition of the regular representation of G , only irreducible representations belonging to the principal series arise. Hence the right-hand side in the Plancherel formula is an integral over the character group A^* of a Cartan subgroup A . Under a suitable normalization of the Haar measures in G and A^* , the Plancherel measure μ of G can be expressed by using the Haar measure dv of A^* :

$$d\mu(v) = w^{-1} \prod_{\alpha} |(\lambda, \alpha) / (\rho, \alpha)|^2 dv,$$

where w is the order of the Weyl group, ρ is the half-sum of all $^+$ positive roots, and α runs over all positive roots (Gel'fand and Naïmark [12]).

X. Real Semisimple Lie Groups

As in the case of a complex semisimple Lie group, a connected real semisimple Lie group G has four series of irreducible unitary representations. However, if G has no parabolic subgroup other than a minimal parabolic subgroup B and G itself, then G has no representation in the degenerate or complementary degenerate series. Examples of such groups are $SL(2, \mathbf{R})$ and higher-dimensional $^+$ Lorentz groups. In general, the classification of irreducible unitary representations in the real semisimple case is more complicated than in the complex semisimple case. Irreducible unitary representations arising from the irreducible decomposition of the regular representation are called representations in the **principal series**. The principal series of G are divided into a finite number of subseries corresponding bijectively to the conjugate classes of the $^+$ Cartan subgroups of G .

A connected semisimple Lie group G has a square integrable representation if and only if G has a compact Cartan subgroup H . The set of all square integrable representations of G is called the **discrete series** of irreducible unitary representations. The discrete series is the subseries in the principal series corresponding to a compact Cartan subgroup H . The representations in the discrete series were classified by Harish-Chandra. Let \mathfrak{h} be the Lie algebra of H , P the set of all positive roots in \mathfrak{h} for a fixed linear order, π the polynomial $\prod_{\alpha \in P} H_{\alpha}$, and \mathcal{F} the set of all real-valued linear forms on $\sqrt{-1}\mathfrak{h}$. Moreover, let L be the set of all linear forms λ in \mathcal{F} such that a single-valued character ξ_{λ} of the group H is defined by the formula $\xi_{\lambda}(\exp X) = e^{\lambda(X)}$, and let L' be the set of all λ in L such that $\pi(\lambda) \neq 0$. Then for each λ in L' , there exists a representation $\omega(\lambda)$ of G in the discrete series, and conversely, every representation in the discrete series is equivalent to $\omega(\lambda)$ for some λ in L' . Two representations

$\omega(\lambda_1)$ and $\omega(\lambda_2)$ ($\lambda_1, \lambda_2 \in L'$) are equivalent if and only if there exists an element s in $W_G = N(H)/H$ such that $\lambda_2 = s\lambda_1$, where $N(H)$ is the normalizer of H in G (W_G can act on \mathcal{F} as a linear transformation group in the natural way). The value of the character χ_{λ} on the subgroup H of the representation $\omega(\lambda)$ ($\lambda \in L'$) is given as follows: Let $\varepsilon(\lambda)$ be the signature of $\pi(\lambda) = \prod_{\alpha \in P} \lambda(H_{\alpha})$, and define q and Δ by $q = (\dim G/K)/2$ and $\Delta(\exp H) = \prod_{\alpha \in P} (e^{\alpha(H)/2} - e^{-\alpha(H)/2})$. Then the character χ_{λ} of the representation $\omega(\lambda)$ has the value $(-1)^q \varepsilon(\lambda) \chi_{\lambda}(h) = \Delta(h)^{-1} \sum_{s \in W_G} (\det s) \xi_{s\lambda}(h)$ on a regular element h in H . The formal degree $d(\omega(\lambda))$ of the representation $\omega(\lambda)$ is given by the formula $d(\omega(\lambda)) = C^{-1} [W_G] |\pi(\lambda)|$, where C is a positive constant (not depending on λ) and $[W_G]$ is the order of the finite group W_G (Harish-Chandra [13]). A formula expressing the character χ_{λ} on the whole set of regular elements in G has been given by T. Hirai [14]. The representations in discrete series are realized on L^2 -cohomology spaces of homogeneous holomorphic line bundles over G/H (W. Schmid [15]). They are also realized on the spaces of harmonic spinors on the $^+$ Riemannian symmetric space G/K (M. Atiyah and Schmid [16]). They are also realized on the eigenspaces of a Casimir operator acting on the sections of vector bundles on G/K (R. Hotta, *J. Math. Soc. Japan*, 23; N. Wallach [17]). An irreducible unitary representation is called **integrable** if at least one of its matrix coefficients belongs to $L^1(G)$. Integrable representations belong to the discrete series. They have been characterized by H. Hecht and Schmid (*Math. Ann.*, 220 (1976)). The theory of the discrete series is easily extended to reductive Lie groups.

The general principal series representations of a connected semisimple Lie group G with finite center are constructed as follows. Let K be a maximal compact subgroup of G . Then there exists a unique involutive automorphism θ of G whose fixed point set coincides with K . θ is called a **Cartan involution** of G . Let H be a θ -stable Cartan subgroup of G . Then H is the direct product of a compact group $T = H \cap K$ and a vector group A . The centralizer $Z(A)$ of A in G is the direct product of a reductive Lie group $M = \theta(M)$ and A . M has a compact Cartan subgroup T . Hence the set \hat{M}_d of the discrete series representations of M is not empty. Let α be an element of the dual space \mathfrak{a}^* of the Lie algebra \mathfrak{a} of A and put $g_{\alpha} = \{X \in \mathfrak{g} | [H, X] = \alpha(H)X (\forall H \in \mathfrak{a})\}$ and $\Delta = \{\alpha \in \mathfrak{a}^* | g_{\alpha} \neq \{0\}\}$. Let Δ^+ be the set of positive elements of Δ in a certain order of \mathfrak{a}^* and put $n = \sum_{\alpha \in \Delta^+} g_{\alpha}$ and $N = \exp n$. Then $P = MAN$ is a closed subgroup of G . P is called a **cuspidal parabolic subgroup** of G . Let $D \in \hat{M}_d$ and $v \in \mathfrak{a}^*$. Then a unitary representation $D \otimes e^{iv}$ of P

is defined by $(D \otimes e^{iv})(man) = D(m)e^{iv(\log a)}$ ($m \in M, a \in A, n \in N$). The unitary representation $\pi_{D,v}$ of G induced by $D \otimes e^{iv}$ is independent of the choice of Δ^+ up to equivalence. Thus $\pi_{D,v}$ depends only on (H, D, v) . The set of representations $\{\pi_{D,v} | D \in \hat{M}_d, v \in \mathfrak{a}^*\}$ is called the **principal H -series**. If v is regular in \mathfrak{a}^* (i.e., $(v, \alpha) \neq 0$ for all $\alpha \in \Delta$), then $\pi_{D,v}$ is irreducible. Every $\pi_{D,v}$ is a finite sum of irreducible representations. The character $\theta_{D,v}$ of $\pi_{D,v}$ is a locally summable function which is supported in the closure of $\bigcup_{g \in G} g(MA)g^{-1}$. If two Cartan subgroups H_1 and H_2 are not conjugate in G , then every H_1 -series representation is disjoint from every H_2 -series representation. Choose a complete system $\{H_1, \dots, H_r\}$ of conjugacy classes of Cartan subgroups of G . Then every H_i can be chosen as θ -stable. The union of the principal H_i -series ($1 \leq i \leq r$) is the principal series of G . The right (or left) regular representation of G is decomposed as the direct integral of the principal series representations. Every complex-valued C^∞ -function on G with compact support has an expansion in terms of the matrix coefficients of the principal series representations. Harish-Chandra [18] proved these theorems and determined explicitly the Plancherel measure by studying the asymptotic behavior of the Eisenstein integral [19, 20].

Y. Spherical Functions

Let G be a locally compact unimodular group and K a compact subgroup of G . The set of all complex-valued continuous functions on G that are invariant under every left translation L_k by elements k in K is denoted by $C(K \backslash G)$. The subset of $C(K \backslash G)$ that consists of all two-sided K -invariant functions is denoted by $C(G, K)$. The subset of $C(G, K)$ consisting of all functions with compact support is denoted by $L = L(G, K)$. L is an algebra over \mathbb{C} if the product of two elements f and g in L is defined by the convolution.

Let λ be an algebra homomorphism from L into \mathbb{C} . Then an element of the eigenspace $F(\lambda) = \{\psi \in C(K, G) | f * \psi = \lambda(f)\psi \ (\forall f \in L)\}$ is called a **spherical function** on $K \backslash G$. If $F(\lambda)$ contains a nonzero element, then $F(\lambda)$ contains a unique two-sided K -invariant element ω normalized by $\omega(e) = 1$, where e is the identity element in G . This function ω is called the **zonal spherical function** associated with λ . In this case, the homomorphism λ is defined by $\lambda(f) = \int_G f(g)\omega(g^{-1})dg$. Hence the eigenspace $F(\lambda)$ is uniquely determined by the zonal spherical function ω . A function $\omega \neq 0$ in $C(G, K)$ is a zonal spherical function on $K \backslash G$ if and only if ω satisfies either of the following two conditions: (i) The mapping $f \mapsto \int f(g)\omega(g^{-1})dg$ is

an algebra homomorphism of L into \mathbb{C} ; (ii) ω satisfies the functional equation

$$\int_K \omega(gkh)dk = \omega(g)\omega(h).$$

When G is a Lie group, every spherical function is a real analytic function on $K \backslash G$.

Z. Expansion by Spherical Functions

In this section, we assume that the algebra L of two-sided K -invariant functions is commutative. In this case there are sufficiently many spherical functions of $K \backslash G$, and two-sided K -invariant functions are expanded by spherical functions. An irreducible unitary representation U of G is called a **spherical representation** with respect to K if the representation space $\mathfrak{H}(U)$ contains a nonzero vector invariant under every U_k , where k runs over K . By the commutativity of L , the K -invariant vectors in $\mathfrak{H}(U)$ form a 1-dimensional subspace. Let x be a K -invariant vector in $\mathfrak{H}(U)$ with the norm $\|x\| = 1$. Then $\omega(g) = (U_g x, x)$ is a zonal spherical function on $K \backslash G$, and for every y in $\mathfrak{H}(U)$, the function $\varphi_y(g) = (U_g x, y)$ is a spherical function associated with ω . Moreover, in this case the zonal spherical function ω is a positive definite function on G . Conversely, every positive definite zonal spherical function ω can be expressed as $\omega(g) = (U_g x, x)$ for some spherical representation U and some K -invariant vector x in $\mathfrak{H}(U)$.

The set of all positive definite zonal spherical functions becomes a locally compact space Ω by the topology of compact convergence. The **spherical Fourier transform** \hat{f} of a function f in $L_1(K \backslash G)$ is defined by

$$\hat{f}(\omega) = \int_G f(g)\omega(g^{-1})dg.$$

There exists a unique \dagger Radon measure μ on Ω such that for every f in L , \hat{f} belongs to $L_2(\Omega, \mu)$. Also, the Plancherel formula

$$\int_G f(s)\overline{g(s)}ds = \int_\Omega \hat{f}(\omega)\overline{\hat{g}(\omega)}d\mu(\omega) \quad (4)$$

holds for every f and g in L , and an inversion formula $f(s) = \int_\Omega \hat{f}(\omega)\omega(s)d\mu(\omega)$ holds for a sufficiently nice two-sided K -invariant function f [21]. Identifying a positive definite zonal spherical function with the corresponding spherical representation, we can regard Ω as a subset of the dual \hat{G} of G . The Plancherel formula for two-sided K -invariant functions is obtained from the general Plancherel formula on G by restricting the domain of the integral from \hat{G} to Ω . When G is a Lie group and L is commutative, a spherical function on $K \backslash G$ can be characterized as a simultaneous eigenfunc-

tion of G -invariant linear differential operators on $K \backslash G$.

AA. Spherical Function on Symmetric Spaces

The most important case where the algebra $L = L(G, K)$ is commutative is when $K \backslash G$ is a \dagger weakly symmetric Riemannian space or, in particular, a \dagger symmetric Riemannian space. When $K \backslash G$ is a compact symmetric Riemannian space, a spherical representation with respect to K is the irreducible component of the regular representation T on $K \backslash G$, and a spherical function on $K \backslash G$ is a function that belongs to the irreducible subspaces in $L_2(K \backslash G)$. In particular, if G is a compact connected semisimple Lie group, the highest weights of spherical representations of G with respect to K are explicitly given by using the **Satake diagram** of $K \backslash G$. The Satake diagram of $K \backslash G$ is the \dagger Satake diagram of the noncompact symmetric Riemannian space $K \backslash G_0$ dual to $K \backslash G$ or the Satake diagram of the Lie algebra of G_0 . If a symmetric space is the underlying manifold of a compact Lie group G , then G can be expressed as $G = K \backslash (G \times G)$, where K is the diagonal subgroup of $G \times G$. In this case, a zonal spherical function ω on $G = K \backslash (G \times G)$ is the normalized character of an irreducible unitary representation U of G : $\omega(g) = (\deg U)^{-1} T_r U_g$. The explicit form of ω is given by \dagger Weyl's character formula (\rightarrow 249 Lie Groups).

The zonal spherical functions on a symmetric Riemannian space $K \backslash G$ of noncompact type are obtained in the following way: Let G be a connected semisimple Lie group with finite center, K be a maximal compact subgroup of G , and $G = NA_+K$ be an \dagger Iwasawa decomposition. Then for any g in G there exists a unique element $H(g)$ in the Lie algebra \mathfrak{a}_+ of A_+ such that g belongs to $N \exp H(g)K$. Let \mathfrak{a} be a Cartan subalgebra containing \mathfrak{a}_+ , P be the set of all positive roots in \mathfrak{a} , and $\rho = (\sum_{\alpha \in P} \alpha)/2$. Then for any complex-valued linear form v on \mathfrak{a}_+ , the function

$$\omega_v(g) = \int_K e^{(iv - \rho)(H(kg))} dk$$

is a zonal spherical function on the symmetric Riemannian space $K \backslash G$. Conversely, every zonal spherical function ω on $K \backslash G$ is equal to ω_v for some v . Two zonal spherical functions ω_v and $\omega_{v'}$ coincide if and only if v and v' are conjugate under the operation of the Weyl group $W_0 = N_K(A)/Z_K(A)$ of $K \backslash G$ (Harish-Chandra [22], S. Helgason [23]). If v is real-valued, then ω_v is positive definite. Such a zonal spherical function ω_v is obtained from a spherical representation belonging to the

principal A -series. Let Ω_0 be the set of all zonal spherical functions ω_v associated with the real-valued linear form v . Then the support of the Plancherel measure μ on $K \backslash G$ is contained in Ω_0 . We can choose v as a parameter on the space Ω_0 . Then the right-hand side of the Plancherel formula can be expressed as an integral over the dual space L of \mathfrak{a}_+ . Moreover, the Plancherel measure μ is absolutely continuous with respect to the Lebesgue measure dv on the Euclidean space L and can be expressed as

$$d\mu(\omega_v) = \omega_0^{-1} |c(v)|^{-2} dv$$

under suitable normalization of μ and dv . The problem of calculating the function $c(v)$ can be reduced to the case of symmetric spaces of rank 1 and can be solved explicitly. Let p_α be the multiplicity of a restricted root α and $I(v)$ be the product

$$I(v) = \prod_{\alpha} B(\tfrac{1}{2}p_\alpha, \tfrac{1}{4}p_\alpha + (v, \alpha)(\alpha, \alpha)^{-1}),$$

where α runs over all positive restricted roots and B is the \dagger beta function. Then $c(v) = I(iv)/I(\rho)$ [20, 24]. Every spherical function f on $K \backslash G$ is expressed as the Poisson integral of its "boundary values" on the Martin boundary $P \backslash G$ of $K \backslash G$, where $P = MA_+N$ is a minimal parabolic subgroup of G . The boundary values of f form a hyperfunction with values in a line bundle over $P \backslash G$ (K. Okamoto et al. [25]).

BB. Spherical Functions and Special Functions

Some important special functions are obtained as the zonal spherical functions on a certain symmetric Riemannian space $M = K \backslash G$ (G is the motion group of M). In particular when M is of rank 1, then the zonal spherical functions are essentially the functions of a single variable. For example, the zonal spherical functions on an n -dimensional Euclidean space can be expressed as

$$\omega_v(r) = 2^m \Gamma(m+1) (vr)^{-m} J_m(vr),$$

where $2m = n - 2$ and J_m is the \dagger Bessel function of the m th order. The zonal spherical function on an $(n-1)$ -dimensional sphere $S^{n-1} = SO(n-1) \backslash SO(n)$ is given by

$$\omega_v(\theta) = \Gamma(v+1) \Gamma(n-2) \Gamma(v+n-2)^{-1} C_v^m(\cos \theta) \\ (v=0, 1, 2, \dots),$$

where $C_v^m(z)$ is the \dagger Gegenbauer polynomial. The zonal spherical functions on an $(n-1)$ -dimensional Lobachevskii space can be expressed as

$$\omega_v(t) = 2^{m-1/2} \Gamma(m+1/2) \sinh^{-m+1/2} t \\ \times \mathfrak{P}_{-1/2-m+v}^{1/2-m}(\cosh t)$$

using a generalized * associated Legendre function \mathfrak{P}_ν^μ . Many properties of special functions can be proved from a group-theoretic point of view. For example, the addition theorem is merely the homomorphism property $U_{gh} = U_g U_h$ expressed in terms of the matrix components of U . The differential equation satisfied by these special functions is derived from the fact that a zonal spherical function ω is an eigenfunction of an invariant differential operator. The integral expression of such a special function can be obtained by constructing a spherical representation U in a certain function space and calculating explicitly the inner product in the expression $\omega(g) = (U_g x, x)$ (N. Ya. Vilenkin [26]).

CC. Generalization of the Theory of Spherical Functions

The theory of spherical functions described in Sections Y–BB can be generalized in several ways. First, spherical functions are related to the trivial representation of K . A generalization is obtained if the trivial representation of K is replaced by an irreducible representation of K . The theory of such zonal spherical functions is useful for representation theory [20]. For example, the Plancherel formula for $SL(2, \mathbf{R})$ can be obtained using such spherical functions (R. Takahashi, *Japan. J. Math.*, 31 (1961)). Harish-Chandra's Eisenstein integral is such a spherical function on a general semisimple Lie group G . He used it successfully to obtain the Plancherel measure of G . Another generalization can be obtained by removing the condition that K is compact. In particular, when $K \backslash G$ is a symmetric homogeneous space of a Lie group G , the algebra \mathcal{D} of all G -invariant linear differential operators is commutative if the space $K \backslash G$ has an invariant volume element. In this case, a spherical function on $K \backslash G$ can be defined as a simultaneous eigenfunction of \mathcal{D} . The character of a semisimple Lie group is a zonal spherical function (distribution) in this sense. The spherical functions and harmonic analysis on symmetric homogeneous space have been studied by T. Oshima and others. T. Oshima and J. Sekiguchi [27] proved the Poisson integral theorem (\rightarrow Section AA) for a certain kind of symmetric homogeneous spaces.

The spherical functions and unitary representations of topological groups that are not locally compact are studied in connection with probability theory and physics. For example, the zonal spherical functions of the rotation group of a real Hilbert space are expressed by Hermite polynomials.

DD. Discontinuous Subgroups and Representations

Let G be a connected semisimple Lie group and Γ be a discrete subgroup of G . Then the regular representation T of G on $\Gamma \backslash G$ is defined by $(T_g f)(x) = f(xg)$ ($f \in L^2(\Gamma \backslash G)$). The problem of decomposing the representation T into irreducible components is important in connection with the theory of * automorphic forms and number theory. First assume that the quotient space $\Gamma \backslash G$ is compact. Then for every function f in $L_1(G)$, the operator $T(f)$ is a compact operator. Hence the regular representation T on $\Gamma \backslash G$ can be decomposed into the discrete sum $T = \sum_{k=1}^{\infty} T^{(k)}$ of irreducible unitary representations $T^{(k)}$, and the multiplicity of every irreducible component is finite. The irreducible unitary representation U of G is related to the automorphic forms of Γ in the following way: Let x be a nonzero element in the representation space $\mathfrak{H} = \mathfrak{H}(U)$ of U . \mathfrak{H} is topologized into a * locally convex topological vector space \mathfrak{H}_x by the set N_x of * seminorms: $N_x = \{P_C(y) = \max_{g \in C} |(U_g x, y)|\}$, where C runs over all compact subsets in G . The topology \mathcal{T}_x of \mathfrak{H}_x is independent of the choice of x provided that $\dim\{T_k x | k \in K\} < \infty$, where K is a maximal compact subgroup of G . Let \mathfrak{H}^* be the completion of \mathfrak{H}_x with respect to the topology \mathcal{T}_x (the completion is independent of the choice of x). \mathfrak{H}^* contains the original Hilbert space \mathfrak{H} as a subspace. Then the representation U of G on \mathfrak{H} can be extended to a representation U^* of G on the space \mathfrak{H}^* . An element f in \mathfrak{H}^* invariant under U_γ^* for every γ in Γ is called an **automorphic form** of Γ of type U . Then the multiplicity of an irreducible representation U in the regular representation T on $\Gamma \backslash G$ is equal to the dimension of the vector space consisting of all automorphic forms of type U . This theorem is called the **Gel'fand–Pyatetskii–Shapiro reciprocity law** [28]. Let $T = \sum_{k=1}^{\infty} T^{(k)}$ be the irreducible decomposition of T and χ_k be the character of the irreducible unitary representation $T^{(k)}$. Then for a suitable function f on G , the integral operator K_f on $\mathfrak{H}(T) = L^2(T \backslash G)$ with kernel $k_f(x, y) = \sum_{\gamma \in \Gamma} f(x^{-1}\gamma y)$ belongs to the trace class. By calculating the trace of K_f in two ways, the following **trace formula** is obtained:

$$\sum_{k=1}^{\infty} \int_G f(g) \chi_k(g) dg = \sum_{\{\gamma\}} \int_{\mathfrak{D}_\gamma} f(x^{-1}\gamma x) dx,$$

where $\{\gamma\}$ is the conjugate class of γ in Γ and \mathfrak{D}_γ is the quotient space of the centralizer G_γ of γ in G by the centralizer Γ_γ of γ in Γ .

When the groups G and Γ are given explicitly, the right-hand side of the trace formula can be expressed in a more explicit form,

and the trace formula leads to useful consequences. A similar trace formula holds for the unitary representation U^L induced by a finite-dimensional unitary representation L of Γ instead of the regular representation T on $\Gamma \backslash G$. When the quotient space $\Gamma \backslash G$ is not compact, the irreducible decomposition of the regular representation T on $\Gamma \backslash G$ contains not only the discrete direct sum but also the direct integral (continuous spectrum). A. Selberg showed that even in this case, there are explicit examples for which the trace formula holds for the part with discrete spectrum. Also, the part with continuous spectrum can be described by the *generalized Eisenstein series. Analytic properties and the functional equation of the generalized Eisenstein series have been studied by R. Langlands [30]. Recent developments are surveyed in [31].

EE. History

Finite-dimensional unitary representations of a finite group were studied by Frobenius and Schur (1896–1905). In 1925, *Weyl studied the finite-dimensional unitary representation of compact Lie groups. The theory of infinite-dimensional unitary representation was initiated in 1939 by E. P. Wigner in his work on the inhomogeneous Lorentz group, motivated by problems of quantum mechanics.

In 1943, Gel'fand and D. A. Raikov proved the existence of sufficiently many irreducible unitary representations of an arbitrary locally compact group. The first systematic studies of unitary representations appeared in 1947 in the work of V. Bargmann on $SL(2, \mathbf{R})$ [31] and the work of Gel'fand and Neumark on $SL(2, \mathbf{C})$. Gel'fand and Naïmark established the theory of unitary representation for complex semisimple Lie groups [12].

Harish-Chandra proved theorems concerning the unitary representations of a general semisimple Lie group; for instance, he proved that a semisimple Lie group G is of type I [7] and defined the character of a unitary representation of G and proved its basic properties [9, III; 10]. Harish-Chandra also determined the discrete series of G and their characters. Harish-Chandra [18] proved the Plancherel formula for an arbitrary connected semisimple Lie group G with finite center. Hence harmonic analysis of square integrable functions on G is established.

Further studies on harmonic analysis on semisimple Lie groups have been carried out. In particular, Paley-Wiener-type theorems, which determine the Fourier transform image

of the space $C_c^\infty(G)$ of C^∞ -functions with compact support, have been proved for the group $PSL(2, \mathbf{R})$ (L. Ehrenpreis and F. Mautner [33]), complex semisimple Lie groups (Zhelobenko [34]), and two-sided K -invariant functions on general semisimple Lie groups (R. Gangolli [35]). A. W. Knap and E. M. Stein [36] studied the intertwining operators.

Concerning the construction of irreducible representations, G. W. Mackey [3] and Bruhat [4] developed the theory of induced representations of locally compact groups and Lie groups, respectively. B. Kostant [37] (see Blattner's article in [38]) noticed a relation between homogeneous *symplectic manifolds and unitary representations and proposed a method of constructing irreducible unitary representations of a Lie group. Selberg's research [29] revealed a connection between unitary representations (or spherical functions) and the theory of automorphic forms and number theory. A number of papers along these lines have since appeared [31]. In connection with number-theoretic investigations of an *algebraic group defined over an algebraic number field, unitary representations of the *adele group of G or an algebraic group over a *p-adic number field have been studied (\rightarrow [31, 38], Gel'fand, M. I. Grayev, and I. I. Pyatetskii-Shapiro [39], and H. M. Jacquet and R. P. Langlands [40]).

For the algebraic approach to the infinite-dimensional representations of semisimple Lie groups and Lie algebras \rightarrow [41].

For surveys of the theory of unitary representations \rightarrow [2, 19, 20, 31, 38].

References

- [1] N. Tatsuuma, A duality theorem for locally compact groups, *J. Math. Kyoto Univ.*, 6 (1967), 187–293.
- [2] J. Dixmier, *Les C^* -algèbres et leurs représentations*, Gauthier-Villars, 1964.
- [3] G. W. Mackey, Induced representations of locally compact groups I, *Ann. Math.*, 55 (1952), 101–139.
- [4] F. Bruhat, Sur les représentations induites des groupes de Lie, *Bull. Soc. Math. France*, 84 (1956), 97–205.
- [5] R. Bott, Homogeneous vector bundles, *Ann. Math.*, 66 (1957), 203–248.
- [6] E. Nelson, Analytic vectors, *Ann. Math.*, 70 (1959), 572–615.
- [7] A. A. Kirillov, Unitary representations of nilpotent Lie groups, *Russian Math. Surveys*, 17, no. 4 (1962), 53–104. (Original in Russian, 1962.)
- [8] L. Auslander and B. Kostant, Polarization

and unitary representations of solvable Lie groups, *Inventiones Math.*, 14 (1971), 255–354.

[9] Harish-Chandra, Representations of a semisimple Lie group on a Banach space I–VI, *Trans. Amer. Math. Soc.*, 75 (1953), 185–243; 76 (1954), 26–65; 76 (1954), 234–253; *Amer. J. Math.*, 77 (1955), 743–777; 78 (1956), 1–41; 78 (1956), 564–628.

[10] Harish-Chandra, Invariant eigendistributions on a semisimple Lie group, *Trans. Amer. Math. Soc.*, 119 (1965), 457–508.

[11] E. M. Stein, Analysis in matrix spaces and some new representations of $SL(N, \mathbb{C})$, *Ann. Math.*, (2) 86 (1967), 461–490.

[12] I. M. Gel'fand and M. A. Naimark (Neumark), *Unitäre Darstellungen der klassischen Gruppen*, Akademische Verlag., 1957. (Original in Russian, 1950.)

[13] Harish-Chandra, Discrete series for semisimple Lie groups I, II, *Acta Math.*, 113 (1965), 241–318; 116 (1966), 1–111.

[14] T. Hirai, The characters of the discrete series for semisimple Lie groups, *J. Math. Kyoto Univ.*, 21 (1981), 417–500.

[15] W. Schmid, L^2 -cohomology and the discrete series, *Ann. Math.*, 102 (1976), 375–394.

[16] M. F. Atiyah and W. Schmid, A geometric construction of the discrete series for semisimple Lie groups, *Inventiones Math.*, 42 (1977), 1–62.

[17] N. Wallach, On the Enright-Varadarajan modules: A construction of the discrete series, *Ann. Sci. Ecole Norm. Sup.*, 9 (1976), 81–102.

[18] Harish-Chandra, Harmonic analysis on real reductive groups I–III, *J. Functional Anal.*, 19 (1975), 104–204; *Inventiones Math.*, 36 (1976), 1–55; *Ann. Math.*, 104 (1976), 117–201.

[19] J. A. Wolf, M. Cahen, and M. De Wilde (eds.), *Harmonic analysis and representations of semisimple Lie groups*, Reidel, 1980.

[20] G. Warner, *Harmonic analysis on semisimple Lie groups I, II*, Springer, 1972.

[21] H. Yoshizawa, A proof of the Plancherel theorem, *Proc. Japan Acad.*, 30 (1954), 276–281.

[22] Harish-Chandra, Spherical functions on a semisimple Lie group I, II, *Amer. J. Math.*, 80 (1958), 241–310, 553–613.

[23] S. Helgason, *Differential geometry and symmetric spaces*, Academic Press, 1962.

[24] J. Rosenberg, A quick proof of Harish-Chandra's Plancherel theorem for spherical functions on a semisimple Lie groups, *Proc. Amer. Math. Soc.*, 63 (1977), 143–149.

[25] M. Kashiwara, A. Kowata, K. Minemura, K. Okamoto, T. Oshima, and M. Tanaka, *Eigenfunctions of invariant differential*

437 Ref.

Unitary Representations

operators on a symmetric space, *Ann. Math.*, 107 (1978), 1–39.

[26] N. Ya Vilenkin, *Special functions and theory of group representations*, *Amer. Math. Soc. Transl. of Math. Monographs*, 22, 1968. (Original in Russian, 1965.)

[27] T. Oshima and J. Sekiguchi, Eigenspaces of invariant differential operators on an affine symmetric space, *Inventiones Math.*, 57 (1980), 1–81.

[28] I. M. Gel'fand and I. I. Pyatetskii-Shapiro, *Theory of representations and theory of automorphic functions*, *Amer. Math. Soc. Transl.*, (2) 26 (1963), 173–200. (Original in Russian, 1959.)

[29] A. Selberg, Harmonic analysis and discontinuous groups in weakly symmetric Riemannian spaces with applications to Dirichlet series, *J. Indian Math. Soc.*, 20 (1956), 47–87.

[30] R. P. Langlands, On the functional equations satisfied by Eisenstein series, *Lecture notes in math.* 544, Springer, 1976.

[31] *Automorphic forms, representations and L-functions*, *Amer. Math. Soc. Proc. Symposia in Pure Math.*, 33, 1979.

[32] V. Bargmann, Irreducible unitary representations of the Lorentz group, *Ann. Math.*, (2) 48 (1947), 568–640.

[33] L. Ehrenpreis and F. I. Mautner, Some properties of the Fourier transform on semisimple Lie groups I–III, *Ann. Math.*, (2) 61 (1955), 406–439; *Trans. Amer. Math. Soc.*, 84 (1957), 1–55; 90 (1959), 431–484.

[34] D. P. Zhelobenko, Harmonic analysis of functions on semisimple Lie groups I, II, *Amer. Math. Soc. Transl.*, (2) 54 (1966), 177–230. (Original in Russian, 1963, 1969.)

[35] R. Gangolli, On the Plancherel formula and the Paley-Wiener theorem for spherical functions on a semisimple Lie group, *Ann. Math.*, 93 (1971), 150–165.

[36] A. W. Knap and E. M. Stein, Intertwining operators for semisimple groups, *Ann. Math.*, 93 (1971), 489–578.

[37] B. Kostant, Quantization and unitary representations, *Lecture notes in math.* 170, Springer, 1970, 87–207.

[38] *Harmonic analysis on homogeneous spaces*, *Amer. Math. Soc. Proc. Symposia in Pure Math.*, 26 (1973).

[39] I. M. Gel'fand, M. I. Grayev, and I. I. Pyatetskii-Shapiro, *Representation theory and automorphic functions*, Saunders, 1969. (Original in Russian, 1966.)

[40] H. Jacquet and R. P. Langlands, *Automorphic forms on $GL(2)$* , *Lecture notes in math.* 114, Springer, 1970.

[41] D. A. Vogan, *Representations of real reductive Lie groups*, Birkhäuser, 1981.

438 (XI.5) Univalent and Multivalent Functions

A. General Remarks

A single-valued [†]analytic function $f(z)$ defined in a domain D of the complex plane is said to be **univalent** (or **simple** or **schlicht**) if it is injective, i.e., if $f(z_1) \neq f(z_2)$ for all distinct points z_1, z_2 in D . A multiple-valued function $f(z)$ is also said to be univalent if its distinct function elements always attain distinct values at their centers. The derivative of a univalent function is never zero. The limit function of a [†]uniformly convergent sequence of univalent functions is univalent unless it reduces to a constant. When $f(z)$ is single-valued, the univalent function $w = f(z)$ gives rise to a one-to-one [†]conformal mapping between D and its image $f(D)$.

B. Univalent Functions in the Unit Disk

A systematic theory of the family of functions [†]holomorphic and univalent in the unit disk originates from a **distortion theorem** obtained by P. Koebe (1909) in connection with the uniformization of analytic functions. In general, distortion theorems are theorems for determining bounds of functionals, such as $|f(z)|$, $|f'(z)|$, $\arg f'(z)$, within the family under consideration. In particular, distortion theorems concerning the bounds of the arguments of $f(z)$ and $f'(z)$ are also called **rotation theorems**. Though results were at first qualitative, they were made quantitative subsequently by L. Bieberbach (1916), G. Faber (1916), and others. Any univalent function $f(z)$ holomorphic in the unit disk and normalized by $f(0) = 0$ and $f'(0) = 1$ satisfies the **distortion inequalities**

$$\frac{|z|}{(1+|z|)^2} \leq |f(z)| \leq \frac{|z|}{(1-|z|)^2},$$

$$\frac{1-|z|}{(1+|z|)^3} \leq |f'(z)| \leq \frac{1+|z|}{(1-|z|)^3}.$$

Here the equality holds only if $f(z)$ is of the form $z/(1-\varepsilon z)^2$ ($|\varepsilon| = 1$). In deriving these inequalities, Bieberbach centered his attention on the family of [†]meromorphic functions $g(\zeta) = \zeta + \sum_{v=0}^{\infty} b_v \zeta^{-v}$ univalent in $|\zeta| > 1$. He established the **area theorem** $\sum_{v=1}^{\infty} v|b_v|^2 \leq 1$, which illustrates the fact that the area of the complementary set of the image domain is nonnegative. Bieberbach, R. Nevanlinna (1919–1920), and others constructed a sys-

tematic theory of univalent functions in the unit disk based on this theorem.

After the area theorem, the chief tools in the theory of univalent functions have been Löwner's method, the method of contour integration, the variational method, and the method of the extremal metric. In contrast to the theory of univalent functions based on Bieberbach's area theorem, K. Löwner (1923) introduced a new method. In view of a theorem on the domain kernel (C. Carathéodory, 1912), it suffices to consider an everywhere dense subfamily in order to estimate a continuous functional within the family of univalent functions holomorphic in the unit disk. Löwner used the subfamily of functions mapping the unit disk onto the so-called bounded slit domains. Namely, the range of a member of this subfamily consists of the unit disk slit along a Jordan arc that starts at a periphery point and does not pass through the origin. A mapping function of this nature is determined as the integral $f(z, t_0)$ of **Löwner's differential equation**

$$\frac{\partial f(z, t)}{\partial t} = -f(z, t) \frac{1 + \kappa(t)f(z, t)}{1 - \kappa(t)f(z, t)}, \quad 0 \leq t \leq t_0,$$

with the initial condition $f(z, 0) = z$, where $\kappa(t)$ is a continuous function with absolute value equal to 1. Any univalent function $f(z)$ holomorphic in the unit disk and satisfying $f(0) = 0$, $f'(0) = 1$ has an arbitrarily close approximation by functions of the form $e^{t_0} f(z, t_0)$. By means of this differential equation Löwner proved that $|a_3| \leq 3$ for any univalent function $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ ($|z| < 1$) and also derived a decisive estimate concerning a coefficient problem for the inverse function [2].

G. M. Golusin (1935) and I. E. Bazilevich (1936) first noticed that Löwner's method is also a powerful tool for deriving several distortion theorems. They showed that classical distortion theorems can be derived in more detailed form (Golusin, *Mat. Sb.*, 2 (1937), 685); in particular, Golusin (1938) obtained a precise estimate concerning the rotation theorem, i.e.,

$$|\arg f'(z)| \leq \begin{cases} 4 \arcsin |z|, & |z| < 1/\sqrt{2}, \\ \pi + \log(|z|^2/(1-|z|^2)), & 1/\sqrt{2} \leq |z| < 1. \end{cases}$$

Löwner's method was also investigated by A. C. Schaeffer and D. C. Spencer (1945) [8].

The method of contour integration was introduced by H. Grunsky. It starts with some 2-dimensional integral which can be shown to be positive. Transforming it into a boundary integral and using the [†]residue theorem, we obtain an appropriate inequality by means of this integral. By this method Grunsky established the following useful inequality (*Math.*

Z , 45 (1939)). For $g(\zeta) = \zeta + \sum_{v=0}^{\infty} b_v \zeta^{-v}$, which is univalent in $|\zeta| > 1$, let

$$\log \frac{g(z) - g(\zeta)}{z - \zeta} = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} c_{mn} z^{-m} \zeta^{-n} \quad (|z| > 1, |\zeta| > 1).$$

The coefficients c_{mn} are polynomials in the coefficients b_v of g . Then **Grunsky's inequality** is: For each integer N and for all complex numbers $\lambda_1, \dots, \lambda_N$,

$$\left| \sum_{m=1}^N \sum_{n=1}^N c_{mn} \lambda_m \lambda_n \right| \leq \sum_{n=1}^N \frac{1}{n} |\lambda_n|^2.$$

It is known that if this inequality holds for an arbitrary integer N and for all complex numbers $\lambda_1, \dots, \lambda_N$, then $g(\zeta)$ is univalent in $|\zeta| > 1$. There are several generalizations of Grunsky's inequality [13].

The variational method was first developed by M. Schiffer for application to the theory of univalent functions. He first used boundary variations (*Proc. London Math. Soc.*, 44 (1938)) and later interior variations (*Amer. J. Math.*, 65 (1943)). The problem of maximizing a given real-valued functional on a family of univalent functions is called an extremal problem, and a function for which the functional attains its maximum is called an extremal function. The **variational method** is used to uncover characteristic properties of an extremal function by comparing it with nearby functions. Typical results are the qualitative information that the extremal function maps the disk $|z| < 1$ onto the complement of a system of analytic arcs satisfying a differential equation and that the extremal function satisfies a differential equation. Following Schiffer, Schaeffer and Spencer [8] and Golusin (*Math. Sb.*, 19 (1946)) gave variants of the method of interior variations.

H. Grötzsch (1928–1934) treated the theory of univalent functions in a unified manner by the method of the ^{*}extremal metric. The idea of this method is to estimate the length of curves and the area of some region swept out by them together with an application of ^{*}Schwarz's inequality (\rightarrow 143 Extremal Length). After Grötzsch, the method of the extremal metric has been used by many authors. In particular, O. Teichmüller, in connection with this method, formulated the principle that the solution of a certain type of extremal problem is in general associated with a ^{*}quadratic differential, although he did not prove any general result realizing this principle in concrete form. J. A. Jenkins gave a concrete expression of the Teichmüller principle; namely, he established the general coefficient theorem and showed that this theorem contains as special cases a great many of the known results on univalent functions [11].

Univalence criteria have been given by various authors. In particular, Z. Nehari (*Bull. Amer. Math. Soc.*, 55 (1949)) proved that if $|\{f(z), z\}| \leq 2(1 - |z|^2)^{-2}$ in $|z| < 1$, then $f(z)$ is univalent in $|z| < 1$, and E. Hille (*Bull. Amer. Math. Soc.*, 55 (1949)) proved that 2 is the best possible constant in the above inequality. Here, $\{f(z), z\}$ denotes the ^{*}Schwarzian derivative of $f(z)$ with respect to z :

$$\{f(z), z\} = \left(\frac{f''(z)}{f'(z)} \right)' - \frac{1}{2} \left(\frac{f''(z)}{f'(z)} \right)^2.$$

C. Coefficient Problems

In several distortion theorems **Koebe's extremal function** $z/(1 - \varepsilon z)^2 = \sum_{n=1}^{\infty} n \varepsilon^{n-1} z^n$ ($|\varepsilon| = 1$) is extensively utilized. Concerning this, Bieberbach stated the following conjecture. If $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ is holomorphic and univalent in $|z| < 1$, then $|a_n| \leq n$ ($n = 2, 3, \dots$), with equality holding only for Koebe's extremal function $z/(1 - \varepsilon z)^2$ ($|\varepsilon| = 1$). This conjecture was solved affirmatively by L. de Branges in 1985 after enormous effort by many mathematicians, as described below.

Bieberbach (1916, [1]) proved $|a_2| \leq 2$ as a corollary to the area theorem. This result can be proved easily by most of the methods. In 1923 Löwner [2] proved $|a_3| \leq 3$, introducing his own method. Schaeffer and Spencer gave a proof of $|a_3| \leq 3$ by the variational method (*Duke Math. J.*, 10 (1943)). Furthermore, Jenkins used the method of the extremal metric to prove a coefficient inequality that implies $|a_3| \leq 3$ (*Analytic Functions*, Princeton Univ. Press, 1960). The problem of the fourth coefficient remained open until 1955, when P. R. Garabedian and Schiffer [3] proved $|a_4| \leq 4$ by the variational method. Their proof was extremely complicated. Subsequently, Z. Charzyński and Schiffer gave an alternative brief proof of $|a_4| \leq 4$ by using the Grunsky inequality (*Arch. Rational Mech. Anal.*, 5 (1960)). M. Ozawa (1969, [4]) and R. N. Pederson (1968, [5]) also used the Grunsky inequality to prove $|a_6| \leq 6$. In 1972, Pederson and Schiffer [6] proved $|a_5| \leq 5$. They applied the Garabedian-Schiffer inequality, a generalization of the Grunsky inequality which Garabedian and Schiffer had derived by the variational method.

On the other hand, W. K. Hayman [7] showed that for each fixed $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$,

$$\lim_{n \rightarrow \infty} \frac{|a_n|}{n} = \alpha \leq 1,$$

with the equality holding only for Koebe's extremal function $z/(1 - \varepsilon z)^2$ ($|\varepsilon| = 1$). Further, it was shown that Koebe's extremal function $z/(1 - z)^2$ gives a local maximum for the n th

coefficient in the sense that $\operatorname{Re}\{a_n\} \leq n$ whenever $|a_2 - 2| < \delta_n$ for some $\delta_n > 0$ (Garabedian, G. G. Ross, and Schiffer, *J. Math. Mech.*, 14 (1964); E. Bombieri, *Inventiones Math.*, 4 (1967); Garabedian and Schiffer, *Arch. Rational Math. Anal.*, 26 (1967)).

In the most general form, the coefficient problem is to determine the region occupied by the points (a_2, \dots, a_n) for all functions $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ univalent in $|z| < 1$. Schaeffer and Spencer [8] found explicitly the region for (a_2, a_3) .

For the coefficients of functions $g(\zeta) = \zeta + \sum_{v=0}^{\infty} b_v \zeta^{-v}$ univalent in $|\zeta| > 1$, the following results are known: $|b_1| \leq 1$ (Bieberbach [1]), $|b_2| \leq 2/3$ (Schiffer, *Bull. Soc. Math. France*, 66 (1938); Golusin, *Mat. Sb.*, 3 (1938)), $|b_3| \leq 1/2 + e^{-6}$ (Garabedian and Schiffer, *Ann. Math.*, (2) 61 (1955)).

D. Other Classes of Univalent Functions

We have discussed the general family of functions univalent in the unit disk. There are also several results on distortion theorems and coefficient problems for subfamilies determined by conditions such as that the images are bounded, *starlike with respect to the origin, or *convex. For instance, if $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ is holomorphic and univalent in $|z| < 1$ and its image is starlike with respect to the origin, then $|a_n| \leq n$ ($n = 2, 3, \dots$). If the image of $f(z)$ is convex, then $f(z)$ satisfies $|a_n| \leq 1$ ($n = 2, 3, \dots$) and the distortion inequalities

$$\frac{|z|}{1+|z|} \leq |f(z)| \leq \frac{|z|}{1-|z|},$$

$$\frac{1}{(1+|z|)^2} \leq |f'(z)| \leq \frac{1}{(1-|z|)^2}.$$

Here the equality sign appears at z_0 ($0 < |z_0| < 1$) if and only if $f(z)$ is of the form $z/(1 + \varepsilon z)$ with $\varepsilon = \pm |z_0|/z_0$.

On the other hand, problems on conformal mappings of multiply connected domains involve essential difficulties in comparison with the simply connected case. Although Bieberbach's method is unsuitable for multiply connected domains, Löwner's method, the method of contour integration, the variational method, and the method of the extremal metric remain useful (\rightarrow 77 Conformal Mappings).

E. Multivalent Functions

Multivalent functions are a natural generalization of univalent functions. There are several results that generalize classical results on univalent functions.

A function $f(z)$ that attains every value at most p times and some values exactly p times in a domain D is said to be **p -valent** in D and is called a **multivalent function** provided that $p > 1$. In order for $f(z) = \sum_{n=0}^{\infty} a_n z^n$, holomorphic in $|z| \leq 1$, to be p -valent there, it is sufficient that it satisfies

$$p-1 < \operatorname{Re}(zf'(z)/f(z)) < p+1$$

on $|z| = 1$. Hence it suffices to have

$$|a_p| - \sum_{n=2}^p n|a_{p+1-n}| > \sum_{n=2}^{\infty} n|a_{p-1+n}|.$$

If $f(z) = (1 + a_1 z + a_2 z^2 + \dots)/z^p$ is holomorphic and p -valent in $0 < |z| \leq 1$, then

$$\frac{d}{dr} \int_0^{2\pi} F(|f(re^{i\theta})|) d\theta \leq 0$$

for any increasing function $F(\rho)$ in $\rho \geq 0$. In particular, if $F(\rho) = \rho^2$, this becomes an area theorem from which follow coefficient estimates, etc., for p -valent functions.

Various subfamilies and generalized families of multivalent functions have been considered. Let $f(z)$ be p -valent in D , and $c_0 + c_1 z + \dots + c_{p-1} z^{p-1} + c_p f(z)$ be at most p -valent in D for any constants c_0, c_1, \dots, c_p . Then $f(z)$ is said to be **absolutely p -valent** in D . If a function $f(z)$ holomorphic in a convex domain K satisfies $\operatorname{Re}(e^{i\alpha} f^{(p)}(z)) > 0$ for a real constant α , then $f(z)$ is absolutely p -valent in K . If $f(z)$ is absolutely p -valent in D , then

$$\left(\sum_{k=0}^{p-1} b_k z^k + b_p f(z) \right) \left/ \left(\sum_{k=0}^{p-1} c_k z^k + c_p f(z) \right) \right.$$

is at most p -valent in D for any constants b_k and c_k .

If $f(z)$ is p -valent in the common part of a domain D and the disk centered at each point of D with a fixed radius ρ , then $f(z)$ is said to be **locally p -valent** in D , and ρ is called its **modulus**. A necessary and sufficient condition for $f(z)$, holomorphic in D , to be at most locally p -valent is that $f'(z), \dots, f^{(p)}(z)$ not vanish simultaneously. In order for $f(z)$, holomorphic in D , to be **locally absolutely p -valent** it is necessary and sufficient that $f^{(p)}(z) \neq 0$. Let the number of $\operatorname{Re} e^{i\varphi}$ -points of $f(z)$ in D be $n(D, \operatorname{Re} e^{i\varphi})$. If $f(z)$ satisfies

$$\frac{1}{2\pi} \int_0^{2\pi} n(D, \operatorname{Re} e^{i\varphi}) d\varphi \leq p,$$

for any $R > 0$, it is said to be **circumferentially mean p -valent** in D . If $f(z)$ satisfies

$$\int_0^R \int_0^{2\pi} n(D, \operatorname{Re} e^{i\varphi}) R dR d\varphi \leq p\pi R^2,$$

it is said to be **areally mean p -valent** in D . If $f(z)^q$ with $q > 1$ is areally mean p -valent in D , then $f(z)$ is areally mean p/q -valent in D . For

$f(z) = (1 + a_1 z + a_2 z^2 + \dots)/z^\lambda$ holomorphic and areally mean λ -valent in $0 < |z| \leq 1$, the following area theorem holds:

$$\sum_{n=1}^{\infty} (n-1) |a_n|^2 \leq \lambda.$$

Let E be a set containing at least three points. If $f(z)$ in D attains every value of E at most p times and a certain value of E exactly p times (it may attain values outside E more than p times), then $f(z)$ is said to be **quasi- p -valent** in D . If $w = f(z)$ is p -valent in D and $g(w)$ is quasi- q -valent in $f(D)$, then $g(f(z))$ is at most quasi- pq -valent in D .

The first success in obtaining sharp inequalities for multivalent functions was attained by Hayman. In his work, an essential role was played by the method of 'symmetrization'. For instance, he obtained the following result. If $f(z) = z^p + a_{p+1} z^{p+1} + \dots$ is holomorphic and circumferentially mean p -valent in $|z| < 1$, then $|a_{p+1}| \leq 2p$, and for $|z| = r$, $0 < r < 1$,

$$\frac{r^p}{(1+r)^{2p}} \leq |f(z)| \leq \frac{r^p}{(1-r)^{2p}},$$

$$|f'(z)| \leq \frac{p(1+r)}{r(1-r)} |f(z)| \leq \frac{pr^{p-1}(1+r)}{(1-r)^{2p+1}}.$$

References

- [1] L. Bieberbach, Über die Koeffizienten derjenigen Potenzreihen, welche eine schlichte Abbildung des Einheitskreises vermitteln, S.-B. Preuss., Akad. Wiss., (1916), 940–955.
- [2] K. Löwner, Untersuchungen über schlichte konforme Abbildungen des Einheitskreises I, Math. Ann., 89 (1923), 103–121.
- [3] P. R. Garabedian and M. Schiffer, A proof of the Bieberbach conjecture for the fourth coefficient, J. Rational Mech. Anal., 4 (1955), 427–465.
- [4] M. Ozawa, On the Bieberbach conjecture for the sixth coefficient, Kôdai Math. Sem. Rep., 21 (1969), 97–128.
- [5] R. N. Pederson, A proof of the Bieberbach conjecture for the sixth coefficient, Arch. Rational Mech. Anal., 31 (1968–69), 331–351.
- [6] R. N. Pederson and M. Schiffer, A proof of the Bieberbach conjecture for the fifth coefficient, Arch. Rational Mech. Anal., 45 (1972), 161–193.
- [7] W. K. Hayman, The asymptotic behavior of p -valent functions, Proc. London Math. Soc., (3) 5 (1955), 257–284.
- [8] A. C. Schaeffer and D. C. Spencer, Coefficient regions for schlicht functions, Amer. Math. Soc. Colloq. Publ., 35 (1950).
- [9] G. M. Golusin, Geometric theory of functions of a complex variable, Amer. Math. Soc.

Transl. of Math. Monographs, 26 (1969).

(Original in Russian, 1952.)

[10] W. K. Hayman, Multivalent functions, Cambridge Univ. Press, 1958.

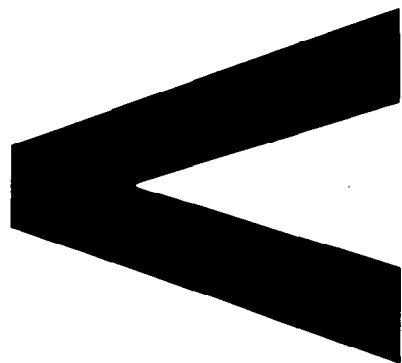
[11] J. A. Jenkins, Univalent functions and conformal mapping, Springer, second edition, 1965.

[12] L. V. Ahlfors, Conformal invariants: Topics in geometric function theory, McGraw-Hill, 1973.

[13] C. Pommerenke, Univalent functions (with a chapter on quadratic differentials by G. Jensen), Vandenhoeck & Ruprecht, 1975.

[14] G. Schober, Univalent functions—Selected topics, Lecture notes in math. 478, Springer, 1975.

[15] P. L. Duren, Coefficients of univalent functions, Bull. Amer. Math. Soc., 83 (1977), 891–911.



439 (III.19) Valuations

A. Introduction

There are two related kinds of valuations, additive (\rightarrow Section B) and multiplicative (\rightarrow Section C). The notion of valuations, originally defined on (commutative) \dagger fields, has been extended to more general cases (\rightarrow Section K); however, we first consider the case of fields.

B. Additive Valuations

In this article, we mean by an **ordered additive group** a **totally ordered additive group**, namely, a commutative group whose operation is addition, which is a \dagger totally ordered set satisfying the condition that $a \geq b$ and $c \geq d$ imply $a + c \geq b + d$ and $-a \leq -b$. Suppose that we are given a field K , an ordered additive group G , and an element ∞ defined to be greater than any element of G . Then a mapping $v: K \rightarrow G \cup \{\infty\}$ is called an **additive valuation** (or simply a **valuation**) of the field K if v satisfies the following three conditions: (i) $v(a) = \infty$ if and only if $a = 0$; (ii) $v(ab) = v(a) + v(b)$ for all $a, b \neq 0$; and (iii) $v(a + b) \geq \min\{v(a), v(b)\}$.

The set $\{v(a) | a \in K - \{0\}\}$ is a submodule of G and is called the **value group** of v , while the set $R_v = \{a \in K | v(a) \geq 0\}$ is a subring of K and is called the **valuation ring** of v . The ring R_v has only one \dagger maximal ideal $\{a | v(a) > 0\}$, called the **valuation ideal** of v (or of R_v), and the \dagger residue class field of R_v modulo the maximal ideal is called the **residue class field** of the valuation v . We have $v(a) \leq v(b)$ if and only if $aR_v \supset bR_v$. Two valuations v and v' of the field K are said to be **equivalent** when $v(a) \leq v(b)$ if and only if $v'(a) \leq v'(b)$; hence v and v' are equivalent if and only if $R_v = R_{v'}$. The **rank** of v is defined to be the \dagger Krull dimension of the valuation ring R_v , and the **rational rank** of v to be the maximum (or supremum) of the numbers of linearly independent elements in the value group. An **extension** (or **prolongation**) of v in a field K' containing K is a valuation v' of K' whose restriction on K is v ; such an extension exists for any given v and K' . Sometimes a valuation of rank 1 is called a **special valuation** (or **exponential valuation**), and a valuation of a general rank is called a **generalized valuation**. On the other hand, if k is a subfield of K such that $v(a) = 0$ for every nonzero element a of k , then v is called a **valuation over the subfield** k .

C. Multiplicative Valuations

A **multiplicative valuation** (or **valuation**) of a field K is a mapping $w: K \rightarrow \Gamma \cup \{0\}$ that satis-

fies the following three conditions, where Γ is the multiplicative group of positive real numbers: (i) $w(a) = 0$ if and only if $a = 0$; (ii) $w(ab) = w(a)w(b)$; and (iii) $w(a + b) \leq C(w(a) + w(b))$, where C is a constant (independent of the choice of a and b , but dependent on the choice of w).

The **value group** of w is defined to be $\{w(a) | a \in K - \{0\}\}$. Extensions of a valuation and equivalence of valuations are defined as in the case of additive valuations. Thus w' is equivalent to w if and only if there is a positive r such that for all $a \in K$, $w(a) = w'(a)^r$. In each equivalence class of valuations of a field, there exists a valuation for which the constant C in condition (iii) can be taken to be 1. A valuation w is said to be a **valuation over a subfield** k if $w(a) = 1$ for any nonzero element a of k .

We call w an **Archimedean valuation** if for any elements $a, b \in K$, $a \neq 0$, there exists a natural number n such that $w(na) > w(b)$; otherwise, w is said to be a **non-Archimedean valuation**. If w is an Archimedean valuation of a field K , then there is an injection σ from K into the complex number field \mathbb{C} such that w is equivalent to the valuation w' defined by $w'(a) = |\sigma(a)|$. If w is a non-Archimedean valuation of a field K , then $w(a + b) \leq \max\{w(a), w(b)\}$. Hence in this case we get an additive valuation v of K when we define $v(a) = -\log w(a)$ ($a \in K$), and either v is of rank 1 or $v(K) = \{1, 0\}$ (in the latter case, v is called **trivial**). Conversely, every additive valuation of rank 1 of K is equivalent to an additive valuation obtained in this way from a non-Archimedean valuation. (This is why an additive valuation of rank 1 is called an exponential valuation.) Therefore a non-Archimedean valuation determines a valuation ring and valuation ideal in a natural manner. Thus we can identify a non-Archimedean valuation with an additive valuation of rank 1.

D. Topology Defined by a Valuation

Let w be a multiplicative valuation of a field K . When the \dagger distance between two elements a, b of K is defined by $w(a - b)$, K becomes a \dagger topological field. (Although this distance may not make K into a \dagger metric space, there exists a valuation w' equivalent to the valuation w such that K becomes a metric space with respect to the distance $w'(a - b)$ between a and b ($a, b \in K$)). If K is \dagger complete under the topology, then we say that K is **complete** with respect to w and w is **complete** on K . On the other hand, suppose that w' is an extension of w in a field K' containing K . If w' is complete and K is \dagger dense in K' under the topology defined by w' , then we say that the valuation w' is a **completion** of w and that the field K' is a

completion of K with respect to w . For any w , a completion exists and is unique up to isomorphism. When w is a non-Archimedean valuation, the valuation ring of the completion of w is called the **completion** of the valuation ring of w .

When v is an additive valuation of a field K , we can introduce a topology on K by taking the set of all nonzero ideals of the valuation ring R_v of v as a base for the neighborhood system of zero. Important cases are given by valuations of rank 1, which are the same as those given by non-Archimedean valuations.

If w is a complete non-Archimedean valuation of a field K , then the valuation ring R_w of w is a Hensel ring, which implies that if K' is a finite algebraic extension of K such that $[K':K] = n$, then w is uniquely extendable to a valuation w' of K' and $w'(a)^n = w(N(a))$, where N is the norm $N_{K'/K}$.

E. Discrete Valuations

For a non-Archimedean valuation (or an additive valuation of rank 1) w , if the valuation ideal of w is a nonzero principal ideal generated by an element p , then we say that p is a **prime element** for w , w is a **discrete valuation**, and the valuation ring for w is a **discrete valuation ring**. The condition on the valuation ideal of w holds if and only if the value group of w is a discrete subgroup of the (multiplicative) group Γ of positive real numbers: In the terminology of additive valuations, a valuation w is discrete if and only if it is equivalent to a valuation w' whose value group is the additive group of integers. Such a valuation w' is called a **normalized valuation** (or **normal valuation**). However, we usually mean normalization of a discrete non-Archimedean valuation as in Section H. Sometimes an additive valuation whose value group is isomorphic to the direct sum of a finite number of copies of \mathbb{Z} (the additive group of integers) with a natural lexicographic order is called a discrete valuation. Concerning a complete discrete valuation w , it is known that if the valuation ring of w contains a field, then it is isomorphic to the ring of formal power series in one variable over a field (for other cases \rightarrow 449 Witt Vectors A).

F. Examples

(1) **Trivial valuations** of a field K are the additive valuation v of K such that $v(a) = 0$ for all $a \in K - \{0\}$ and the multiplicative valuation w of K such that $w(a) = 1$ for all $a \in K - \{0\}$.

(2) If K is isomorphic to a subfield of the complex number field, then we get an Archi-

medean valuation using the absolute value, and as stated in Section C, every Archimedean valuation of K is equivalent to a valuation obtained in this way.

(3) Let \mathfrak{p} be a prime ideal of a Dedekind domain R , $\pi \in \mathfrak{p}$ be such that $\pi \notin \mathfrak{p}^2$, and K be the field of quotients of R . Then each nonzero element α of K can be expressed in the form $\pi^r ab^{-1}$ ($r \in \mathbb{Z}$; $a, b \in R$; $a, b \notin \mathfrak{p}$), where r , the **degree** of α with respect to \mathfrak{p} , is uniquely determined by α . Hence, letting c be a constant greater than 1, we obtain a non-Archimedean valuation w defined by $w(\alpha) = c^{-r}$. This valuation w is called a **\mathfrak{p} -adic valuation**. We also get an additive valuation v defined by $v(\alpha) = r$, called a **\mathfrak{p} -adic exponential valuation**. The completion $K_{\mathfrak{p}}$ of K with respect to v is called the **\mathfrak{p} -adic extension** of K . If K is a finite algebraic number field, the $K_{\mathfrak{p}}$ is called a **\mathfrak{p} -adic number field**. If \mathfrak{p} is generated by an element p , then “ \mathfrak{p} -adic” is replaced by “ p -adic.” For instance, given a rational prime number p , we have a **p -adic valuation** of the rational number field \mathbb{Q} , and we obtain the p -adic extension \mathbb{Q}_p of \mathbb{Q} , which is called the **p -adic number field**. Every nonzero element α of \mathbb{Q}_p can be written as a uniquely determined expansion $\sum_{n=-r}^{\infty} a_n p^n$ ($a_r \neq 0, r \in \mathbb{Z}, a_n \in \mathbb{Z}, 0 \leq a_n < p$). Then we obtain a valuation v of \mathbb{Q}_p defined by $v(\alpha) = r$. This valuation v is a discrete additive valuation, and \mathbb{Q}_p is complete with respect to v . The valuation ring of v is usually denoted by \mathbb{Z}_p , which is called the **ring of p -adic integers**. Each element of \mathbb{Q}_p (\mathbb{Z}_p) is called a **p -adic number** (**p -adic integer**).

(4) Consider the field of power series $k((t))$ in one variable t over a field k . For $0 \neq \alpha \in k((t))$, we define $v(\alpha) = r$ if $\alpha = \sum_{n=-r}^{\infty} a_n t^n$ ($a_n \in k, a_r \neq 0$). Then v is a discrete valuation of $k((t))$, and $k((t))$ is complete with respect to this valuation.

(5) Let v be an additive valuation of a field K with the valuation ring R_v and the valuation ideal \mathfrak{m}_v . Let v' be an additive valuation of the field R_v/\mathfrak{m}_v with the valuation ring $R_{v'}$. Then $R'' = \{a \in R_v \mid (a \bmod \mathfrak{m}_v) \in R_{v'}\}$ is a valuation ring of K . A valuation v'' whose valuation ring coincides with R'' is called the **composite** of v and v' .

G. The Approximation Theorem and the Independence Theorem

The **approximation theorem** states: Let w_1, \dots, w_n be mutually nonequivalent and nontrivial multiplicative valuations of a field K . Then for any given n elements a_1, \dots, a_n of K and a positive number ε , there exists an element a of K such that $w_i(a - a_i) < \varepsilon$ ($i = 1, 2, \dots, n$).

From this follows the **independence theorem**: Let e_1, \dots, e_n be real numbers, and let w_i and K be as in the approximation theorem. If $\prod_i w_i(a)^{e_i} = 1$ for all $a \in K - \{0\}$, then $e_1 = \dots = e_n = 0$.

Similar theorems hold for additive valuations. The following independence theorem is basic: Let v_1, \dots, v_n be additive valuations of a field K , R_1, \dots, R_n their valuation rings, and $\mathfrak{m}_1, \dots, \mathfrak{m}_n$ their maximal ideals. Let $D = \bigcap_i R_i$, $\mathfrak{p}_i = \mathfrak{m}_i \cap D$, and consider the rings of quotients $D_{\mathfrak{p}_i}$. Then $D_{\mathfrak{p}_i} = R_i$. If $R_i \not\subseteq R_j$ (for $i \neq j$), then D has exactly n maximal ideals $\mathfrak{p}_1, \dots, \mathfrak{p}_n$.

H. Prime Divisors

Let K be an \dagger algebraic number field (algebraic function field of one variable over a field k). An equivalence class of nontrivial multiplicative valuations (over k) is called a **prime divisor** (**prime spot**) of K .

If K is an algebraic number field of degree n , there are exactly n mutually distinct injections $\sigma_1, \dots, \sigma_n$ of K into the complex number field \mathbb{C} . We may assume that $\sigma_i(K)$ is contained in the real number field if and only if $i \leq r_1$ and $\sigma_{n-i+1}(a)$ and $\sigma_{r_1+i}(a)$ are conjugate complex numbers ($n - r_1 \geq i > 0$, $a \in K$). For $i \leq r_1$, let $v_i(a) = |\sigma_i(a)|$, and for $1 \leq i \leq (n - r_1)/2$, let $v_{r_1+i}(a) = |\sigma_{r_1+i}(a)|^2$. Then $v_1, \dots, v_{r_1+r_2}$ ($r_2 = (n - r_1)/2$) is a maximal set of mutually non-equivalent Archimedean valuations of K . Equivalence classes of v_1, \dots, v_{r_1} are called **real (infinite) prime divisors**, and those of $v_{r_1+1}, \dots, v_{r_1+r_2}$ are called **imaginary (infinite) prime divisors**; all of them are called **infinite prime divisors**. An equivalence class of non-Archimedean valuations of K is called a **finite prime divisor**.

An Archimedean valuation of K is said to be **normal** if it is one of the valuations v_i . If v is non-Archimedean, then v is a p -adic valuation, where \mathfrak{p} is a prime ideal of the principal order \mathfrak{o} of K (\rightarrow Section F, example (3)). Hence if a is an element of K , there exists a constant c ($c > 1$) such that $v(a) = c^{-r}$, where r is the degree of a with respect to \mathfrak{p} . In particular, if c is the norm of \mathfrak{p} (i.e., c is the cardinality of the set $\mathfrak{o}/\mathfrak{p}$), then the valuation v is called **normal**. Any finite prime divisor is represented by a normal valuation. Then we have the **product formula** $\prod_w w(a) = 1$ for all $a \in K - \{0\}$, where w ranges over all normal valuations of K .

For a function field, a **normal valuation** is defined similarly, using e^f instead of the norm of \mathfrak{p} , where e is a fixed real number greater than 1 and f is the degree of the residue class field of the valuation over k . In this case we also have the product formula.

I. Extending Valuations to an Algebraic Extension of Finite Degree

Assume that a field K' is a finite algebraic extension of a field K . Let v be an additive valuation of K and v' be an extension of v to K' . We denote the valuation rings, valuation ideals, and value groups of v and v' by $R_v, R_{v'}$, $\mathfrak{m}_v, \mathfrak{m}_{v'}$, and G, G' , respectively. Then the degree of the extension $f_{v'} = [R_{v'}/\mathfrak{m}_{v'} : R_v/\mathfrak{m}_v]$ is called the **degree** of v' over v . The group index $e_{v'} = [G' : G]$ is called the **ramification index** of v' over v . If v' ranges over all extensions of v in K' , then the sum $\sum f_{v'} e_{v'}$ is not greater than $[K' : K]$ and the equality holds when v is a discrete valuation and either K' is \dagger separable over K or v is complete.

J. Places

Let k, K , and L be fields, and suppose that $k \subset K$. Let f be a mapping of K onto $L \cup \{\infty\}$ such that $f(ab) = f(a)f(b)$ and $f(a+b) = f(a) + f(b)$, whenever the right member is meaningful, and such that the restriction of f to k is an injection. Here ∞ is an element adjoined to L and satisfying $\infty + a = a + \infty = \infty$, $\infty a = a\infty = \infty$ (for any nonzero element a of K), $1/\infty = 0$, and $1/0 = \infty$. Then f is called a **place** of K over k . In this case $R = \{x \in K \mid f(x) \neq \infty\}$ is a valuation ring of K containing k . Let \mathfrak{m} be the maximal ideal of R . Then f can be identified with the mapping $g: K \rightarrow R/\mathfrak{m} \cup \{\infty\}$ defined as follows: If $a \in R$, then $g(a) = (a \bmod \mathfrak{m})$; otherwise, $g(a) = \infty$. Places of K over k can be classified in a natural way, and there exists a one-to-one correspondence between the set of classes of places of K over k and the set of equivalence classes of additive valuations over k . When K is an \dagger algebraic function field, we usually consider the case where k is the \dagger ground field. Then if $a_1, \dots, a_n \in R$, $(a_1, \dots, a_n) \rightarrow (g(a_1), \dots, g(a_n))$ gives a \dagger specialization of points over k . Conversely, if $a_i, b_j \in K$ are such that $(a_1, \dots, a_n) \rightarrow (b_1, \dots, b_n)$ is a specialization over k , then there is a place f of K over k such that (b_1, \dots, b_n) is isomorphic to $(f(a_1), \dots, f(a_n))$ (usually there are infinitely many such f 's).

K. Pseudovaluations

A **pseudovaluation** φ of a ring A (not necessarily commutative) is a mapping of A into the set of nonnegative real numbers satisfying the following four conditions: (i) $\varphi(a) = 0$ if and only if $a = 0$; (ii) $\varphi(ab) \leq \varphi(a)\varphi(b)$; (iii) $\varphi(a+b) \leq \varphi(a) + \varphi(b)$; and (iv) $\varphi(-a) = \varphi(a)$. These conditions are weaker than those for multiplicative valuations, but with them a topology

can be introduced into A as in Section D, with respect to which A becomes a topological ring.

L. History

The theory of valuations was originated by K. Hensel when he introduced p -adic numbers and applied them to number theory [1]. J. Kürschák (*J. Reine Angew. Math.*, 142 (1913)) first treated the theory of multiplicative valuations axiomatically; it was then developed remarkably by A. Ostrowski (*Acta Math.*, 41 (1918)). However, in their theory condition (iii) (\rightarrow Section C) was given only in the case $C = 1$, thus excluding the normal valuation of an imaginary prime divisor in an algebraic number field. A valuation with general C was introduced by E. Artin [3]. The theory of additive valuations was originated by W. Krull (*J. Reine Angew. Math.*, 167 (1932)), although the concept of exponential valuations existed before. The theory of valuations is used to simplify \dagger class field theory and the theory of algebraic function fields in one variable. For these purposes, the notion of multiplicative valuations is sufficient (\rightarrow 9 Algebraic Curves; 59 Class Field Theory). The idea is also used in the theory of normal rings and in algebraic geometry, for both of which the concept of additive valuations is also necessary. Pseudovaluations were used by M. Deuring (*Erg. Math.*, Springer, 1935) in the arithmetic of algebras.

References

- [1] K. Hensel, *Theorie der algebraischen Zahlen*, Teubner, 1908.
 - [2] O. F. G. Schilling, *The theory of valuations*, Amer. Math. Soc. Math. Surveys, 1950.
 - [3] E. Artin, *Algebraic numbers and algebraic functions*, Gordon & Breach, 1967.
 - [4] O. Zariski and P. Samuel, *Commutative algebra II*, Van Nostrand, 1960.
- Also \rightarrow references to 67 Commutative Rings.

440 (X.35) Variational Inequalities

A. Introduction

Variational inequalities arise when we consider extremal problems of functionals under **unilateral constraints**. Some problems in physics and engineering are studied by formulating them as elliptic, parabolic, and hyperbolic variational inequalities [1–8].

B. Stationary Variational Inequality

Let D be a bounded domain in m -dimensional Euclidean space and $f \in L_2(D)$ be a given real-valued function. Consider the variational problem of minimizing the following functional J with the argument function v :

$$J[v] = \int_D |\text{grad } v|^2 dx - 2 \int_D f v dx.$$

Here, we suppose the set of admissible functions to be the closed convex subset

$$K = \{v \in H_0^1(D) \mid v \leq 0 \text{ a.e. in } D\}$$

of the Hilbert space $H_0^1(D)$ (\rightarrow 168 Function Spaces). It can be shown by choosing a minimizing sequence that there exists a minimum value of J which is realized by a unique $u \in K$. Since the stationary function u belongs to $H_0^1(D)$, it can be shown that the boundary condition $u|_{\partial D} = 0$ is satisfied in the sense that the \dagger trace $\gamma_0 u \in H^{1/2}(\partial D)$ (\rightarrow 224 Interpolation of Operators) of u on ∂D vanishes a.e. on ∂D . In view of the fact that $J[u] \leq J[v]$ is valid for any $v \in K$, it can be verified that the **stationary variational inequality**

$$\left. \begin{aligned} -\Delta u - f &\leq 0 \\ u &\leq 0 \\ (-\Delta u - f) \cdot u &= 0 \end{aligned} \right\} \quad (1)$$

is satisfied in D in the sense of differentiation of distributions (\rightarrow 125 Distributions and Hyperfunctions). The problem (1) is a **Dirichlet problem with obstacle**. Moreover, we can prove the regularity of $u \in H^2(D)$ under an assumption of suitable smoothness for ∂D by establishing the boundedness of the solutions u_ε in $H^2(D)$ of the **penalized problems** associated with (1):

$$-\Delta u_\varepsilon + \frac{1}{\varepsilon} u_\varepsilon^+ = f \quad (\varepsilon > 0),$$

$$u_\varepsilon|_{\partial D} = 0.$$

Here we note that the u_ε are the stationary functions of the ordinary variational problems of minimization in $H_0^1(D)$ of the functionals

$$J_\varepsilon[v] = \int_D |\text{grad } v|^2 dx - 2 \int_D f v dx + \frac{1}{\varepsilon} \int_D |v^+|^2 dx$$

with the **penalty term** (the third term of the right-hand side of the equality above). We have thus found that the stationary variational inequality (1) is the Euler equation of a conditional problem of variation (\rightarrow 46 Calculus of Variations).

C. Variational Inequality of Evolution

Let $\psi \in H^1(D)$ be a given function on D such that $\psi|_{\partial D} \geq 0$ and $\Delta \psi \in L_2(D)$. The **variational**

inequality of evolution

$$\frac{\partial u}{\partial t} - \Delta u \leq 0,$$

$$u \leq \psi,$$

$$\left(\frac{\partial u}{\partial t} - \Delta u \right) \cdot (u - \psi) = 0 \quad (t > 0, x \in D),$$

$$u(0, x) = a(x) \quad (x \in D),$$

$$u(t, x)|_{\partial D} = 0 \quad (t > 0)$$

can be formulated as an abstract Cauchy problem (\rightarrow 286 Nonlinear Functional Analysis X)

$$\frac{du}{dt} \in Au \quad (t > 0),$$

$$u(+0) = a$$

in a Hilbert space with a multivalued operator $A = -\partial\varphi$, where $\partial\varphi$ is the subdifferential of the following lower semicontinuous proper convex function on the Hilbert space $L_2(D)$:

$$\varphi(v) = \begin{cases} \frac{1}{2} \int_D |\text{grad } v|^2 dx & \text{if } v \in H_0^1(D) \text{ and } v \leq \psi, \\ +\infty & \text{otherwise.} \end{cases}$$

Thus the solution u is given by the vector-valued function

$$u(t) = e^{tA} a.$$

Here e^{tA} is the \dagger nonlinear semigroup generated by A (\rightarrow 88 Convex Analysis, 378 Semigroups of Operators and Evolution Equations).

D. Optimal Stopping Time Problem and Variational Inequalities

Let $\{X_t\}_{t \geq 0}$ be an m -dimensional Brownian motion (\rightarrow 45 Brownian Motion) and consider the problem of finding a \dagger stopping time σ that minimizes

$$J_x[\sigma] = E_x \left(\int_0^\sigma f(X_t) dt \right) \quad (x \in \mathbf{R}^m)$$

under the restriction that $0 \leq \sigma \leq \sigma_{\partial D}$, where $\sigma_{\partial D}$ is the \dagger hitting time for the boundary ∂D . Let us define

$$u(x) = \min_{\sigma} J_x[\sigma].$$

Then the \dagger principle of optimality in dynamic programming gives the stationary variational inequality (1) with Δ replaced by $\frac{1}{2}\Delta$, and we can show by the \dagger Dynkin formula that an optimal stopping time $\hat{\sigma}$ is the hitting time for the set $\{x \in \bar{\Omega} | u(x) = 0\}$ (\rightarrow 127 Dynamic Programming). We can systematically discuss problems in mathematical programming and

operations research by introducing quasivariational inequalities, which are slight generalizations of variational inequalities (\rightarrow 227 Inventory Control, 408 Stochastic Programming). The above-mentioned facts are applicable to general \dagger diffusion processes described by \dagger stochastic differential equations (\rightarrow 115 Diffusion Processes, 406 Stochastic Differential Equations). We have thus found the relation

free boundary problem \leftrightarrow variational inequality
 $\swarrow \quad \searrow$
 optimal stopping time problem

(\rightarrow 405 Stochastic Control and Stochastic Filtering).

E. Numerical Solution of Variational Inequalities

Since the solution u of the variational inequality (1) is the stationary function for the variational problem, we can apply to the evaluation of the function u numerical methods based on the direct method of the calculus of variations (\rightarrow 300 Numerical Methods). The \dagger finite element method, which can be regarded as a type of Ritz-Galerkin method, is extensively employed to calculate numerical solutions. In view of the unilateral constraint $u \leq 0$, iteration methods, such as the Gauss-Seidel iteration method, are used with modifications. An algorithm of relaxation with projection is proposed in [3] (\rightarrow 304 Numerical Solution of Partial Differential Equations).

References

- [1] H. Brézis, Problèmes unilatéraux, J. Math. Pures Appl., 51 (1972), 1–168.
- [2] V. Barbu, Nonlinear semigroups and differential equations in Banach spaces, Noordhoff, 1976.
- [3] J.-L. Lions, Sur quelques questions d'analyse, de mécanique et de contrôle optimal, Les Presses de l'Université de Montréal, 1976.
- [4] J.-L. Lions and G. Stampacchia, Variational inequalities, Comm. Pure Appl. Math., 20 (1967), 493–519.
- [5] C. Baiocchi, Su un problema di frontiera libera connesso a questioni di idraulica, Ann. Mat. Pura Appl. 92 (1972), 107–127.
- [6] G. Duvaut and J.-L. Lions, Les inéquations en mécanique et en physique, Dunod, 1972; English translation, Springer, 1975.
- [7] A. Friedman and D. Kinderlehrer, A one phase Stefan problem, Indiana Univ. Math. J., 24 (1975), 1005–1035.
- [8] R. Glowinski, J.-L. Lions, and R. Trémoières, Analyse numérique des inéquations variationnelles I, II, Dunod, 1976; English translation, North-Holland, 1981.

441 (XX.3) Variational Principles

A. General Remarks

Among the principles that appear in physics are those expressed not in terms of differential forms but in terms of variational forms. These principles, describing the conditions under which certain quantities attain extremal values, are generally called **variational principles**. Besides Hamilton's principle in classical mechanics (\rightarrow Section B) and Fermat's principle in geometric optics (\rightarrow Section C), examples are found in \dagger electromagnetism, \dagger relativity theory, \dagger quantum mechanics, \dagger field theory, etc. Independence of the choice of coordinate system is an important characteristic of variational principles. Originally these principles had theological and metaphysical connotations, but a variational principle is now regarded simply as a postulate that precedes a theory and furnishes its foundation. Thus a variational principle is considered to be the supreme form of a law of physics.

B. Mechanics

In 1744 P. L. Maupertuis published an almost theological thesis, dealing with the **principle of least action**. This was the beginning of the search for a single, universal principle of mechanics, contributions to which were made successively by L. Euler, C. F. Gauss, W. R. Hamilton, H. R. Hertz, and others.

Let $\{q_r\}$ be the \dagger generalized coordinates of a system of particles, and consider the integral of a function $L(q_r, \dot{q}_r, t)$ taken from time t_0 to t_1 . If we compare the values of the integral taken along any arbitrary path starting from a fixed point P_0 in the coordinate space at time t_0 and arriving at another fixed point P_1 at time t_1 , then the actual motion $q_r(t)$ (which obeys the laws of mechanics) is given by the condition that the integral is an \dagger extremum (\dagger stationary value), that is, $\delta \int_{t_0}^{t_1} L dt = 0$, provided that the function L is properly chosen. This is **Hamilton's principle**, and L is the \dagger Lagrangian function. In \dagger Newtonian mechanics, the \dagger kinetic energy T of a system of particles is expressed as a \dagger quadratic form in \dot{q}_r . Furthermore, if the forces acting on the particles can be given by $-\text{grad } V$, where the potential V does not depend explicitly on \dot{q}_r , we can choose $L = T - V$. Also, for a charged particle in \dagger special relativity, we can take $L = -m_0 c^2 (1 - v^2/c^2)^{1/2} - e\phi + e(\mathbf{v} \cdot \mathbf{A})$, where m_0 is the rest mass of the particle, e is the charge, \mathbf{v} is the velocity (with v its magnitude), c is the speed of light in

vacuum, and ϕ and \mathbf{A} are the scalar and vector potentials of the electromagnetic field, respectively.

In general relativity theory, the motion of a particle can be derived from the variational principle $\delta \int ds = 0$ (ds is the Riemannian line element). Hence, geometrically, the particle moves along a \dagger geodesic curve in 4-dimensional space-time.

C. Geometric Optics

The path of a light ray between two points P_0 and P_1 (subject to reflection and refraction) is such that the time of transit along the path among all neighboring virtual paths is an extremum (stationary value). This is called **Fermat's principle**. If the index of refraction is n , Fermat's principle can be expressed as $\delta \int_{P_0}^{P_1} n ds = 0$ (ds is the Euclidean line element). The laws of reflection and refraction of light, as well as the law of rectilinear propagation of light in homogeneous media, can be derived from this principle.

D. Field Theory

Not only the equations of motion of a system of particles, but also various field equations (\dagger Maxwell's equations of the electromagnetic field, \dagger Dirac's equation of the electron field, the meson field equation, the gravitational field equation, etc.) can be derived from variational principles in terms of appropriate Lagrangian functions. In \dagger field theory the essential virtue of the variational principle appears in the fact that the properties of various possible fields as well as conservation laws can be systematically discussed by assuming relativistic invariance and gauge invariance of the Lagrangian functions adopted. In particular, for an electromagnetic field in vacuum, the Lagrangian function density is $L = (\mathbf{H}^2 - \mathbf{E}^2)/2$, and the integration is carried out over a certain 4-dimensional domain.

E. Quantum Mechanics

If H is the \dagger Hamiltonian operator for any quantum-mechanical system, the eigenfunction ψ can be determined by the variational principle

$$\delta \int \bar{\psi} H \psi d\tau = 0, \quad \text{with} \quad \int \bar{\psi} \psi d\tau = 1,$$

where $\bar{\psi}$ is the complex conjugate of ψ and $d\tau$ is the volume element. Based on this variational principle, the \dagger direct method of the calculus of variations is often employed for

an approximate numerical calculation of the energy eigenvalues and eigenfunctions. In particular, by restricting the functional form of ψ to the product of one-body wave functions, we can obtain Hartree's equation. A further suitable symmetrization of ψ leads to Fock's equation.

F. Statistical Mechanics

Let φ be a statistical-mechanical state of a system, and let $S(\varphi)$ and $E(\varphi)$ be the state's entropy and energy (mean entropy and mean energy for an infinitely extended system); T is the thermodynamical temperature, and $f(\varphi) = E(\varphi) - TS(\varphi)$ is the free energy. Then the equilibrium state for $T \geq 0$ is determined as the state φ that gives the minimum value of $f(\varphi)$ (maximum for $T < 0$).

References

- [1] R. Courant and D. Hilbert, *Methods of mathematical physics*, Interscience, I, 1953; II, 1962.
 - [2] B. L. Moiseiwitsch, *Variational principles*, Interscience, 1966.
 - [3] R. Weinstock, *Calculus of variations, with applications to physics and engineering*, McGraw-Hill, 1952.
- Also \rightarrow references to 46 *Calculus of Variations*.

442 (VI.12) Vectors

A. Definitions

The **vector** concept originated in physics from such well-known notions as velocity, acceleration, and force. These physical quantities are supplied with length and direction; they can be added or multiplied by scalars. In the Euclidean space E^n (or, in general, an \dagger affine space), a vector \mathbf{a} is represented by an **oriented segment** \overrightarrow{pq} . Two oriented segments $\overrightarrow{p_1q_1}$ and $\overrightarrow{p_2q_2}$ are considered to represent the same vector \mathbf{a} if and only if the following two conditions are satisfied: (1) The four points p_1, q_1, p_2, q_2 lie in the same plane π . (2) p_1q_1 / p_2q_2 and p_1q_2 / q_1q_2 . Hence a vector in E^n is an equivalence class of oriented segments \overrightarrow{pq} , where the equivalence relation $\overrightarrow{p_1q_1} \sim \overrightarrow{p_2q_2}$ is defined by the two conditions just given. Hereafter, we denote the vector by $[\overrightarrow{pq}]$, or simply \overrightarrow{pq} . The points p and q are called the **initial point** and **terminal point** of the vector \overrightarrow{pq} .

Given a vector $\mathbf{a} = \overrightarrow{pq}$ and a real number λ , we define the **scalar multiple** $\lambda\mathbf{a}$ as the vector \overrightarrow{pr} , where r is the point on the straight line containing both p and q such that the ratio $[\overrightarrow{pr} : \overrightarrow{pq}]$ is equal to λ (if $p = q$, then we put $r = p$). The operation $(\lambda, \mathbf{a}) \rightarrow \lambda\mathbf{a}$ is called **scalar multiplication**. Given two vectors $\mathbf{a} = \overrightarrow{pq}$ and $\mathbf{b} = \overrightarrow{qr}$, the vector $\mathbf{c} = \overrightarrow{pr}$ is called the **sum** of \mathbf{a} and \mathbf{b} and is denoted by $\mathbf{c} = \mathbf{a} + \mathbf{b}$. The vector $\overrightarrow{pp} = \mathbf{0}$ is called the **zero vector**. If $\mathbf{a} = \overrightarrow{pq}$, we put $-\mathbf{a} = \overrightarrow{qp}$.

Scalar multiplication and addition of vectors satisfy the following seven conditions: (1) $\mathbf{a} + \mathbf{b} = \mathbf{b} + \mathbf{a}$ (commutative law); (2) $\mathbf{a} + (\mathbf{b} + \mathbf{c}) = (\mathbf{a} + \mathbf{b}) + \mathbf{c}$ (associative law); (3) $\mathbf{a} + \mathbf{0} = \mathbf{a}$; (4) for each \mathbf{a} there is $-\mathbf{a}$ such that $\mathbf{a} + (-\mathbf{a}) = \mathbf{0}$; (5) $\lambda(\mathbf{a} + \mathbf{b}) = \lambda\mathbf{a} + \lambda\mathbf{b}$, $(\lambda + \mu)\mathbf{a} = \lambda\mathbf{a} + \mu\mathbf{a}$ (distributive laws); (6) $\lambda(\mu\mathbf{a}) = (\lambda\mu)\mathbf{a}$ (associative law for scalar multiplication); and (7) $1\mathbf{a} = \mathbf{a}$. Hence the set V of all vectors in E^n forms a \dagger real linear space. Sometimes, a set satisfying (1)–(7), that is, by definition, a linear space, is called a **vector space**, and its elements are called **vectors**.

The pair consisting of a vector \overrightarrow{pq} and a specific initial point p of \overrightarrow{pq} is sometimes called a **fixed vector**. An illustration of this is given by the force vector with its initial point being where the force is applied. By contrast, a vector \overrightarrow{pq} is sometimes called a **free vector**. If we fix the origin o in E^n , then for any point p in E^n , the vector \overrightarrow{op} is called the **position vector** of p .

If two vectors $\mathbf{a} = \overrightarrow{op}$ and $\mathbf{b} = \overrightarrow{oq}$ are \dagger linearly dependent, they are sometimes said to be **collinear**. If three vectors $\mathbf{a} = \overrightarrow{op}$, $\mathbf{b} = \overrightarrow{oq}$, and $\mathbf{c} = \overrightarrow{or}$ are linearly dependent, they are sometimes said to be **coplanar**.

If a set of vectors $\mathbf{e}_1, \dots, \mathbf{e}_n$ forms a \dagger basis of a vector space V , then the vectors \mathbf{e}_i are called **fundamental vectors** in V . Each vector $\mathbf{a} \in V$ is uniquely expressed as $\mathbf{a} = \sum \alpha_i \mathbf{e}_i$ ($\alpha_i \in \mathbb{R}$). We call $(\alpha_1, \dots, \alpha_n)$ the **components** of the vector \mathbf{a} with respect to the fundamental vectors $\mathbf{e}_1, \dots, \mathbf{e}_n$.

B. Inner Product

In the Euclidean space E^n , the length of the line segment \overrightarrow{pq} is called the **absolute value** (or **magnitude**) of the vector $\mathbf{a} = \overrightarrow{pq}$ and is denoted by $|\mathbf{a}|$. A vector of length one is called a **unit vector**. For two vectors $\mathbf{a} = \overrightarrow{op}$ and $\mathbf{b} = \overrightarrow{oq}$, the value $(\mathbf{a}, \mathbf{b}) = |\mathbf{a}||\mathbf{b}|\cos\theta$ is called the **inner product** (or **scalar product**) of \mathbf{a} and \mathbf{b} , where θ is the angle $\angle poq$. Instead of (\mathbf{a}, \mathbf{b}) , the notations $\mathbf{a} \cdot \mathbf{b}$, or \mathbf{ab} are also used. If neither vector \mathbf{a} nor vector \mathbf{b} is equal to $\mathbf{0}$, then $(\mathbf{a}, \mathbf{b}) = 0$ implies $\angle poq = \pi/2$, that is, the orthogonality of the two vectors \overrightarrow{op} and \overrightarrow{oq} . If we take an

*orthonormal basis $(\mathbf{e}_1, \dots, \mathbf{e}_n)$ in E^n (i.e., a set of fundamental vectors with $|\mathbf{e}_i| = 1$, $(\mathbf{e}_i, \mathbf{e}_j) = 0$ ($i \neq j$)), the inner product of vectors $\mathbf{a} = \sum \alpha_i \mathbf{e}_i$, $\mathbf{b} = \sum \beta_i \mathbf{e}_i$ is equal to $\sum_{i=1}^n \alpha_i \beta_i$. The inner product has the following three properties (i) $(\mathbf{x}, \mathbf{x}) \geq 0$ and is zero if and only if $\mathbf{x} = \mathbf{0}$; (ii) $(\mathbf{x}, \mathbf{y}) = (\mathbf{y}, \mathbf{x})$; (iii) $(\mathbf{x}_1 + \mathbf{x}_2, \mathbf{y}) = (\mathbf{x}_1, \mathbf{y}) + (\mathbf{x}_2, \mathbf{y})$, $(\alpha \mathbf{x}, \mathbf{y}) = \alpha(\mathbf{x}, \mathbf{y})$ ($\alpha \in \mathbf{R}$). Similar linearity holds for \mathbf{y} .

Generally, an \mathbf{R} -valued *bilinear form (\mathbf{x}, \mathbf{y}) on a linear space V satisfying the previous three conditions is also called an inner product. If a linear space V is equipped with an inner product, the space is called an **inner product space** (\rightarrow 256 Linear Spaces H; 197 Hilbert Spaces). If V is an inner product space, the absolute value $|\mathbf{x}|$ of $\mathbf{x} \in V$ is defined to be $\sqrt{(\mathbf{x}, \mathbf{x})}$.

C. Vector Product

In the 3-dimensional Euclidean space E^3 , we take an orthonormal basis $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$. Let \mathbf{a} and \mathbf{b} be vectors in E^3 whose components with respect to $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ are $(\alpha_1, \alpha_2, \alpha_3)$, $(\beta_1, \beta_2, \beta_3)$. The vector

$$\begin{vmatrix} \alpha_2 & \alpha_3 \\ \beta_2 & \beta_3 \end{vmatrix} \mathbf{e}_1 + \begin{vmatrix} \alpha_3 & \alpha_1 \\ \beta_3 & \beta_1 \end{vmatrix} \mathbf{e}_2 + \begin{vmatrix} \alpha_1 & \alpha_2 \\ \beta_1 & \beta_2 \end{vmatrix} \mathbf{e}_3,$$

which is symbolically written as

$$\begin{vmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \\ \alpha_1 & \alpha_2 & \alpha_3 \\ \beta_1 & \beta_2 & \beta_3 \end{vmatrix},$$

is called the **exterior product** or **vector product** of \mathbf{a} and \mathbf{b} and is denoted by $[\mathbf{a}, \mathbf{b}]$ or $\mathbf{a} \times \mathbf{b}$.

The vector $[\mathbf{a}, \mathbf{b}]$ is determined uniquely up to its sign by \mathbf{a} and \mathbf{b} and is independent of the choice of the orthonormal basis.

Suppose that we have $\mathbf{a} = \overrightarrow{op}$, $\mathbf{b} = \overrightarrow{oq}$. Then $|[\mathbf{a}, \mathbf{b}]| = |\mathbf{a}| \cdot |\mathbf{b}| \sin \theta$, where $\theta = \angle poq$. Also $|[\mathbf{a}, \mathbf{b}]|$ is equal to the area of the parallelogram determined by \mathbf{a} and \mathbf{b} . To illustrate the orientation of $[\mathbf{a}, \mathbf{b}]$, we sometimes use the idea of a turning screw. That is, the direction of a right-handed screw advancing while turning at o from p to q (within the angle less than 180°) coincides with the direction of $[\mathbf{a}, \mathbf{b}]$ (Fig. 1). The exterior product has the following three properties: (1) $[\mathbf{a}, \mathbf{b}] = -[\mathbf{b}, \mathbf{a}]$ (antisymmetric law); (2) $[\lambda \mathbf{a}, \mathbf{b}] = \lambda [\mathbf{a}, \mathbf{b}]$ (associative law for

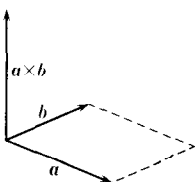


Fig. 1

scalar multiplication); (3) $[\mathbf{a}, \mathbf{b} + \mathbf{c}] = [\mathbf{a}, \mathbf{b}] + [\mathbf{a}, \mathbf{c}]$ (distributive law). The vector product does not satisfy the associative law, but it does satisfy the *Jacobi identity $[\mathbf{a}, [\mathbf{b}, \mathbf{c}]] + [\mathbf{b}, [\mathbf{c}, \mathbf{a}]] + [\mathbf{c}, [\mathbf{a}, \mathbf{b}]] = \mathbf{0}$. The vector $[\mathbf{a}, [\mathbf{b}, \mathbf{c}]]$ is sometimes called the **vector triple product**, and for this we have **Lagrange's formula** $[\mathbf{a}, [\mathbf{b}, \mathbf{c}]] = (\mathbf{a}, \mathbf{c})\mathbf{b} - (\mathbf{a}, \mathbf{b})\mathbf{c}$.

Let $\mathbf{a}, \mathbf{b}, \mathbf{c}$ be vectors in E^3 whose components with respect to an orthonormal fundamental basis are $(\alpha_1, \alpha_2, \alpha_3)$, $(\beta_1, \beta_2, \beta_3)$, and $(\gamma_1, \gamma_2, \gamma_3)$. Then $(\mathbf{a}, [\mathbf{b}, \mathbf{c}]) = (\mathbf{b}, [\mathbf{c}, \mathbf{a}]) = (\mathbf{c}, [\mathbf{a}, \mathbf{b}]) = [\mathbf{a}, \mathbf{b}, \mathbf{c}]$, and the common value is equal to the determinant of the 3×3 matrix

$$\begin{vmatrix} \alpha_1 & \alpha_2 & \alpha_3 \\ \beta_1 & \beta_2 & \beta_3 \\ \gamma_1 & \gamma_2 & \gamma_3 \end{vmatrix}.$$

The value denoted by $[\mathbf{a}, \mathbf{b}, \mathbf{c}]$ is called the **scalar triple product** of $\mathbf{a}, \mathbf{b}, \mathbf{c}$ and is equal to the volume of the parallelotope whose three edges are $\mathbf{a} = \overrightarrow{op}$, $\mathbf{b} = \overrightarrow{oq}$, and $\mathbf{c} = \overrightarrow{or}$ with common initial point o . The triple $\mathbf{a}, \mathbf{b}, \mathbf{c}$ is called a right-hand system or a left-hand system according as $[\mathbf{a}, \mathbf{b}, \mathbf{c}]$ is positive or negative. We have $[\mathbf{a}, \mathbf{b}, \mathbf{c}] = 0$ if and only if \mathbf{a}, \mathbf{b} and \mathbf{c} are coplanar. (For the *exterior product of vectors in E^n and the concept of * p -vectors \rightarrow 256 Linear Spaces O.)

D. Vector Fields

In this section we consider the case of a 3-dimensional Euclidean space E^3 (for the general case \rightarrow 105 Differentiable Manifolds). A scalar-valued or a vector-valued function defined on a set D in E^3 is called a **scalar field** or a **vector field**, respectively. The continuity or the differentiability of a vector field is defined by the continuity or the differentiability of its components.

For a differentiable scalar field $f(x, y, z)$, the vector field with the components $(\partial f / \partial x, \partial f / \partial y, \partial f / \partial z)$ is called the **gradient** of f and is denoted by **grad** f . For a differentiable vector field $\mathbf{V}(x, y, z)$ whose components are $(u(x, y, z), v(x, y, z), w(x, y, z))$, the vector field with components

$$\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}, \frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}, \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right)$$

is called the **rotation** (or **curl**) of \mathbf{V} and is denoted by **rot** \mathbf{V} (or **curl** \mathbf{V}). Also, for a differentiable vector field \mathbf{V} , the scalar field defined by $\partial u / \partial x + \partial v / \partial y + \partial w / \partial z$ is called the **divergence** of \mathbf{V} and is denoted by **div** \mathbf{V} . Utilizing the vector operator ∇ having differential operators $(\partial / \partial x, \partial / \partial y, \partial / \partial z)$ as its components, we may write simply **grad** $f = \nabla f$, **div** $\mathbf{V} = (\nabla, \mathbf{V})$, **rot** $\mathbf{V} =$

$[\nabla, \nabla]$. The symbol ∇ is called **nabla**, **atled** (inverse of delta), or **Hamiltonian**.

A vector field \mathbf{V} with $\text{rot } \mathbf{V} = 0$ is said to be **irrotational**, (**lamellar**, or **without vortex**). A vector field \mathbf{V} with $\text{div } \mathbf{V} = 0$ is said to be **solenoidal** (or **without source**). Thus $\text{grad } f$ is irrotational and $\text{rot } \mathbf{V}$ is solenoidal. In a small neighborhood or in a \dagger simply-connected domain, an irrotational field is a gradient, a solenoidal field is a rotation, and an arbitrary vector field \mathbf{V} is the sum of these two kinds of vector fields: $\mathbf{V} = \text{grad } \varphi + \text{rot } \mathbf{u}$ (**Helmholtz theorem**); the function φ is called the **scalar potential** of \mathbf{V} , and the vector field \mathbf{u} is called the **vector potential** of \mathbf{V} . Furthermore, the operator $\nabla^2 = \nabla \nabla = \text{div grad} = \partial^2/\partial x^2 + \partial^2/\partial y^2 + \partial^2/\partial z^2$ is called the **Laplace operator** (or **Laplacian**) and is denoted by Δ . A function that satisfies $\Delta \varphi = 0$ is called a \dagger harmonic function. Locally, an irrotational and solenoidal vector field is the gradient of a harmonic function. If \mathbf{A} is a vector field whose components are $(\varphi_1, \varphi_2, \varphi_3)$ (i.e., $\mathbf{A}(\mathbf{v}) = (\varphi_1(\mathbf{v}), \varphi_2(\mathbf{v}), \varphi_3(\mathbf{v}))$), we can let Δ operate on \mathbf{A} by setting $\Delta \mathbf{A} = (\Delta \varphi_1, \Delta \varphi_2, \Delta \varphi_3)$. We then have $\Delta \mathbf{A} = \nabla^2 \mathbf{A} = \text{grad div } \mathbf{A} - \text{rot rot } \mathbf{A}$.

Suppose that we are given a vector field \mathbf{V} and a curve C such that the vector $\mathbf{V}(p)$ is tangent to the curve at each point $p \in C$. The curve C is the \dagger integral curve of the vector field \mathbf{V} and is called the **vector line** of the vector field \mathbf{V} . The set of all vector lines intersecting with a given closed curve C is called a **vector tube**. Given a closed curve C and a vector field \mathbf{V} , the \dagger curvilinear integral $\int_C (\mathbf{V}, d\mathbf{s})$ (where $d\mathbf{s}$ is the line element of C) is called the **circulation** (of \mathbf{V}) along the closed curve C . A vector field is irrotational if its circulation along every closed curve vanishes; the converse is true in a simply connected domain. Further, let v_n be the \dagger normal component of a vector field \mathbf{V} with respect to a surface S , and let dS be the volume element of the surface. We put $\mathbf{n} dS = d\mathbf{S}$, where \mathbf{n} is the unit normal vector in the positive direction of the surface S . Then the \dagger surface integral $\int_S v_n dS = \int_S (\mathbf{V}, d\mathbf{S})$ is called the **vector flux** through the surface S . A vector field whose vector flux vanishes for every closed surface is solenoidal. (For the corresponding formulas \rightarrow 94 Curvilinear Integrals and Surface Integrals. For generalizations to higher-dimensional manifolds \rightarrow 105 Differentiable Manifolds; 194 Harmonic Integrals; Appendix A, Table 3.)

References

[1] H. Weyl, *Raum, Zeit, Materie*, Springer, fifth edition, 1923; English translation, *Space, time, matter*, Dover, 1952.

[2] S. Banach, *Mechanics*, Warsaw, 1951.

[3] H. Flanders, *Differential forms, with applications to the physical sciences*, Academic Press, 1963.

[4] H. K. Nickerson, D. C. Spencer, and N. E. Steenrod, *Advanced calculus*, Van Nostrand, 1959.

443 (XII.8) Vector-Valued Integrals

A. General Remarks

Integrals whose values are elements (or subsets) of \dagger topological linear spaces are generally called **vector-valued integrals** or **vector integrals**. As in the scalar case, there are vector-valued integrals of Riemann type (\rightarrow 37 Banach Spaces K) and of Lebesgue type. In this article we consider only the latter. There are cases where integrands are vector-valued, where measures are vector-valued, and where both are vector-valued. The methods of integration are also divided into the strong type, in which the integrals are defined by means of the original topology of the topological linear space X , and the weak type, in which they are reduced to numerical integrals by applying continuous linear functionals on X . Combining these we can define many kinds of integrals.

Historically, D. Hilbert's \dagger spectral resolution is the first example of vector-valued integrals, but the general theory of vector-valued integrals started only after S. Bochner [1] defined in 1933 an integral of strong type for functions with values in a Banach space with respect to numerical measures. Then G. Birkhoff [2] defined a more general integral by replacing absolutely convergent sums with unconditionally convergent sums. At approximately the same time, N. Dunford introduced integrals equivalent to these. Later, R. S. Phillips (*Trans. Amer. Math. Soc.*, 47 (1940)) generalized the definition to the case where values of functions are in a \dagger locally convex topological linear space, and C. E. Rickart (*Trans. Amer. Math. Soc.*, 52 (1942)) to the case where functions take subsets of a locally convex topological linear space as their values. The theory of integrals of weak type for functions with values in a Banach space and numerical measures was constructed by I. M. Gel'fand [3], Dunford [4], B. J. Pettis [5], and others (1936–1938). N. Bourbaki [6] dealt with the case where integrands take values in a locally convex topological linear space. As for integrals of numerical functions by vector-valued

measures, a representative of strong type integrals is the integral of R. G. Bartle, Dunford, and J. T. Schwartz [7] (1955). Weak type integrals have been discussed by Bourbaki [6], D. R. Lewis (*Pacific J. Math.*, 33 (1970)), and I. Kluvnek (*Studia Math.*, 37 (1970)). The bilinear integral of Bartle (*Studia Math.*, 15 (1956)) is typical of integrals in the case where both integrands and measures are vector-valued. For interrelations of these integrals — the papers by Pettis and Bartle cited above and T. H. Hildebrandt's report in the *Bulletin of the American Mathematical Society*, 59 (1953).

Since the earliest investigations [1–3] the main aim of the theory of vector-valued integrals has been to obtain integral representations of vector-valued (set) functions and various linear operators [8]. However, there is the fundamental difficulty of the nonvalidity of the Radon-Nikodm theorem. Whatever definition of integrals we take, the theorem does not hold for vector-valued set functions unconditionally. Many works sought conditions for functions, operators, or spaces such that the conclusion of the theorem would be restored; the works of Dunford and Pettis [9] and Phillips (*Amer. J. Math.*, 65 (1943)) marked a summit of these attempts. Later, after A. Grothendieck's investigations (1953–1956), this problem began to be studied again, beginning in the late 1960s, by many mathematicians (→ J. Diestel and J. J. Uhl, Jr., *Rocky Mountain J. Math.*, 6 (1976); [10]).

Recently, integrals of multivalued vector-valued functions have also been employed in mathematical statistics, economics, control theory, and many other fields. Some contributions are, besides Rickart cited above, G. B. Price (*Trans. Amer. Math. Soc.*, 47 (1940)), H. Kudo (*Sci. Rep. Ochanomizu Univ.*, 4 (1953)), H. Richter (*Math. Ann.*, 150 (1963)), R. J. Aumann [11], G. Debreu [12], and M. Hukuhara (*Funkcial. Ekvac.*, 10 (1967)). Furthermore, C. Castaing and M. Varadier [13] have defined weak type integrals of multivalued functions and introduced many results concerning them. In the following we shall give explanations of typical vector-valued integrals with values in a Banach space only.

B. Measurable Vector-Valued Functions

Let $x(s)$ be a function defined on a σ -finite measure space (S, \mathfrak{S}, μ) with values in a Banach space X . This is called a **simple function** or **finite-valued function** if there exists a partition of S into a finite number of mutually disjoint measurable sets A_1, A_2, \dots, A_n in each of which $x(s)$ takes a constant value c_j . Then

$x(s)$ can be written as $\sum_{j=1}^n c_j \chi_{A_j}(s)$, where $\chi_{A_j}(s)$ is the \dagger characteristic function of A_j . A function $x(s)$ is said to be **measurable** or **strongly measurable** if it is the strong limit of a sequence of simple functions almost everywhere, that is, $\lim_{n \rightarrow \infty} \|x_n(s) - x(s)\| = 0$ a.e. Then the numerical function $\|x(s)\|$ is measurable. If μ is a \dagger Radon measure on a compact Hausdorff space S , then the measurable functions can be characterized by \dagger Luzin's property (→ 270 Measure Theory I).

A function $x(s)$ is said to be **scalarly measurable** or **weakly measurable** if the numerical function $\langle x(s), x' \rangle$ is measurable for any \dagger continuous linear functional $x' \in X'$. A function $x(s)$ is measurable if and only if it is scalarly measurable and there are a \dagger null set $E_0 \subset S$ and a \dagger separable closed subspace $Y \subset X$ such that $x(s) \in Y$ whenever $s \notin E_0$ (**Pettis measurability theorem**).

C. Bochner Integrals

A measurable vector-valued function $x(s)$ is said to be **Bochner integrable** if the norm $\|x(s)\|$ is \dagger integrable. If $x(s)$ is a Bochner integrable simple function $\sum c_j \chi_{A_j}(s)$, then its Bochner integral is defined by

$$\int_S x(s) d\mu = \sum \mu(A_j) c_j.$$

For a general Bochner integrable function $x(s)$ there exists a sequence of simple functions satisfying the following conditions: (i) $\lim_{n \rightarrow \infty} \|x_n(s) - x(s)\| = 0$ a.e. (ii) $\lim_{n \rightarrow \infty} \int_S \|x_n(s) - x(s)\| d\mu = 0$. Then $\int_S x_n(s) d\mu$ converges strongly and its limit does not depend on the choice of the sequence $\{x_n(s)\}$. We call the limit the **Bochner integral** of $x(s)$ and denote it by $\int_S x(s) d\mu$ or by $(Bn) \int_S x(s) d\mu$ to distinguish it from other kinds of integrals. A Bochner integrable function on S is Bochner integrable on every measurable subset of S . The Bochner integral has the basic properties of Lebesgue integrals, such as linearity, \dagger complete additivity, and \dagger absolute continuity, with absolute values replaced by norms. \dagger Lebesgue's convergence theorem and \dagger Fubini's theorem also hold. However, the Radon-Nikodm theorem does not hold in general (→ Section H). Let T be a \dagger closed linear operator from X to another Banach space Y . If both $x(s)$ and $Tx(s)$ are Bochner integrable, then the integral of $x(s)$ belongs to the domain of T and

$$T\left(\int_S x(s) d\mu\right) = \int_S Tx(s) d\mu.$$

If, in particular, T is bounded, then the assumption is always satisfied. If μ is the \dagger Le-

besgue measure on the Euclidean space \mathbf{R}^n , then Lebesgue's differentiability theorem holds for the Bochner integrals regarded as a set function on the regular closed sets (\rightarrow 380 Set Functions D).

D. Unconditionally Convergent Series

Let $\sum_{j=1}^{\infty} x_j$ be a series of elements x_j of a Banach space X . It is said to be **absolutely convergent** if $\sum \|x_j\| < \infty$. It is called **unconditionally convergent** if for any rearrangement α the resulting series $\sum x_{\alpha(j)}$ converges strongly. Then the sum does not depend on α . Clearly, an absolutely convergent series is unconditionally convergent. If X is the number space or is finite-dimensional, then the converse holds. However, if X is infinite-dimensional, there is always an unconditionally convergent series which is not absolutely convergent (**Dvoretzky-Rogers theorem**).

A series $\sum x_j$ is unconditionally convergent if and only if each subseries converges weakly (**Orlicz-Pettis theorem**). If $\sum x_j$ is an unconditionally convergent series, then $\sum \langle x_j, x' \rangle$ converges absolutely for any continuous linear functional $x' \in X'$. If X is a Banach space containing no closed linear subspace isomorphic to the * sequence space c_0 , then conversely a series $\sum x_j$ converges unconditionally whenever $\sum |\langle x_j, x' \rangle| < \infty$ for any $x' \in X'$ (**Bessaga-Pelczyński theorem**). A Banach space that is * sequentially complete relative to the weak topology, such as a * reflexive Banach space, and a separable Banach space that is the dual of another Banach space, such as l_1 and the * Hardy space $H_1(\mathbf{R}^n)$, satisfy the assumption, while c_0 , l_{∞} , and $L_{\infty}(\Omega)$ for an infinitely divisible Ω do not. The totality of absolutely convergent series (resp. unconditionally convergent series) in X is identified with the * topological tensor product $l_1 \hat{\otimes} X$ (resp. $l_1 \hat{\otimes} X$) (Grothendieck).

E. Birkhoff Integrals

We say that a series $\sum B_j$ of subsets of X converges unconditionally if for any $x_j \in B_j$ the series $\sum x_j$ converges unconditionally. Then $\sum B_j$ denotes the set of such sums. A vector-valued function $x(s)$ is said to be **Birkhoff integrable** if there is a countable partition $\Delta: S = \bigcup_{j=1}^{\infty} A_j$ ($A_j \in \mathfrak{S}$, $A_j \cap A_k = \emptyset$ ($j \neq k$), $\mu(A_j) < \infty$) such that the set $x(A_j)$ of values on A_j are bounded and $\sum \mu(A_j)x(A_j)$ converges unconditionally and if the sum converges to an element of X as the partition is subdivided. The limit is called the **Birkhoff integral** of $x(s)$ and is denoted by $(Bk) \int_S x(s) d\mu$ or simply by

$\int_S x(s) d\mu$. A Birkhoff integrable function is Birkhoff integrable on any measurable set. The Birkhoff integral has, as a set function, complete additivity and absolute continuity in μ . It is linear in the integrand but Fubini's theorem and the Radon-Nikodým theorem do not hold. A Bochner integrable function is Birkhoff integrable, and the integrals coincide. The converse does not hold.

F. Gel'fand-Pettis Integrals

A scalarly measurable function $x(s)$ is said to be **scalarly integrable** or **weakly integrable** if for each $x' \in X'$, $\langle x(s), x' \rangle$ is integrable. Then the linear functional x^* on X' defined by

$$\int_S \langle x(s), x' \rangle d\mu = \langle x', x^* \rangle$$

is called the **scalar integral** of $x(s)$. Gel'fand [3] and Dunford [4] proved that x^* belongs to the bidual X'' . Hence scalarly integrable functions are often called **Dunford integrable** and the integrals x^* the **Dunford integrals**. More generally, Gel'fand [3] showed that if $x'(s)$ is a function with values in the dual X' of a Banach space X such that $\langle x, x'(s) \rangle$ is integrable for any $x \in X$, then there is an $x' \in X'$ satisfying

$$\int_S \langle x, x'(s) \rangle d\mu = \langle x, x' \rangle.$$

This element is sometimes called the **Gel'fand integral** of $x'(s)$. A scalarly integrable function $x(s)$ is scalarly integrable on any measurable subset A . If the scalar integral is always in X , i.e., for each A there is an $x_A \in X$ such that

$$\int_A \langle x, x'(s) \rangle d\mu = \langle x_A, x' \rangle, \quad x' \in X',$$

then $x(s)$ is said to be **Pettis integrable** or **Gel'fand-Pettis integrable** and x_A is called the **Pettis integral** or **Gel'fand-Pettis integral** on A and is denoted by $(P) \int_A x(s) d\mu$ or simply by $\int_A x(s) d\mu$. The Pettis integral has complete additivity and absolute continuity as a set function, similarly to the Birkhoff integral. Again, Fubini's theorem and the Radon-Nikodým theorem do not hold. The scalar integral on measurable sets of a scalarly integrable function $x(s)$ is completely additive and absolutely continuous with respect to the * weak* topology of X'' as the dual to X' . It is completely additive or absolutely continuous in the norm topology if and only if $x(s)$ is Pettis integrable (Pettis [5]; [10]). If $x(s)$ is Pettis integrable and $f(s)$ is a numerical function in $L_{\infty}(S)$, then the product $f(s)x(s)$ is Pettis integrable. Birkhoff integrable functions are Pettis integrable, and the integrals coin-

cide. Conversely, if a measurable function is Pettis integrable, then it is Birkhoff integrable. When X satisfies the Bessaga-Pelczyński condition (\rightarrow Section D), a measurable scalarly integrable function is Pettis integrable.

G. Vector Measures

Let Φ be a set function defined on a completely additive class \mathfrak{E} of subsets of the space S and with values in a Banach space X . It is called a **finitely additive vector measure** (resp. a **completely additive vector measure** or simply a **vector measure**) if $\Phi(A_1 \cup A_2) = \Phi(A_1) + \Phi(A_2)$ whenever A_1 and $A_2 \in \mathfrak{E}$ are disjoint (resp. $\Phi(\bigcup_{j=1}^{\infty} A_j) = \sum_{j=1}^{\infty} \Phi(A_j)$ in the norm topology for all $A_j \in \mathfrak{E}$ such that $A_j \cap A_k = \emptyset$ ($j \neq k$)). We remark that the latter sum always converges unconditionally. A set function Φ is completely additive if and only if $\langle \Phi(A), x' \rangle$ is completely additive for all $x' \in X'$ (**Pettis complete additivity theorem**).

Let Φ be a finitely additive vector measure and E be a measurable set. The **total variation** of Φ on E and the **semivariation** of Φ on E are defined by

$$V(\Phi)(E) = \sup \sum_{j=1}^n \|\Phi(A_j)\| \quad (1)$$

and

$$\|\Phi\|(E) = \sup \left\| \sum_{j=1}^n \alpha_j \Phi(A_j) \right\|, \quad (2)$$

respectively, where the suprema are taken over all finite partitions of $E: E = \bigcup A_j$ ($A_j \in \mathfrak{E}$, $A_j \cap A_k = \emptyset$ ($j \neq k$)) and all numbers α_j with $|\alpha_j| \leq 1$. If $V(\Phi)(S) < \infty$, then Φ is called a **measure of bounded variation**. $\|\Phi\|(S) < \infty$ if and only if $\sup \{ \|\Phi(A)\| \mid A \in \mathfrak{E} \} < \infty$. Then Φ is said to be **bounded**. The function $V(\Phi)(E)$ of E is finitely additive but $\|\Phi\|(E)$ is only subadditive: $\|\Phi\|(A \cup B) \leq \|\Phi\|(A) + \|\Phi\|(B)$. If Φ is a vector measure of bounded variation, then $V(\Phi)$ is a positive measure. Every vector measure is bounded. A completely additive vector measure on a \dagger finitely additive class \mathfrak{Q} can uniquely be extended to a vector measure on the completely additive class \mathfrak{E} generated by \mathfrak{Q} (Klůvák).

Let μ be a positive measure and Φ be a vector measure. Then we have $\Phi(A) \rightarrow 0$ as $\mu(A) \rightarrow 0$ if and only if Φ vanishes on every A with $\mu(A) = 0$. Then Φ is said to be **absolutely continuous** with respect to μ . For every vector measure Φ there is a measure μ such that $\|\Phi\|(A) \rightarrow 0$ as $\mu(A) \rightarrow 0$ and that $0 \leq \mu(A) \leq \|\Phi\|(A)$ (Bartle, Dunford, and Schwartz). As a set function, the Bochner integral is a vector measure of bounded variation and the Pettis integral is a bounded vector measure. Both

are absolutely continuous with respect to the integrating measure. Let X be $L_p(0, 1)$ for $1 \leq p \leq \infty$, and define $\Phi(E)$ for a Lebesgue measurable set E to be the characteristic function of E . If $p = 1$, Φ is a vector measure of bounded variation. If $1 < p < \infty$, Φ is a bounded vector measure, but it is not of bounded variation on any set E with $\mu(E) > 0$. If $p = \infty$, then Φ is no longer completely additive. These vector measures are absolutely continuous with respect to the Lebesgue measure, but they cannot be represented as the Bochner integral or the Pettis integral.

Let Φ be a vector measure on \mathfrak{E} . An \mathfrak{E} -measurable numerical function $f(s)$ is said to be Φ -integrable if there exists a sequence of simple functions $f_n(s)$ such that $f_n(s) \rightarrow f(s)$ a.e. and that for each $E \in \mathfrak{E}$, $\int_E f_n(s) d\Phi$ converges in the norm of X . Then the limit is independent of the choice of f_n . It is called the **Bartle-Dunford-Schwartz integral** and is denoted by $\int_E f(s) d\Phi$. Lebesgue's convergence theorem holds for this integral. If Φ is absolutely continuous with respect to the measure μ , then every $f \in L_\infty(\mu)$ is Φ -integrable, and the operator that maps f to $\int_S f d\Phi$ is continuous with respect to the weak* topology in $L_\infty(\mu)$ and the weak topology of X . Hence it is a \dagger weakly compact operator. In particular, the range of a vector measure is relatively compact in the weak topology [7]. If Φ is the vector measure of the Pettis integral of a vector-valued function $x(s)$, then the above integral is equal to the Pettis integral of $f(s)x(s)$.

A vector measure Φ is said to be **nonatomic** if for each set A with $\Phi(A) \neq 0$ there is a subset B of A such that $\Phi(B) \neq 0$ and $\Phi(A \setminus B) \neq 0$. If X is finite-dimensional, then the range of a nonatomic vector measure is a compact convex set (**Lyapunov convexity theorem**). This has been generalized to the infinite-dimensional case in many ways, but the conclusion does not hold in the original form (\rightarrow Klůvák and G. Knowles [15]; [10]).

H. The Radon-Nikodým Theorem

As the above examples show, the \dagger Radon-Nikodým theorem does not hold for vector measures in the original form. From 1967 to 1971, M. Metivier, M. A. Rieffel, and S. Moedomo and Uhl improved the classical result of Phillips (1943) and proved the following theorem.

Radon-Nikodým theorem for vector measures. The following conditions are equivalent for μ -absolutely continuous vector measures Φ defined on a finite measure space (S, \mathfrak{E}, μ) : (i) There is a Pettis integrable measurable func-

tion $x(s)$ such that

$$\Phi(A) = (P) \int_A x(s) d\mu.$$

(ii) For each $\varepsilon > 0$ there is an $E \in \mathfrak{S}$ such that $\mu(S \setminus E) < \varepsilon$ and such that $\{\Phi(A)/\mu(A) \mid A \in \mathfrak{S}, A \subset E\}$ is relatively compact. (iii) For each $E \in \mathfrak{S}$ with $\mu(E) > 0$ there is a subset F of E with $\mu(F) > 0$ such that $\{\Phi(A)/\mu(A) \mid A \in \mathfrak{S}, A \subset F\}$ is relatively weakly compact. Then Φ is of bounded variation if and only if $x(s)$ is Bochner integrable.

On the other hand, since Birkhoff and Gel'fand it has been known that for special Banach spaces X every μ -absolutely continuous vector measure of bounded variation with values in X can be represented as a Bochner integral with respect to μ . Such spaces are said to have the **Radon-Nikodým property**. Separable dual spaces (Gel'fand, Pettis; Dunford and Pettis), reflexive spaces (Gel'fand, Pettis, Phillips), and $l_1(\Omega)$, Ω arbitrary, etc., have the Radon-Nikodým property, while $L_\infty(0, 1)$ (Bochner), c_0 (J. A. Clarkson), $L_1(\Omega)$ on a nonatomic Ω (Clarkson, Gel'fand), and $C(\Omega)$ on an infinite compact Hausdorff space Ω , etc., do not. Gel'fand proved that $L_1(0, 1)$ (and c_0) is not a dual by means of this fact. From 1967 to 1974, Rieffel, H. B. Maynard, R. E. Huff, and W. J. Davis and R. P. Phelps succeeded in characterizing geometrically the Banach spaces with the Radon-Nikodým property. We know today that the following conditions for Banach spaces X are equivalent [10]: (i) X has the Radon-Nikodým property. (ii) Every separable closed linear subspace of X has the Radon-Nikodým property. (iii) Every function $f: [0, 1] \rightarrow X$ of bounded variation is (strongly or weakly) differentiable a.e. (iv) For any finite measure space (S, \mathfrak{S}, μ) and bounded linear operator $T: L_1(S) \rightarrow X$, there is an essentially bounded measurable function $x(s)$ with values in X such that

$$Tf = \int_S f(s)x(s) d\mu, \quad f \in L_1(S).$$

(v) Each nonvoid bounded closed convex set K in X is the $\bar{*}$ closed convex hull of the set of its strongly exposed points, where a point $x_0 \in K$ is called a **strongly exposed point** of K if there is an $x' \in X'$ such that $\langle x_0, x' \rangle > \langle x, x' \rangle$ for all $x \in K \setminus \{x_0\}$ and that any sequence $x_n \in K$ with $\lim \langle x_n, x' \rangle = \langle x_0, x' \rangle$ converges to x_0 strongly.

A Banach space X is said to have the **Kreĭn-Mil'man property** if each bounded closed convex set in X is the closed convex hull of its $\bar{*}$ extreme points. A Banach space X with the Radon-Nikodým property has the Kreĭn-Mil'man property (J. Lindenstrauss). If X is a dual space, then the converse holds (Huff and P. D. Morris). A Banach space with the Kreĭn-

Mil'man property clearly has no closed linear space isomorphic to c_0 , but there are Banach spaces that do not contain c_0 and do not have the Kreĭn-Mil'man property. The dual X' of a Banach space X has the Radon-Nikodým property if and only if the dual of every separable closed linear subspace of X is separable (Uhl, C. Stegall).

I. Integrals of Multivalued Vector Functions

Let $\Gamma(s)$ be a multivalued function defined on a σ -finite complete measure space (S, \mathfrak{S}, μ) with values that are nonempty closed subsets of a separable Banach space X . The inverse image of a subset E of X under $\Gamma(s)$ is, by definition, the set of all s such that $\Gamma(s) \cap E \neq \emptyset$. $\Gamma(s)$ is said to be **measurable** or **strongly measurable** if the inverse image of each open set in X under $\Gamma(s)$ belongs to \mathfrak{S} . Let $\mathfrak{B}(X)$ be the $\bar{*}$ Borel field of X , and $\mathfrak{S} \times \mathfrak{B}(X)$ be the product completely additive class, that is, the smallest completely additive class containing all direct products $A \times B$ of $A \in \mathfrak{S}$ and $B \in \mathfrak{B}(X)$. Then the measurability of $\Gamma(s)$ is equivalent to each of the following: (i) The graph $\{(s, x) \mid x \in \Gamma(s), s \in \mathfrak{S}\}$ of $\Gamma(s)$ belongs to $\mathfrak{S} \times \mathfrak{B}(X)$. (ii) The inverse image of every Borel set in X under $\Gamma(s)$ belongs to \mathfrak{S} . (iii) For each $x \in X$, the distance $d(x, \Gamma(s)) = \inf\{\|x - y\| \mid y \in \Gamma(s)\}$ between x and $\Gamma(s)$ is measurable as a function on S .

A measurable function $x(s)$ on S with values in X is called a **measurable selection** of $\Gamma(s)$ if $x(s)$ is in $\Gamma(s)$ for all s . (X being separable, we need not discriminate between strong and weak measurability.) The measurability of $\Gamma(s)$ is also equivalent to the following important statement on the existence of measurable selections of $\Gamma(s)$: (iv) There are a countable number of measurable selections $\{x_n(s)\}$ of $\Gamma(s)$ such that the closure of the set $\{x_n(s) \mid n = 1, 2, \dots\}$ coincides with $\Gamma(s)$ for all $s \in S$. $\Gamma(s)$ is said to be **scalarly measurable** or **weakly measurable** if the support function $\delta'(x', \Gamma(s)) = \sup\{\langle x, x' \rangle \mid x \in \Gamma(s)\}$ is measurable on S for all $x' \in X'$. The strong measurability of $\Gamma(s)$ clearly implies the weak one. If the values of $\Gamma(s)$ are nonempty weakly compact convex sets, then the measurabilities are equivalent. Hereafter we shall assume that $\Gamma(s)$ takes the values in the weakly compact convex sets. If the support function $\delta'(x', \Gamma(s))$ is integrable on S for all $x' \in X'$, then $\Gamma(s)$ is said to be **scalarly integrable**. Then the **scalar integral** of $\Gamma(s)$ is defined to be the set in X'' of all scalar integrals of its measurable selections, i.e.,

$$\int_S \Gamma(s) d\mu = \left\{ \int_S x(s) d\mu \mid x(s) \text{ is a measurable selection of } \Gamma(s) \right\}.$$

If $\|\Gamma(s)\| = \sup\{\|x\| \mid x \in \Gamma(s)\}$ is integrable, then every measurable selection is Bochner integrable and the integral $\int_S \Gamma(s) d\mu$ becomes a nonempty weakly compact convex set in X . When the values of $\Gamma(s)$ are nonempty compact convex sets, there is another method, by G. Debreu, of defining the integral. Let \mathcal{Q} be the class of all nonempty compact convex sets in X and δ be the Hausdorff metric, i.e., for K_1 and $K_2 \in \mathcal{Q}$ define $\delta(K_1, K_2) = \max[\sup\{d(x, K_2) \mid x \in K_1\}, \sup\{d(x, K_1) \mid x \in K_2\}]$. Further, for $K_1, K_2 \in \mathcal{Q}$ and $\alpha \geq 0$ define the sum and the nonnegative scalar multiple by $K_1 + K_2 = \{x_1 + x_2 \mid x_1 \in K_1, x_2 \in K_2\}$ and $\alpha \cdot K_1 = \{\alpha x \mid x \in K_1\}$, respectively. Then \mathcal{Q} endowed with the Hausdorff metric and the above addition and scalar multiplication is isometrically embedded in a closed convex cone in a separable Banach space Y by the Rådström embedding theorem (*Proc. Amer. Math. Soc.*, 3 (1952)). Let φ be this isometry. Then the (strong) measurability and the (strong) integrability of $\Gamma(s)$ are defined by the measurability and the Bochner integrability of the Y -valued function $\varphi(\Gamma(s))$, respectively, and its (strong) integral as the inverse image of the Bochner integral of $\varphi(\Gamma(s))$ under φ :

$$\int_S \Gamma(s) d\mu = \varphi^{-1} \left(\int_S \varphi(\Gamma(s)) d\mu \right).$$

This definition of integral for strongly measurable $\Gamma(s)$ is shown to be compatible with that mentioned before. It is clear by the definition that the integral value in this case is a nonempty compact convex set and that most properties of Bochner integrals also hold for this integral.

References

- [1] S. Bochner, Integration von Funktionen, deren Werte die Elemente eines Vektorraumes sind, *Fund. Math.*, 20 (1933), 262–276.
- [2] G. Birkhoff, Integration of functions with values in a Banach space, *Trans. Amer. Math. Soc.*, 38 (1935), 357–378.
- [3] I. Gel'fand, Abstrakte Funktionen und lineare Operatoren, *Mat. Sb.*, 4 (46) (1938), 235–286.
- [4] N. Dunford, Uniformity in linear spaces, *Trans. Amer. Math. Soc.*, 44 (1938), 305–356.
- [5] B. J. Pettis, On integration in vector spaces, *Trans. Amer. Math. Soc.*, 44 (1938), 277–304.
- [6] N. Bourbaki, *Éléments de mathématique*, Intégration, Hermann, ch. 6, 1959.
- [7] R. G. Bartle, N. Dunford, and J. Schwartz, Weak compactness and vector measures, *Canad. J. Math.*, 7 (1955), 289–305.

444 Ref.

Viète, François

- [8] N. Dunford and J. T. Schwartz, *Linear operators I*, Interscience, 1958.
- [9] N. Dunford and B. J. Pettis, Linear operations on summable functions, *Trans. Amer. Math. Soc.*, 47 (1940), 323–392.
- [10] J. Diestel and J. J. Uhl, Jr., *Vector measures*, *Amer. Math. Soc. Math. Surveys* 13 (1977).
- [11] R. J. Aumann, Integrals of set-valued functions, *J. Math. Anal. Appl.*, 12 (1965), 1–22.
- [12] G. Debreu, Integration of correspondences, *Proc. Fifth Berkeley Symp. Math. Statist. Probab.*, II, pt. I (1967), 351–372.
- [13] C. Castaing and M. Valadier, *Convex analysis and measurable multifunctions*, Springer, 1977.
- [14] N. Dinculeanu, *Vector measures*, Pergamon, 1967.
- [15] I. Kluvánek and G. Knowles, *Vector measures and control systems*, North-Holland, 1975.

444 (XXI.42)

Viète, François

François Viète (1540–December 13, 1603) was born in Fontenay-le-Comte, Poitou, in western France. He served under Henri IV, first as a lawyer and later as a political advisor. His mathematics was done in his leisure time. He used symbols for known variables for the first time and established the methodology and principles of symbolic algebra. He also systematized the algebra of the time and used it as a method of discovery. He is often called the father of algebra. He improved the methods of solving equations of the third and fourth degrees obtained by G. Cardano and L. Ferrari. Realizing that solving the algebraic equation of the 45th degree proposed by the Belgian mathematician A. van Roomen can be reduced to searching for $\sin(\alpha/45)$ knowing $\sin \alpha$, he was able to solve it almost immediately. However, he would not acknowledge negative roots and refused to add terms of different degrees because of his belief in the Greek principle of homogeneity of magnitudes. He also contributed to trigonometry and represented the number π as an infinite product.

References

- [1] Francisci Vietae, *Opera mathematica*, F. van Schooten (ed.), Leyden, 1646 (Georg Olms, 1970).
- [2] Jacob Klain, *Die griechische Logistik und die Entstehung der Algebra I, II, Quellen und*

Studien zur Gesch. Math., (B) 3 (1934), 18–105; (B) 3 (1936), 122–235.

445 (XXI.43) Von Neumann, John

John von Neumann (December 28, 1903–February 8, 1957) was born in Budapest, Hungary, the son of a banker. By the time he graduated from the university there in 1921, he had already published a paper with M. Fekete. He was later influenced by H. Weyl and E. Schmidt at the universities of Zürich and Berlin, respectively, and he became a lecturer at the universities of Berlin and Hamburg. He moved to the United States in 1930 and in 1933 became professor at the Institute for Advanced Study at Princeton. In 1954 he was appointed a member of the US Atomic Energy Commission. The fields in which he was first interested were [†]set theory, theory of [†]functions of real variables, and [†]foundations of mathematics. He made important contributions to the axiomatization of set theory. At the same time, however, he was deeply interested in theoretical physics, especially in the mathematical foundations of quantum mechanics. From this field, he was led into research on the theory of [†]Hilbert spaces, and he obtained basic results in the theory of [†]operator rings of Hilbert spaces. To extend the theory of operator rings, he introduced [†]continuous geometry. Among his many famous works are the theory of [†]almost periodic functions on a group and the solving of [†]Hilbert's fifth problem for compact groups. In his later years, he contributed to [†]game theory and to the design of computers, thus playing a major role in all fields of applied mathematics.

References

- [1] J. von Neumann, Collected works I–VI, Pergamon, 1961–1963.
- [2] J. von Neumann, 1903–1957, J. C. Oxtoby, B. J. Pettis, and G. B. Price (eds.), Bull. Amer. Math. Soc., 64 (1958), 1–129.
- [3] J. von Neumann, Mathematische Grundlagen der Quantenmechanik, Springer, 1932.
- [4] J. von Neumann, Functional operators I, II, Ann. Math. Studies, Princeton Univ. Press, 1950.
- [5] J. von Neumann, Continuous geometry, Princeton Univ. Press, 1960.
- [6] J. von Neumann and O. Morgenstern, Theory of games and economic behavior, Princeton Univ. Press, third edition, 1953.

W

446 (XX.13) Wave Propagation

A disturbance originating at a point in a medium and propagating at a finite speed in the medium is called a **wave**. For example, a sound wave propagates a change of density or stress in a gas, liquid, or solid. A wave in an elastic solid body is called an elastic wave.

Surface waves appear near the surface of a medium, such as water or the earth. When electromagnetic disturbances are propagated in a gas, liquid, or solid or in a vacuum, they are called **electromagnetic waves**. Light is a kind of electromagnetic wave. According to [†]general relativity theory, gravitational action can also be propagated as a wave.

It many cases waves can be described by the **wave equation**:

$$\frac{\partial^2 \psi}{\partial t^2} = c^2 \left(\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} + \frac{\partial^2 \psi}{\partial z^2} \right).$$

Here t is time, x, y, z are the Cartesian coordinates of points in the space, c is the propagation velocity, and ψ represents the state of the medium.

If we take a closed surface surrounding the origin of the coordinate system, the state $\psi(0, t)$ at the origin at time t can be determined by the state at the points on the closed surface at time $t - r/c$, with r the distance of the point from the origin. More precisely, we have

$$\psi(0, t) = \frac{1}{4\pi} \int \left(\psi \frac{\partial}{\partial n} \left(\frac{1}{r} \right) - \frac{1}{r} \frac{\partial \psi}{\partial n} - \frac{1}{cr} \frac{\partial \psi}{\partial t} \frac{\partial r}{\partial n} \right)_{t-r/c} df.$$

Here n is the inward normal at any point of the closed surface, and the integral is taken over the surface, while the value of the integrand is taken at time $t - r/c$. This relation is a mathematical representation of **Huygens's principle**, which is valid for the 3-dimensional case but does not hold for the 2-dimensional case (\rightarrow 325 Partial Differential Equations of Hyperbolic Type).

A **plane wave** propagating in the direction of a unit vector \mathbf{n} can be represented by $\psi = F(t - \mathbf{n} \cdot \mathbf{r}/c)$, where F is an arbitrary function and $\mathbf{r}(x, y, z)$ is the position vector. The simplest case is given by a **sine wave** (**sinusoidal wave**): $\psi = A \sin(\omega t - \mathbf{k} \cdot \mathbf{r} + \delta)$. Here A (**amplitude**) and δ (**phase constant**) are arbitrary constants, \mathbf{k} is in the direction of wave propagation and satisfies the relation $|\mathbf{k}|c = \omega$. ω is the **angular frequency**, $\omega/2\pi$ the **frequency**, \mathbf{k} the **wave number vector**, $|\mathbf{k}|$ the **wave number**, $2\pi/\omega$

the **period**, and $2\pi/|\mathbf{k}|$ the **wavelength**. The velocity with which the crest of the wave advances is equal to $\omega/|\mathbf{k}| = c$ and is called the **phase velocity**.

A **spherical wave** radiating from the origin can generally be represented by

$$\psi = \sum_n \varphi_n \left(\frac{d}{r dr} \right)^n \frac{1}{r} F \left(t - \frac{r}{c} \right),$$

where φ_n is the [†]solid harmonic of order n .

Waves are not restricted to those governed by the wave equation. In general, ψ is not a scalar, but has several components (e.g., ψ may be a vector), which satisfy a set of simultaneous differential equations of various kinds. Usually they have solutions in the form of sinusoidal waves, but the phase velocity $c = \omega/|\mathbf{k}|$ is generally a function of the wavelength λ . Such a wave, called a **dispersive wave**, has a propagation velocity (velocity of propagation of the disturbance through the medium) that is not equal to the phase velocity. A disturbance of finite extent that can be approximately represented by a plane wave is propagated with a velocity $c - \lambda dc/d\lambda$, called the **group velocity**. Often there exists a definite relationship between the amplitude vector \mathbf{A} (and the corresponding phase constant δ) and wave number vector \mathbf{k} , in which case the wave is said to be **polarized**. In particular, when \mathbf{A} and \mathbf{k} are parallel (perpendicular), the wave is called a **longitudinal** (**transverse**) **wave**. Usually equations governing the wave are linear, and therefore superposition of two solutions gives a new solution ([†]principle of superposition). Superposition of two sinusoidal waves traveling in opposite directions gives rise to a wave whose crests do not move (e.g., $\psi = A \sin \omega t \sin \mathbf{k} \cdot \mathbf{r}$). Such a wave is called a **stationary wave**. Since the energy of a wave is proportional to the square of ψ , the energy of the resultant wave formed by superposition of two waves is not equal to the sum of the energies of the component waves. This phenomenon is called **interference**. When a wave reaches an obstacle it propagates into the shadow region of the obstacle, where there is formed a special distribution of energy dependent on the shape and size of the obstacle. This phenomenon is called **diffraction**.

For aerial sound waves and water waves, if the amplitude is so large that the wave equation is no longer valid, we are faced with [†]nonlinear problems. For instance, **shock waves** appear in the air when surfaces of discontinuity of density and pressure exist. They appear in explosions and for bodies traveling at high speeds. Concerning wave mechanics dealing with atomic phenomena \rightarrow 351 Quantum Mechanics.

References

- [1] H. Lamb, *Hydrodynamics*, Cambridge Univ. Press, sixth edition, 1932.
- [2] Lord Rayleigh, *The theory of sound*, Macmillan, second revised edition, I, 1937; II, 1929.
- [3] M. Born and E. Wolf, *Principles of optics*, Pergamon, fourth edition, 1970.
- [4] F. S. Crawford, Jr., *Waves*, Berkeley physics course III, McGraw-Hill, 1968.
- [5] C. A. Coulson, *Waves; A mathematical theory of the common type of wave motion*, Oliver & Boyd, seventh edition, 1955.
- [6] L. Brillouin, *Wave propagation and group velocity*, Academic Press, 1960.
- [7] I. Tolstoy, *Wave propagation*, McGraw-Hill, 1973.
- [8] J. D. Achenbach, *Wave propagation in elastic solids*, North-Holland, 1973.
- [9] K. F. Graff, *Wave motion in elastic solids*, Ohio State Univ. Press, 1975.
- [10] J. Lighthill, *Waves in fluids*, Cambridge Univ. Press, 1978.
- [11] R. Courant and D. Hilbert, *Methods of mathematical physics II*, Interscience, 1962.

447 (XXI.44) Weierstrass, Karl

Karl Weierstrass (October 31, 1815–February 19, 1897) was born into a Catholic family in Ostenfelde, in Westfalen, Germany. From 1834 to 1838 he studied law at the University of Bonn. In 1839 he moved to Münster, where he came under the influence of C. Gudermann, who was then studying the theory of elliptic functions. From this time until 1855, he taught in a parochial junior high school; during this period he published an important paper on the theory of analytic functions. Invited to the University of Berlin in 1856, he worked there with L. Kronecker and E. E. Kummer. In 1864, he was appointed to a full professorship, which he held until his death.

His foundation of the theory of analytic functions of a complex variable at about the same time as Riemann is his most fundamental work. In contrast to Riemann, who utilized geometric and physical intuition, Weierstrass stressed the importance of rigorous analytic formulation. Aside from the theory of analytic functions, he contributed to the theory of functions of real variables by giving examples of continuous functions that were nowhere differentiable. With his theory of \dagger minimal surfaces, he also contributed to geometry. His lectures at the University of Berlin drew many

448 Ref. Weyl, Hermann

listeners, and in his later years he was a respected authority in the mathematical world.

References

- [1] K. Weierstrass, *Mathematische Werke I–VII*, Mayer & Miller, 1894–1927.
- [2] F. Klein, *Vorlesungen über die Entwicklung der Mathematik im 19. Jahrhundert I*, Springer, 1926 (Chelsea, 1956).

448 (XXI.45) Weyl, Hermann

Hermann Weyl (November 9, 1885–December 8, 1955) was born in Elmshorn in the state of Schleswig-Holstein in Germany. Entering the University of Göttingen in 1904, he also audited courses for a time at the University of Munich. In 1908, he obtained his doctorate from the University of Göttingen with a paper on the theory of integral equations, and by 1910 he was a lecturer at the same university. In 1913, he became a professor at the Federal Technological Institute at Zürich; in 1928–1929, a visiting professor at Princeton University; in 1930, a professor at the University of Göttingen; and in 1933, a professor at the Institute for Advanced Study at Princeton. He retired from his professorship there in 1951, when he became professor emeritus. He died in Zürich in 1955.

Weyl contributed fresh and fundamental works covering all aspects of mathematics and theoretical physics. Among the most notable are results on problems in \dagger integral equations, \dagger Riemann surfaces, the theory of \dagger Diophantine approximation, the representation of groups, in particular compact groups and \dagger semisimple Lie groups (whose structure he elucidated), the space-time problem, the introduction of \dagger affine connections in differential geometry, \dagger quantum mechanics, and the foundations of mathematics. In his later years, with his son Joachim he studied meromorphic functions. In addition to his many mathematical works he left works in philosophy, history, and criticism.

References

- [1] H. Weyl, *Gesammelte Abhandlungen I–IV*, Springer, 1968.
- [2] H. Weyl, *Die Idee der Riemannschen Fläche*, Teubner, 1913, revised edition, 1955; English translation, *The concept of a Riemann surface*, Addison-Wesley, 1964.

- [3] H. Weyl, *Raum, Zeit, Materie*, Springer, 1918, fifth edition, 1923; English translation, *Space, time, matter*, Dover, 1952.
 [4] H. Weyl, *Das Kontinuum*, Veit, 1918.
 [5] H. Weyl, *Gruppentheorie und Quantenmechanik*, Hirzel, 1928.
 [6] H. Weyl, *Classical groups*, Princeton Univ. Press, 1939, revised edition, 1946.
 [7] H. Weyl and F. J. Weyl, *Meromorphic functions and analytic curves*, Princeton Univ. Press, 1943.
 [8] H. Weyl, *Philosophie der Mathematik und Naturwissenschaften*, Oldenbourg, 1926; English translation, *Philosophy of mathematics and natural science*, Princeton Univ. Press, 1949.
 [9] H. Weyl, *Symmetry*, Princeton Univ. Press, 1952.

449 (III.18) Witt Vectors

A. General Remarks

Let Γ be an \dagger integral domain of characteristic 0, and p a fixed prime number. For each infinite-dimensional vector $x = (x_0, x_1, \dots)$ with components in Γ , we define its **ghost components** $x^{(0)}, x^{(1)}, \dots$ by $x^{(0)} = x_0, x^{(n)} = x_0^{p^n} + px_1^{p^{n-1}} + \dots + p^n x_n$. We define the sum of the vectors x and $y = (y_0, y_1, \dots)$ to be the vector with ghost components $x^{(0)} + y^{(0)}, x^{(1)} + y^{(1)}, \dots$, and their product to be the vector with ghost components $x^{(0)}y^{(0)}, x^{(1)}y^{(1)}, \dots$. The sum and product are uniquely determined vectors with components in Γ . Writing their first two terms explicitly, we have

$$x + y = \left[x_0 + y_0, x_1 + y_1 - \sum_{v=1}^{p-1} \frac{1}{p} \binom{p}{v} x_0^v y_0^{p-v}, \dots \right],$$

$$xy = (x_0 y_0, x_1 y_0^p + y_1 x_0^p + p x_1 y_1, \dots).$$

In general, it can be proved that the n th components $\sigma_n(x, y)$ and $\pi_n(x, y)$ of the sum and product are polynomials in $x_0, y_0, x_1, y_1, \dots, x_n, y_n$ whose coefficients are rational integers. With these operations of addition and multiplication, the set of these vectors forms a \dagger commutative ring, of which the zero element is $(0, 0, \dots)$ and the unity element is $(1, 0, \dots)$. Let k be a field of characteristic p . For vectors (ξ_0, ξ_1, \dots) , and (η_0, η_1, \dots) with components in k , we define their sum and product by $(\xi_0, \xi_1, \dots) + (\eta_0, \eta_1, \dots) = (\dots, \sigma_n(\xi, \eta), \dots)$ and $(\xi_0, \xi_1, \dots)(\eta_0, \eta_1, \dots) = (\dots, \pi_n(\xi, \eta), \dots)$. Since the coefficients of σ_n and π_n are rational in-

tegers, these operations are well defined. With these operations, the set of such vectors becomes an integral domain $W(k)$ of characteristic 0. Elements of $W(k)$ are called **Witt vectors** over k .

If we put $V(\xi_0, \xi_1, \dots) = (0, \xi_0, \xi_1, \dots)$ and $(\xi_0, \xi_1, \dots)^p = (\xi_0^p, \xi_1^p, \dots)$, we get the formula $p\xi = V\xi^p$. (Note that this ξ^p is not the p th power of ξ in $W(k)$ in the usual sense.) Therefore, if we put $|\xi| = p^{-n}$ for a vector ξ whose first nonzero component is ξ_n , then this absolute value $|\cdot|$ gives a \dagger valuation of $W(k)$. In particular, when k is a \dagger perfect field, denoting the vector $(\xi_0, 0, \dots)$ by $\{\xi_0\}$ we get $(\xi_0, \xi_1, \dots) = \sum p^i \{\xi_i p^{-i}\}$, and $W(k)$ is a \dagger complete valuation ring with respect to this valuation. Therefore the \dagger field of quotients of $W(k)$ is a complete valuation field of which p is a prime element and k is the \dagger residue class field. Conversely, let K be a field of characteristic 0 that is complete under a \dagger discrete valuation v , \mathfrak{o} be the valuation ring of v , and k be the residue class field of v . Assume that k is a perfect field of characteristic p . If p is a prime element of \mathfrak{o} , then $\mathfrak{o} = W(k)$. If $v(p) = v(\pi^e)$ ($e > 1$) with a prime element π of \mathfrak{o} , we have $\mathfrak{o} = W(k)[\pi]$, and π is a root of an \dagger Eisenstein polynomial $X^e + a_1 X^{e-1} + \dots + a_e$ ($a_i \in pW(k)$, $a_e \notin p^2 W(k)$). In this way we can determine explicitly the structure of a $\dagger p$ -adic number field (\rightarrow 257 Local Fields).

B. Applications to Abelian p -Extensions and Cyclic Algebras of Characteristic p

Next we consider $W_n(k) = W(k)/V^n W(k)$. The elements of $W_n(k)$ can be viewed as the n -dimensional vectors $(\xi_0, \dots, \xi_{n-1})$, but their laws of composition are defined as in the previous section. They are called **Witt vectors of length n** . We define an operator \wp by $\wp \xi = \xi^p - \xi$. Using it, we can generalize the theory of \dagger Artin-Schreier extensions (\rightarrow 172 Galois Theory) to the case of Abelian extensions of exponent p^n over a field of characteristic p . Indeed, let k be a field of characteristic p and $\xi = (\xi_0, \dots, \xi_{n-1})$ an element of $W(k)$. If $\eta = (\eta_0, \dots, \eta_{n-1})$ is a root of the vector equation $\wp X - \xi = 0$, then the other roots are of the form $\eta + \alpha(\alpha = (\alpha_0, \dots, \alpha_{n-1}), \alpha_i \in \mathbb{F}_p)$. In particular, if $\xi_0 \notin \wp k = \{\alpha^p - \alpha \mid \alpha \in k\}$, the field $K = k(\eta_0, \dots, \eta_{n-1})$ is a cyclic extension of degree p^n over k , and conversely, every cyclic extension of k of degree p^n is obtained in this way. Let $(1/\wp)\xi$ denote the set of all roots of $\wp X - \xi = 0$. Then more generally, any finite Abelian extension of exponent p^n of k can be obtained as $K = k((1/\wp)\xi \mid \xi \in H)$ with a suitable finite subgroup $H/\wp W_n(k)$ of $W_n(k)/\wp W_n(k)$, and

the Galois group of K/k is isomorphic to $H/\varphi W_n(k)$.

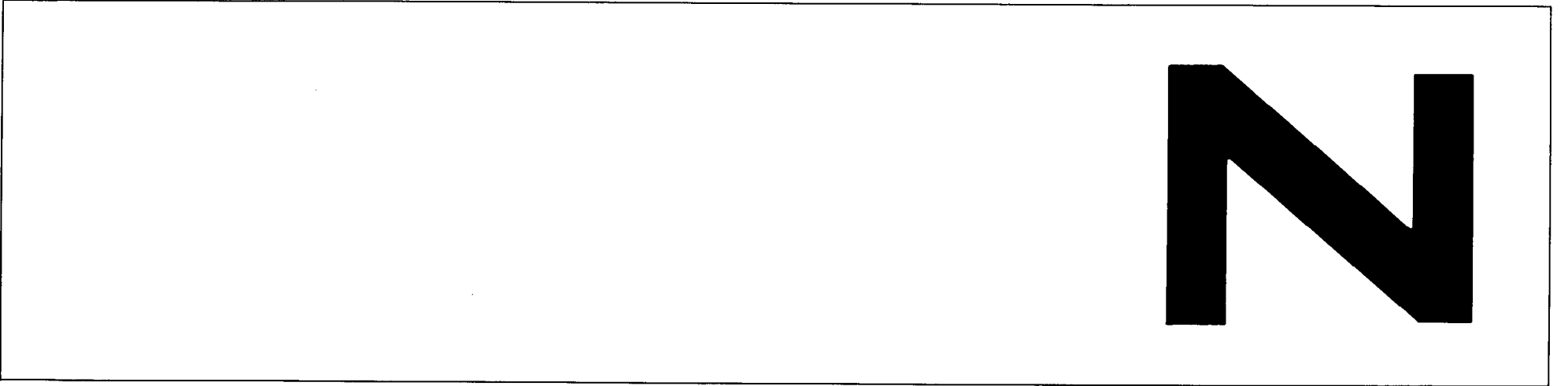
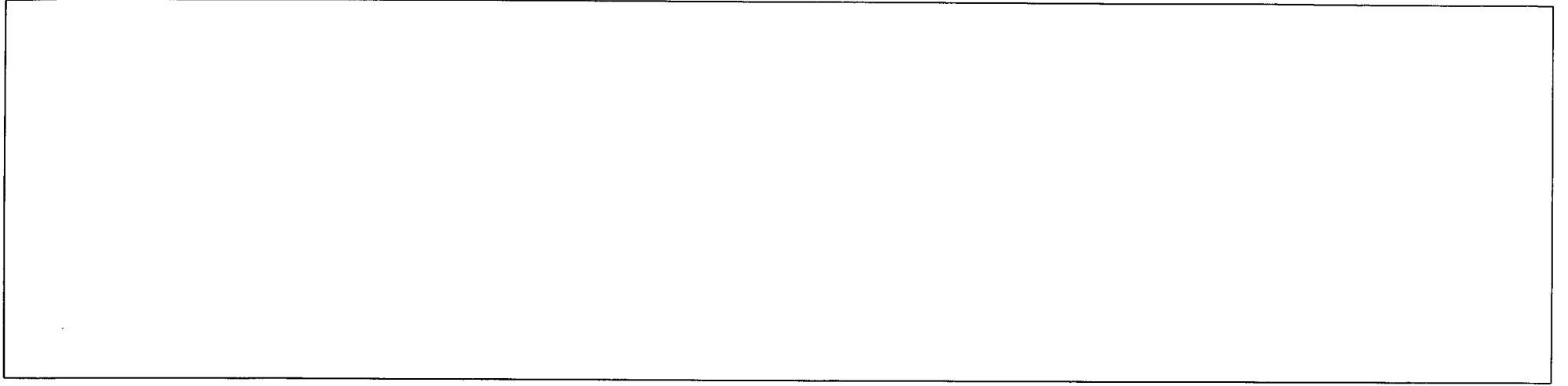
Moreover, for a t cyclic extension $K = k((1/\varphi)\beta)$ of exponent p^n over k and for $\alpha \in k (\alpha \neq 0)$, we can define a t cyclic algebra $(\alpha, \beta]$ generated by an element u over K by the fundamental relations $u^{p^n} = \alpha$, $\varphi\theta = \beta$, $u\theta u^{-1} = \theta + (1, 0, \dots, 0)$ (where $\theta = (\theta_0, \dots, \theta_{n-1})$, $u\theta u^{-1} = (u\theta_0 u^{-1}, \dots, u\theta_{n-1} u^{-1})$), and $(\alpha, \beta]$ is a central simple algebra over k .

Using these results, we can develop the structure theory of the t Brauer group of exponent p^n of a t field of power series in one variable with coefficients in a finite field F_q (of a t field of algebraic functions in one variable over F_q) exactly as in the case of a p -adic field (of an algebraic number field) (E. Witt [1]; \rightarrow 29 Associative Algebras G).

On the other hand, $W_n(k)$ is a commutative t algebraic group over k and is important in the theories of algebraic groups and t formal groups (\rightarrow 13 Algebraic Groups).

References

- [1] E. Witt, Zyklische Körper und Algebren der Charakteristik p vom Grad p^n , J. Reine Angew. Math., 176 (1937), 126–140.
- [2] H. Hasse, Zahlentheorie, Akademie-Verlag., 1949.
- [3] N. Jacobson, Lectures in abstract algebra III, Van Nostrand, 1964.



450 (V.19) Zeta Functions

A. Introduction

Since the 19th century, many special functions called ζ -functions (zeta functions) have been defined and investigated. The four main problems concerning ζ -functions are: (1) Methods of defining ζ -functions. (2) Investigation of the properties of ζ -functions. Generally, ζ -functions have the following four properties in common: (i) They are meromorphic on the whole complex plane; (ii) they have † Dirichlet series expansions; (iii) they have Euler product expansions; and (iv) they satisfy certain functional equations. Also, it is an important problem to find the poles, residues, and zeros of ζ -functions. (3) Application to number theory, in particular to the theory of decomposition of prime ideals in finite extensions of algebraic number fields (\rightarrow 59 Class Field Theory). (4) Study of the relations between different ζ -functions.

Most of the functions called ζ -functions or L -functions have the four properties of problem (2). The following is a classification of the important types of ζ -functions that are already known, which will be discussed later in this article:

(1) The ζ - and L -functions of algebraic number fields: the Riemann ζ -function, Dirichlet L -functions (study of these functions gave impetus to the theory of ζ -functions), Dedekind ζ -functions, Hecke L -functions, Hecke L -functions with † Grössencharakteren, Artin L -functions, and Weil L -functions. (2) The p -adic L -functions related to the works of H. W. Leopoldt, T. Kubota, K. Iwasawa, etc. (3) The ζ -functions of quadratic forms: Epstein ζ -functions, ζ -functions of indefinite quadratic forms (C. L. Siegel), etc. (4) The ζ - and L -functions of algebras: Hey ζ -functions and the ζ -functions given by R. Godement, T. Tamagawa, etc. (5) The ζ -functions associated with Hecke operators, related to the work of E. Hecke, M. Eichler, G. Shimura, H. Jacquet, R. P. Langlands, etc. (6) The congruence ζ - and L -functions attached to algebraic varieties defined over finite fields (E. Artin, A. Weil, A. Grothendieck, P. Deligne), ζ - and L -functions of schemes. (7) Hasse ζ -functions attached to the algebraic varieties defined over algebraic number fields. (8) The ζ -functions attached to discontinuous groups: Selberg ζ -functions, the Eisenstein series defined by A. Selberg, Godement, and I. M. Gel'fand, etc. (9) Y. Ihara's ζ -function related to non-Abelian class field theory over a function field over a finite field.

(10) ζ -functions associated with prehomogeneous vector spaces (M. Sato, T. Shintani).

B. The Riemann ζ -Function

Consider the series

$$\zeta(s) = 1 + \frac{1}{2^s} + \frac{1}{3^s} + \dots + \frac{1}{n^s} + \dots,$$

which converges for all real numbers $s > 1$. It was already recognized by L. Euler that $\zeta(s)$ can also be expressed by a convergent infinite product $\prod_p (1 - p^{-s})^{-1}$, where p runs over all prime numbers (*Werke*, ser. I, vol. VII, ch. XV, § 274). This expansion is called **Euler's infinite product expansion** or simply the **Euler product**. However, Riemann was the first to treat $\zeta(s)$ successfully as a function of a complex variable s (1859) [R1]; for this reason, it is called the **Riemann ζ -function**. As can be seen from its Euler product expansion, $\zeta(s)$ is holomorphic and has no zeros in the domain $\text{Re } s > 1$. Riemann proved, moreover, that it has an analytic continuation to the whole complex plane, is meromorphic everywhere, and has a unique pole $s = 1$. The functions $(s-1)\zeta(s)$ and $\zeta(s) - 1/(s-1)$ are † integral functions of s . This can be seen by considering the integral expression

$$\begin{aligned} \pi^{-s/2} \Gamma\left(\frac{s}{2}\right) \zeta(s) &= \int_0^{\infty} x^{s/2-1} \left(\sum_{n=1}^{\infty} e^{-n^2 \pi x} \right) dx \\ &= -\frac{1}{s} - \frac{1}{1-s} + \int_1^{\infty} (x^{(1-s)/2-1} + x^{(s/2-1)}) \\ &\quad \times \left(\sum_{n=1}^{\infty} e^{-n^2 \pi x} \right) dx. \end{aligned}$$

From this last formula, we also obtain an equality

$$\zeta(s) = \zeta(1-s),$$

where

$$\zeta(s) = \pi^{-s/2} \Gamma(s/2) \zeta(s).$$

This equality is called the **functional equation** for the ζ -function. The residue of $\zeta(s)$ at $s = 1$ is 1, and around $s = 1$,

$$\zeta(s) = \frac{1}{s-1} + C + O(|s-1|),$$

where C is † Euler's constant. This is called the **Kronecker limit formula** for $\zeta(s)$.

The function $\zeta(s)$ has no zeros in $\text{Re } s \geq 1$, and its only zeros in $\text{Re } s \leq 0$ are simple zeros at $s = -2, -4, \dots, -2n, \dots$. But $\zeta(s)$ has infinitely many zeros in $0 < \text{Re } s < 1$, which are called the nontrivial zeros. B. Riemann conjectured

tured that all nontrivial zeros lie on the line $\text{Re } s = 1/2$ (1859). This is called the **Riemann hypothesis**, which has been neither proved nor disproved (\rightarrow Section I).

If $N(T)$ denotes the number of zeros of $\zeta(s)$ in the rectangle $0 < \text{Re } s < 1$, $0 < \text{Im } s < T$, we have an asymptotic formula

$$N(T) = \frac{1}{2\pi} T \log T - \frac{1 + \log 2\pi}{2\pi} T + O(\log T)$$

(H. von Mangoldt, 1905). Also, $\zeta(s)$ has the following infinite product expansion:

$$(s-1)\zeta(s) = \frac{1}{2} e^{bs} \frac{1}{\Gamma\left(\frac{s}{2} + 1\right)} \prod_{\rho} \left(1 - \frac{s}{\rho}\right) e^{s/\rho},$$

where b is a constant and ρ runs over all nontrivial zeros of $\zeta(s)$ (J. Hadamard, 1893).

Hadamard and C. de La Vallée-Poussin proved the \dagger prime number theorem, almost simultaneously, by using some properties of $\zeta(s)$ (\rightarrow 123 Distribution of Prime Numbers B).

The following **approximate functional equation** is important in investigating the values of $\zeta(s)$:

$$\zeta(s) = \sum_{n \leq x} \frac{1}{n^s} + \varphi(s) \sum_{n < y} \frac{1}{n^{1-s}} + O(x^{-\sigma}) + O(y^{\sigma-1} |t|^{(1/2)-\sigma}),$$

where φ is \dagger Euler's function, and $\zeta(s) = \varphi(s)\zeta(1-s)$, $s = \sigma + it$, $2\pi xy = |t|$, and the approximation is uniform for $-h \leq \sigma \leq h$, $x > k$, $y > k$ with h and k positive constants (G. H. Hardy and J. E. Littlewood, 1921).

Euler obtained the values of $\zeta(s)$ for positive even integers s :

$$\zeta(2m) = \frac{2^{2m-1} \pi^{2m} B_{2m}}{(2m)!}$$

($m = 1, 2, 3, \dots$, and the B_{2m} are \dagger Bernoulli numbers). The values of $\zeta(s)$ for positive odd integers s , however, have not been expressed in such a simple form. The values of $\zeta(s)$ for negative integers s are given by $\zeta(0) = B_1(0)$

$$= -\frac{1}{2}, \zeta(1-n) = -\frac{B_n(0)}{n}, n = 2, 3, \dots, \text{ where}$$

the $B_n(x)$ are \dagger Bernoulli polynomials.

As a slight generalization of $\zeta(s)$, A. Hurwitz (1862) considered

$$\zeta(s, a) = \sum_{n=0}^{\infty} \frac{1}{(n+a)^s}, \quad 0 < a \leq 1.$$

This is called the **Hurwitz ζ -function**. Thus $\zeta(s, 1) = \zeta(s)$, and $\zeta(s, 1/2) = (2^s - 1)\zeta(s)$. This function $\zeta(s, a)$ can also be continued analytically to the whole complex plane and satisfies a certain functional equation. But in general it has no Euler product expansion.

C. Dirichlet L -Functions

Let m be a positive integer, and classify all rational integers modulo m . The set of all classes coprime to m forms a multiplicative Abelian group of order $h = \varphi(m)$. Let χ be a \dagger character of this group. Call (n) the residue class of $n \bmod m$, and put $\chi(n) = \chi((n))$ when $(n, m) = 1$ and $\chi(n) = 0$ when $(n, m) \neq 1$. Now, the function of a complex variable s defined by

$$L(s, \chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s}$$

is called a **Dirichlet L -function**. This function converges absolutely for $\text{Re } s > 1$ and has an Euler product expansion

$$L(s, \chi) = \prod_p \frac{1}{1 - \chi(p)p^{-s}}.$$

If there exist a divisor f of m ($f \neq m$) and a character χ^0 modulo f such that $\chi(n) = \chi^0(n)$ for all n with $(n, m) = 1$, we call χ a **nonprimitive character**. Otherwise, χ is called a **primitive character**. If χ is nonprimitive, there exists such a unique primitive χ^0 . In this situation, the divisor f of m associated with χ^0 is called the **conductor** of χ (and of χ^0). We have

$$L(s, \chi) = L(s, \chi^0) \prod_{p|m} (1 - \chi^0(p)p^{-s}).$$

Let χ be primitive. If the conductor $f = 1$, then χ is a trivial character ($\chi = 1$), and $L(s)$ is equal to the Riemann ζ -function $\zeta(s)$. On the other hand, if $f > 1$, then $L(s)$ is an entire function of s . In particular, if χ is a nontrivial primitive character, $L(1) = L(1, \chi)$ is finite and nonzero. P. G. L. Dirichlet proved the theorem of existence of prime numbers in arithmetic progressions using this fact (\rightarrow 123 Distribution of Prime Numbers D).

$L(s, \chi)$ has a functional equation similar to that of $\zeta(s)$; namely, if χ is a primitive character with conductor f and we put

$$\xi(s, \chi) = (f/\pi)^{s/2} \Gamma((s+a)/2) L(s, \chi),$$

where $a = 0$ for $\chi(-1) = 1$ and $a = 1$ for $\chi(-1) = -1$, then we have

$$\xi(s, \chi) = W(\chi) \xi(1-s, \bar{\chi}),$$

where

$$W(\chi) = (-i)^a f^{-1/2} \tau(\chi), \quad \tau(\chi) = \sum_{n \bmod f} \chi(n) \zeta_f^n$$

($\zeta_f = \exp(2\pi i/f)$). The latter sum is called the **Gaussian sum**. Note that $|W(\chi)| = 1$.

The values of $L(s)$ for negative integers s are given by $L(1-m, \chi) = -B_{\chi, m}/m$ ($m = 1, 2, 3, \dots$), where the $B_{\chi, m}$ are defined by

$$\sum_{\mu=1}^f \frac{\chi(\mu) t e^{\mu t}}{e^{f t} - 1} = \sum_{m=0}^{\infty} B_{\chi, m} t^m.$$

Moreover, if $\chi(-1) = -1$, we have

$$L(1, \chi) = \frac{\pi}{i} \frac{\tau(\chi)}{f} \sum_{x=1}^f (-\chi(\overline{x}) \cdot x) \\ = \pi i \tau(\chi) B_{\overline{x}, 1},$$

and if $\chi(-1) = 1$, $\chi \neq 1$, we have

$$L(1, \chi) = 2 \frac{\tau(\chi)}{f} \sum_{x=1}^{f/2} (-\chi(x) \log |1 - \zeta_f^x|).$$

In certain cases, the functional equation can be utilized to obtain the values of $L(m, \chi)$ from those of $L(1 - m, \chi)$. Actually, if $\chi(-1) = 1$, $m = 2n = 2, 4, 6, \dots$, we have

$$L(2n, \chi) = \frac{(-1)^n}{(2n)!} \left(\frac{2\pi}{f} \right)^{2n} \tau(\chi) (-B_{\chi, 2n}),$$

and if $\chi(-1) = -1$, $m = 2n + 1 = 3, 5, 7, \dots$, we have

$$L(2n + 1, \chi) \\ = (-i) \frac{(-1)^n}{(2n + 1)!} \left(\frac{2\pi}{f} \right)^{2n+1} \tau(\chi) (-B_{\chi, 2n+1}).$$

Dirichlet L -functions are important not only in the arithmetic of rational number fields but also in the arithmetic of quadratic or cyclotomic fields.

D. ζ -Functions of Algebraic Number Fields (Dedekind ζ -Functions)

The Riemann ζ -function can be generalized to ζ -functions of algebraic number fields (\rightarrow 14 Algebraic Number Fields). Let k be an algebraic number field of degree n , and let \mathfrak{a} run over all integral ideals of k . Consider the sequence $\zeta_k(s) = \sum_{\mathfrak{a}} N(\mathfrak{a})^{-s}$. This sequence converges for $\text{Re } s > 1$ and has an Euler product expansion $\zeta_k(s) = \prod_{\mathfrak{p}} (1 - N(\mathfrak{p})^{-s})^{-1}$, where \mathfrak{p} runs over all prime ideals of k . This function, which is continued analytically to the whole complex plane as a meromorphic function, is called a **Dedekind ζ -function**. Its only pole is a simple pole at $s = 1$, with the residue $h_k \kappa_k$. Here h_k is the class number of k , and $\kappa_k = 2^{r_1 + r_2} \pi^{r_2} R / (w |d|^{1/2})$, where r_1 ($2r_2$) is the number of isomorphisms of k into the real (complex) number field, w is the number of roots of unity in k , d is the discriminant of k , and R is the regulator of k (R. Dedekind, 1877) [D1].

The function $\zeta_k(s)$ has no zeros in $\text{Re } s \geq 1$, while in $\text{Re } s \leq 0$ it has zeros of order r_2 at $-1, -3, -5, \dots$, zeros of order $r_1 + r_2$ at $-2, -4, -6, \dots$, and a zero of order $r_1 + r_2 - 1$ at $s = 0$. All other zeros lie in the open strip $0 < \text{Re } s < 1$, which actually contains infinitely many zeros. It is conjectured that all these zeros lie on the line $\text{Re } s = 1/2$ (the Riemann hypothesis for Dedekind ζ -functions). To obtain a generali-

zation of the functional equation for the Riemann $\zeta(s)$ to the case of $\zeta_k(s)$, we put

$$\Xi_k(s) = \left(\frac{\sqrt{|d|}}{2^{r_2} \pi^{n/2}} \right)^s \Gamma\left(\frac{s}{2}\right)^{r_1} \Gamma(s)^{r_2} \zeta_k(s).$$

Then $\Xi_k(s) = \Xi_k(1 - s)$ (Hecke, 1917). If K is a Galois extension of k , then $\zeta_K(s)/\zeta_k(s)$ is an integral function (H. Aramata, 1933; R. Brauer, 1947).

E. Hecke L -Functions

As a generalization of Dirichlet L -functions to algebraic number fields, Hecke (1917) defined the following L -function $L_k(s, \chi)$: Let k be an algebraic number field of finite degree, and let $\mathfrak{m} = \mathfrak{m} \prod_{\infty}$ be an integral divisor (\mathfrak{m} the finite part, \prod_{∞} the infinite part). Consider the ideal class group of k modulo \mathfrak{m} and its character χ (here we put $\chi(\mathfrak{a}) = 0$ for $(\mathfrak{a}, \mathfrak{m}) \neq 1$). Then the L -functions are defined by

$$L_k(s, \chi) = \sum_{\mathfrak{a}} \chi(\mathfrak{a}) / N(\mathfrak{a})^s$$

[H2], where \mathfrak{a} runs over all integral ideals of k . $L_k(s, \chi)$ is called a **Hecke L -function**. It converges for $\text{Re } s > 1$ and has an Euler product expansion

$$L_k(s, \chi) = \prod_{\mathfrak{p}} \frac{1}{1 - \chi(\mathfrak{p}) N(\mathfrak{p})^{-s}}.$$

Here \mathfrak{p} runs over all prime ideals of k . If there is a divisor $\tilde{\mathfrak{f}} | \mathfrak{m}$ ($\tilde{\mathfrak{f}} \neq \mathfrak{m}$) and a character χ^0 modulo $\tilde{\mathfrak{f}}$ such that $\chi^0(\mathfrak{a}) = \chi(\mathfrak{a})$ for all \mathfrak{a} with $(\mathfrak{a}, \mathfrak{m}) = 1$, then χ is called **nonprimitive**; otherwise, χ is called a **primitive character**. In general, there exist unique such $\tilde{\mathfrak{f}}$ and χ^0 . In this situation, $\tilde{\mathfrak{f}}$ is called the **conductor** of χ . If χ is primitive and the conductor $\tilde{\mathfrak{f}}$ is (1) , then χ is a trivial character and $L_k(s, \chi)$ coincides with $\zeta_k(s)$. If χ is primitive and $\chi \neq 1$, then $L_k(s, \chi)$ is an integral function of s , and $L_k(1, \chi) \neq 0$. Utilizing this fact, it can be proved that there exist infinitely many prime ideals in each class of the ideal class group modulo an integral divisor \mathfrak{m} of k .

Let χ be a primitive character with the conductor $\tilde{\mathfrak{f}}$, d be the discriminant of k , $\sigma_1, \dots, \sigma_{r_1}$ be all distinct isomorphisms of k into the real number field \mathbf{R} , and \mathfrak{f} be the finite part of $\tilde{\mathfrak{f}}$. Then if ξ is an integer of k such that $\xi \equiv 1 \pmod{\mathfrak{f}}$, we have

$$\chi((\xi)) = (\text{sgn } \xi^{\sigma_1})^{a_1} \cdots (\text{sgn } \xi^{\sigma_{r_1}})^{a_{r_1}},$$

where a_m ($m = 1, \dots, r_1$) is either 0 or 1, depending on χ . By putting

$$\xi_k(s, \chi) = \left(\frac{\sqrt{|d| N(\mathfrak{f})}}{2^{r_2} \pi^{n/2}} \right)^s \cdot \prod_{m=1}^{r_1} \Gamma\left(\frac{s + a_m}{2}\right) \Gamma(s)^{r_2} \cdot L_k(s, \chi),$$

we have the following functional equation for the Hecke L -function:

$$\zeta_k(s, \chi) = W(\chi) \zeta_k(1-s, \bar{\chi}),$$

where $W(\chi)$ is a complex number with absolute value 1 and the exact value of $W(\chi)$ is given as a Gaussian sum. Just as some properties concerning the distribution of prime numbers can be proved using the Riemann ζ -function and Dirichlet L -functions, some properties concerning the distribution of prime ideals can be proved using the Hecke L -functions (\rightarrow 123 Distribution of Prime Numbers F).

T. Takagi used Hecke L -functions in founding his \dagger class field theory. In the other direction, this theory implies $L(1, \chi) \neq 0$ ($\chi \neq 1$).

Let K be a \dagger class field over k that corresponds to an ideal class group H of k with index h . By using class field theory, we obtain $\zeta_K(s) = \prod_{\chi} L_k(s, \chi)$, where the product is over all characters χ of ideal class groups of k , such that $\chi(H) = 1$. This formula can be regarded as an alternative formulation of the decomposition theorem of class field theory (\rightarrow 59 Class Field Theory). By taking the residues of both sides of the formula at $s = 1$, we obtain $h_K \kappa_K = h_k \kappa_k \prod_{\chi \neq 1} L_k(1, \chi)$.

In particular, if $k = \mathbf{Q}$ (the rational number field) and K is a quadratic number field $\mathbf{Q}(\sqrt{d})$ (d is the discriminant of K), then we have

$$\zeta_K(s) = \zeta(s) \cdot L(s), \quad L(s) = \sum_{n=1}^{\infty} \left(\frac{d}{n}\right) n^{-s},$$

where (d/n) is the \dagger Kronecker symbol, and we put $(d/n) = 0$ when $(n, d) \neq 1$. From this, we obtain the class number formula for quadratic number fields (\rightarrow 347 Quadratic Fields). A similar method is used for computation of class numbers of cyclotomic fields K (\rightarrow 14 Algebraic Number Fields L).

In general, the computation of the relative class number h_K/h_k when K/k is an Abelian extension is reduced to the evaluation of $L(1, \chi)$. This computation has been made successfully for the following cases (besides for the examples in the previous paragraph): k is imaginary quadratic and K is the absolute class field of k or the class field corresponding to \dagger ray $S(m)$; k is totally real and K is a totally imaginary quadratic extension of k . H. M. Stark and T. Shintani made conjectures about the values of $L(1, \chi)$ [S25, S19].

Let $L(s, \chi)$ be a Hecke L -function for the character χ . Then it follows from the functional equation that the values of $L(s, \chi)$ at $s = 0, -1, -2, -3, \dots$ are zero if k is not totally real. Furthermore, if k is a totally real finite algebraic number field, then these values of $L(s, \chi)$ are algebraic numbers (C. L. Siegel, H. Klingen, T. Shintani).

F. Hecke L -Functions with Grössencharakteren

E. Hecke (1918, 1920) extended the notion of characters by introducing the \dagger Grössencharakter χ and defined L -functions with such characters:

$$L_k(s, \chi) = \sum_{\mathfrak{a}} \frac{\chi(\mathfrak{a})}{N(\mathfrak{a})^s}.$$

He also proved the existence of their Euler product expansions and showed that they satisfy certain functional equations [H3]. Moreover, by estimating the sum $\sum_{N(\mathfrak{p}) < x} \chi(\mathfrak{p})$, he obtained some results on the distribution of prime ideals.

Later, Iwasawa and J. Tate independently gave clearer definitions of the Grössencharakter χ and $L_k(s, \chi)$ by using harmonic analysis on the adèle and idele groups of k (\rightarrow 6 Adeles and Ideles) [L3].

Let \mathbf{J}_k be the idele group of k , \mathbf{P}_k be the group of \dagger principal ideles, and $\mathbf{C}_k = \mathbf{J}_k/\mathbf{P}_k$ be the idele class group. Then a Grössencharakter is a continuous character χ of \mathbf{C}_k , and χ induces a character of \mathbf{J}_k , which is also denoted by χ . Let $\mathbf{J}_k = \mathbf{J}_{\infty} \times \mathbf{J}_0$ be the decomposition of \mathbf{J}_k into the infinite part \mathbf{J}_{∞} and the finite part \mathbf{J}_0 . Let \mathbf{U}_0 be the unit group of \mathbf{J}_0 , and for each integral ideal \mathfrak{m} of k , put $\mathbf{U}_{\mathfrak{m},0} = \{u \in \mathbf{U}_0 \mid u \equiv 1 \pmod{\mathfrak{m}}\}$, so that $\{\mathbf{U}_{\mathfrak{m},0}\}$ forms a base for the neighborhood system of 1 in \mathbf{J}_0 . Put $\mathbf{J}_{\mathfrak{m},0} = \{a \in \mathbf{J}_0 \mid a_{\mathfrak{p}} = 1 \text{ for all } \mathfrak{p} \mid \mathfrak{m}\}$, and with each $a \in \mathbf{J}_{\mathfrak{m},0}$, associate an ideal $\tilde{a} = \prod_{\mathfrak{p}} \mathfrak{p}^{v_{\mathfrak{p}}(a)}$, where $a = (a_{\mathfrak{p}})$ and the ideal in $k_{\mathfrak{p}}$ generated by $a_{\mathfrak{p}}$ is equal to $\mathfrak{p}^{v_{\mathfrak{p}}(a)}$. Then the mapping $a \rightarrow \tilde{a}$ gives a homomorphism of $\mathbf{J}_{\mathfrak{m},0}$ into the group $G(\mathfrak{m}) = \{\tilde{a} \mid (a, \mathfrak{m}) = 1\}$, and its kernel is contained in $\mathbf{U}_{\mathfrak{m},0}$. Since χ is continuous, $\chi(\mathbf{U}_{\mathfrak{m},0}) = 1$ for some \mathfrak{m} . The greatest common divisor \mathfrak{f} of all such ideals \mathfrak{m} is called the **conductor** of χ . For each $a \in \mathbf{J}_{\mathfrak{f},0}$, $\chi(a)$ depends only on the ideal \tilde{a} ($\in G(\mathfrak{f})$); hence by putting $\chi(a) = \tilde{\chi}(\tilde{a})$, we obtain a character $\tilde{\chi}$ of $G(\mathfrak{f})$. Now put $L_k(s, \chi) = \sum \tilde{\chi}(\tilde{a})/N(\tilde{a})^s$, where the sum is over all integral ideals $\tilde{a} \in G(\mathfrak{f})$. This is called a **Hecke L -function with Grössencharakter** χ . For $\chi \neq 1$, it is an entire function. On the other hand, if we restrict χ to $\mathbf{J}_{\infty} = \mathbf{R}^{*r_1} \times \mathbf{C}^{*r_2}$, then for $u = (a_1, \dots, a_{r_1}, a_{r_1+1}, \dots, a_{r_1+r_2}) \in \mathbf{J}_{\infty}$, we have

$$\chi(u) = \prod_{j=1}^{r_1+r_2} |a_j|^{\lambda_j \sqrt{-1}} \cdot \prod_{j=1}^{r_1} (\operatorname{sgn} a_j)^{e_j} \cdot \prod_{l=r_1+1}^{r_1+r_2} \left(\frac{a_l}{|a_l|}\right)^{e_l},$$

where $e_j = 0$ or 1 , $e_l \in \mathbf{Z}$, $\lambda_j \in \mathbf{R}$. The numbers e_j , e_l , λ_j are determined uniquely by χ . Putting

$$\begin{aligned} \zeta_k(s, \chi) &= \left(\frac{\sqrt{|d|N(\mathfrak{f})}}{2^{r_2} \pi^{n/2}}\right)^s \cdot \prod_{j=1}^{r_1} \Gamma\left(\frac{s + e_j + \lambda_j \sqrt{-1}}{2}\right) \\ &\quad \times \prod_{l=r_1+1}^{r_1+r_2} \Gamma\left(s + \frac{|e_l| + \lambda_l \sqrt{-1}}{2}\right) \cdot L(s, \chi), \end{aligned}$$

we have a functional equation

$$\xi_k(s, \chi) = W(\chi) \xi_k(1-s, \bar{\chi}),$$

where $W(\chi)$ is a complex number with absolute value 1.

We can express $\xi_k(s, \chi)$ by an integral form on \mathbf{J}_k as

$$\xi_k(s, \chi) = c \int_{\mathbf{J}_k} \varphi(r) \chi(r) V(r)^s d^*r,$$

where $V(r)$ is the total volume of the idele r , c is a constant that depends on the Haar measure d^*r of \mathbf{J}_k , and $\varphi(r)$ is defined by

$$\varphi(r) = \prod_p \varphi_p(x_p), \quad r = (\dots x_p \dots),$$

$$\begin{aligned} \varphi_{p_{x,i}}(x) &= x^{e_i} e^{-\pi x^2}, \quad i \leq r_1, \quad k_{p_{x,i}} = \mathbf{R}, \\ &= \frac{1}{2\pi} \bar{x}^{e_i} e^{-2\pi|x|^2}, \quad e_i \geq 0, \\ &= \frac{1}{2\pi} x^{-e_i} e^{-2\pi|x|^2}, \quad e_i < 0, \end{aligned} \left\{ \begin{array}{l} i > r_1, \quad k_{p_{x,i}} = \mathbf{C}, \end{array} \right.$$

$$\begin{aligned} \varphi_p(x) &= e^{2\pi i \lambda(x)}, \quad x \in (\mathfrak{bf})_p^{-1}, \\ &= 0, \quad x \notin (\mathfrak{bf})_p^{-1} \end{aligned} \left\{ \begin{array}{l} p \text{ finite.} \end{array} \right.$$

Hence $(\mathfrak{bf})_p^{-1}$ is the p -component (\rightarrow 6 Adeles and Ideles B) of the ideal $(\mathfrak{bf})^{-1}$ (\mathfrak{b} is the different of k/\mathbf{Q}) and $\lambda(x)$ is an additive character of k_p defined as follows. \mathbf{Q}_p is the p -adic field, \mathbf{Z}_p is the ring of p -adic integers, λ_0 is the mapping $\mathbf{Q}_p \rightarrow \mathbf{Q}_p/\mathbf{Z}_p \subset \mathbf{Q}/\mathbf{Z} \subset \mathbf{R}/\mathbf{Z}$, and $\lambda = \lambda_0 \circ \text{Tr}_{k_p/\mathbf{Q}_p}$. By putting $\chi(r) = \prod_p \chi_p(x_p)$, $r = (\dots x_p \dots)$, we have

$$\xi_k(s, \chi) = c \prod_p \int_{k_p} \varphi_p(x) \chi_p(x) V_p(x)^{-s} d^*x,$$

where p runs over all prime divisors of k , finite or infinite. Moreover, with a constant C_p , we have

$$\begin{aligned} &\int_{k_{p_{x,i}}} \varphi_{p_{x,i}}(x) \chi_{p_{x,i}}(x) V_{p_{x,i}}(x)^{-s} d^*x \\ &= C_{p_{x,i}} \cdot \pi^{-(s + \sqrt{-1} \lambda_i + e_i)/2} \\ &\quad \times \Gamma((s + \sqrt{-1} \lambda_i + e_i)/2), \quad k_{p_{x,i}} = \mathbf{R}, \\ &= C_{p_{x,i}} \cdot (2\pi)^{-(s + \sqrt{-1} \lambda_i + |e_i|)/2} \\ &\quad \times \Gamma(s + (\sqrt{-1} \lambda_i + |e_i|)/2), \quad k_{p_{x,i}} = \mathbf{C}, \\ &\int_{k_p} \varphi(x) \chi_p(x) V_p(x)^{-s} d^*x \\ &= C_p N(\mathfrak{bf})^{s-1/2} \tilde{\chi}(\mathfrak{bf}^{-1}) \frac{1}{1 - \tilde{\chi}(\mathfrak{p})/N(\mathfrak{p})^s}, \\ &= C_p N((\mathfrak{bf})_p)^s \tau_p(\chi_p) \cdot \mu(U_{\mathfrak{f},p}), \quad p \nmid \mathfrak{f}. \end{aligned}$$

Here $\tau_p(\chi_p)$ is a constant called the **local Gaussian sum**, and $\mu(U_{\mathfrak{f},p})$ is the volume of $\{u \in k_p \mid u \equiv 1 \pmod{\mathfrak{f}}\}$. These integrals over k_p are the Γ -factors and Euler factors of $\xi_k(s, \chi)$, according as p is infinite or finite. The func-

tional equation is obtained by applying the * Poisson summation formula for $\varphi(x)$ and its * Fourier transform on the adèle group $\mathbf{A}_k (\rightarrow$ 6. Adeles and Ideles).

Let \mathbf{D}_k be the connected component of 1 in \mathbf{C}_k . If $\chi(\mathbf{D}_k) = 1$, the corresponding $\tilde{\chi}$ is a character of an ideal class group of k with a conductor \mathfrak{f} . Conversely, all such characters can be obtained in this manner.

As stated in Section E, the Hecke L -functions with characters (of ideal class groups) can be used to describe the decomposition law of prime divisors in class field theory. However, for L -functions with Grössencharakter, such arithmetic implications have not been found yet, except that in the case of Grössencharakter of A_0 type, Y. Taniyama discovered, following the suggestion of A. Weil, that the L -function has a deep connection with the arithmetic of a certain infinite Abelian extension of k [T2, W7]. In particular, when $L(s, \chi)$ is a factor of the * Hasse ζ -function of an Abelian variety A with * complex multiplication, it describes the arithmetic of the field generated by the coordinates of the division points of A .

G. Artin L -Functions

Let K be a finite Galois extension of an algebraic number field k (of degree n), $G = G(K/k)$ be its Galois group, $\sigma \rightarrow A(\sigma)$ be a matrix representation (characteristic 0) of G , and χ be its character. Let \mathfrak{p} be a prime ideal of k , and define $L_p(s, \chi)$ by

$$\log L_p(s, \chi) = \sum_{m=1}^{\infty} \frac{\chi(\mathfrak{p}^m)}{mN(\mathfrak{p}^m)^s}, \quad \text{Re } s > 1,$$

with $\chi(\mathfrak{p}^m) = (1/e) \sum_{\tau \in T} \chi(\sigma^m \tau)$, where T is the * inertia group of \mathfrak{p} , $|T| = e$, and σ is a * Frobenius automorphism of \mathfrak{p} . Then we have

$$L_p(s, \chi) = \det(E - A_p \cdot N(\mathfrak{p})^{-s})^{-1},$$

$$A_p = \frac{1}{e} \sum_{\tau \in T} A(\sigma \tau).$$

In particular, if $T = \{1\}$ (i.e., \mathfrak{p} is * unramified in K/k), then

$$L_p(s, \chi) = \det(E - A(\sigma) \cdot N(\mathfrak{p})^{-s})^{-1}.$$

Now put

$$L(s, \chi, K/k) = \prod_p L_p(s, \chi), \quad \text{Re } s > 1,$$

and call $L(s, \chi, K/k)$ an **Artin L -function** [A2].

(1) The most important property of $L(s, \chi, K/k)$ is that if K/k is an Abelian extension and χ is a linear character, it follows from class field theory that $\chi(\mathfrak{p})$ is the character of the ideal class group of k (modulo the conductor of K/k) and that the Artin L -function equals

a Hecke L -function. This equality is equivalent to Artin's reciprocity law, and in fact Artin obtained his reciprocity law after he conjectured the equality.

(2) If $K' \supset K \supset k$ and K'/k is a Galois extension, then $L(s, \chi, K/k) = L(s, \chi, K'/k)$.

(3) If $K \supset \Omega \supset k$ and ψ is a character of $G(K/\Omega)$, then $L(s, \psi, K/\Omega) = L(s, \chi_\psi, K/k)$, where χ_ψ is the character of $G(K/k)$ induced from ψ .

(4) If $\chi_1 = 1$, then $L(s, \chi_1, K/k) = \zeta_k(s)$.

(5) $L(s, \chi_1 + \chi_2, K/k) = L(s, \chi_1, K/k) \cdot L(s, \chi_2, K/k)$.

Conversely, the Artin L -function $L(s, \chi, K/k)$ is characterized by properties (1)–(5).

(6) If χ_R is the regular representation of G , then $L(s, \chi_R, K/k) = \zeta_K(s)$; hence

$$\zeta_K(s) = \zeta_k(s) \prod_{\chi \neq 1} L(s, \chi, K/k)^{d(\chi)},$$

where χ runs over all irreducible characters $\neq 1$ of G .

(7) Every character of a finite group G can be expressed as $\chi = \sum m_i \chi_{\psi_i}$ ($m_i \in \mathbb{Z}$), where each χ_{ψ_i} is an induced character from a certain linear character ψ_i of an elementary subgroup of G (**Brauer's theorem**). (Here an elementary subgroup is a subgroup that is the direct product of a cyclic group and a p -group for some prime p .) Hence (3) and (5) imply that an Artin L -function is the product of integral powers (positive or negative) of Hecke L -functions $L_{\Omega_i}(s, \psi_i)$:

$$L(s, \chi, K/k) = \prod_i L_{\Omega_i}(s, \psi_i)^{m_i}.$$

Hence an Artin L -function is a univalent meromorphic function defined over the whole complex plane. Artin made the still open conjecture that if χ is irreducible and $\chi \neq 1$, then $L(s, \chi, K/k)$ is an entire function (**Artin's conjecture**).

This conjecture holds obviously if all m_i are nonnegative. Except for such a case, Artin's conjecture had no affirmative examples until 1974, when Deligne and Serre [D9] proved that each "new cusp form" of weight 1 gives rise to an entire Artin L -function $L(s, \chi, K/k)$ with $\chi(1) = 2$ and $\chi(\rho) = 0$ (ρ is the complex conjugation); by this method, some nontrivial examples were computed by J. Tate and J. Buhler (*Lecture notes in math.* 654 (1978)). Then R. P. Langlands [L5] constructed nontrivial examples of Artin's conjecture for certain 2-dimensional representations

$$\text{Gal}(K/k) \ni \sigma \mapsto A(\sigma) \in \text{GL}(2, \mathbb{C})$$

by using ideas of H. Saito and T. Shintani [S1, S20]. This method works for all representations for which the image of the $A(\sigma)$ in $\text{PGL}(2, \mathbb{C})$ is the tetrahedral group. It also works for some octahedral cases, but a new idea is needed in the icosahedral case.

(8) Let $\mathfrak{p}_{x,i}$ ($i = 1, \dots, r_1 + r_2$) be the infinite primes of k . Put

$$\gamma(s, \chi, \mathfrak{p}_{x,i}, K/k) = (\Gamma(s/2) \Gamma((s+1)/2))^{x(1)}$$

for complex $\mathfrak{p}_{x,i}$,

$$= \Gamma(s/2)^{(x(1) + \chi(\sigma))/2} \Gamma((s+1)/2)^{(x(1) - \chi(\sigma))/2}$$

for real $\mathfrak{p}_{x,i}$,

where $\sigma \in G$ is the complex conjugation determined by a prime factor of $\mathfrak{p}_{x,i}$ in K . Next we introduce the notion of the **conductor** \mathfrak{f}_χ with the **group character** χ defined by Artin (*J. Reine Angew. Math.*, 164 (1931)). First, for any subset $m \subset G$, we put $\chi(m) = \sum_{g \in m} \chi(g)$; then \mathfrak{f}_χ is given by

$$\mathfrak{f}_\chi = \mathfrak{f}(\chi, K/k) = \prod_{\mathfrak{p}} \mathfrak{p}^{f(\mathfrak{p})},$$

where

$$f(\mathfrak{p}) = \frac{1}{e} [\{e\chi(1) - \chi(T)\} + \{p^{e_1}\chi(1) - \chi(V_1)\} + \{p^{e_2}\chi(1) - \chi(V_2)\} + \dots],$$

and where V_1, V_2, \dots , are the higher ramification groups of prime factors of \mathfrak{p} in K (in lower numbering) and $p^{e_i} = |V_i|$ (\rightarrow 14 Algebraic Number Fields I).

Now put

$$\xi(s, \chi, K/k) = \left(\frac{|d|^{x(1)} N_k(\mathfrak{f}_\chi)}{\pi^{n_{x(1)}}} \right)^{s/2} \times \prod_{\mathfrak{p}_{x,i}} \gamma(s, \chi, \mathfrak{p}_{x,i}, K/k) \cdot L(s, \chi, K/k).$$

Then the functional equation is written

$$\xi(1-s, \bar{\chi}, K/k) = W(\chi) \xi(s, \chi, K/k), \quad |W(\chi)| = 1.$$

The known proof of this functional equation depends on (7) and the functional equations of Hecke L -functions discussed in Section E. As for the constants $W(\chi)$, there are significant results by B. Dwork, Langlands, and Deligne [D6].

(9) There are some applications to the theory of the distribution of prime ideals.

H. Weil L -Functions

Weil defined a new L -function that is a generalization of both Artin L -functions and Hecke L -functions with Grössencharakter [W5]. Let K be a finite Galois extension of an algebraic number field k , let C_K be the idele class group K_A^\times / K^\times of K , and let $\alpha_{K,k} \in H^2(\text{Gal}(K/k), C_K)$ be the canonical cohomology class of class field theory. Then this $\alpha_{K,k}$ determines an extension $W_{K,k}$ of $\text{Gal}(K/k)$ by $C_K: 1 \rightarrow C_K \rightarrow W_{K,k} \rightarrow \text{Gal}(K/k) \rightarrow 1$ (exact), and

the transfer induces an isomorphism $W_{K/k}^{ab} \simeq C_k$, where ab denotes the topological commutator quotient. If L is a Galois extension of k containing K , then there is a canonical homomorphism $W_{L/k} \rightarrow W_{K/k}$. Hence we define the **Weil group** W_k for \bar{k}/k as the \dagger projective limit group $\text{proj}_K \lim W_{K/k}$ of the $W_{K/k}$. It is obvious that we have a surjective homomorphism $\varphi: W_k \rightarrow \text{Gal}(\bar{k}/k)$ and an isomorphism $r_k: C_k \rightarrow W_k^{ab}$, where W_k^{ab} is the maximal Abelian Hausdorff quotient of W_k . For $w \in W_k$, let $\|w\|$ be the adelic norm of $r_k^{-1}(w)$.

If k_v is a \dagger local field, then we define the Weil group W_{k_v} for \bar{k}_v/k_v by replacing the idele class group C_k with the multiplicative group K_w^\times in the above definition, where K_w denotes a Galois extension of k_v . If k_v is the completion of a finite algebraic number field k at a place v , then we have natural homomorphisms $k_v^\times \rightarrow C_k$ and $\text{Gal}(\bar{k}_v/k_v) \rightarrow \text{Gal}(\bar{k}/k)$. Accordingly, we have a homomorphism $W_{k_v} \rightarrow W_k$ that commutes with these homomorphisms.

Let W_k be the Weil group of an algebraic number field k , and let $\rho: W_k \rightarrow GL(V)$ be a continuous representation of W_k on a complex vector space V . Let $v = \mathfrak{p}$ be a finite prime of k , and let ρ_v be the representation of W_{k_v} induced from ρ . Let Φ be an element of W_{k_v} such that $\varphi(\Phi)$ is the inverse Frobenius element of \mathfrak{p} in $\text{Gal}(\bar{k}_v/k_v)$, and let I be the subgroup of W_{k_v} consisting of elements w such that $\varphi(w)$ belongs to the \dagger inertia group of \mathfrak{p} in $\text{Gal}(\bar{k}_v/k_v)$. Let V^I be the subspace of elements in V fixed by $\rho_v(I)$, let $N_{\mathfrak{p}}$ be the norm of \mathfrak{p} , and let

$$L_{\mathfrak{p}}(V, s) = \det(1 - (N_{\mathfrak{p}})^{-s} \rho_v(\Phi) | V^I)^{-1}.$$

We can define $L_v(V, s)$ for each Archimedean prime v also, and let

$$L(V, s) = \prod_v L_v(V, s).$$

Then this product converges for s in some right half-plane and defines a function $L(V, s)$. We call $L(V, s)$ the **Weil L -function** for the representation $\rho: W_k \rightarrow GL(V)$. This function $L(V, s)$ can be extended to a meromorphic function on the complex plane and satisfies the functional equation

$$L(V, s) = \varepsilon(V, s) L(V^*, 1 - s)$$

(T. Tamagawa), where V^* is the dual of V , and $\varepsilon(V, s)$ is an exponential function of s of the form ab^s [T6].

P. Deligne generalized these results in the following manner: Let W'_k be a \dagger group scheme over \mathbb{Q} which is the \dagger semidirect product of W_k by the additive group G_a , on which W_k acts by the rule $wxw^{-1} = \|w\|x$. We can define the notion of representations of W'_k and the L -functions of them in the natural manner [T6].

I. The Riemann Hypothesis

As mentioned in Section B, the Riemann hypothesis asserts that all zeros of the Riemann ζ -function in $0 < \text{Re } s < 1$ lie on the line $\text{Re } s = 1/2$. In his celebrated paper [R1], Riemann gave six conjectures (including this), and assuming these conjectures, proved the \dagger prime number theorem:

$$\pi(x) \sim \frac{x}{\log x} \sim \text{Li}(x) = \int_2^x \frac{dx}{\log x}, \quad x \rightarrow \infty.$$

Here $\pi(x)$ denotes the number of prime numbers smaller than x . Among his six conjectures, all except the Riemann hypothesis have been proved (a detailed discussion is given in [L1]). The prime number theorem was proved independently by Hadamard and de La Vallée-Poussin without using the Riemann hypothesis (— Section B; 123 Distribution of Prime Numbers B).

R. S. Lehman showed that there are exactly 2,500,000 zeros of $\zeta(\sigma + it)$ for which $0 < t < 170,571.35$, all of which lie on the critical line $\sigma = 1/2$ and are simple (*Math. Comp.*, 20 (1966)). Later R. P. Brent extended this computation up to 75,000,000 first zeros (1979).

Hardy proved that there are infinitely many zeros of $\zeta(s)$ on the line $\text{Re } s = 1/2$ (1914). Furthermore, A. Selberg [S6] proved that if $N_0(T)$ is the number of zeros of $\zeta(s)$ on the line with $0 < \text{Im } s < T$, then $N_0(T) > AT \log T$ (A is a positive constant) (1942). Thus if $N(T)$ is the number of zeros of $\zeta(s)$ in the rectangle $0 < \text{Re } s < 1$, $0 < \text{Im } s < T$, then $\liminf_{T \rightarrow \infty} N_0(T)/N(T) > 0$. N. Levinson proved $\liminf_{T \rightarrow \infty} N_0(T)/N(T) > 1/3$ (*Advances in Math.*, 13 (1974)). If $N_{\varepsilon}(T)$ is the number of zeros of $\zeta(s)$ in $1/2 - \varepsilon < \text{Re } s < 1/2 + \varepsilon$, $0 < \text{Im } s < T$, then $\lim_{T \rightarrow \infty} N_{\varepsilon}(T)/N(T) = 1$ for any positive number ε (H. Bohr and E. Landau, 1914). Bohr studied the distribution of the values of $\zeta(s)$ in detail and initiated the theory of \dagger almost periodic functions (1925).

D. Hilbert remarked in his lecture at the Paris Congress that the Riemann hypothesis is equivalent to

$$\pi(x) = \text{Li}(x) + O(\sqrt{x} \log x), \quad x \rightarrow \infty$$

(H. von Koch, 1901). It is also equivalent to

$$\sum_{n=1}^N \mu(n) = O(N^{1/2+\varepsilon}), \quad N \rightarrow \infty,$$

for any $\varepsilon > 0$, where $\mu(n)$ is the Möbius function. Assuming the Riemann hypothesis, we get

$$N(T) = \frac{1}{2\pi} T \log T - \frac{1 + \log 2\pi}{2\pi} T + o(\log T)$$

(Littlewood, 1924).

The computation of the zeros of the ζ -functions and the L -functions of general algebraic number fields is more difficult, but conjectures similar to the Riemann hypothesis have been proposed.

Weil showed that a necessary and sufficient condition for the validity of the Riemann hypothesis for all Hecke L -functions $L(s, \chi)$ is that a certain \dagger distribution on the idele group J_K be positive definite [W1 (1952b)].

It is not known whether the general ζ - and L -functions of algebraic number fields have any zeros in the interval $(0, 1)$ on the real axis (see the works of A. Selberg and S. Chowla). Similar problems are considered for the various ζ -functions given in Sections P, Q, and T.

J. p -Adic L -Functions

Let χ be a \dagger primitive Dirichlet character with conductor f , and let $L(s, \chi)$ be the \dagger Dirichlet L -function for χ . Then the values $L(1-n, \chi)$ of $L(s, \chi)$ at nonpositive integers $1-n$ ($n=1, 2, \dots$) are algebraic numbers (\rightarrow Section E). Let p be a prime number, let \mathbf{Q}_p be the $\dagger p$ -adic number field, and let \mathbf{C}_p be the completion of the algebraic closure $\bar{\mathbf{Q}}_p$ of \mathbf{Q}_p . It is known that \mathbf{C}_p is also algebraically closed. Since $\mathbf{Q} \subset \mathbf{Q}_p$, we fix an embedding $\bar{\mathbf{Q}} \subset \bar{\mathbf{Q}}_p$ and consider $\{L(1-n, \chi)\}_{n=1}^\infty$ as a sequence in \mathbf{C}_p .

Let $| \cdot |_p$ be the extension to \mathbf{C}_p of the standard p -adic valuation of \mathbf{Q}_p . Let q be p or 4 according as $p \neq 2$ or $p=2$, and let ω be the primitive Dirichlet character with conductor q satisfying $\omega(n) \equiv n \pmod{q}$ for any integer n prime to p . Then T. Kubota and H. W. Leopoldt proved that there exists a unique function $L_p(s, \chi)$ satisfying the conditions [K5]:

- (1) $L_p(s, \chi) = \frac{a_{-1}}{s-1} + \sum_{n=0}^\infty a_n(s-1)^n \quad (a_n \in \mathbf{C}_p)$;
- (2) $a_{-1} = 0$ if $\chi \neq 1$ and the series $\sum_{n=0}^\infty a_n(s-1)^n$ converges for $|s-1|_p < |q^{-1}p^{1/(p-1)}|_p$;
- (3) $L_p(1-n, \chi) = (1 - \chi\omega^{-n}p^{1-n})L(1-n, \chi\omega^{-n})$ holds for $n=1, 2, 3, \dots$.

The function $L_p(s, \chi)$ satisfying these three conditions is called the **p -adic L -function** for the character χ . It is easy to see that $L_p(s, \chi)$ is identically zero if $\chi(-1) = -1$, but $L_p(s, \chi)$ is nontrivial if $\chi(-1) = 1$.

Let B_n be the Bernoulli number. Then B_n satisfies the conditions: (1) B_n/n is p -integral if $(p-1) \nmid n$ (von Staudt) and (2) $(1/n)B_n \equiv (1/(n+p-1))B_{n+p-1} \pmod{p}$ holds in this case (Kummer). The generalization of these results for the generalized Bernoulli number $B_{\chi, n}$ was obtained by Leopoldt. Since $L(1-n, \chi) = -(1/n)B_{\chi, n}$, such p -integrabilities and congruences can be naturally interpreted and

generalized in terms of the p -adic L -functions $L(s, \chi)$.

We assume $\chi(-1) = 1$. Then $L_p(0, \chi) = (1 - \chi\omega^{-1}(p))L(0, \chi\omega^{-1})$ and $\chi\omega^{-1}(-1) = -1$. Hence we can express the first factor h_N^- of the class number of a cyclotomic field $\mathbf{Q}(\exp(2\pi i/N))$ as a product of some $L_p(0, \chi)$'s. By using this fact, K. Iwasawa proved [I7] that the p -part $p^{e_n^-}$ of the \dagger first factor $h_{Np^n}^-$ ($N \in \mathbf{N}$) satisfies

$$e_n^- = \lambda n + \mu p^n + v \quad (\lambda, \mu, v \in \mathbf{Z}; \lambda, \mu \geq 0)$$

for any sufficiently large n . Here Iwasawa conjectured $\mu = 0$, which was proved by B. Ferrero and L. Washington [F1]. Also, we can obtain some congruences involving the first factor h_N^- of $\mathbf{Q}(\exp(2\pi i/N))$ from this formula.

Let χ be a nontrivial primitive Dirichlet character with conductor f , let

$$\tau(\chi) = \sum_{a=1}^f \chi(a) e^{2\pi i a/f}$$

be the \dagger Gaussian sum for χ , and let \log_p be the p -adic logarithmic function. Then Leopoldt [L6] calculated the value $L(1, \chi)$ and obtained

$$L_p(1, \chi) = - \left(1 - \frac{\chi(p)}{p} \right) \frac{\tau(\chi)}{f} \sum_{a=1}^f \chi(a) \log_p(1 - e^{-2\pi i a/f}).$$

As an application of this formula, Leopoldt obtained a p -adic \dagger class number formula for the maximal real subfield $F = \mathbf{Q}(\cos(2\pi/N))$ of $\mathbf{Q}(\exp(2\pi i/N))$: Let $\zeta_p(s, F)$ be the product of the $L_p(s, \chi)$ for all primitive Dirichlet characters χ such that (1) $\chi(-1) = 1$ and (2) the conductor of χ is a divisor of N . We define the **p -adic regulator R_p** by replacing the usual log by the p -adic logarithmic function \log_p . Let h be the class number of F , $m = [F:\mathbf{Q}]$, and let d be the discriminant of F . Then the residue of $\zeta_p(s, F)$ at $s=1$ is

$$\prod_{\chi} \left(1 - \frac{\chi(p)}{p} \right) \frac{2^{m-1} h R_p}{\sqrt{d}}.$$

Hence $\zeta_p(s, F)$ has a simple pole at $s=1$ if and only if $R_p \neq 0$. In general, for any totally real finite algebraic number field F , Leopoldt conjectured that the p -adic regulator R_p of F is not zero (**Leopoldt's conjecture**). This conjecture was proved by J. Ax and A. Brumer for the case when F is an Abelian extension of \mathbf{Q} [A4, B7].

By making use of the Stickelberger element, Iwasawa gave another proof of the existence of the p -adic L -function [I7]. In particular, he obtained the following result: Let χ be a primitive Dirichlet character with conductor f . Then there exists a primitive Dirichlet character θ such that the p -part of the conductor of θ is

either 1 or q and such that the conductor and the order of $\chi\theta^{-1}$ are both powers of p . Let \mathfrak{o}_θ be the ring generated over the ring \mathbf{Z}_p of p -adic integers by the values of θ . Then there exists a unique element $f(x, \theta)$ of the quotient field of $\mathfrak{o}_\theta[[x]]$ depending only on θ and satisfying

$$L_p(s, \chi) = 2f(\zeta(1 + q_0)^s - 1, \theta),$$

where q_0 is the least common multiple of q and the conductor of θ , and $\zeta = \chi(1 + q_0)^{-1}$. Furthermore, Iwasawa proved that $f(x, \theta)$ belongs to $\mathfrak{o}_\theta[[x]]$ if θ is not trivial.

Let $P = \mathbf{Q}(\exp(2\pi i/q))$ and, for any $n \geq 1$, let $P_n = \mathbf{Q}(\exp(2\pi i/q^n))$. Let $P_\infty = \bigcup_{n \geq 1} P_n$. Then P_∞ is a Galois extension of \mathbf{Q} satisfying $\text{Gal}(P_\infty/\mathbf{Q}) \cong \mathbf{Z}_p^\times$ (the multiplicative group of p -adic units), and P is the subfield of P_∞/\mathbf{Q} corresponding to the subgroup $1 + q\mathbf{Z}_p$ of \mathbf{Z}_p^\times .

Let ψ be a \mathbf{C}_p -valued primitive Dirichlet character such that (1) $\psi(-1) = -1$ and (2) the p -part of the conductor f_ψ of ψ is either 1 or q . Let K_ψ be the cyclic extension of \mathbf{Q} corresponding to ψ by class field theory. Let $K = K_\psi \cdot P$, $K_n = K \cdot P_n$ and $K_\infty = K \cdot P_\infty$. Let A_n be the p -primary part of the ideal class group of K_n , let $A_n \rightarrow A_m$ ($n \geq m$) be the mapping induced by the \dagger relative norm N_{K_n/K_m} , and let $X_K = \varprojlim A_n$. Since each A_n is a finite p -group, X_K is a \mathbf{Z}_p -module. Let $V_K = X_K \otimes_{\mathbf{Z}_p} \mathbf{C}_p$, and let $V_\psi = \{v \in V_K \mid \delta(v) = \psi(\delta)v \text{ for all } \delta \in \text{Gal}(K/\mathbf{Q})\}$.

Let q_0 be the least common multiple of f_ψ and q , and let γ_0 be the element of $\text{Gal}(K_\infty/K)$ that corresponds to

$$1 + q_0 \in 1 + q\mathbf{Z}_p = \text{Gal}(P_\infty/P)$$

by the restriction mapping $\text{Gal}(K_\infty/K) \hookrightarrow \text{Gal}(P_\infty/P)$. Let $f_\psi(x)$ be the characteristic polynomial of $\gamma_0 - 1$ acting on V_ψ . Hence $f_\psi(x)$ is an element of $\mathfrak{o}_\psi[[x]]$.

We assume that $\omega\psi^{-1}$ is not trivial. Let $f(x, \omega\psi^{-1})$ be as before. Then $f(x, \omega\psi^{-1})$ is an element of $\mathfrak{o}_\psi[[x]]$. Iwasawa conjectured that $f_\psi(x)$ and $f(x, \omega\psi^{-1})$ coincide up to a unit of $\mathfrak{o}_\psi[[x]]$ (**Iwasawa's main conjecture**). This conjecture was proved recently by B. Mazur and A. Wiles in the case where ψ is a power of ω .

Let F be a totally real finite algebraic number field, let K be a totally real Abelian extension of F , and let χ be a character of $\text{Gal}(K/F)$. Let $L(s, \chi)$ be the \dagger Artin L -function for χ . Then we can construct the p -adic analog $L_p(s, \chi)$ of $L(s, \chi)$ (J.-P. Serre, J. Coates, W. Sinnott, P. Deligne, K. Ribet, P. Cassou-Nogués). But, at present, we have no formula for $L_p(1, \chi)$. Coates generalized Iwasawa's main conjecture to this case, but it has not yet been proven.

p -adic L -functions have been defined in some other cases (e.g. \rightarrow [K3, M1, M3]).

K. ζ -Functions of Quadratic Forms

Dirichlet defined a Dirichlet series associated with a binary quadratic form and also considered a sum of such Dirichlet series extended over all classes of binary quadratic forms with a given discriminant D , which is actually equivalent to the Dedekind ζ -function of a quadratic field. Dirichlet obtained a formula for the class numbers of binary quadratic forms. The formula is interpreted nowadays as a formula for the class numbers of quadratic fields in the narrow sense.

According as the binary quadratic form is definite or indefinite, we apply different methods to obtain its class number.

Epstein ζ -functions: P. Epstein generalized the definition of the ζ -function of a positive definite binary quadratic form to the case of n variables (*Math. Ann.*, 56 (1903), 63 (1907)). Let V be a real vector space of dimension m with a positive definite quadratic form Q . Let M be a \dagger lattice in V , and put

$$\zeta_Q(s, M) = \sum_{\substack{x \in M \\ x \neq 0}} \frac{1}{Q(x)^s}, \quad \text{Re } s > \frac{m}{2}.$$

This series is absolutely convergent in $\text{Re } s > m/2$, and

$$\lim_{s \rightarrow m/2} \left(s - \frac{m}{2}\right) \zeta_Q(s, M) = D(M)^{-1/2} \pi^{m/2} \Gamma\left(\frac{m}{2}\right)^{-1},$$

$$D(M) = \det|Q(x_i, x_j)|,$$

where x_1, \dots, x_m is a basis of M and $Q(x, y) = (Q(x+y) - Q(x) - Q(y))/2$. If the $Q(x)$ ($x \in M, x \neq 0$) are all positive integers, we can write

$$\zeta_Q(s, M) = \sum_{n=1}^{\infty} \frac{a(n)}{n^s},$$

where $a(n)$ is the number of distinct $x \in M$ with $Q(x) = n$. In general, let x_1, \dots, x_m be a basis of the lattice M and x_1^*, \dots, x_m^* be its dual basis ($Q(x_i, x_j^*) = \delta_{ij}$). Call $M^* = \sum_i x_i^* \mathbf{Z}$ the **dual lattice** of M . If we consider the ϑ -series (\dagger theta series)

$$\vartheta_Q(u, M) = \sum_{x \in M} \exp(-\pi u Q(x)) \quad (\text{Re } u > 0),$$

then

$$\vartheta_Q(u, M) = (u^{-m/2} D(M)^{-1/2}) \vartheta_Q(u^{-1}, M^*).$$

With $\zeta_Q(s, M) = \pi^{-s} \Gamma(s) \zeta_Q(s, M)$, the displayed equality leads to the functional equation

$$\zeta_Q(s, M) = D(M)^{-1/2} \cdot \zeta_Q\left(\frac{m}{2} - s, M^*\right).$$

In general, $\zeta_Q(s, M)$ has no Euler product expansion.

Consider the case where $M = \sum \mathbf{Z}x_i$ ($x_i =$

$(0, \dots, 0, 1, 0, \dots, 0)$, $Q(x) = \sum_{i=1}^m u_i^2$, for $x = (u_1, \dots, u_m)$. If we put $\zeta_m(s) = \zeta_Q(s, M)$, $L(s) = \sum_{n=1}^\infty (-4/n)n^{-s}$, then we have

$$\begin{aligned}\zeta_1(s) &= 2\zeta(2s), \\ \zeta_2(s) &= 4\zeta(s) \cdot L(s) = 4 \\ &\quad \times (\text{the Dedekind } \zeta\text{-function of } \mathbf{Q}(\sqrt{-1})), \\ \zeta_4(s) &= 8(1 - 2^{2-2s})\zeta(s)\zeta(s-1), \\ \zeta_6(s) &= -4(\zeta(s)L(s-2) - 4\zeta(s-2)L(s)), \\ \zeta_8(s) &= 16(1 - 2^{1-s} + 2^{4-2s})\zeta(s)\zeta(s-3), \\ \zeta_{10}(s) &= (4/5)(\zeta(s)L(s-4) + 4^2\zeta(s-4)L(s)) \\ &\quad - 2 \sum_{\substack{\mu \in \mathbf{Z}[\sqrt{-1}] \\ \mu \neq 0}} \frac{\mu^4}{(\mu\bar{\mu})^s}, \\ \zeta_{12}(s) &= c_1 2^{-s}\zeta(s)\zeta(s-5)(2^6 - 2^{6-s}) \\ &\quad + c_2 \varphi\{\sqrt{\Delta(\tau)}\},\end{aligned}$$

where $\varphi\{\sqrt{\Delta(\tau)}\}$ is the Dirichlet series corresponding to $\sqrt{\Delta(\tau)}$ by the * Mellin transform and $\Delta(\tau) = z \{\prod_{n=1}^\infty (1 - z^n)\}^{24}$ with $z = e^{2\pi i \tau}$. $\zeta_m(s)$ has zeros on the line $\text{Re } s = \sigma = m/4$, given explicitly for $m = 4, 8$ as follows:

$$\begin{aligned}m=4: \quad s &= 1 + l\pi i / \log 2, \quad l = 1, 2, \dots, \\ m=8: \quad s &= 2 + (i/\log 2)(2l\pi \pm \arctan \sqrt{15}), \\ &\quad l = 0, \pm 1, \dots\end{aligned}$$

Regarding the Epstein ζ -function of binary quadratic forms

$$\zeta_Q(s) = \sum_{m,n} Q(m, n)^{-s},$$

with

$$\begin{aligned}Q(x, y) &= ax^2 + bxy + cy^2, \\ a, b, c &\in \mathbf{R}, a > 0, c > 0, \Delta = 4ac - b^2 > 0,\end{aligned}$$

we have the Chowla-Selberg formula (1949):

$$\begin{aligned}\zeta_Q(s) &= \left(2\zeta(2s)a^{-s} + \frac{2^{2s}a^{s-1}\sqrt{\pi}}{\Gamma(s)\Delta^{s-1/2}} \right. \\ &\quad \times \zeta(2s-1)\Gamma\left(s-\frac{1}{2}\right) \Bigg) \\ &\quad + \left(\frac{\pi^s 2^{s+3/2}}{a^{1/2}\Gamma(s)\Delta^{s/2-1/4}} \right. \\ &\quad \times \sum_{n=1}^\infty n^{s-1/2} \sigma_{1-2s}(n) \cos \frac{n\pi b}{a} \Bigg) \\ &\quad \times \int_0^\infty \varphi^{s-3/2} \exp \left\{ -\frac{\pi n \Delta^{1/2}}{2a} (\varphi + \varphi^{-1}) \right\} d\varphi,\end{aligned}$$

where $\sigma_k(n) = \sum_{d|n} d^k$ and $\zeta(s)$ is the Riemann ζ -function. By using this formula, we can give another proof of the following result of H. Heilbronn: Let $h(-\Delta)$ be the class number of the imaginary quadratic field with discriminant $-\Delta$. Then $h(-\Delta) \rightarrow \infty$ ($\Delta \rightarrow \infty$).

The following generalization of this result was obtained by C. L. Siegel [S22]: Let k be a fixed finite algebraic number field. Let K be a finite Galois extension of k , and let $d = d(K)$, $h = h(K)$, and $R = R(K)$ be the discriminant of K , the class number of K , and the regulator of K , respectively. We assume that K runs over extensions of k such that $[K:k]/\log d \rightarrow 0$; then we have

$$\log(hR) \sim \log \sqrt{|d|}.$$

Siegel ζ -functions of indefinite quadratic forms: Siegel defined and investigated some ζ -functions attached to nondegenerate indefinite quadratic forms, which are also meromorphic on the whole complex plane and satisfy certain functional equations [S24].

The case of quadratic forms with irrational algebraic coefficients was treated by Tamagawa and K. G. Ramanathan.

L. ζ -Functions of Algebras

K. Hey defined the ζ -function of a * simple algebra A over the rational number field \mathbf{Q} (M. Deuring [D10]) (\rightarrow 27 Arithmetic of Associative Algebras). Consider an arbitrary * maximal order \mathfrak{o} of A , and let

$$\zeta_A(s) = \sum_{\mathfrak{a}} \frac{1}{N(\mathfrak{a})^s}, \quad \text{Re } s > 1,$$

with the summation taken over all left integral ideals \mathfrak{a} of \mathfrak{o} . Then ζ_A is independent of the choice of a maximal order \mathfrak{o} . Let k be the * center of A , and put $[A:k] = n^2$. First, ζ_A is decomposed into Euler's infinite product expansion $\zeta_A(s) = \prod_{\mathfrak{p}} Z_{\mathfrak{p}}(s)$ (\mathfrak{p} runs over the prime ideals of k). For \mathfrak{p} not dividing the discriminant \mathfrak{d} of A , $Z_{\mathfrak{p}}(s)$ coincides with the \mathfrak{p} -component of $\prod_{j=0}^{n-1} \zeta_k(ns-j)$. Hence $\zeta_A(s)$ coincides with $\prod_{j=0}^{n-1} \zeta_k(ns-j)$ up to a product of \mathfrak{p} -factors for $\mathfrak{p} | \mathfrak{d}$ which are explicit rational functions of $N(\mathfrak{p})^{-ns}$.

Moreover, if A is the total matrix algebra of degree r over the division algebra \mathfrak{D} , then we have $\zeta_A(s) = \prod_{j=0}^{r-1} \zeta_{\mathfrak{D}}(rs-j)$, and $\zeta_{\mathfrak{D}}(s)$ satisfies a functional equation similar to that of $\zeta_k(s)$ (Hey). Also, $\zeta_A(s)$ is meromorphic over the whole complex plane, and at $s = 1, (n-1)/n, \dots, 1/n$, it has poles of order 1. Using analytic methods, M. Zorn (1931) showed that the simple algebra A with center k such that $A_{\mathfrak{p}}$ is a matrix algebra over $k_{\mathfrak{p}}$ for every finite or infinite prime divisor \mathfrak{p} of k is itself a matrix algebra over k (\rightarrow 27 Arithmetic of Associative Algebras D). A purely algebraic proof of this was given by Brauer, H. Hasse, and E. Noether. G. Fujisaki (1958) gave another proof using the Iwasawa-Tate method. As a direct

application of the ζ -function, the computation of the residue at $s=1$ of ζ_A leads to the formula containing the class number of maximal order \mathfrak{D} .

Godement defined the ζ -function of fairly general algebras [G1], and Tamagawa investigated in detail the explicit ζ -functions of division algebras, deriving their functional equations [T1].

Let $\tilde{A} = \prod_p A_p$ be the adèle ring of A , and let $G = \prod_p G_p$ be the idele group (of A). We take a maximal order \mathfrak{D}_p of A_p and a maximal compact subgroup U_p of G_p . Let ω_p be a \dagger zonal spherical function of G_p with respect to U_p ; that is, ω_p is a function in G_p and satisfies

$$\omega_p(ugv) = \omega_p(g) \quad (u, v \in U_p), \quad \omega_p(1) = 1,$$

$$\int_{U_p} \omega_p(guh) du = \omega_p(g) \omega_p(h).$$

In addition, we define the weight function φ_p on A_p by

$$\varphi_p(x) = \begin{cases} \text{the characteristic function of } \mathfrak{D}_p \\ \text{when } p \text{ is finite,} \\ \exp(-\pi T_p(xx^*)) \\ \text{when } p \text{ is infinite,} \end{cases}$$

where T_p is the \dagger reduced trace of A_p/\mathbf{R} and $*$ is a positive \dagger involution. Tamagawa gave an explicit form of the local ζ -function with the character ω_p defined by

$$\zeta_p(s, \omega_p) = \int_{G_p} \varphi_p(g) \omega_p(g^{-1}) |N_p(g)|_p^s dg,$$

where N_p is the \dagger reduced norm of A_p/k_p , and $|\cdot|_p$ is the valuation of k_p . Then $\omega = \prod_p \omega_p$ is the zonal spherical function of G with respect to $\prod U_p = U$. In particular, if ω is a positive definite zonal spherical function belonging to the spectrum of the discrete subgroup $\Gamma = A^* = \{\text{all the invertible elements of } A\}$ of G , then the **Tamagawa ζ -function** with character ω is given by

$$\zeta(s, \omega) = \prod_p \zeta_p(s, \omega_p) = \int_G \varphi(g) \omega(g^{-1}) \|g\|^s dg,$$

where $\varphi(g) = \prod \varphi_p(g_p)$ and $\|\cdot\|$ is the volume of the element g of G . When A is a division algebra, $\zeta(s, \omega)$ is analytically continued to a meromorphic function over the whole complex plane and satisfies the functional equation. The Tamagawa ζ -function may also be considered as one type of ζ -function of the Hecke operator. When A is an indefinite quaternion algebra over a totally real algebraic number field Φ , the groups of units of various orders of A operate discontinuously on the product of complex upper half-planes. Thus the spaces of holomorphic forms are naturally associated with A . The investigation of ζ -functions asso-

ciated with these holomorphic automorphic forms was initiated by M. Eichler and extended by G. Shimura, H. Shimizu, and others. Eichler investigated the case $\Phi = \mathbf{Q}$, and Shimura and Shimizu investigated the case for an arbitrary totally real field Φ by defining general holomorphic automorphic forms, Hecke operators, and corresponding ζ -functions. The functional equations of these ζ -functions were proved by Shimizu. Shimizu generalized Eichler's work and found relations among ζ -functions of orders of various quaternion algebras belonging to different discriminants and levels [S10]. For the related results, see, e.g., the work of K. Doi and H. Naganuma [D12].

M. ζ -Functions Defined by Hecke Operators

The ζ -functions of algebraic number fields, algebras, or quadratic forms, and the L -functions are all defined by Dirichlet series, are analytically continued to univalent functions on the complex plane, and satisfy functional equations. One problem is to characterize the functions having such properties.

(1) H. Hamburger (1921–1922) characterized the Riemann ζ -function (up to constant multiples) by the following three properties: (i) It can be expanded as $\zeta(s) = \sum_{n=1}^{\infty} a_n/n^s$ ($\text{Re } s \gg 0$); (ii) it is holomorphic on the complex plane except as $s=1$, and $(s-1)\zeta(s)$ is an entire function of finite \dagger genus; (iii) $G(s) = G(1-s)$, where $G(s) = \pi^{-s/2} \Gamma(s/2) \zeta(s)$.

(2) E. Hecke's theory [H4]: Fixing $\lambda > 0$, $k > 0$, $\gamma = \pm 1$, and putting

$$R(s) = (2\pi/\lambda)^{-s} \Gamma(s) \varphi(s)$$

for an analytic function $\varphi(s)$, we make the following three assumptions: (i) $(s-k)\varphi(s)$ is an entire function of finite genus; (ii) $R(s) = \gamma R(k-s)$; (iii) $\varphi(s)$ can be expanded as $\varphi(s) = \sum_{n=1}^{\infty} a_n/n^s$ ($\text{Re } s > \sigma_0$). Then we call $\varphi(s)$ a function belonging to the sign (λ, k, γ) .

The functions $\zeta(2s)$, $L(2s)$, and $L(2s-1)$ satisfy assumptions (i)–(iii), where L may be either a Dirichlet L -function, an L -function with Grössencharakter of an imaginary quadratic field, or an L -function with class character of a real quadratic form whose Γ -factors are of the form $\Gamma(s/2)\Gamma((s+1)/2) \sim \Gamma(s)$. If $\varphi(s)$ belongs to the sign (λ, k, γ) , then $n^{-s}\varphi(s)$ belongs to the sign $(n\lambda, k, \gamma)$. To each Dirichlet series $\varphi(s) = \sum_{n=1}^{\infty} a_n/n^s$ with the sign (λ, k, γ) , we attach the series $f(\tau) = a_0 + \sum_{n=1}^{\infty} a_n e^{2\pi i n \tau / \lambda}$, where

$$\begin{aligned} a_0 &= \gamma (2\pi/\lambda)^{-k} \Gamma(k) \text{Res}_{s=k}(\varphi(s)) \\ &= \gamma \text{Res}_{s=k}(R(s)). \end{aligned}$$

This correspondence $\varphi(s) \rightarrow f(\tau)$ may also be

realized by the †Mellin transform

$$R(s) = \int_0^\infty \left(\sum_{n=1}^\infty a_n e^{-2\pi n y / \lambda} \right) y^{s-1} dy \\ = \int_0^\infty (f(iy) - a_0) y^{s-1} dy,$$

$$f(iy) - a_0 = \frac{1}{2\pi i} \int_{\text{Re } s = \sigma_0} R(s) y^{-s} ds.$$

In this case, (i) $f(\tau)$ is holomorphic in the upper half-plane and $f(\tau + \lambda) = f(\tau)$, (ii) $f(-1/\tau)/(-i\tau)^k = \gamma f(\tau)$, and (iii) $f(x + iy) = O(y^{\text{const}})$ ($y \rightarrow +0$) uniformly for all x .

Conversely, the Dirichlet series $\varphi(s) = \sum_{n=1}^\infty a_n n^{-s}$ formed by the transformation in the previous paragraph from $f(\tau)$ satisfying (i)–(iii) belongs to the sign (λ, k, γ) . We also say that the function $f(\tau)$ belongs to the sign (λ, k, γ) .

If k is an even integer, then the functions $f(\tau)$ belonging to $(1, k, (-1)^{k/2})$ are the †modular forms of level 1 and weight k . A necessary and sufficient condition for a function $\varphi(s)$ belonging to $(1, k, (-1)^{k/2})$ to have an Euler product is that the corresponding modular form $f(\tau)$ be a simultaneous eigenfunction of the ring formed by the †Hecke operators T_n ($n = 1, 2, \dots$). In this case, the coefficient a_n of $\varphi(s) = \sum a_n n^{-s}$ coincides with the eigenvalue of T_n . Namely, if $f|T_n = t_n f$, we have $\varphi(s) = a_1 (\sum_{n=1}^\infty t_n n^{-s})$, and this is decomposed into the Euler product $\varphi(s) = a_1 \prod_p (1 - t_p p^{-s} + p^{k-1-2s})^{-1}$. We call $\varphi(s)/a_1$ a **ζ -function defined by Hecke operators** (Hecke [H5]). For example, $\zeta(s) \cdot \zeta(s - k + 1)$ and the Ramanujan function

$$\sum_{n=1}^\infty \tau(n) n^{-s} = \prod_p (1 - \tau(p) p^{-s} + p^{1-2s})^{-1}$$

are ζ -functions defined by Hecke operators. Hecke applied the theory of Hecke operators to study the group $\Gamma(N)$ [H5]; the situation is more complicated than the case of $\Gamma(1) = SL(2, \mathbf{Z})$. The space of automorphic forms of weight k belonging to the †congruence subgroup

$$\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbf{Z}) \mid c \equiv 0 \pmod{N} \right\}$$

is denoted by $\mathfrak{M}_k(\Gamma_0(N))$. The essential part of $\mathfrak{M}_k(\Gamma_0(N))$ is spanned by the functions $f(\tau) = \sum a_n e^{2\pi i n \tau}$ satisfying the conditions: (1) $\varphi(s) = \sum a_n n^{-s}$ has the Euler product expansion

$$\varphi(s) = \prod_{p|N} (1 - a_p p^{-s})^{-1} \\ \times \prod_{p \nmid N} (1 - a_p p^{-s} + p^{k-1-2s})^{-1}.$$

(2) The functional equation $R(s) = \gamma R(k - s)$ holds, where $R(s) = (2\pi/\sqrt{N})^{-s} \Gamma(s) \varphi(s)$. (3) When χ is an arbitrary primitive character of \mathbf{Z} such that the conductor f is coprime to N ,

then

$$R(s, \chi) = (2\pi/\sqrt{N})^{-s} \Gamma(s) \sum a_n \chi(n) n^{-s}$$

extends to an entire function satisfying the functional equation $R(s, \chi) = \omega R(k - s, \bar{\chi})$ ($|\omega| = 1$) (Shimura). Conversely, (2) and (3) characterize the Dirichlet series $\varphi(s)$ corresponding to $f(\tau) \in \mathfrak{M}_k(\Gamma_0(N))$ (Weil [W1 (1967a)]).

Considering the correspondence $f(\tau) = \sum a_n q^n \rightarrow \varphi(s) = \sum a_n n^{-s}$ not as a Mellin transformation but rather as a correspondence effected through Hecke operators, we can derive the ζ -function defined by Hecke operators. When the Hecke operator T_n is defined with respect to a discontinuous group Γ and we have a representation space \mathfrak{M} of the Hecke operator ring \mathcal{H} , we denote the matrix of the operation of $T_n \in \mathcal{H}$ on \mathfrak{M} by $(T_n) = (T_n)_{\mathfrak{M}}$ and call the matrix-valued function $\Sigma_n(T_n)_{\mathfrak{M}} n^{-s}$ the ζ -function defined by Hecke operators. The equation $\varphi(s) = \sum a_n n^{-s}$ is a specific instance of the correspondence in the first sentence, where $\Gamma = \Gamma(N)$, $\mathfrak{M} \subset \mathfrak{M}_k(\Gamma_0(N))$, $\dim \mathfrak{M} = 1$. One advantage of this definition is that it may be applied whenever the concept of Hecke operators can be defined with respect to the group Γ (for instance, even for the Fuchsian group without a †cusp). Thus when Γ is a Fuchsian group given by the unit group of a quaternion algebra Φ over the rational number field \mathbf{Q} and \mathfrak{M} is the space of automorphic forms with respect to Γ , the ζ -function $\Sigma(T_n)_{\mathfrak{M}} n^{-s}$ is defined (Eichler). Moreover, by using its integral expression over the idele group \mathbf{J}_Φ of Φ , we can obtain its functional equation following the Iwasawa-Tate method (Shimura). Furthermore, by algebrogeometric consideration of T_n , it can be shown that

$$\zeta(s) \zeta(s-1) \det(\Sigma(T_n)_{\mathfrak{E}_2} n^{-s}) \\ = \zeta(s) \zeta(s-1) \text{dct} \left(\prod_p (1 - (T_p)_{\mathfrak{E}_2} p^{-s} \right. \\ \left. + (R_p)_{\mathfrak{E}_2} p^{1-2s})^{-1} \right)$$

coincides (up to a trivial factor) with the Hasse ζ -function of some model of the Riemann surface defined by Γ when \mathfrak{M} is the space \mathfrak{E}_2 of all †cusp forms of weight 2 (Eichler [E1], Shimura [S12]).

The algebrogeometric meaning of $\det(\Sigma(T_n)_{\mathfrak{E}_k} n^{-s})$, when \mathfrak{M} is the space \mathfrak{E}_k of all cusp forms of weight k , has been made clear for the case where Γ is obtained from $\Gamma_0(N)$, $\Gamma(N)$, and the quaternion algebra (M. Kuga, M. Sato, Shimura, Y. Ihara). From these facts, it becomes possible to express $(T_p)_{\mathfrak{E}_k}$, the decomposition of the prime number p in some type of Galois extension (Shimura [S14], Kuga), in terms of Hecke operators. These works gave the first examples of non-Abelian class field

theory. Note that this type of ζ -function may be regarded as the analog (or generalization) of L -functions of algebraic number fields, as can be seen from the comparison in Table 1.

Table 1

Algebraic number field	k	Ideal group	Character χ	$\sum \chi(n)n^{-s}$
	\updownarrow	\updownarrow	\updownarrow	\updownarrow
Algebraic group	G	Hecke ring	Representation space \mathfrak{M}	$\sum (T_n)_{\mathfrak{M}} n^{-s}$

As for special values of ζ -functions defined by Hecke operators, the following fact is known: Let $f(\tau) = \sum a_n q^n \in \mathfrak{M}_k(SL(2, \mathbb{Z}))$ be a common eigenfunction of the Hecke operators, and let $\varphi(s) = \sum a_n n^{-s}$ be the corresponding Dirichlet series. Let K_f be the field generated over the rational number field \mathbb{Q} by the coefficients a_n of f . Then, for any two integers m and m' satisfying $0 < m, m' < k$ and $m \equiv m' \pmod{2}$, the ratio $(R(m):R(m'))$ of the special values of

$$R(s) = \frac{\Gamma(s)}{(2\pi)^s} \varphi(s) = \int_0^\infty (f(iy) - a_0) y^{s-1} dy$$

at m and m' belongs to the field K_f .

G. Shimura discovered this fact for Ramanujan's function $\Delta(\tau)$ (*J. Math. Soc. Japan*, 11 (1959)), and then Yu. I. Manin generalized it to the above case and, by constructing a p -adic analog of $\varphi(s)$ from it, pointed out the importance of such results [M1]. R. M. Damerell also used such results to study special values of Hecke's L -function with Grössencharakter of an imaginary quadratic field (*Acta Arith.*, 17 (1970), 19 (1971)). Furthermore, Shimura generalized these results to congruence subgroups of $SL(2, \mathbb{Z})$ (*Comm. Pure Appl. Math.*, 29 (1976)), and to Hilbert modular groups (*Ann. Math.*, 102 (1975)). The connection between special values of ζ -functions and the periods of integrals has been studied further by Shimura, Deligne, and others.

In addition, in connection with nonholomorphic automorphic forms H. Maass considered L -functions of real quadratic fields (with class characters) having $\Gamma(s/2)^2$ or $\Gamma((s+1)/2)^2$ as Γ -factors. Furthermore, T. Kubota studied the relation of ζ -functions $\zeta_k(s)$ of an arbitrary algebraic field k or ζ -functions of simple rings to (nonanalytic) automorphic forms of several variables and considered the reciprocity law for the Gaussian sum from a new viewpoint.

N. L -Functions of Automorphic Representations (I)

R. P. Langlands reconstructed the theory of * Hecke operators from the viewpoint of repre-

sentation theory and defined very general L -functions. He proposed many conjectures about them in [L4], and he and H. Jacquet proved most of them in [J1] for the case $G = GL_2$.

First Langlands defined the L -group ${}^L G$ for any connected reductive algebraic group G defined over a field k in the following manner [B6].

There is a canonical bijection between isomorphism classes of connected * reductive algebraic groups defined over a fixed algebraically closed field \bar{k} and isomorphism classes of * root systems. It is defined by associating to G the root data $\Psi(G) = (X^*(T), \Phi, X_*(T), \Phi^v)$, where T is a * maximal torus of G , $X_*(T)$ ($X^*(T)$) the group of characters (* 1-parameter subgroups) of T , Φ (Φ^v) the set of roots (coroots) of G with respect to T .

Since the choice of a * Borel subgroup B of G containing T is equivalent to that of a basis Δ of Φ , the aforementioned bijection yields one between isomorphism classes of triples (G, B, T) and isomorphism classes of based root data $\Psi_0(G) = (X^*(T), \Delta, X_*(T), \Delta^v)$. There is a split exact sequence

$$1 \rightarrow \text{Int } G \rightarrow \text{Aut } G \rightarrow \text{Aut } \Psi_0(G) \rightarrow 1.$$

and this mapping induces a canonical bijection $\text{Aut } \Psi_0(G) \cong \text{Aut}(G, B, T, \{x_\alpha\}_{\alpha \in \Delta})$ if $x_\alpha \in G_\alpha$ ($\alpha \in \Delta$) are fixed.

Let G be a connected reductive algebraic group defined over \bar{k} . Let T be a maximal torus of G , and let B be a Borel subgroup of G containing T . Let $\Psi_0(G) = (X^*(T), \Delta, X_*(T), \Delta^v)$ be as before. Then there is a connected reductive algebraic group ${}^L G^0$ over \mathbb{C} such that $\Psi_0(G)^v = (X_*(T), \Delta^v, X^*(T), \Delta)$ corresponds to the triple $({}^L G^0, {}^L B^0, {}^L T^0)$, where ${}^L B^0$ and ${}^L T^0$ are a Borel subgroup of ${}^L G^0$ and the maximal torus of ${}^L B^0$. For example, (1) if $G = GL_n$, then ${}^L G^0 = GL_n(\mathbb{C})$; (2) if $G = Sp_{2n}$, then ${}^L G^0 = SO_{2n+1}(\mathbb{C})$.

We assume that \bar{k} is the algebraic closure of k and G is defined over k . Then $\gamma \in \text{Gal}(\bar{k}/k)$ induces an automorphism of the \bar{k} -group $G \times_k \bar{k}$. Hence γ defines an element of $\text{Aut}({}^L G^0, {}^L B^0, {}^L T^0)$ because it is a holomorphic image of $\text{Aut } \Psi_0(G \times_k \bar{k}) = \text{Aut } \Psi_0(G \times_k \bar{k})^v$. Hence we can define the * semidirect product ${}^L G = {}^L G^0 \rtimes \text{Gal}(\bar{k}/k)$, and call it the L -group of G .

Let k be a * local field, and let G be a connected reductive algebraic group defined over k . We identify G with the group of its k -rational points. Let W_k be the Weil-Deligne group of k (\rightarrow Section H), and let $\Phi(G)$ be the set of homomorphisms $\varphi: W_k \rightarrow {}^L G$ over $\text{Gal}(\bar{k}/k)$. Let $\Pi(G)$ be the set of infinitesimal equivalence classes of irreducible **admissible** representations of G . If k is a non-Archimedean field, then

$\Pi(G)$ is the set consisting of equivalence classes of irreducible representations $\pi: G \rightarrow \text{Aut } V$ on complex vector spaces V such that the space V^K of vectors invariant by K is finite dimensional for every compact open subgroup K of G and such that $V = \bigcup V^K$, where K runs over the compact open subgroups of G . If k is an Archimedean field, then $\Pi(G)$ is the set consisting of equivalence classes of representations π of the pair (\mathfrak{g}, K) of the Lie algebra \mathfrak{g} of G and a maximal compact subgroup K satisfying similar conditions [B6]. Then Langlands conjectured that we can parametrize $\Pi(G)$ by $\Phi(G)$ as $\Pi(G) = \bigcup_{\varphi \in \Phi(G)} \Pi(G)_{\varphi}$. Let $\pi \in \Pi(G)_{\varphi}$ ($\varphi \in \Phi(G)$), and let r be a representation of ${}^L G$. Then we can define the L -function $L(s, \pi, r)$ and the ε -factor $\varepsilon(s, \pi, r)$ of π by

$$L(s, \pi, r) = L(s, r \circ \varphi), \quad \varepsilon(s, \pi, r) = \varepsilon(s, r \circ \varphi, \psi),$$

where the right-hand sides are those of the Weil-Deligne group (\rightarrow Section H) and ψ is a nontrivial character of k .

Let G be a connected reductive group over a global field k (i.e., an algebraic number field of finite degree or an algebraic function field of one variable over a finite field), let π be an irreducible admissible representation of G_A , where G_A is the group of rational points of G over the a -adele ring k_A of k , and let r be a finite-dimensional representation of ${}^L G$. Let ψ be a nontrivial character of k_A which is trivial on k . For any place v of k , let r_v be the representation of the L -group of $G_v = G \times_k k_v$ induced by r , and let ψ_v be the additive character of k_v associated with ψ . It is known that π is decomposed into the tensor product $\bigotimes \pi_v$ of $\pi_v \in \Pi(G(k_v))$ [B6]. Hence we put

$$L(s, \pi, r) = \prod_v L(s, \pi_v, r_v),$$

$$\varepsilon(s, \pi, r) = \prod_v \varepsilon(s, \pi_v, r_v).$$

The local factor $L(s, \pi_v, r_v)$ is in fact defined if v is Archimedean, or G is a a -torus, or φ is unramified (i.e., G_v is quasisplit and splits over an unramified extension of k_v , and $G(\mathfrak{o}_v)$ is a special maximal compact subgroup of $G(k_v)$, and π_v is of class one with respect to $G(\mathfrak{o}_v)$, where \mathfrak{o}_v is the integer ring of k_v). It follows that the right-hand side $\prod L(s, \pi_v, r_v)$ is defined up to a finite number of non-Archimedean places v . Furthermore, Langlands proved that $\prod \varepsilon(s, \pi_v, r_v)$ is in fact a finite product, and the infinite product $\prod L(s, \pi_v, r_v)$ converges in some right half-plane if π is automorphic (i.e., if π is a subquotient of the right regular representation of G_A in $G_k \backslash G_A$). It is conjectured that $L(s, \pi, r)$ admits a meromorphic continuation to the whole complex plane and satisfies a functional equation

$$L(s, \pi, r) = \varepsilon(s, \pi, r) L(1-s, \tilde{\pi}, \tilde{r})$$

if π is automorphic, where $\tilde{\pi}$ is the a -contragredient representation of π . Furthermore, if $G = GL_n$ and r is the standard representation of GL_n , then we can construct $L(s, \pi, r)$ and $\varepsilon(s, \pi, r)$ by generalizing the Iwasawa-Tate method. We can also show in this case that $L(s, \pi, r)$ is entire if π is cuspidal. The conjectures are studied in some other cases [B6].

O. L -Functions of Automorphic Representations (II)

A. Weil generalized the theory of a -Hecke operators and the corresponding L -functions to the case of a -automorphic forms (for holomorphic and nonholomorphic cases together) of GL_2 over a global field [W9]. Then H. Jacquet and Langlands developed a theory from the viewpoint of a -representation theory [J1, J2]. They attached L -functions not to automorphic forms but to a -automorphic representations of $GL_2(k)$.

Let k be a non-Archimedean local field, and let \mathfrak{o}_k be the maximal order of k . Let \mathcal{H}_k be the space of functions on $G_k = GL_2(k)$ that are locally constant and compactly supported. Then \mathcal{H}_k becomes an algebra with the convolution product

$$(f_1 * f_2)(h) = \int_{G_k} f_1(g) f_2(g^{-1}h) dg,$$

where dg is the a -Haar measure of G_k that assigns 1 to the maximal compact subgroup $K_k = GL_2(\mathfrak{o}_k)$. Let π be a representation of \mathcal{H}_k on a complex vector space V . Then we say that π is **admissible** if and only if π satisfies the following two conditions: (1) For every v in V , there is an f in \mathcal{H}_k so that $\pi(f)v = v$; (2) Let σ_i ($i = 1, \dots, r$) be a family of inequivalent irreducible finite-dimensional representations of K_k , and let

$$\xi(g) = \sum_{i=1}^r \dim(\sigma_i)^{-1} \text{tr } \sigma_i(g^{-1}).$$

Then ξ is an idempotent of \mathcal{H}_k . We call such a ξ an **elementary idempotent** of \mathcal{H}_k . Then for every elementary idempotent ξ of \mathcal{H}_k , the operator $\pi(\xi)$ has a finite-dimensional range. If π is an admissible representation of $GL_2(k)$ (\rightarrow Section N), then

$$\pi(f) = \int_{G_k} f(g) \pi(g) dg \quad (f \in \mathcal{H}_k)$$

gives an admissible representation of \mathcal{H}_k in this sense. Furthermore, any admissible representation of \mathcal{H}_k can be obtained from an admissible representation of $GL_2(k)$.

Let k be the real number field. Let \mathcal{H}_1 be the

space of infinitely differentiable compactly supported functions on $G_k (= GL_2(k))$ that are $K_k (= O(2, k))$ finite on both sides, let \mathcal{H}_2 be the space of functions on K_k that are finite sums of matrix elements of irreducible representations of K_k , and let $\mathcal{H}_k = \mathcal{H}_1 \oplus \mathcal{H}_2$. Then \mathcal{H}_1 , \mathcal{H}_2 , and \mathcal{H}_k become algebras with the convolution product. Let π be a representation of \mathcal{H}_k on a complex vector space V . Then π is **admissible** if and only if the following three conditions are satisfied: (1) Every vector v in V is of the form $v = \sum_{i=1}^r \pi(f_i)v_i$ with $f_i \in \mathcal{H}_1$ and $v_i \in V$; (2) for every elementary idempotent $\xi(g) = \sum_{i=1}^r \dim(\sigma_i)^{-1} \text{tr } \sigma_i(g^{-1})$, where the σ_i are a family of inequivalent irreducible representations of K_k , the range of $\pi(\xi)$ is finite-dimensional; (3) for every elementary idempotent ξ of \mathcal{H}_k and for every vector v in $\pi(\xi)V$, the mapping $f \mapsto \pi(f)v$ of $\xi\mathcal{H}_1\xi$ into the finite-dimensional space $\pi(\xi)V$ is continuous. We can define the Hecke algebra \mathcal{H}_k and the notion of admissible representations also in the case $k = \mathbb{C}$. In these cases, an admissible representation of \mathcal{H}_k comes from a representation of the universal enveloping algebra of $GL_2(k)$ but may not come from a representation of $GL_2(k)$. It is known that for any local field k , the \dagger character of each irreducible representation is a locally integrable function.

Let k be a global field, $G_k = GL_2(k)$, and let $G_A = GL_2(k_A)$ be the group of rational points of G_k over the adèle ring k_A of k . For any place v of k , let k_v be the completion of k at v , let $G_v = GL_2(k_v)$, and let K_v be the standard maximal compact subgroup of G_v . Let \mathcal{H}_v be the Hecke algebra \mathcal{H}_{K_v} of G_v , and let ε_v be the normalized Haar measure of K_v . Then ε_v is an elementary idempotent of \mathcal{H}_v . Let $\mathcal{H} = \bigotimes_{\varepsilon_v} \mathcal{H}_v$ be the restricted tensor product of the local Hecke algebra \mathcal{H}_v with respect to the family $\{\varepsilon_v\}$. We call \mathcal{H} the **global Hecke algebra** of G_A .

Let π be a representation of \mathcal{H} on a complex vector space V . We define the notion of admissibility of π as before. Then we can show that, for any irreducible admissible representation π of \mathcal{H} and for any place v of k , there exists an irreducible admissible representation π_v of \mathcal{H}_v on a complex vector space V_v such that (1) for almost all v , $\dim V_v^{K_v} = 1$ and (2) π is equivalent to the restricted tensor product $\bigotimes \pi_v$ of the π_v with respect to a family of nonzero $x_v \in V_v^{K_v}$. Furthermore, the factors $\{\pi_v\}$ are unique up to equivalence.

Let k be a local field, let ψ be a nontrivial character of k , and let \mathcal{H}_k be the Hecke algebra of $G_k = GL_2(k)$. Let π be an infinite-dimensional admissible irreducible representation of \mathcal{H}_k . Then there is exactly one space $W(\pi, \psi)$ of continuous functions on G_k with the following three properties: (1) If W is in

$W(\pi, \psi)$, then for all g in G_k and for all x in k ,

$$W\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} g\right) = \psi(x)W(g);$$

(2) $W(\pi, \psi)$ is invariant under the right translations of \mathcal{H}_k , and the representation on $W(\pi, \psi)$ is equivalent to π ; (3) if k is Archimedean and if W is in $W(\pi, \psi)$, then there is a positive number N such that

$$W\left(\begin{pmatrix} t & 0 \\ 0 & 1 \end{pmatrix}\right) = O(|t|^N)$$

as $|t| \rightarrow \infty$. We call $W(\pi, \psi)$ the **Whittaker model** of π . The Whittaker model exists in the global case if and only if each factor π_v of $\pi = \bigotimes \pi_v$ is infinite-dimensional.

Let k be a local field, and let π be as before. Then the L -function $L(s, \pi)$ and the ε -factor $\varepsilon(s, \pi, \psi)$ are defined in the following manner: Let ω be the quasicharacter of k^\times (i.e., the continuous homomorphism $k^\times \rightarrow \mathbb{C}^\times$) defined by

$$\pi\left(\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}\right) = \omega(a)id_V.$$

Then the \dagger contragredient representation $\tilde{\pi}$ of π is equivalent to $\omega^{-1} \otimes \pi$. For any g in G_k and W in $W(\pi, \psi)$, let

$$\Psi(g, s, W) = \int_{k^\times} W\left(\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} g\right) |a|^{s-1/2} d^\times a,$$

$$\tilde{\Psi}(g, s, W) = \int_{k^\times} W\left(\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} g\right) |a|^{s-1/2} \omega^{-1}(a) d^\times a.$$

Then there is a real number s_0 such that these integrals converge for $\text{Re}(s) > s_0$ for any $g \in G_k$ and $W \in W(\pi, \psi)$. If k is a non-Archimedean local field with \mathbb{F}_q as its residue field, then there is a unique factor $L(s, \pi)$ such that $L(s, \pi)^{-1}$ is a polynomial of q^{-s} with constant term 1,

$$\Phi(g, s, W) = \Psi(g, s, W)/L(s, \pi)$$

is a holomorphic function of s for all g and W , and there is at least one W in $W(\pi, \psi)$ so that $\Phi(e, s, W) = a^s$ with a positive constant a . If k is an Archimedean local field, then we can define the gamma factor $L(s, \pi)$ in the same manner. Furthermore, for any local field k , if

$$\tilde{\Phi}(g, s, W) = \tilde{\Psi}(g, s, W)/L(s, \tilde{\pi}),$$

then there is a unique factor $\varepsilon(s, \psi, \pi)$ which, as a function of s , is an exponential such that

$$\tilde{\Phi}\left(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} g, 1-s, W\right) = \varepsilon(s, \pi, \psi)\Phi(g, s, W)$$

for all $g \in G_k$ and $W \in W(\pi, \psi)$.

Let π and π' be two infinite-dimensional irreducible admissible representations of G_k . Then π and π' are equivalent if and only if the

quasicharacters ω and ω' are equal and

$$\frac{L(1-s, \chi^{-1} \otimes \tilde{\pi}) \varepsilon(s, \chi \otimes \pi, \psi)}{L(s, \chi \otimes \pi)} = \frac{L(1-s, \chi^{-1} \otimes \tilde{\pi}') \varepsilon(s, \chi \otimes \pi', \psi)}{L(s, \chi \otimes \pi')}$$

holds for any quasicharacter χ . In particular, the set $\{L(s, \chi \otimes \pi) \text{ and } \varepsilon(s, \chi \otimes \pi, \psi) \text{ for all } \chi\}$ characterizes the representation π .

Let k be a global field, $G_k = GL_2(k)$, $G_A = GL_2(k_A)$, and let $K_A = \prod K_v$ be the standard maximal compact subgroup of G_A . Then the \dagger global Hecke algebra \mathcal{H} acts on the space of continuous functions on $G_k \backslash G_A$ by the right translations. Let φ be a continuous function on $G_k \backslash G_A$. Then φ is an **automorphic form** if and only if (1) φ is K_A -finite on the right, (2) for every \dagger elementary idempotent ξ in \mathcal{H} , the space $(\xi \mathcal{H})\varphi$ is finite-dimensional, and (3) φ is slowly increasing if k is an algebraic number field. An automorphic form φ is a **cusp form** if and only if

$$\int_{k \backslash k_A} \varphi \left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} g \right) dx = 0$$

for all g in G_A . Let \mathcal{A} be the space of automorphic forms on $G_k \backslash G_A$, and let \mathcal{A}_0 be the space of cusp forms on $G_k \backslash G_A$. They are \mathcal{H} -modules. Let $\psi = \prod \psi_v$ be a nontrivial character of $k \backslash k_A$, and let π be an irreducible admissible representation $\pi = \otimes_v \pi_v$ of the global Hecke algebra $\mathcal{H} = \otimes_v \mathcal{H}_v$. If π is a \dagger constituent of the \mathcal{H} -module \mathcal{A} , then we can define the local factors $L(s, \pi_v)$ and $\varepsilon(s, \pi_v, \psi_v)$ for all v , although π_v may not be infinite-dimensional. Further, the infinite products

$$L(s, \pi) = \prod L(s, \pi_v) \text{ and } L(s, \tilde{\pi}) = \prod L(s, \tilde{\pi}_v)$$

converge absolutely in a right half-plane, and the functions $L(s, \pi)$ and $L(s, \tilde{\pi})$ can be analytically continued to the whole complex plane as meromorphic functions of s . If π is a constituent of \mathcal{A}_0 , then all π_v are infinite-dimensional, $L(s, \pi)$ and $L(s, \tilde{\pi})$ are entire functions, and π is contained in \mathcal{A}_0 with multiplicity one. If k is an algebraic number field, then they have only a finite number of poles and are bounded at infinity in any vertical strip of finite width. If k is an algebraic function field of one variable with field of constant F_q , then they are rational functions of q^{-s} . In either case, $\varepsilon(s, \pi_v, \psi_v) = 1$ for almost all v , and hence

$$\varepsilon(s, \pi) = \prod \varepsilon(s, \pi_v, \psi_v)$$

is well defined. Furthermore, the functional equation

$$L(s, \pi) = \varepsilon(s, \pi) L(1-s, \tilde{\pi})$$

is satisfied.

As for the condition for π being a constituent of \mathcal{A}_0 , we have the following: Let $\pi = \otimes \pi_v$ be an irreducible admissible representation of \mathcal{H} . Then π is a constituent of \mathcal{A}_0 if and only if (1) for every v , π_v is infinite-dimensional; (2) the quasicharacter η defined by

$$\pi \left(\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} \right) = \eta(a) id.$$

is trivial on k^\times ; (3) π satisfies a certain condition so that, for any quasicharacter ω of $k^\times \backslash k_A^\times$, $L(s, \omega \otimes \pi) = \prod L(s, \omega_v \otimes \pi_v)$ and $L(s, \omega^{-1} \otimes \tilde{\pi}_v) = \prod L(s, \omega_v^{-1} \otimes \tilde{\pi}_v)$ converge on a right half-plane; and (4) for any quasicharacter ω of $k^\times \backslash k_A^\times$, $L(s, \omega \otimes \pi)$ and $L(s, \omega^{-1} \otimes \tilde{\pi})$ are entire functions of s which are bounded in vertical strips and satisfy the functional equation

$$L(s, \omega \otimes \pi) = \varepsilon(s, \omega \otimes \pi) L(1-s, \omega^{-1} \otimes \tilde{\pi}).$$

P. Congruence ζ -Functions of Algebraic Function Fields of One Variable or of Algebraic Curves

Let K be an \dagger algebraic function field of one variable over $k = F_q$ (finite field with q elements). The ζ -function of the algebraic function field K/k , denoted by $\zeta_K(s)$, is defined by the infinite sum $\sum_{\mathfrak{A}} N(\mathfrak{A})^{-s}$, where the summation is over all integral divisors \mathfrak{A} of K/k and where the norm $N(\mathfrak{A})$ equals $q^{\deg(\mathfrak{A})}$. Equivalently, $\zeta_K(s)$ is defined by the infinite product $\prod_p (1 - N(\mathfrak{p})^{-s})^{-1}$, where \mathfrak{p} runs over all prime divisors of K/k . By the change of variable $u = q^{-s}$, $\zeta_K(s) = Z_K(u)$ becomes a formal power series in u . $\zeta_K(s)$ and $Z_K(u)$ are sometimes called the **congruence ζ -functions** of K/k .

The fundamental theorem states that (i) (Rationality) $Z_K(u)$ is a rational function of u of the form $Z_K(u) = P(u)/(1-u)(1-qu)$, where $P(u) \in \mathbb{Z}[u]$ is a polynomial of degree $2g$, g being the genus of K ; (ii) (Functional equation) $Z_K(u)$ satisfies the functional equation

$$Z_K(1/u) = q^{g-1} u^{2-2g} Z_K(u/q);$$

and (iii) if $P(u)$ is decomposed into linear factors in $\mathbb{C}[u]$: $P(u) = \prod_{i=1}^{2g} (1 - \alpha_i u)$, then all the reciprocal roots α_i are complex numbers of absolute value \sqrt{q} . Statement (iii) is the analog of the **Riemann hypothesis** because it is equivalent to saying that all the zeros of $\zeta_K(s) = Z_K(q^{-s})$ lie on the line $\operatorname{Re} s = 1/2$.

The congruence ζ -function was introduced by E. Artin [A1 (1924)] as an analog of the Riemann or Dedekind ζ -functions. Of its fundamental properties, the rationality (i) and the functional equation (ii) were proven by

F. K. Schmidt (1931), using the [†]Riemann-Roch theorem for the function field K/k . The Riemann hypothesis (iii) was verified first in the elliptic case ($g=1$) by H. Hasse [H1] and then in the general case by A. Weil [W2 (1948)]. For the proof of (iii), it was essential to consider the geometry of algebraic curves that correspond to given function fields.

Let C be a nonsingular complete curve over k with function field K . Then $Z_K(u)$ coincides with the ζ -function of C/k , denoted by $Z(u, C)$, which is defined by the formal power series $\exp(\sum_{m=1}^{\infty} N_m u^m/m)$. Here N_m is the number of rational points of C over the extension k_m of k of degree m . The rationality of $Z_K(u)$ is then equivalent to the formula

$$N_m = 1 + q^m - \sum_{i=1}^{2g} \alpha_i^m \quad (m \in \mathbf{N}),$$

and the Riemann hypothesis for $Z_K(u)$ is equivalent to the estimate

$$(*) \quad |N_m - 1 - q^m| \leq 2g q^{m/2} \quad (m \in \mathbf{N}).$$

Now if F is the q th power morphism of C to itself (the **Frobenius morphism** of C relative to k), then an important observation is that N_m is the number of fixed points of the m th iterate F^m of F . In other words, N_m is equal to the intersection number of the graph of F^m with the diagonal on the surface $C \times C$, and is related to the "trace" of the Frobenius correspondence. Then (*) follows from [†]Castelnuovo's lemma in the theory of correspondences on a curve. This is Weil's proof of the Riemann hypothesis in [W2]; compare the proof by A. Mattuck and J. Tate (*Abh. Math. Sem. Hamburg* 22 (1958)) and A. Grothendieck (*J. Reine Angew. Math.*, 200 (1958)) using the Riemann-Roch theorem for an algebraic surface.

On the other hand, let J be the [†]Jacobian variety of C over k . For each prime number l different from the characteristic of k , let $M_l(x)$ denote the [†] l -adic representation of an endomorphism α of J obtained from its action on points of J of order l^n ($n=1, 2, \dots$). Letting π be the endomorphism of J induced from F (which is the same as the Frobenius morphism of J), we have $P(u) = \det(1 - M_l(\pi)u)$, i.e., the numerator of the ζ -function coincides with the characteristic polynomial of $M_l(\pi)$. In this setting, the Riemann hypothesis is a consequence of the positivity of the Rosati antiautomorphism [E1]. This is the second proof given by Weil [W2], and applies to arbitrary Abelian varieties.

Recently E. Bombieri, inspired by Stepanov's idea, gave an elementary proof of (*) using only the Riemann-Roch theorem for a curve (*Sém. Bourbaki*, no. 430 (1973)).

Q. ζ -Functions of Algebraic Varieties over Finite Fields

Let V be an algebraic variety over the finite field with q elements \mathbf{F}_q , and let N_m be the number of \mathbf{F}_q^m -rational points of V . Then the ζ -function of V over \mathbf{F}_q is the formal power series in $\mathbf{Z}[[u]]$ defined by

$$Z(u, V) = \exp\left(\sum_{m=1}^{\infty} N_m u^m/m\right);$$

alternatively it can be defined by the infinite product $\prod_P (1 - u^{\deg P})^{-1}$, where P runs over the set of prime divisors of V and $\deg P$ is the degree of the residue field of P over \mathbf{F}_q (in other words, P runs over prime rational 0-cycles of V over \mathbf{F}_q).

Weil Conjecture. In 1949, the following properties of the ζ -function were conjectured by Weil [W3]. Let V be an n -dimensional complete nonsingular (absolutely irreducible) variety over \mathbf{F}_q . Then (1) $Z(u, V)$ is a rational function of u . (2) $Z(u, V)$ satisfies the functional equation

$$Z((q^n u)^{-1}, V) = \pm q^{n\chi/2} u^{\chi} Z(u, V),$$

where the integer χ is the intersection number (the degree of $\Delta_V \cdot \Delta_V$) of the diagonal subvariety Δ_V with itself in the product $V \times V$, which is called the Euler-Poincaré characteristic of V . (3) Moreover, we have

$$Z(u, V) = \frac{P_1(u) \cdot P_3(u) \cdot \dots \cdot P_{2n-1}(u)}{P_0(u) \cdot P_2(u) \cdot \dots \cdot P_{2n}(u)},$$

where $P_h(u) = \prod_{j=1}^{B_h} (1 - \alpha_j^{(h)} u)$ is a polynomial with \mathbf{Z} -coefficients such that $\alpha_j^{(h)}$ are algebraic integers of absolute value $q^{h/2}$ ($0 \leq h \leq 2n$); the latter statement is the **Riemann hypothesis for V/\mathbf{F}_q** . (4) When V is the reduction mod p of a complete nonsingular variety V^* of characteristic 0, then the degree B_h of $P_h(u)$ is the h th Betti number of V^* considered as a complex manifold.

This conjecture, called the **Weil conjecture**, has been completely proven. To give a brief history, first B. Dwork [D13] proved the rationality of the ζ -function for any (not necessarily complete or nonsingular) variety over \mathbf{F}_q . Then A. Grothendieck [A3, G2, G3] developed the l -adic étale cohomology theory with M. Artin and others, and proved the above statements (1)–(4) (except for the Riemann hypothesis) with $P_h(u)$ replaced by some $P_{h,l}(u) \in \mathbf{Q}_l[[u]]$; and S. Lubkin [L7] obtained similar results for liftable varieties. Finally Deligne [D4] proved the Riemann hypothesis and the independence of l of $P_{h,l}(u)$. More details will be given below. Before the final solution for the general case was obtained, the

conjecture had been verified for some special types of varieties. For curves and Abelian varieties, its truth was previously shown by Weil (\rightarrow Section P). In the paper [W3] in which he proposed the above conjecture, Weil verified it for Fermat hypersurfaces, i.e., those defined by the equation $a_0 x_0^m + \dots + a_{n+1} x_{n+1}^m = 0$ ($a_i \in \mathbb{F}_q^\times$); in this case, the ζ -function is of the form $P(u)^{(-1)^{n+1}} / \prod_{j=0}^n (1 - q^j u)$ with a polynomial $P(u)$ that can be explicitly described in terms of Jacobi sums. Dwork [D14] studied by p -adic analysis the case of hypersurfaces in a projective space, verifying the conjecture for them except for the Riemann hypothesis. Further nontrivial examples were provided by *K3 surfaces (Deligne [D2], Pyatetskii-Shapiro, Shafarevich [P1]) and cubic 3-folds (E. Bombieri, H. Swinnerton-Dyer [B5]); in these cases the proof of the Riemann hypothesis was reduced to that of certain Abelian varieties naturally attached to these varieties. It can be said that the Weil conjecture has greatly influenced the development of algebraic geometry, as regards both the foundations and the methods of proof of the conjecture itself; see the expositions by N. Katz [K2] or B. Mazur [M2].

Weil Cohomology, l -Adic Cohomology. The Weil conjecture suggested the possibility of a good cohomology theory for algebraic varieties over a field of arbitrary characteristic. We first formulate the desired properties of a good cohomology (S. Kleiman [K4]). Let \bar{k} be an algebraically closed field and K a field of characteristic 0, which is called the coefficient field. A contravariant functor $V \rightarrow H^*(V)$ from the category of complete connected smooth varieties over \bar{k} to the category of augmented \mathbb{Z}^+ -graded finite-dimensional anticommutative K -algebras (cup product as multiplication) is called a **Weil cohomology** with coefficients in K if it has the following three properties. (1) **Poincaré duality:** If $n = \dim V$, then a canonical isomorphism $H^{2n}(V) \cong K$ exists and the cup product $H^j(V) \times H^{2n-j}(V) \rightarrow H^{2n}(V) \cong K$ induces a perfect pairing. (2) **Künneth formula:** For any V_1 and V_2 the mapping $H^*(V_1) \otimes H^*(V_2) \rightarrow H^*(V_1 \times V_2)$ defined by $a \otimes b \rightarrow \text{Proj}_1^*(a) \cdot \text{Proj}_2^*(b)$ is an isomorphism. (3) **Good relation with algebraic cycles:** Let $C^j(V)$ be the group of algebraic cycles of codimension j on V . There exists a fundamental-class homomorphism **FUND**: $C^j(V) \rightarrow H^{2j}(V)$ for all j , which is functorial in V , compatible with products via Künneth's formula, has compatibility of the intersection with the cup product, and maps 0-cycle $\in C^0(V)$ to its degree as an element of $K \cong H^{2n}(V)$. If a Weil cohomology theory H exists for the V 's over \bar{k} , we can

prove the **Lefschetz fixed-point formula**:

$$((\text{graph of } F) \cdot (\text{diagonal}))_{V \times V} = \sum_{j=0}^{2n} (-1)^j \text{tr}(F^* | H^j(V))$$

for a morphism $F: V \rightarrow V$.

If $\bar{k} = \mathbb{C}$ (the field of complex numbers), the classical cohomology $V \rightarrow H^*(V^{an}, \mathbb{Q})$, where V^{an} denotes the complex manifold associated with V , gives a Weil cohomology. If \bar{k} is an arbitrary algebraically closed field and if l is a prime number different from the characteristic of \bar{k} , then the principal results in the theory of the étale cohomology state that the l -adic cohomology $V \rightarrow H_{\text{ét}}^*(V, \mathbb{Q}_l)$ is a Weil cohomology with coefficient field \mathbb{Q}_l (the field of l -adic numbers) [A3, D5, G3, M4]. In defining this, Grothendieck introduced a new concept of topology, which is now called Grothendieck topology. In the étale topology of a variety V , for example, any étale covering of a Zariski open subset is regarded as an "open set." With respect to the étale topology, the cohomology group $H^*(V, \mathbb{Z}/n)$ of V with coefficients in \mathbb{Z}/n is defined in the usual manner and is a finite \mathbb{Z}/n -module. If l is a prime number as above, $\varprojlim_v H^*(V, \mathbb{Z}/l^v)$ is a module over $\mathbb{Z}_l = \varprojlim_v \mathbb{Z}/l^v$ of finite rank, and

$$H_{\text{ét}}^*(V, \mathbb{Q}_l) = (\varprojlim_v H^*(V, \mathbb{Z}/l^v)) \otimes_{\mathbb{Z}_l} \mathbb{Q}_l$$

defines the l -adic cohomology group, giving rise to a Weil cohomology.

For the characteristic p of k , p -adic étale cohomology does not give Weil cohomology; but the crystalline cohomology (Grothendieck and P. Berthelot [B2, B3]) takes the place of p -adic cohomology and is almost a Weil cohomology: in this theory the fundamental class is defined only for smooth subvarieties.

Now fix a Weil cohomology for $\bar{k} = \bar{\mathbb{F}}_q$, an algebraic closure of a finite field \mathbb{F}_q . Given an algebraic variety V over \mathbb{F}_q , let $\bar{V} = V \otimes \bar{k}$ denote the base extension of V to \bar{k} ; then \mathbb{F}_q^m -rational points of V can be identified with the fixed points of the m th iterate of the Frobenius morphism F of V relative to \mathbb{F}_q . Then the Lefschetz fixed-point formula implies the rationality of $Z(u, V)$; more precisely, letting $P_j(u) = \det(1 - uF^* | H^j(\bar{V}))$ be the characteristic polynomial of the automorphism F^* of $H^j(\bar{V})$ induced by F , we have

$$Z(u, V) = \prod_{j=0}^{2n} P_j(u)^{(-1)^{j+1}}.$$

The functional equation of the ζ -function then follows from the Poincaré duality. This proves (1), (2), and a part of (3) in the statement of the Weil conjecture. Further, in the case of l -adic cohomology, (4) means that $\deg P_j(u) =$

$\dim_{\mathbf{Q}_l} H^j(\bar{V}, \mathbf{Q}_l)$ is equal to the j th Betti number of a lifting of V to characteristic 0; this follows from the comparison theorem of M. Artin for the l -adic cohomology and the classical cohomology, combined with the invariance of l -adic cohomology under specialization.

Proof of the Riemann Hypothesis. In 1974, Deligne [D4, I] completed the proof of the Weil conjecture for projective nonsingular varieties by proving that, given such a V over \mathbf{F}_q , any eigenvalue of F^* on $H_{\text{ét}}^j(\bar{V}, \mathbf{Q}_l)$ is an algebraic integer, all the conjugates of which are of absolute value $q^{j/2}$. (This implies that $P(u) = \det(1 - uF^* | H_{\text{ét}}^j(\bar{V}, \mathbf{Q}_l))$ is in $\mathbf{Z}[u]$ and is independent of l .) The proof is done by induction on $n = \dim V$; by the general results in l -adic cohomology (the weak Lefschetz theorem on a hyperplane section, the Poincaré duality, and the Künneth formula), the proof is reduced to the assertion that (*) any eigenvalue α of F^* on $H_{\text{ét}}^n(\bar{V}, \mathbf{Q}_l)$ is an algebraic integer such that $|\alpha'| \leq q^{(n+1)/2}$ for all conjugates α' of α . The main ingredients in proving (*) are (1) Grothendieck's theory of L -functions, based on the étale cohomology with compact support and with coefficients in a \mathbf{Q}_l -sheaf [G2, G3]; (2) the theory of Lefschetz pencils (Deligne and Katz [D7]), and the Kajdan-Margulis theorem on the monodromy of a Lefschetz pencil (J. L. Verdier, *Sém. Bourbaki*, no. 423 (1972)); and (3) Rankin's methods to estimate the coefficients of modular forms, as adapted to the Grothendieck's L -series. By means of these geometric and arithmetic techniques, Deligne achieved the proof of the Riemann hypothesis for projective nonsingular varieties. For the generalization to complete varieties, see Deligne [D4, II].

Applications of the (Verified) Weil Conjecture.

(1) The Ramanujan conjecture (\rightarrow 32 Automorphic Functions D): The connection of this conjecture and the Weil conjecture for certain fiber varieties over a modular curve was observed by M. Sato and partially verified by Y. Ihara [I1] and then established by Deligne [D3]. The Weil conjecture as proven above implies the truth of the Ramanujan conjecture and its generalization by H. Petersson.

(2) Estimation of trigonometric sums: Let q be the power of a prime number p . Then

$$\left| \sum_{(x_1, \dots, x_n) \in \mathbf{F}_q^n} \exp \frac{2\pi i}{p} \operatorname{tr}_{\mathbf{F}_q/\mathbf{F}_p}(F(x_1, \dots, x_n)) \right| \leq (d-1)^n q^{n/2},$$

where $F(X_1, \dots, X_n) \in \mathbf{F}_q[X_1, \dots, X_n]$ is a polynomial of degree d that is not divisible by p , and the homogeneous part of the highest degree of F defines a smooth irreducible

hypersurface in \mathbf{P}^{n-1} . This is a generalization of the Weil estimation of the Kloosterman sum ([D4, W1 (1948c)]; \rightarrow 4 Additive Number Theory D).

(3) The **hard Lefschetz theorem**: Let $L \in H^2(V)$ be the class of a hyperplane section of an n -dimensional projective nonsingular variety V over an algebraically closed field. Then the cup product by $L^i: H^{n-i}(V) \rightarrow H^{n+i}(V)$ is an isomorphism for all $i \leq n$. Deligne [D4, II] proved this for l -adic cohomology, from which N. Katz and W. Messing [K1] deduced its validity in any Weil cohomology or in the crystalline cohomology.

Also some geometric properties of an algebraic variety V are reflected in the properties of $Z(u, V)$. The ζ -function $Z(u, A)$ of an Abelian variety A determines the isogeny class of A [T4]. For any algebraic integer α , every conjugate of which has absolute value $q^{1/2}$, there exists an Abelian variety A/\mathbf{F}_q such that α is a root of $\det(1 - uF^* | H^1(A)) = 0$ [H6]. J. Tate [T3] conjectured that the rank of the space cohomology classes of algebraic cycles of codimension r is equal to the order of the pole at $u = 1/q^r$ of $Z(u, V)$. This conjecture is still open but has been verified in certain nontrivial cases, e.g., (1) products of curves and Abelian varieties, $r = 1$ (Tate [T4]), (2) Fermat hypersurfaces of dimension $2r$ with some condition on the degree and the characteristic (Tate [T3], T. Shioda, *Proc. Japan Acad.* 55 (1979)), and (3) elliptic $K3$ surfaces, $r = 1$ (M. Artin and Swinnerton-Dyer, *Inventiones Math.* 20 (1973)).

R. ζ - and L -Functions of Schemes

Let X be a \ast scheme of finite type over \mathbf{Z} , and let $|X|$ denote the set of closed points of X ; for each $x \in |X|$, the residue field $k(x)$ is finite, and its cardinality is called the norm $N(x)$ of x . The ζ -function of a scheme X is defined by the product $\zeta(s, X) = \prod_{x \in |X|} (1 - N(x)^{-s})^{-1}$. This converges absolutely for $\operatorname{Re} s > \dim X$, and it is conjectured to have an analytic continuation in the entire s -plane (Serre [S7]). It reduces to the Riemann (resp. Dedekind) ζ -function if $X = \operatorname{Spec}(\mathbf{Z})$ (resp. $\operatorname{Spec}(\mathfrak{o})$, \mathfrak{o} being the ring of integers of an algebraic number field), and to the ζ -function $Z(q^{-s}, X)$ (\rightarrow Section Q) if X is a variety over a finite field \mathbf{F}_q . The case of varieties defined over an algebraic number field is discussed in Section S.

Let G be a finite group of automorphisms of a scheme X , and assume that the quotient $Y = X/G$ exists (e.g., X is quasiprojective). For an element x in $|X|$, let y be its image in $|Y|$, and let $D(x) = \{g \in G | g(x) = x\}$, the decomposition group of x over y . The natural mapping $D(x) \rightarrow \operatorname{Gal}(k(x)/k(y))$ is surjective, and its

kernel $I(x)$ is called the inertia group at x . An element of $D(x)$ is called a Frobenius element at x if its image in $\text{Gal}(k(x)/k(y))$ corresponds to the $N(y)$ th-power automorphism of $k(x)$. Now let R be a representation of G with character χ . The **Artin L -function** $L(s, X, \chi)$ is defined by

$$L(s, X, \chi) = \exp \left(\sum_{y \in |Y|} \sum_{n=1}^{\infty} \chi(y^n) N(y)^{-ns}/n \right) \\ = \prod_{y \in |Y|} \det(1 - R(F_y) N(y)^{-s})^{-1},$$

where $\chi(y^n)$ denotes the mean value of χ on the n th power of Frobenius elements F_x at x (any point of $|X|$ over y), and similarly $R(F_y)$ denotes the mean value of $R(F_x)$; it converges absolutely for $\text{Re } s > \dim X$. Again this is reduced to the usual Artin L -function (\rightarrow Section G) if X is the spectrum of the ring of integers of an algebraic number field. The Artin L -functions of a scheme have many formal properties analogous to those of Artin L -functions of a number field (Serre [S7]).

Let us consider the case where X is an algebraic variety over a finite field F_q and elements of G are automorphisms of X over F_2 ; in this case, $L(s, X, \chi)$ is a formal power series in $u = q^{-s}$, which is called a congruence Artin L -function. For the case where X is a complete nonsingular algebraic curve and χ is an irreducible character of G different from the trivial one, Weil [W2] proved that $L(s, X, \chi)$ is a polynomial in $u = q^{-s}$; thus the analog of \dagger Artin's conjecture holds here. More generally, for any algebraic variety X over F_q , Grothendieck [G2, G3] proved the rationality of L -functions together with the alternating product expression by polynomials in u , as in the case of ζ -functions, by the methods of l -adic cohomology. Actually, Grothendieck treated a more general type of L -function associated with l -adic sheaves on X , which also play an important role in Deligne's proof of the Riemann hypothesis (\rightarrow Section Q).

S. Hasse ζ -Functions

For a nonsingular complete algebraic variety V defined over a finite algebraic number field K , let V_p be the reduction of V modulo a prime ideal p of K , K_p be the residue field of p , and $Z(u, V_p)$ be the ζ -function of V_p over K_p . The ζ -function $\zeta(s, V)$ of the complex variable s , determined by the infinite product (excluding the finite number of p 's where V_p is not defined),

$$\zeta(s, V) = \prod_p Z(N(p)^{-s}, V_p),$$

is called the **Hasse ζ -function** of V over the algebraic number field K . For this function,

we have **Hasse's conjecture** [W4]: $\zeta(s, V)$ is a meromorphic function over the whole complex plane of s and satisfies the functional equation of ordinary type. Sometimes it is more natural to consider

$$\zeta_j(s, V) = \prod_p P_j(N(p)^{-s}, V_p)^{-1} \quad (0 \leq j \leq 2 \dim V),$$

where $P_j(u, V_p)$ is the j th factor of $Z(u, V_p)$, and we have a similar conjecture for them. For the definition of $\zeta_j(s, V)$ taking into account the factors for bad primes and the precise form of the conjectural functional equation, see Serre [S8]. Note that $\zeta_j(s, V)$ converges absolutely for $\text{Re } s > j/2 + 1$ as a consequence of the Weil conjecture.

Hasse's conjecture remains unsolved for the general case, but has been verified when V is one of the following varieties:

(I_a) Algebraic curves defined by the equation $y^e = \gamma x^f + \delta$ and Fermat hypersurfaces (Weil [W6]).

(I_b) Elliptic curves with complex multiplication (Deuring [D11]).

(I_c) Abelian varieties with complex multiplication (Taniyama [T2], Shimura and Taniyama [S11], Shimura, H. Yoshida).

(I_d) Singular K3 surfaces, i.e., K3 surfaces with 20 Picard numbers (Shafarevich and Pyatetskii-Shapiro [P1], Deligne [D2], T. Shioda and H. Inose [S21]).

(II_a) Algebraic curves that are suitable models of the elliptic modular function fields (Eichler [E1], Shimura [S12]).

(II_b) Algebraic curves that are suitable models of the automorphic function fields obtained from a quaternion algebra (Shimura [S13, S15]).

(II_c) Certain fiber varieties of which the base is a curve of type (II_a) or (II_b) and the fibers are Abelian varieties (Kuga and Shimura [K6], Ihara [I1], Deligne [D3]).

(II_d) Certain Shimura varieties of higher dimension (Langlands and others; \rightarrow [B6]).

In these cases, $\zeta(s, V)$ can be expressed by known functions, i.e., by Hecke L -functions with Grössencharakteren of algebraic number fields in cases (I) or by Dirichlet series corresponding to modular forms in cases (II). This fact has an essential meaning for the arithmetic properties of these functions. For example, the extended \dagger Ramanujan conjecture concerning the Hecke operator of the automorphic form reduces to Weil's conjecture on varieties related to those in cases II. Moreover, for (II_a)–(II_c) the essential point is the congruence relation $\hat{T}_p = \Pi + \Pi^*$ (Kronecker, Eichler [E1], Shimura). In particular, for (II_b) this formula is related to the problem of constructing class fields over totally imaginary quadratic extensions of a totally real field F utilizing special

values of automorphic functions and class fields over F . Actually, the formula is equivalent to the reciprocity law for class fields (Shimura).

One of the facts that makes the Hasse ζ -function important is that it describes the decomposition law of prime ideals of algebraic number fields when V is an algebraic curve or an Abelian variety (Weil, Shimura [S14], Taniyama [T2], T. Honda [H6]). In that case, its Hasse ζ -function has the following arithmetic meaning.

Let C be a complete, nonsingular algebraic curve defined over an algebraic number field K , and let J be the Jacobian variety of C defined over K . For a prime number l , fix an l -adic coordinate system Σ_l on J , and let $K(J, l^\infty)$ be the extension field of K obtained by adjoining to K all the coordinates of the l^n th division points ($n = 1, 2, \dots$) of J . Then $K(J, l^\infty)/K$ is an infinite Galois extension of K . The corresponding Galois group $G(J, l^\infty)$ has the l -adic representation $\sigma \rightarrow M_l^*(\sigma)$ by the l -adic coordinates Σ_l . Almost all prime ideals \mathfrak{p} of K are unramified in $K(J, l^\infty)/K$. Thus when we take an arbitrary prime factor \mathfrak{P} of \mathfrak{p} in $K(J, l^\infty)$, the Frobenius substitution of \mathfrak{P} ,

$$\sigma_{\mathfrak{P}} = \left[\frac{K(J, l^\infty)/K}{\mathfrak{P}} \right],$$

is uniquely determined. Furthermore, the characteristic polynomial $\det(1 - M_l^*(\sigma_{\mathfrak{P}})u)$ is determined only by \mathfrak{p} and does not depend on the choice of the prime factor \mathfrak{P} ; we denote this polynomial by $P_p(u, C)$. In this case, for almost all \mathfrak{p} , $P_p(u, C)$ is a polynomial with rational integral coefficients independent of l ; namely, the numerator of the ζ -function of the reduction of $C \bmod p$. Thus

$$\zeta_1(s, C) = \prod_{\mathfrak{p}}' P_p(N(\mathfrak{p})^{-s}, C)^{-1} \\ \sim \prod_{\mathfrak{p}}' \det(1 - M_l^*(\sigma_{\mathfrak{P}})N(\mathfrak{p})^{-s})^{-1}.$$

Here the product $\prod' \det(1 - M_l^*(\sigma_{\mathfrak{P}})N(\mathfrak{p})^{-s})^{-1}$ has the same expression as the Artin L -function if we ignore the fact that M_l^* is the l -adic representation and $K(J, l^\infty)$ is the infinite extension. Thus if we can describe $\zeta(s, C)$ explicitly, then the decomposition process of the prime ideal for intermediate fields between $K(J, l^\infty)$ and K can be made fairly clear. In fact, this is the case for examples $(I_a) - (I_c)$ and $(II_a) - (II_c)$, from which the relations between the arithmetic of the field of division points $K(J, l^\infty)/K$ and the eigenvalues of the Hecke operator have been obtained. Thus for curves and Abelian varieties, $\zeta(s, V)$ is related to the arithmetic of some number fields; but it is not known whether similar arithmetical relations exist for other kinds of varieties except in a few cases.

Tate's Conjecture. For a projective nonsingular variety V over a finite algebraic number field K , let $\mathfrak{A}^r(\bar{V})$ denote the group of algebraic cycles of codimension r on $\bar{V} = V \otimes_K \mathbb{C}$ modulo homological equivalence and let $\mathfrak{A}^r(V)$ be the subgroup of $\mathfrak{A}^r(\bar{V})$ generated by algebraic cycles rational over K . Then Tate [T3] conjectured that the rank of $\mathfrak{A}^r(V)$ is equal to the order of the pole of $\zeta_{2r}(s, V)$ at $s = r + 1$. This conjecture is closely connected with **Hodge's conjecture** that the space of rational cohomology classes of type (r, r) on \bar{V} is spanned by $\mathfrak{A}^r(\bar{V})$; in fact, the equivalence of these conjectures is known for Abelian varieties of \dagger CM type (H. Pohlmann, *Ann. Math.*, 88 (1968)) and for Fermat hypersurfaces of dimension $2r$ (Tate [T3], Weil [W6]). Thus, when $r = 1$, Tate's conjecture for these varieties holds by Lefschetz's theorem, and when $r > 1$, it holds in certain cases where the Hodge conjecture is verified (Shioda, *Math. Ann.*, 245 (1979); Z. Ran, *Compositio Math.*, 42 (1981)). Further examples are given by $K3$ surfaces with large Picard numbers (Shioda and Inose [S21]; T. Oda, *Proc. Japan Acad.*, 56 (1980)).

L -Functions of Elliptic Curves. Let E be an elliptic curve (with a rational point) over the rational number field \mathbb{Q} , and let N be its conductor; a prime number p divides N if and only if E has bad reduction $\bmod p$ (Tate [T5]). The L -function of E over \mathbb{Q} is defined as follows:

$$L(s, E) = \prod_{p|N} (1 - \varepsilon_p p^{-s})^{-1} \prod_{p \nmid N} (1 - a_p p^{-s} + p^{1-2s})^{-1},$$

where $\varepsilon_p = 0$ or ± 1 and $1 - a_p u + pu^2 = P_1(u, E \bmod p)$. There are many interesting results and conjectures concerning $L(s, E)$ [T5]:

(1) Functional equation. Let

$$\xi(s, E) = N^{s/2} (2\pi)^{-s} \Gamma(s) L(s, E).$$

Then it is conjectured that $\xi(s, E)$ is holomorphic in the entire s -plane and satisfies the functional equation $\xi(s, E) = \pm \xi(2-s, E)$. This is true if E has complex multiplication (Deuring) or E is a certain modular curve (Eichler, Shimura).

(2) **Taniyama-Weil conjecture.** Weil [W1 (1967a)] conjectured that, if $L(s, E) = \sum_{n=1}^{\infty} a_n n^{-s}$, then $f(\tau) = \sum_{n=1}^{\infty} a_n e^{2\pi i n \tau}$ is a cusp form of weight 2 for the congruence subgroup $\Gamma_0(N)$ which is an eigenfunction for Hecke operators; moreover E is isogenous to a factor of the Jacobian variety of the modular curve for $\Gamma_0(N)$ in such a way that $f(\tau) d\tau$ corresponds to the differential of the first kind on E . If this conjecture is true, then the statements in (1) follow.

(3) **Birch–Swinnerton-Dyer conjecture.** Assuming analytic continuation of $L(s, E)$, B. Birch and H. Swinnerton-Dyer [B4] conjectured that the order of the zero of $L(s, E)$ at $s = 1$ is equal to the rank r of the group $E(\mathbf{Q})$ of rational points of E which is finitely generated by the Mordell-Weil theorem. They verified this for many examples, especially for curves of the type $y^2 = x^3 - ax$. J. Coates and A. Wiles (*Inventiones Math.*, 39 (1977)) proved that if E has complex multiplication and if $r > 0$ then $L(s, E)$ vanishes at $s = 1$. This conjecture has a refinement which extends also to Abelian varieties over a global field (Tate, *Sém. Bourbaki*, no. 306 (1966)).

(4) **Sato's conjecture.** Let

$$1 - a_p u + pu^2 = (1 - \pi_p u)(1 - \bar{\pi}_p u),$$

with $\pi_p = \sqrt{p} e^{i\theta_p}$ ($0 < \theta_p < \pi$). When E has complex multiplication, the distribution of θ_p for half of p is uniform in the interval $[0, \pi]$, and θ_p is $\pi/2$ for the remaining half of p . Suppose that E does not have complex multiplication. Then Sato conjectured that

$$\lim_{x \rightarrow \infty} \frac{(\text{the number of prime numbers } p \text{ less than } x \text{ such that } \theta_p \in [\alpha, \beta])}{(\text{the number of prime numbers less than } x)} = \frac{2}{\pi} \int_{\alpha}^{\beta} \sin^2 \theta \, d\theta \quad (0 < \alpha < \beta < \pi)$$

(Tate [T3]).

H. Yoshida [Y1] posed an analog of Sato's conjecture for elliptic curves defined over function fields with finite constant fields and proved it in certain cases.

(5) **Formal groups.** Letting $L(s, E) = \sum a_n n^{-s}$ as before, set $f(x) = \sum_{n=1}^{\infty} a_n x^n / n$. Honda [H6] showed that $f^{-1}(f(x) + f(y))$ is a * formal (Lie) group with coefficients in \mathbf{Z} and that this group is isomorphic over \mathbf{Z} to a formal group obtained by power series expansion of the group law of E with respect to suitable * local uniformizing coordinates at the origin. Such an interpretation of the ζ -function also applies to other cases in which ζ -functions of * group varieties may be characterized as Dirichlet series whose coefficients give a normal form of the group law; e.g., the case of algebraic tori (T. Ibukiyama, *J. Fac. Sci. Univ. Tokyo*, (IA) 21 (1974)).

T. Selberg ζ -Functions and ζ -Functions Associated with Discontinuous Groups

Let $\Gamma \subset SL(2, \mathbf{R})$ be a * Fuchsian group operating on the complex upper half-plane $H = \{z = x + iy \mid y > 0\}$. When the two eigenvalues of an element $\gamma \in \Gamma$ are distinct real numbers ξ_1 ,

ξ_2 ($\xi_1 \xi_2 = 1, \xi_1 < \xi_2$), we call γ * hyperbolic. Then the number ξ_2^2 is denoted by $N(\gamma)$ and is called the norm of γ . When γ is hyperbolic, γ^n ($n = 1, 2, 3, \dots$) is also hyperbolic. When $\pm \gamma$ is not a positive power of other hyperbolic elements, γ is called a primitive hyperbolic element. The elements conjugate to primitive hyperbolic elements are also primitive hyperbolic elements and have the same norm as γ . Let P_1, P_2, \dots be the conjugacy classes of primitive hyperbolic elements of Γ , and let $\gamma_i \in P_i$ be their representatives. Suppose that a matrix representation $\gamma \rightarrow M(\gamma)$ of Γ is given. Then the analytic function given by

$$Z_{\Gamma}(s, M) = \prod_i \prod_{n=0}^{\infty} \det(I - M(\gamma_i) N(\gamma_i)^{-s-n})$$

is called the **Selberg ζ -function** (Selberg [S5]). When $\Gamma \backslash H$ is compact and Γ is torsion-free, then $Z_{\Gamma}(s, M)$ has the following properties.

(1) It can be analytically continued to the whole complex plane of s and gives an * integral function of genus at most 2.

(2) It has zeros of order $(2n+1)(2g-2)v$ at $-n$ ($n = 0, 1, 2, 3, \dots$). Here g is the genus of the Riemann surface $\Gamma \backslash H$ and v is the degree of the representation M . All other zeros lie on the line $\text{Re } s = 1/2$, except for a finite number that lie on the interval $(0, 1)$ of the real axis.

(3) It satisfies the functional equation

$$Z_{\Gamma}(1-s, M) = Z_{\Gamma}(s, M) \exp \left(-v A(\Gamma \backslash H) \times \int_0^{s-1/2} v \tan(\pi v) dv \right),$$

where

$$A(\Gamma \backslash H) = \iint_{\Gamma \backslash H} \frac{dx dy}{y^2} = 2\pi(2g-2), \quad x + iy \in H.$$

Property (2) shows that the Riemann hypothesis is almost valid for $Z_{\Gamma}(s, M)$. The proof is based on the following fact concerning the eigenvalue problem for the variety $\Gamma \backslash H$: The eigenvalue λ of the equation

$$y^2(\partial^2/\partial x^2 + \partial^2/\partial y^2)u + \lambda u = 0, \quad u \in L^2(\Gamma \backslash H)$$

cannot be a negative number.

Using this function, T. Yamada (1965) investigated the unit distribution of real quadratic fields.

Selberg ζ -functions are defined similarly when $\Gamma \backslash G$ has finite volume but is noncompact. In this case, however, the decomposition of $L_2(\Gamma \backslash G)$ into irreducible representation spaces has a continuous spectrum; hence the properties of the Selberg ζ -function of Γ are quite different from the case when $\Gamma \backslash G$ is compact. Selberg defined the **generalized Eisenstein series** to give the eigenfunctions of this continuous spectrum explicitly. When $\Gamma =$

$SL(2, \mathbf{Z})$, the series is given by $\sum_{(c,d)=1} \frac{y^s}{|c\tau+d|^{2s}}$.

This type of generalized Eisenstein series is also defined for the general semisimple algebraic group G and its arithmetic subgroup. It has been studied by Selberg, Godement, Gelfand, Harish-Chandra, Langlands, D. Zagier, and others.

U. Ihara ζ -Functions

Let k_p be a p -adic field, \mathfrak{o}_p the ring of integers in k_p , and $G = PSL_2(\mathbf{R}) \times PSL_2(k_p)$. Suppose that Γ is a subgroup of G such that (1) Γ is discrete, (2) $\Gamma \backslash G$ is compact, (3) Γ has no torsion element except the identity, (4) $\Gamma_{\mathbf{R}}$ (the projection of Γ in $PSL_2(\mathbf{R})$) is dense in $PSL_2(\mathbf{R})$, and (5) Γ_p (the projection of Γ in $PSL_2(k_p)$) is dense in $PSL_2(k_p)$. Then $\Gamma \cong \Gamma_{\mathbf{R}} \cong \Gamma_p$. Let $X = \{x + iy | y > 0\}$ be the upper half-plane, and let Γ act on X via $\Gamma_{\mathbf{R}}$. The action of Γ on X is not discontinuous, but the subgroup $\Gamma_0 = \{\gamma \in \Gamma | \text{projection of } \gamma \text{ to } \Gamma_p \in PSL_2(\mathfrak{o}_p)\}$ operates on X properly discontinuously. For each $z \in X$, define $\Gamma_z = \{\gamma \in \Gamma | \gamma(z) = z\}$. Then Γ_z is isomorphic to \mathbf{Z} or $\{1\}$. Let $\tilde{\mathbf{P}}(\Gamma) = \{z \in X | \Gamma_z \cong \mathbf{Z}\}$. The group Γ acts on $\tilde{\mathbf{P}}(\Gamma)$, since Γ_z and $\Gamma_{\gamma z}$ are conjugate in Γ . Let $\mathbf{P}(\Gamma) = \tilde{\mathbf{P}}(\Gamma)/\Gamma$. Suppose that $P \in \mathbf{P}(\Gamma)$ is represented by $z \in X$. Choose a generator γ of Γ_z and project γ to Γ_p . Then γ is equivalent to a diagonal matrix $\begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}$ with $\lambda \in k_p$. We denote the valuation of k_p by ord_p and consider $|\text{ord}_p(\lambda)|$. This value depends only on P and we denote it by $\deg(P)$. The **Ihara ζ -function** of Γ is defined by

$$Z_{\Gamma}(u) = \prod_{P \in \mathbf{P}(\Gamma)} (1 - u^{\deg(P)})^{-1}.$$

Ihara proved that

$$Z_{\Gamma}(u) = \frac{\prod_{i=1}^g (1 - \pi_i u)(1 - \pi_i' u)}{(1 - u)(1 - q^2 u)} (1 - u)^H,$$

where q is the number of elements in the residue class field of p , and g is the genus of the Riemann surface $\Gamma_0 \backslash X$ and $H = (g - 1)q(q - 1)$. Similar results hold even if Γ has torsion elements and the quotient $\Gamma \backslash G$ is only assumed to have finite volume.

Aside from the factor $(1 - u)^H$, this looks like Weil's formula for the congruence ζ -function of an algebraic curve defined over \mathbf{F}_{q^2} . Ihara conjectured that the first factor of $Z_{\Gamma}(u)$ is always the congruence ζ -function of some algebraic curve over \mathbf{F}_{q^2} , and furthermore that Γ could be regarded as the fundamental group of a certain Galois covering of this curve which describes the decomposition law of

prime divisors in this covering [12, 13, 14]. He verified the conjecture in the case $\Gamma = PGL_2(\mathbf{Z}[1/p])$ by using the moduli of elliptic curves. Related results have been obtained by Shimura, Ihara, Y. Morita, and others.

V. ζ -Functions Associated with Prehomogeneous Vector Spaces

M. Sato posed a notion of prehomogeneous vector spaces and defined ζ -functions associated with them. Sato's program has been carried on by himself and T. Shintani [S2, S3, S17, S18]. Let G be a linear algebraic group, V a finite-dimensional linear space of dimension n , and ρ a rational representation $G \rightarrow GL(V)$, where G , V , and ρ are defined over \mathbf{Q} . The triple (G, ρ, V) is called a **prehomogeneous vector space** if there exists a proper algebraic subset S of $V_{\mathbf{C}}$ such that $V_{\mathbf{C}} - S$ is a single $G_{\mathbf{C}}$ -orbit. The algebraic set S is called the set of singular points of V . We also assume that G is reductive and S is an irreducible hypersurface of V . Let V^* be the dual vector space of V , and ρ^* the dual (contragredient) representation of G . Then (G, ρ^*, V^*) is again a prehomogeneous vector space, and we denote its set of singular points by S^* . There are homogeneous polynomials P and Q of the same degree d on V and V^* , respectively, such that $S = \{x \in X | P(x) = 0\}$ and $S^* = \{x^* \in V^* | Q(x^*) = 0\}$. P and Q are relative invariants of G , i.e., $P(\rho(g)x) = \chi(g)P(x)$ and $Q(\rho^*(g)x^*) = \chi(g)^{-1}Q(x^*)$ (for $g \in G$, $x \in V$, and $x^* \in V^*$) hold with a rational character χ of G . Put $G^1 = \ker \chi = \{g \in G | \chi(g) = 1\}$. Denote by $G_{\mathbf{R}}^+$ the connected component of 1 of the Lie group $G_{\mathbf{R}}$. Let $V_{\mathbf{R}} - S = V_1 \cup \dots \cup V_l$, $V_{\mathbf{R}}^* - S^* = V_1^* \cup \dots \cup V_l^*$ be the decompositions of $V_{\mathbf{R}} - S$ and $V_{\mathbf{R}}^* - S^*$ into their topologically connected components. Then V_i and V_j^* are $G_{\mathbf{R}}^+$ -orbits. We further assume that $V_{\mathbf{R}} \cap S$ decomposes into the union of a finite number of $G_{\mathbf{R}}^1$ -orbits. Set $\Gamma = G_{\mathbf{R}}^+ \cap G_{\mathbf{Z}}^1$, and take Γ -invariant lattices L and L^* in $V_{\mathbf{Q}}$ and $V_{\mathbf{Q}}^*$, respectively. Consider the following functions in s :

$$\Phi_i(f, s) = \int_{V_i} f(x) |P(x)|^s dx,$$

$$\Phi_j^*(f, s) = \int_{V_j^*} f^*(x^*) |Q(x^*)|^s dx^*,$$

and

$$Z_i(f, L, s) = \int_{G_{\mathbf{R}}^+/\Gamma} \chi(g)^s \sum_{x \in L \cap V_i} f(\rho(g)x) dg,$$

$$Z_i^*(f^*, L^*, s)$$

$$= \int_{G_{\mathbf{R}}^+/\Gamma} \chi(g)^{-s} \sum_{x^* \in L^* \cap V_j^*} f^*(\rho^*(g)x^*) dg,$$

where f and f^* are rapidly decreasing functions on V_R and V_R^* , respectively, dx and dx^* are Haar measures of V_R and V_R^* , respectively, and dg is a Haar measure of G . Then the ratios

$$\frac{Z_i(f, L, s)}{\Phi_i(f, s - n/d)} = \xi_i(s, L),$$

$$\frac{Z_j^*(f^*, L^*, s)}{\Phi_j^*(f^*, s - n/d)} = \xi_j^*(s, L^*)$$

are independent of the choice of f and f^* and are Dirichlet series in s . These Dirichlet series $\xi_i(s, L)$ and $\xi_j^*(s, L^*)$ are called **ξ -functions associated with the prehomogeneous space**.

Considering Fourier transforms of $|P(x)|^s$ and $|Q(x^*)|^s$, we obtain functional equations for ξ_i and ξ_j^* under some additional (but mild) conditions on (G, ρ, V) as follows. The Dirichlet series ξ_i and ξ_j^* are analytically continuable to meromorphic functions on the whole s -plane, and they satisfy

$$\begin{aligned} v(L^*)\xi_j^*(n/d - s, L^*) \\ = \gamma(s - n/d)(2\pi)^{-ds}|b_0|^s \exp(\pi d\sqrt{-1}s/2) \\ \times \sum_{j=1}^l u_{ij}(s)\xi_i(s, L), \end{aligned}$$

with a Γ -factor $\gamma(s) = \prod_{i=1}^d \Gamma(s - c_i + 1)$.

Here $u_{ij}(s)$ ($1 \leq i, j \leq l$) are polynomials in $\exp(-\pi\sqrt{-1}s)$ with degree $\leq d$, and b_0 and c_i are constants depending only on (G, ρ, V) .

Epstein's ζ -functions and Siegel's Dirichlet series associated with indefinite quadratic forms are examples of the above-defined ζ -functions. Shintani defined such ζ -functions related to integral binary cubic forms and obtained asymptotic formulas concerning the class numbers of irreducible integral binary cubic forms with discriminant n , which are improvements on the results of Davenport [S17].

Recently M. Sato studied ζ -functions of prehomogeneous vector spaces without assuming the conditions that G is reductive and S is irreducible. In this case, ζ -functions of several complex variables are obtained. For examples and classification of prehomogeneous vector spaces \rightarrow [S4].

References

- [A1] E. Artin, Collected papers, S. Lang and J. T. Tate (eds.), Addison-Wesley, 1965.
[A2] E. Artin, Zur Theorie der L -Reihen mit allgemeinen Gruppencharakteren, Abh. Math. Sem. Univ. Hamburg, 8 (1930), 292–306 [A1, 165–179].

- [A3] M. Artin, A. Grothendieck, and J. L. Verdier, Théorie des topos et cohomologie étale des schémas (SGA 4), Lecture notes in math. 269, 270, 305, Springer, 1972–1973.
[A4] J. Ax, On the units of an algebraic number fields, Illinois J. Math., 9 (1965), 584–589.
[B1] P. Bayer and J. Neukirch, On values of zeta functions and l -adic Euler characteristics, Inventiones Math., 50 (1978), 35–64.
[B2] P. Berthelot, Cohomologie cristalline des schéma de caractéristique $p > 0$, Lecture notes in math. 407, Springer, 1974.
[B3] P. Berthelot and A. Ogus, Notes on crystalline cohomology, Math. notes 21, Princeton Univ. Press, 1978.
[B4] B. Birch and H. Swinnerton-Dyer, Notes on elliptic curves II, J. Reine Angew. Math., 218 (1965), 79–108.
[B5] E. Bombieri and H. P. F. Swinnerton-Dyer, On the local zeta function of a cubic threefold, Ann. Scuola Norm. Sup. Pisa, (3) 21 (1967), 1–29.
[B6] A. Borel and W. Casselman (eds.), Automorphic forms, representations, and L -functions, Amer. Math. Soc. Proc. Symp. Pure Math., 33 (pts. 1 and 2) (1979).
[B7] A. Brumer, On the units of algebraic number fields, Mathematika, 14 (1967), 121–124.
[C1] K. Chandrasekharan, Lectures on the Riemann zeta-function, Tata Inst., 1953.
[C2] S. Chowla and A. Selberg, On Epstein's zeta function I, Proc. Nat. Acad. Sci. US, 35 (1949), 371–374.
[C3] J. Coates, p -adic L -functions and Iwasawa's theory, Algebraic Number Fields (L -functions and Galois Properties), A. Fröhlich (ed.), Academic Press, 1977.
[D1] R. Dedekind, Über die Theorie der ganzen algebraischen Zahlen, Dirichlet's Vorlesungen über Zahlentheorie, Supplement XI, fourth edition, 1894, secs. 184, 185, 186 (Gesammelte mathematische Werke III, Braunschweig, 1932, 297–314; Chelsea, 1968).
[D2] P. Deligne, La conjecture de Weil pour les surfaces $K3$, Inventiones Math., 75 (1972), 206–226.
[D3] P. Deligne, Formes modulaires et représentations l -adiques, Sémin. Bourbaki, exp. 355, Lecture notes in math. 349, Springer, 1973.
[D4] P. Deligne, La conjecture de Weil, I, II, Publ. Math. Inst. HES, 43 (1974), 273–307; 52 (1980), 137–252.
[D5] P. Deligne, Cohomologie étale (SGA 4 $\frac{1}{2}$), Lecture notes in math. 569, Springer, 1977.
[D6] P. Deligne, Les constantes des équations fonctionnelles des fonctions L , Lecture notes in math. 349, Springer, 1973.
[D7] P. Deligne and N. Katz, Groupes de monodromie en géométrie algébrique (SGA

- 7II), Lecture notes in math. 340, Springer, 1973.
- [D8] P. Deligne and K. Ribet, Values of Abelian L -functions at negative integers over totally real fields, *Inventiones Math.*, 59 (1980), 227–286.
- [D9] P. Deligne and J.-P. Serre, Formes modulaires de poids 1, *Ann. Sci. École Norm. Sup. (4)*, 7 (1974), 507–530.
- [D10] M. Deuring, *Algebren*, Erg. Math., Springer, second edition, 1968.
- [D11] M. Deuring, Die Zetafunktionen einer algebraischen Kurve vom Geschlechte Eins I, II, III, IV, *Nachr. Akad. Wiss. Göttingen* (1953), 85–94; (1955), 13–42; (1956), 37–76; (1957), 55–80.
- [D12] K. Doi and H. Naganuma, On the functional equation of certain Dirichlet series, *Inventiones Math.*, 9 (1969–1970), 1–14.
- [D13] B. Dwork, On the rationality of the zeta-function of an algebraic variety, *Amer. J. Math.*, 82 (1960), 631–648.
- [D14] B. Dwork, On the zeta function of a hypersurface I, II, III, *Publ. Math. Inst. HES*, 1962, 5–68; *Ann. Math.*, (2) 80 (1964), 227–299; (2) 83 (1966), 457–519.
- [E1] M. Eichler, *Einführung in die Theorie der algebraischen Zahlen und Funktionen*, Birkhäuser, 1963; English translation, *Introduction to the theory of algebraic numbers and functions*, Academic Press, 1966.
- [F1] B. Ferrero and L. C. Washington, The Iwasawa invariant μ_p vanishes for Abelian number fields, *Ann. Math.*, (2) 109 (1979), 377–395.
- [G1] R. Godement, Les fonctions ζ des algèbres simples I, II, *Sém. Bourbaki*, exp. 171, 176, 1958–1959.
- [G2] A. Grothendieck, Formule de Lefschetz et rationalité des fonction L , *Sém. Bourbaki*, exp. 379, 1964–1965 (Benjamin, 1966).
- [G3] A. Grothendieck, *Cohomologie l -adique et fonctions L* (SGA 5), Lecture notes in math. 589, Springer, 1977.
- [H1] H. Hasse, Zur Theorie der abstrakten elliptischen Funktionenkörper I, II, III, *J. Reine Angew. Math.*, 175 (1936), 55–62, 69–88, 193–208.
- [H2] E. Hecke, *Mathematische Werke*, Vandenhoeck & Ruprecht, 1959.
- [H3] E. Hecke, Eine neue Art von Zetafunktionen und ihre Beziehung zur Verteilung der Primzahlen I, II, *Math. Z.*, 1 (1918), 357–376; 6 (1920), 11–51 [H2, 215–234, 249–289].
- [H4] E. Hecke, Über die Bestimmung Dirichletscher Reihen durch ihre Funktionalgleichung, *Math. Ann.*, 112 (1936), 664–699 [H2, 591–626].
- [H5] E. Hecke, Über Modulformen und die Dirichletschen Reihen mit Eulerscher Produktentwicklung I, II, *Math. Ann.*, 114 (1937), 1–28; 316–351 [H2, 644–671, 672–707].
- [H6] T. Honda, Formal groups and zeta functions, *Osaka J. Math.*, 5 (1968), 199–213.
- [H7] T. Honda, Isogeny classes of Abelian varieties over finite fields, *J. Math. Soc. Japan*, 20 (1968), 83–95.
- [I1] Y. Ihara, Hecke polynomials as congruence ζ functions in elliptic modular case (To validate Sato's identity), *Ann. Math.*, (2) 85 (1967), 267–295.
- [I2] Y. Ihara, On congruence monodromy problems, I, II, Lecture notes, Univ. of Tokyo, 1968–1969.
- [I3] Y. Ihara, Non-Abelian class fields over function fields in special cases, *Actes Congr. Intern. Math.*, 1970, Nice, Gauthier-Villars, p. 381–389.
- [I4] Y. Ihara, Congruence relations and Shimura curves, *Amer. Math. Soc. Proc. Symp. Pure Math.*, 33 (1979), pt. 2, 291–311.
- [I5] K. Iwasawa, On Γ -extensions of algebraic number fields, *Bull. Amer. Math. Soc.*, 65 (1959), 183–226.
- [I6] K. Iwasawa, On p -adic L -functions, *Ann. Math.*, (2) 89 (1969), 198–205.
- [I7] K. Iwasawa, *Lectures on p -adic L -functions*, *Ann. math. studies* 74, Princeton Univ. Press, 1972.
- [J1] H. Jacquet and R. P. Langlands, Automorphic forms on $GL(2)$, Lecture notes in math. 114, Springer, 1970.
- [J2] H. Jacquet, Automorphic forms on $GL(2)$, II, Lecture notes in math. 278, Springer, 1972.
- [K1] N. Katz and W. Messing, Some consequences of the Riemann hypothesis for varieties over finite fields, *Inventiones Math.*, 23 (1974), 73–77.
- [K2] N. Katz, An overview of Deligne's proof of the Riemann hypothesis for varieties over finite fields, *Amer. Math. Soc. Proc. Symp. Pure Math.*, 28 (1976), 275–305.
- [K3] N. Katz, p -adic L -functions for CM-fields, *Inventiones Math.*, 49 (1978), 199–297.
- [K4] S. Kleiman, Algebraic cycles and the Weil conjectures, *Dix exposés sur la cohomologies des schémas*, North-Holland, 1968, 359–386.
- [K5] T. Kubota and H. W. Leopoldt, Eine p -adische Theorie der Zetawerte, *J. Reine Angew. Math.*, 214/215 (1964), 328–339.
- [K6] M. Kuga and G. Shimura, On the zeta function of a fiber variety whose fibers are Abelian varieties, *Ann. Math.*, (2) 82 (1965), 478–539.
- [L1] E. G. H. Landau, *Handbuch der Lehre von der Verteilung der Primzahlen I, II*, Teubner, 1909 (Chelsea, 1953).
- [L2] S. Lang, Sur les series L d'une variété algébrique, *Bull. Soc. Math. France*, 84 (1956), 385–407.

- [L3] S. Lang, Algebraic number theory, Addison-Wesley, 1970.
- [L4] R. P. Langlands, Problems in the theory of automorphic forms, Lecture notes in math. 170, Springer, 1970, 18–86.
- [L5] R. P. Langlands, Base change for $GL(2)$, Ann. math. studies 96, Princeton Univ. Press, 1980.
- [L6] H. W. Leopoldt, Eine p -adische Theorie der Zetawerte II, J. Reine Angew. Math., 274/275 (1975), 224–239.
- [L7] S. Lubkin, A p -adic proof of the Weil conjecture, Ann. Math., (2) 87 (1968), 105–194; (2) (1968), 195–225.
- [M1] Yu. I. Manin, Periods of parabolic forms and p -adic Hecke series, Math. USSR-Sb., 21 (1973), 371–393.
- [M2] B. Mazur, Eigenvalues of Frobenius acting on algebraic varieties over finite fields, Amer. Math. Soc. Proc. Symp. Pure Math., 29 (1975).
- [M3] B. Mazur and H. Swinnerton-Dyer, On the p -adic L -series of an elliptic curve, Inventiones Math., 25 (1974), 1–61.
- [M4] J. S. Milne, Etale cohomology, Princeton series in mathematics 33, Princeton Univ. Press, 1980.
- [P1] I. I. Pyatetskii-Shapiro and I. R. Shafarevich, The arithmetic of $K3$ surfaces, Proc. Steklov Inst. Math., 132 (1973), 45–57.
- [P2] K. Prachar, Primzahlverteilung, Springer, 1957.
- [R1] B. Riemann, Über die Anzahl der Primzahlen unter einer gegebenen Grösse, Gesammelte mathematische Werke, 1859, 145–153 (Dover, 1953).
- [S1] H. Saito, Automorphic forms and algebraic extensions of number fields, Lectures in math. 8, Kinokuniya, Tokyo, 1975.
- [S2] M. Sato, Theory of prehomogeneous vector spaces (notes by T. Shintani) (in Japanese), Sûgaku no Ayumi, 15 (1970), 85–157.
- [S3] M. Sato and T. Shintani, On zeta functions associated with prehomogeneous vector spaces, Ann. Math., (2) 100 (1974), 131–170.
- [S4] M. Sato and T. Kimura, A classification of irreducible prehomogeneous vector spaces and their invariants, Nagoya Math. J., 65 (1977), 1–155.
- [S5] A. Selberg, Harmonic analysis and discontinuous groups on weakly symmetric Riemannian spaces with applications to Dirichlet series, J. Indian Math. Soc., 20 (1956), 47–87.
- [S6] A. Selberg, On the zeros of Riemann's zeta-function, Skr. Norske Vid. Akad. Oslo, no. 10 (1942), 1–59.
- [S7] J.-P. Serre, Zeta and L functions, Arithmetical Algebraic Geometry, Harper & Row, 1965, 82–92.
- [S8] J.-P. Serre, Facteurs locaux des fonctions zêta des variétés algébriques (Définitions et conjectures), Sém. Delange-Pisot-Poitou, 11 (1969/70), no. 19.
- [S9] J.-P. Serre, Formes modulaires et fonctions zêta p -adiques, Lecture notes in math. 317, Springer, 1973, 319–338.
- [S10] H. Shimizu, On zeta functions of quaternion algebras, Ann. Math., (2) 81 (1965), 166–193.
- [S11] G. Shimura and Y. Taniyama, Complex multiplication of Abelian varieties and its applications to number theory, Publ. Math. Soc. Japan, no. 6, 1961.
- [S12] G. Shimura, Introduction to the arithmetic theory of automorphic functions, Publ. Math. Soc. Japan, no. 11, 1971.
- [S13] G. Shimura, On the zeta-functions of the algebraic curves uniformized by certain automorphic functions, J. Math. Soc. Japan, 13 (1961), 275–331.
- [S14] G. Shimura, A reciprocity law in non-solvable extensions, J. Reine Angew. Math., 221 (1966), 209–220.
- [S15] G. Shimura, Construction of class fields and zeta functions of algebraic curves, Ann. Math., (2) 85 (1967), 58–159.
- [S16] G. Shimura, On canonical models of arithmetic quotients of bounded symmetric domains, I, II, Ann. Math., (2) 91 (1970) 144–222, 92 (1970), 528–549.
- [S17] T. Shintani, On Dirichlet series whose coefficients are class numbers of integral binary cubic forms, J. Math. Soc. Japan, 24 (1972), 132–188.
- [S18] T. Shintani, On zeta functions associated with the vector space of quadratic forms, J. Fac. Sci. Univ. Tokyo, (IA) 22 (1975), 25–65.
- [S19] T. Shintani, On a Kronecker limit formula for real quadratic fields, J. Fac. Sci. Univ. Tokyo, (IA) 24 (1977), 167–199.
- [S20] T. Shintani, On liftings of holomorphic automorphic forms, Amer. Math. Soc. Proc. Symp. Pure Math., 33 (1979), pt. 2, 97–110.
- [S21] T. Shioda and H. Inose, On singular $K3$ surfaces, Complex Analysis and Algebraic Geometry, Iwanami and Cambridge Univ. Press, 1977, 119–136.
- [S22] C. L. Siegel, Gesammelte Abhandlungen I, II, III, IV, Springer, 1966, 1979.
- [S23] C. L. Siegel, Über die Classenzahl quadratischer Zahlkörper, Acta Arith., 1 (1935), 83–86 [S22, I. 406–409].
- [S24] C. L. Siegel, Über die Zetafunktionen indefiniter quadratischer Formen I, II, Math. Z., 43 (1938), 682–708; 44 (1939), 398–426 [S22, II, 41–67, 68–96].
- [S25] H. M. Stark, L -functions at $s = 1$, II, III, Advances in Math., 17 (1975), 60–92; 22 (1976), 64–84.
- [T1] T. Tamagawa, On the ζ -functions of a

- division algebra, *Ann. Math.*, (2) 77 (1963), 387–405.
- [T2] Y. Taniyama, *L*-functions of number fields and zeta functions of Abelian varieties, *J. Math. Soc. Japan*, 9 (1957), 330–366.
- [T3] J. Tate, Algebraic cycles and poles of zeta functions, *Arithmetical Algebraic Geometry*, Harper & Row, 1965, 93–110.
- [T4] J. Tate, Endomorphisms of Abelian varieties over finite fields, *Inventiones Math.*, 2 (1966), 134–144.
- [T5] J. Tate, The arithmetic of elliptic curves, *Inventiones Math.*, 23 (1974), 179–206.
- [T6] J. Tate, Number theoretic background, *Amer. Math. Soc. Proc. Symp. Pure Math.*, 33 (1979), pt. 2, 3–26.
- [T7] T. Tatzuwa, On the Hecke-Landau *L*-series, *Nagoya Math. J.*, 16 (1960), 11–20.
- [T8] E. C. Titchmarsh, *The theory of the Riemann zeta-function*, Clarendon Press, 1951.
- [W1] A. Weil, *Collected papers, I–III*, Springer, 1980.
- [W2] A. Weil, *Courbes algébriques et variétés abéliennes*, Hermann, 1971 (orig., 1948).
- [W3] A. Weil, Numbers of solutions of equations in finite fields, *Bull. Amer. Math. Soc.*, 55 (1949), 497–508 [W1, I, 399–400].
- [W4] A. Weil, Number theory and algebraic geometry, *Proc. Intern. Congr. Math.*, 1950, Cambridge, vol. 2, 90–100 [W1, I, 442–449].
- [W5] A. Weil, Sur la théorie du corps de classes, *J. Math. Soc. Japan*, 3 (1951), 1–35 [W1, I, 483–517].
- [W6] A. Weil, Jacobi sums as “Größencharaktere,” *Trans. Amer. Math. Soc.*, 73 (1952), 487–495 [W1, II, 63–71].
- [W7] A. Weil, On a certain type of characters of the idele-class group of an algebraic number field, *Proc. Intern. Symposium on Algebraic Number Theory*, Tokyo-Nikko, 1955, 1–7 [W1, II, 255–261].
- [W8] A. Weil, *Basic number theory*, Springer, 1967.
- [W9] A. Weil, *Dirichlet series and automorphic forms*, *Lecture notes in math.* 189, Springer, 1971.
- [Y1] H. Yoshida, On an analogue of the Sato conjecture, *Inventiones Math.*, 19 (1973), 261–277.

Appendix A

Tables of Formulas

1	Algebraic Equations
2	Trigonometry
3	Vector Analysis and Coordinate Systems
4	Differential Geometry
5	Lie Algebras, Symmetric Riemannian Spaces, and Singularities
6	Algebraic Topology
7	Knot Theory
8	Inequalities
9	Differential and Integral Calculus
10	Series
11	Fourier Analysis
12	Laplace Transforms and Operational Calculus
13	Conformal Mappings
14	Ordinary Differential Equations
15	Total and Partial Differential Equations
16	Elliptic Integrals and Elliptic Functions
17	Gamma Functions and Related Functions
18	Hypergeometric Functions and Spherical Functions
19	Functions of Confluent Type and Bessel Functions
20	Systems of Orthogonal Functions
21	Interpolation
22	Distribution of Typical Random Variables
23	Statistical Estimation and Statistical Hypothesis Testing

1. Algebraic Equations (→ 10 Algebraic Equations)

(I) Quadratic Equation $ax^2 + bx + c = 0$ ($a \neq 0$)

The roots are

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-b' \pm \sqrt{b'^2 - ac}}{a} \quad (b \equiv 2b').$$

The discriminant is $b^2 - 4ac$.

(II) Cubic Equation $ax^3 + bx^2 + cx + d = 0$ ($a \neq 0$)

By the translation $\xi = x + b/3a$, the equation is transformed into $\xi^3 + 3p\xi + q = 0$, where

$$p \equiv (3ac - b^2)/9a^2, \quad q \equiv (2b^3 - 9abc + 27a^2d)/27a^3.$$

Its discriminant is $-27(q^2 + 4p^3)$. The roots of the latter equation are

$$\xi = \sqrt[3]{\alpha} + \sqrt[3]{\beta}, \quad \omega \sqrt[3]{\alpha} + \omega^2 \sqrt[3]{\beta}, \quad \omega^2 \sqrt[3]{\alpha} + \omega \sqrt[3]{\beta},$$

where

$$\omega = e^{2\pi i/3} = \frac{-1 + \sqrt{3}i}{2}, \quad \left. \begin{matrix} \alpha \\ \beta \end{matrix} \right\} = \frac{-q \pm \sqrt{q^2 + 4p^3}}{2} \quad (\text{Cardano's formula}).$$

Casus irreducibilis (the case when $q^2 + 4p^3 < 0$). Putting $\alpha \equiv re^{i\theta}$ ($\beta = \bar{\alpha}$), the roots are

$$\xi = 2\sqrt[3]{r} \cos(\theta/3), \quad 2\sqrt[3]{r} \cos[(\theta + 2\pi)/3], \quad 2\sqrt[3]{r} \cos[(\theta + 4\pi)/3].$$

(III) Quartic Equation (Biquadratic Equation) $ax^4 + bx^3 + cx^2 + dx + e = 0$ ($a \neq 0$)

By the translation $\xi = x + b/4a$, the equation is transformed into

$$\xi^4 + p\xi^2 + q\xi + r = 0.$$

The cubic resolvent of the latter is $t^3 - pt^2 - 4rt + (4pr - q^2) = 0$. If t_0 is one of the roots of the cubic resolvent, the roots ξ of the above equation are the solutions of two quadratic equations

$$\xi^2 \pm \sqrt{t_0 - p} [\xi - q/2(t_0 - p)] + t_0/2 = 0 \quad (\text{Ferrari's formula}).$$

2. Trigonometry

(I) Trigonometric Functions (→ 432 Trigonometry)

(1) In Fig. 1, $OA = OB = OP = 1$, and

$$MP = \sin \theta, \quad OM = \cos \theta, \quad AT = \tan \theta,$$

$$BL = \cot \theta, \quad OT = \sec \theta, \quad OL = \operatorname{cosec} \theta.$$

(2) $\sin^2 \theta + \cos^2 \theta = 1$,

$$\tan \theta = \sin \theta / \cos \theta, \quad \cot \theta = 1 / \tan \theta, \quad \sec \theta = 1 / \cos \theta,$$

$$\operatorname{cosec} \theta = 1 / \sin \theta, \quad 1 + \tan^2 \theta = \sec^2 \theta, \quad 1 + \cot^2 \theta = \operatorname{cosec}^2 \theta.$$

(3)

	θ	$-\theta$	$\pi/2 \pm \theta$	$\pi \pm \theta$	$n\pi \pm \theta$
sin	s	$-s$	c	$\mp s$	$\pm (-1)^n s$
cos	c	c	$\mp s$	$-c$	$(-1)^n c$
tan	t	$-t$	$\mp 1/t$	$\pm t$	$\pm t$

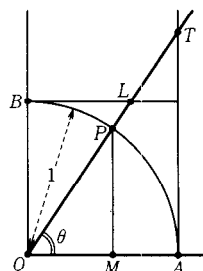


Fig. 1

(4)	α	0°	15°	18°	22.5°	30°	36°	45°	
	\curvearrowright	0	$\pi/12$	$\pi/10$	$\pi/8$	$\pi/6$	$\pi/5$	$\pi/4$	
	$\sin \alpha$	0	$\frac{\sqrt{3}-1}{2\sqrt{2}}$	$\frac{\sqrt{5}-1}{4}$	$\frac{\sqrt{2-\sqrt{2}}}{2}$	$\frac{1}{2}$	$\frac{\sqrt{10-2\sqrt{5}}}{4}$	$\frac{1}{\sqrt{2}}$	$\cos \alpha$
	$\cos \alpha$	1	$\frac{\sqrt{3}+1}{2\sqrt{2}}$	$\frac{\sqrt{10+2\sqrt{5}}}{4}$	$\frac{\sqrt{2+\sqrt{2}}}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{5}+1}{4}$	$\frac{1}{\sqrt{2}}$	$\sin \alpha$
		$\pi/2$	$5\pi/12$	$2\pi/5$	$3\pi/8$	$\pi/3$	$3\pi/10$	$\pi/4$	α
		90°	75°	72°	67.5°	60°	54°	45°	

(5) Addition Formulas

$$\sin(\alpha \pm \beta) = \sin \alpha \cos \beta \pm \cos \alpha \sin \beta, \quad \cos(\alpha \pm \beta) = \cos \alpha \cos \beta \mp \sin \alpha \sin \beta,$$

$$\tan(\alpha \pm \beta) = (\tan \alpha \pm \tan \beta) / (1 \mp \tan \alpha \tan \beta).$$

(6) $\sin 2\alpha = 2 \sin \alpha \cos \alpha, \quad \cos 2\alpha = \cos^2 \alpha - \sin^2 \alpha = 2 \cos^2 \alpha - 1 = 1 - 2 \sin^2 \alpha,$

$$\tan 2\alpha = 2 \tan \alpha / (1 - \tan^2 \alpha).$$

$$\sin 3\alpha = 3 \sin \alpha - 4 \sin^3 \alpha, \quad \cos 3\alpha = 4 \cos^3 \alpha - 3 \cos \alpha,$$

$$\tan 3\alpha = (3 \tan \alpha - \tan^3 \alpha) / (1 - 3 \tan^2 \alpha).$$

$$\sin n\alpha = \sum_{i=0}^{[(n-1)/2]} \binom{n}{2i+1} (-1)^i \sin^{2i+1} \alpha \cos^{n-(2i+1)} \alpha,$$

$$\cos n\alpha = \sum_{i=0}^{[n/2]} \binom{n}{2i} (-1)^i \sin^{2i} \alpha \cos^{n-2i} \alpha.$$

(7) $\sin^2(\alpha/2) = (1 - \cos \alpha)/2, \quad \cos^2(\alpha/2) = (1 + \cos \alpha)/2,$

$$\tan^2(\alpha/2) = (1 - \cos \alpha) / (1 + \cos \alpha).$$

(8) $2 \sin \alpha \cos \beta = \sin(\alpha + \beta) + \sin(\alpha - \beta), \quad 2 \cos \alpha \sin \beta = \sin(\alpha + \beta) - \sin(\alpha - \beta),$

$$2 \cos \alpha \cos \beta = \cos(\alpha + \beta) + \cos(\alpha - \beta), \quad -2 \sin \alpha \sin \beta = \cos(\alpha + \beta) - \cos(\alpha - \beta).$$

$$\sin \alpha + \sin \beta = 2 \sin[(\alpha + \beta)/2] \cos[(\alpha - \beta)/2],$$

$$\sin \alpha - \sin \beta = 2 \cos[(\alpha + \beta)/2] \sin[(\alpha - \beta)/2],$$

$$\cos \alpha + \cos \beta = 2 \cos[(\alpha + \beta)/2] \cos[(\alpha - \beta)/2],$$

$$\cos \alpha - \cos \beta = -2 \sin[(\alpha + \beta)/2] \sin[(\alpha - \beta)/2].$$

(II) Plane Triangles

As shown in Fig. 2, we denote the interior angles of a triangle ABC by α, β, γ ; the corresponding side lengths by a, b, c ; the area by S ; the radii of inscribed, circumscribed, and escribed circles by r, R, r_A , respectively; the perpendicular line from the vertex A to the side BC by AH ; the midpoint of the side BC by M ; bisector of the angle A by AD ; and the lengths of AH, AM, AD by h_A, m_A, f_A , respectively. Similar notations are used for B and C . Put $s \equiv (a + b + c)/2$. The symbol ... means similar formulas by the cyclic permutation of the letters A, B, C , and corresponding quantities.

$$\frac{a}{\sin \alpha} = \frac{b}{\sin \beta} = \frac{c}{\sin \gamma} = 2R \quad (\text{law of sines}).$$

$$a = b \cos \gamma + c \cos \beta, \quad \dots \quad (\text{the first law of cosines}).$$

$$a^2 = b^2 + c^2 - 2bc \cos \alpha, \quad \dots \quad (\text{the second law of cosines}).$$

$$\sin^2(\alpha/2) = (s-b)(s-c)/bc, \quad \dots; \quad \cos^2(\alpha/2) = s(s-a)/bc, \quad \dots.$$

$$(b+c)\sin(\alpha/2) = a \cos[(\beta-\gamma)/2], \quad \dots; \quad (b-c)\cos(\alpha/2) = \alpha \sin[(\beta-\gamma)/2], \quad \dots.$$

$$\frac{a+b}{a-b} = \frac{\tan[(\alpha+\beta)/2]}{\tan[(\alpha-\beta)/2]}, \quad \dots \quad (\text{Napier's rule}).$$

$$\begin{aligned}
 S &= ah_A/2 = (1/2)bc \sin \alpha = (1/2)a^2 \sin \beta \sin \gamma / \sin \alpha = abc/4R = 2R^2 \sin \alpha \sin \beta \sin \gamma \\
 &= rs = r_A(s-a) = \sqrt{rr_A r_B r_C} \\
 &= \sqrt{s(s-a)(s-b)(s-c)} \quad (\text{Heron's formula}). \\
 r &= (s-a) \tan(\alpha/2) = 4R \sin(\alpha/2) \sin(\beta/2) \sin(\gamma/2). \\
 r_A &= s \tan(\alpha/2) = (s-b) \cot(\gamma/2) = 4R \sin(\alpha/2) \cos(\beta/2) \cos(\gamma/2). \\
 1/r &= (1/h_A) + (1/h_B) + (1/h_C). \\
 m_A^2 &= (2b^2 + 2c^2 - a^2)/4 = (b^2 + c^2 + 2bc \cos \alpha)/4. \\
 f_A &= 2bc \cos(\alpha/2)/(b+c) = 2\sqrt{bcs(s-a)}/(b+c). \\
 f_A f_B f_C &= 8abcrs^2/(b+c)(c+a)(a+b).
 \end{aligned}$$

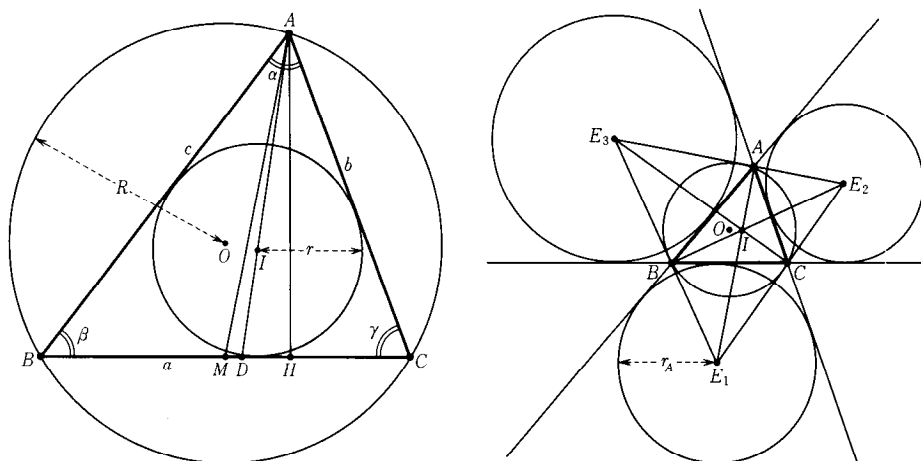


Fig. 2

(III) Spherical Triangles

We denote the interior angles of a spherical triangle by α, β, γ ; the corresponding sides by a, b, c ; the area by S ; and the radius of the supporting sphere by ρ . We have

$$\sin a : \sin b : \sin c = \sin \alpha : \sin \beta : \sin \gamma \quad (\text{law of sines}).$$

$$\cos a = \cos b \cos c + \sin b \sin c \cos \alpha, \quad \dots; \quad \cos \alpha = -\cos \beta \cos \gamma + \sin \beta \sin \gamma \cos a, \quad \dots$$

(law of cosines).

$$\sin a \cos \beta = \cos b \sin c - \sin b \cos c \cos \alpha, \quad \dots \quad (\text{law of sines and cosines}).$$

$$\cot a \sin b = \cos b \cos \gamma + \cot \alpha \sin \gamma, \quad \dots \quad (\text{law of cotangents}).$$

$$\tan[(a+b)/2]/\tan[(a-b)/2] = \tan[(\alpha+\beta)/2]/\tan[(\alpha-\beta)/2], \quad \dots \quad (\text{law of tangents}).$$

$$\tan[(\alpha+\beta)/2] \tan(\gamma/2) = \cos[(a-b)/2]/\cos[(a+b)/2], \quad \dots;$$

$$\tan[(\alpha-\beta)/2] \tan(\gamma/2) = \sin[(a-b)/2]/\sin[(a+b)/2], \quad \dots;$$

$$\tan[(a+b)/2] \cot(c/2) = \cos[(\alpha-\beta)/2]/\cos[(\alpha+\beta)/2], \quad \dots;$$

$$\tan[(a-b)/2] \cot(c/2) = \sin[(\alpha-\beta)/2]/\sin[(\alpha+\beta)/2], \quad \dots \quad (\text{Napier's analogies}).$$

$$S = (\alpha + \beta + \gamma - \pi)\rho^2 = 2\rho^2 \arccos \frac{\cos^2(a/2R) + \cos^2(b/2R) + \cos^2(c/2R)}{2 \cos(a/2R) \cos(b/2R) \cos(c/2R)} \quad (\text{Heron's formula}).$$

For a right triangle ($\gamma = \pi/2$), we have Napier's rule of circular parts: taking the subscripts modulo 5 in Fig. 3,

$$\sin \theta_i = \tan \theta_{i+1} \tan \theta_{i-1} = \cos \theta_{i+2} \cos \theta_{i-2}.$$

For example, we have

$$\cos c = \cos a \cos b = \cot \alpha \cot \beta,$$

$$\cos \beta = \tan a \cot c = \cos b \sin \alpha,$$

$$\sin a = \tan b \cot \beta = \sin c \sin \alpha.$$

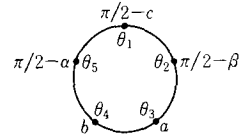


Fig. 3

3. Vector Analysis and Coordinate Systems

We denote a 3-dimensional vector by $\mathbf{A} \equiv (A_x, A_y, A_z) = A_x \mathbf{i} + A_y \mathbf{j} + A_z \mathbf{k}$, $|\mathbf{A}| = \sqrt{A_x^2 + A_y^2 + A_z^2}$.

(I) Vector Algebra (\rightarrow 442 Vectors)

Scalar product $\mathbf{A} \cdot \mathbf{B} \equiv \mathbf{AB} \equiv (\mathbf{A}, \mathbf{B}) = A_x B_x + A_y B_y + A_z B_z = |\mathbf{A}| |\mathbf{B}| \cos \theta$
(where θ is the angle between \mathbf{A} and \mathbf{B}).

Vector product

$$\mathbf{A} \times \mathbf{B} \equiv [\mathbf{A}, \mathbf{B}] = (A_y B_z - A_z B_y) \mathbf{i} + (A_z B_x - A_x B_z) \mathbf{j} + (A_x B_y - A_y B_x) \mathbf{k} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}.$$

$$|\mathbf{A} \times \mathbf{B}| = |\mathbf{A}| |\mathbf{B}| \sin \theta.$$

$$\mathbf{A} \cdot \mathbf{B} = \mathbf{B} \cdot \mathbf{A}. \quad \mathbf{A} \cdot \mathbf{A} \equiv A^2 = |\mathbf{A}|^2. \quad \mathbf{A} \times \mathbf{A} = 0. \quad \mathbf{A} \cdot (\mathbf{A} \times \mathbf{B}) = 0. \quad (\mathbf{A} \times \mathbf{B})^2 = |\mathbf{A}|^2 |\mathbf{B}|^2 - (\mathbf{A} \cdot \mathbf{B})^2.$$

$$\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = (\mathbf{A} \cdot \mathbf{C}) \mathbf{B} - (\mathbf{A} \cdot \mathbf{B}) \mathbf{C}. \quad \mathbf{A} \times (\mathbf{B} \times \mathbf{C}) + \mathbf{B} \times (\mathbf{C} \times \mathbf{A}) + \mathbf{C} \times (\mathbf{A} \times \mathbf{B}) = 0.$$

$$(\mathbf{A} \times \mathbf{B}) \cdot (\mathbf{C} \times \mathbf{D}) = \mathbf{A} \cdot \{\mathbf{B} \times (\mathbf{C} \times \mathbf{D})\} = (\mathbf{A} \cdot \mathbf{C})(\mathbf{B} \cdot \mathbf{D}) - (\mathbf{B} \cdot \mathbf{C})(\mathbf{A} \cdot \mathbf{D}).$$

$$\text{Scalar triple product} \quad [\mathbf{ABC}] \equiv \mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) = \mathbf{B} \cdot (\mathbf{C} \times \mathbf{A}) = \mathbf{C} \cdot (\mathbf{A} \times \mathbf{B}) = \begin{vmatrix} A_x & A_y & A_z \\ B_x & B_y & B_z \\ C_x & C_y & C_z \end{vmatrix}.$$

$$[\mathbf{BCD}]\mathbf{A} + [\mathbf{ACD}]\mathbf{B} + [\mathbf{ABD}]\mathbf{C} = [\mathbf{ABC}]\mathbf{D}. \quad [\mathbf{ABC}][\mathbf{EFG}] = \begin{vmatrix} \mathbf{A} \cdot \mathbf{E} & \mathbf{A} \cdot \mathbf{F} & \mathbf{A} \cdot \mathbf{G} \\ \mathbf{B} \cdot \mathbf{E} & \mathbf{B} \cdot \mathbf{F} & \mathbf{B} \cdot \mathbf{G} \\ \mathbf{C} \cdot \mathbf{E} & \mathbf{C} \cdot \mathbf{F} & \mathbf{C} \cdot \mathbf{G} \end{vmatrix}.$$

(II) Differentiation of a Vector Field (\rightarrow 442 Vectors)

$$\nabla \equiv \mathbf{i} \frac{\partial}{\partial x} + \mathbf{j} \frac{\partial}{\partial y} + \mathbf{k} \frac{\partial}{\partial z} \quad (\text{Nabla}),$$

$$\text{grad } \varphi \equiv \nabla \varphi = \frac{\partial \varphi}{\partial x} \mathbf{i} + \frac{\partial \varphi}{\partial y} \mathbf{j} + \frac{\partial \varphi}{\partial z} \mathbf{k} \quad (\text{gradient of } \varphi),$$

$$\text{rot } \mathbf{A} \equiv \nabla \times \mathbf{A} = \left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z} \right) \mathbf{i} + \left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x} \right) \mathbf{j} + \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} \right) \mathbf{k} \quad (\text{rotation of } \mathbf{A}),$$

$$\text{div } \mathbf{A} \equiv \nabla \cdot \mathbf{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z} \quad (\text{divergence of } \mathbf{A}),$$

$$\Delta \varphi \equiv \nabla^2 \varphi \equiv \text{div grad } \varphi = \frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial y^2} + \frac{\partial^2 \varphi}{\partial z^2} \quad (\text{Laplacian of } \varphi).$$

$$\text{grad}(\varphi \psi) = \varphi \text{ grad } \psi + \psi \text{ grad } \varphi,$$

$$\text{grad}(\mathbf{A} \cdot \mathbf{B}) = (\mathbf{B} \cdot \text{grad}) \mathbf{A} + (\mathbf{A} \cdot \text{grad}) \mathbf{B} + \mathbf{A} \times \text{rot } \mathbf{B} + \mathbf{B} \times \text{rot } \mathbf{A},$$

$$\text{rot}(\varphi \mathbf{A}) = \varphi \text{ rot } \mathbf{A} - \mathbf{A} \times \text{grad } \varphi, \quad \text{rot}(\mathbf{A} \times \mathbf{B}) = (\mathbf{B} \cdot \text{grad}) \mathbf{A} - (\mathbf{A} \cdot \text{grad}) \mathbf{B} + \mathbf{A} \text{ div } \mathbf{B} - \mathbf{B} \text{ div } \mathbf{A},$$

$$\text{div}(\varphi \mathbf{A}) = \varphi \text{ div } \mathbf{A} + \mathbf{A} \cdot \text{grad } \varphi, \quad \text{div}(\mathbf{A} \times \mathbf{B}) = \mathbf{B} \cdot \text{rot } \mathbf{A} - \mathbf{A} \cdot \text{rot } \mathbf{B}.$$

$$\text{rot grad } \varphi = 0, \quad \text{div rot } \mathbf{A} = 0. \quad \Delta \mathbf{A} = \text{grad div } \mathbf{A} - \text{rot rot } \mathbf{A}.$$

$$\Delta(f \circ \varphi) = (df/d\varphi) \Delta \varphi + (d^2 f/d\varphi^2) (\text{grad } \varphi)^2, \quad \Delta(\varphi \psi) = \varphi \Delta \psi + \psi \Delta \varphi + 2(\text{grad } \varphi \cdot \text{grad } \psi).$$

(III) Integration of a Vector Field (→ 94 Curvilinear Integrals and Surface Integrals, 442 Vectors)

Let D be a 3-dimensional domain, B its boundary, dV the volume element of D , dS the surface element of B , and $d\mathbf{S} = \mathbf{n} dS$, where \mathbf{n} is the outer normal vector of the surface B . We have

$$\begin{aligned} \text{Gauss's formula} \quad & \iiint_D \operatorname{div} \mathbf{A} dV = \iint_B d\mathbf{S} \cdot \mathbf{A} = \iint_B (\mathbf{n} \cdot \mathbf{A}) dS, \\ & \iiint_D \operatorname{rot} \mathbf{A} dV = \iint_B d\mathbf{S} \times \mathbf{A} = \iint_B (\mathbf{n} \times \mathbf{A}) dS, \\ & \iiint_D \operatorname{grad} \varphi dV = \iint_B \varphi d\mathbf{S}; \\ \text{Green's formula} \quad & \iint_B \varphi \frac{\partial \psi}{\partial n} dS = \iiint_D (\varphi \Delta \psi + \operatorname{grad} \varphi \cdot \operatorname{grad} \psi) dV, \\ & \iint_B \left(\varphi \frac{\partial \psi}{\partial n} - \psi \frac{\partial \varphi}{\partial n} \right) dS = \iiint_D (\varphi \Delta \psi - \psi \Delta \varphi) dV, \\ & 4\pi \varphi(x_0) = - \iiint_D \frac{\Delta \varphi}{r} dV + \iint_B \left\{ \frac{1}{r} \frac{\partial \varphi}{\partial n} - \varphi \frac{\partial}{\partial n} \left(\frac{1}{r} \right) \right\} dS, \end{aligned}$$

where r is the distance from the point x_0 .

Let B be a bordered surface with a boundary curve Γ , ds the line element of Γ , dS the surface element of B , and $ds = \mathbf{t} ds$, $d\mathbf{S} = \mathbf{n} dS$, for \mathbf{t} the unit tangent vector of Γ and under the proper choice of the positive direction for the surface normal \mathbf{n} . We have

$$\text{Stokes's formula} \quad \iint_B d\mathbf{S} \cdot \operatorname{rot} \mathbf{A} = \oint_{\Gamma} \mathbf{A} \cdot d\mathbf{s} = \oint_{\Gamma} (\mathbf{t} \cdot \mathbf{A}) ds, \quad \iint_B d\mathbf{S} \times \operatorname{grad} \varphi = \oint_{\Gamma} \varphi d\mathbf{s}.$$

If the domain D is simply connected, and the vector field \mathbf{V} tends sufficiently rapidly to 0 near the boundary of D and at infinity, we have

$$\text{Helmholtz's theorem} \quad \mathbf{V} = \operatorname{grad} \varphi + \operatorname{rot} \mathbf{A}, \quad \varphi = - \iiint_D \frac{\operatorname{div} \mathbf{V}}{4\pi r} dV, \quad \mathbf{A} = \iiint_D \frac{\operatorname{rot} \mathbf{V}}{4\pi r} dV.$$

(IV) Moving Coordinate System

Denote differentiation with respect to the rest and the moving systems by d/dt , d^*/dt , respectively. Let the relative velocity of the systems be \mathbf{v} . Then we have

$$\frac{d\varphi}{dt} = \frac{d^*\varphi}{dt} - \mathbf{v} \cdot \operatorname{grad} \varphi, \quad \frac{d\mathbf{A}}{dt} = \frac{d^*\mathbf{A}}{dt} - [\mathbf{v} \cdot \operatorname{grad} \mathbf{A} - (\mathbf{A} \cdot \operatorname{grad}) \mathbf{v}].$$

With respect to rotating coordinates we have

$$\begin{aligned} \mathbf{v} &= \mathbf{w} \times \mathbf{r}, \\ \frac{d\mathbf{A}}{dt} &= \frac{d^*\mathbf{A}}{dt} + [\mathbf{w} + \mathbf{A} - ((\mathbf{w} \times \mathbf{r}) \cdot \operatorname{grad}) \mathbf{A}] \end{aligned}$$

When the domain of integration is also a function of t ,

$$\begin{aligned} \frac{d}{dt} \int \mathbf{A} \cdot d\mathbf{s} &= \int \left\{ \frac{\partial \mathbf{A}}{\partial t} + \operatorname{grad} (\mathbf{v} \cdot \mathbf{A}) - \mathbf{v} \times \operatorname{rot} \mathbf{A} \right\} \cdot d\mathbf{s}, \\ \frac{d}{dt} \iint \mathbf{A} \cdot d\mathbf{S} &= \iint \left\{ \frac{\partial \mathbf{A}}{\partial t} + \operatorname{rot} (\mathbf{A} \times \mathbf{v}) + \mathbf{v} \operatorname{div} \mathbf{A} \right\} \cdot d\mathbf{S}, \\ \frac{d}{dt} \iiint \varphi dV &= \iiint \left\{ \frac{\partial \varphi}{\partial t} + (\mathbf{v} \cdot \operatorname{grad} \varphi) + \varphi \operatorname{div} \mathbf{v} \right\} dV = \iiint \frac{\partial \varphi}{\partial t} dV + \iint \varphi \mathbf{v} \cdot d\mathbf{S}. \end{aligned}$$

(V) Curvilinear Coordinates (→ 90 Coordinates)

Let (x_1, \dots, x_n) be rectangular coordinates in an n -dimensional Euclidean space. If

$$x_j = \varphi_j(u_1, \dots, u_n) \quad (j = 1, \dots, n), \quad J \equiv \det(\partial \varphi_j / \partial u_k) \neq 0,$$

the system (u_1, \dots, u_n) may be taken as a coordinate system of an n -dimensional space, and the

original space is a Riemannian manifold with the first fundamental form

$$g_{jk} = \sum_{i=1}^n \frac{\partial \varphi_i}{\partial u_j} \frac{\partial \varphi_i}{\partial u_k} \quad (j, k = 1, \dots, n),$$

$$g \equiv \det(g_{jk}) = J^2.$$

When the metric is of the diagonal form $g_{jk} = g_j^2 \delta_{jk}$, the coordinate system (u_1, \dots, u_n) is called an orthogonal curvilinear coordinate system or an isothermal curvilinear coordinate system. In such a case we have $J = g_1 \dots g_n$, and the line element is given by $ds^2 = \sum_{j=1}^n g_j^2 du_j^2$.

For a scalar f and a vector $\xi = (\xi_1, \dots, \xi_n)$, we have

$$(\text{grad} f)_j = \frac{1}{g_j} \frac{\partial f}{\partial u_j} \quad (j = 1, \dots, n), \quad \Delta f = \frac{1}{J} \sum_{j=1}^n \frac{\partial}{\partial u_j} \left(\frac{J}{g_j^2} \frac{\partial f}{\partial u_j} \right),$$

$$\text{div} \xi = \frac{1}{J} \sum_{j=1}^n \frac{\partial}{\partial u_j} \left(\frac{J}{g_j} \xi_j \right), \quad (\text{rot} \xi)_{jk} = \frac{1}{g_j g_k} \left[\frac{\partial (g_k \xi_k)}{\partial u_j} - \frac{\partial (g_j \xi_j)}{\partial u_k} \right] \quad (j, k = 1, \dots, n).$$

When $n=2$, the rot may be considered a scalar, $\text{rot} \xi = (\text{rot} \xi)_{12}$, and when $n=3$, the rot may be considered a vector, with components

$$\text{rot} \xi = ((\text{rot} \xi)_{23}, (\text{rot} \xi)_{31}, (\text{rot} \xi)_{12}).$$

The following are examples of orthogonal coordinates.

(1) Planar Curvilinear Coordinates. In the present Section (1), we put

$$x_1 = x, \quad x_2 = y, \quad u_1 = u, \quad u_2 = v, \quad g_1 = p, \quad g_2 = q.$$

$$ds^2 = p^2 dx^2 + q^2 dy^2, \quad J \equiv \partial(x, y) / \partial(u, v) = \sqrt{pq}.$$

Planar orthogonal curvilinear coordinates may be represented in the form $x + iy = F(U + iV)$, F being a complex analytic function, by suitable choice of the functions $U = U(u)$, $V = V(v)$.

(i) Polar Coordinates (r, θ) (Fig. 4).

$$x = r \cos \theta, \quad y = r \sin \theta; \quad x + iy = \exp(\log r + i\theta).$$

$$r = \sqrt{x^2 + y^2}, \quad \theta = \arctan(y/x).$$

$$p = 1, \quad q = r, \quad J = r, \quad ds^2 = dr^2 + r^2 d\theta^2.$$

$$\Delta f = \frac{\partial^2 f}{\partial r^2} + \frac{1}{r} \frac{\partial f}{\partial r} + \frac{1}{r^2} \frac{\partial^2 f}{\partial \theta^2}.$$

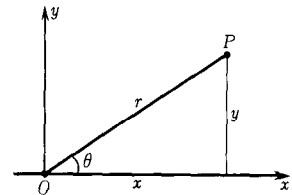


Fig. 4

(ii) Elliptic Coordinates (μ, ν) (Fig. 5). Among the family of confocal conics

$$\frac{x^2}{a^2 + \rho} + \frac{y^2}{b^2 + \rho} = 1 \quad (a > b),$$

there are two values of ρ for which the curve passes through a given point $P(x, y)$. Denote the two values of ρ by μ and ν , where $\mu > -b^2 > \nu > -a^2$. The curve corresponding to $\rho = \mu$ or $\rho = \nu$ is an ellipse or a hyperbola, respectively. Then we have the relations

$$x^2 = (\mu + a^2)(\nu + a^2)/(a^2 - b^2), \quad y^2 = (\mu + b^2)(\nu + b^2)/(b^2 - a^2).$$

Let the common foci be $(\pm c, 0)$ ($c^2 = a^2 - b^2$). Then we have

$$r_1 = \sqrt{(x - c)^2 + y^2}, \quad r_2 = \sqrt{(x + c)^2 + y^2}$$

where r_1, r_2 are the distances from the two foci as in Fig. 5, and

$$4(a^2 + \mu) = (r_1 + r_2)^2, \quad 4(a^2 + \nu) = (r_1 - r_2)^2.$$

$$p = \frac{1}{2} \sqrt{\frac{\mu - \nu}{(\mu + a^2)(\mu + b^2)}}, \quad q = \frac{1}{2} \sqrt{\frac{\nu - \mu}{(\nu + a^2)(\nu + b^2)}}$$

(iii) Parabolic Coordinates (α, β) (Fig. 6). Among the family of parabolas $y^2 = 4\rho(x + \rho)$ with the focus at the origin and having the x -axis as the principal axis, there are two values of ρ for which the curve passes through a given point $P(x, y)$. Denote the two values of ρ by α, β ($\alpha > 0 > \beta$). We have $x = -(\alpha + \beta)$, $y = \sqrt{-4\alpha\beta}$.

(iv) Equilateral (or Rectangular) Hyperbolic Coordinates (u, v) (Fig. 7). This is a system that

replaces $x/2, y/2$ in (iii) by $-y$ and x , respectively, with $\sqrt{\alpha}=u, \sqrt{-\beta}=v$. The relations are

$$x=uv, \quad y=(u^2-v^2)/2; \quad x+iy=i(u-iv)^2/2, \quad u^2, v^2=\sqrt{x^2+y^2} \pm y, \quad p=q=\sqrt{u^2+v^2}.$$

The curves $x=\text{constant}$ or $y=\text{constant}$ are equilateral hyperbolas.

(v) Bipolar Coordinates (ξ, η) (Fig. 8). These coordinates represent a point $P(x, y)$ on a plane as the intersection of the family of circles passing through two fixed points $(\pm a, 0)$ and the family of loci on which the ratio of distances from the same two fixed points $(\pm a, 0)$ is constant. The latter is the set of Apollonius' circles. The relations are

$$x = \frac{a \sinh \xi}{\cosh \xi + \cos \eta}, \quad y = \frac{a \sin \eta}{\cosh \xi + \cos \eta} \quad (-\infty < \xi < \infty, 0 \leq \eta \leq 2\pi).$$

$$p = q = \frac{a}{\cosh \xi + \cos \eta}.$$

(2) Curvilinear Coordinates in 3-Dimensional Space. In the present Section (2), we put $x_1 = x$, $x_2 = y$, $x_3 = z$.

(i) Circular Cylindrical Coordinates (Cylindrical Coordinates) (ρ, φ, z) (Fig. 9).

$$x = \rho \cos \varphi, \quad y = \rho \sin \varphi, \quad z = z.$$

$$ds^2 = d\rho^2 + \rho^2 d\varphi^2 + dz^2, \quad J = \rho. \quad \Delta f = \frac{1}{\rho} \frac{\partial}{\partial \rho} \left(\rho \frac{\partial f}{\partial \rho} \right) + \frac{1}{\rho^2} \frac{\partial^2 f}{\partial \varphi^2} + \frac{\partial^2 f}{\partial z^2}.$$

(ii) Polar Coordinates (Spherical Coordinates) (Fig. 9).

$$x = r \sin \theta \cos \varphi, \quad y = r \sin \theta \sin \varphi, \quad z = r \cos \theta.$$

$$r = \sqrt{x^2 + y^2 + z^2}, \quad \varphi = \arctan(y/x), \quad \theta = \arctan(\sqrt{x^2 + y^2}/z).$$

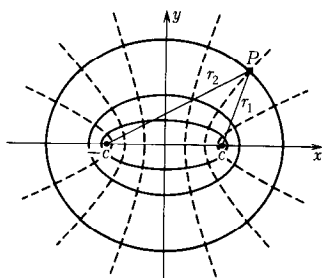


Fig. 5

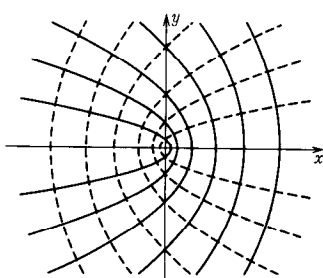


Fig. 6

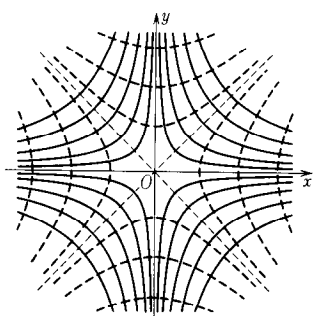


Fig. 7

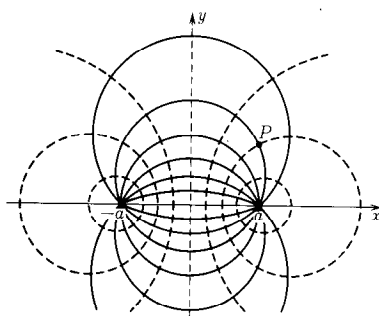


Fig. 8

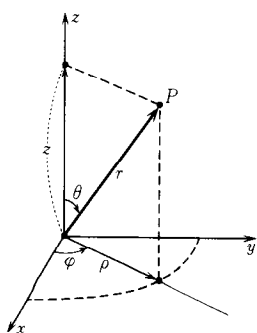


Fig. 9

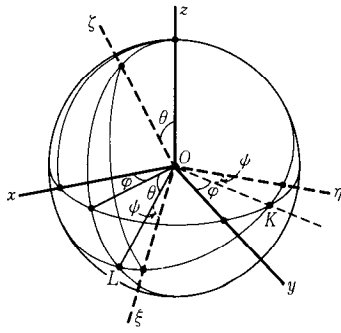


Fig. 10

The angles φ and θ are called azimuth and zenith angle, respectively. We further have

$$ds^2 = dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\varphi^2, \quad J = r^2 \sin \theta.$$
$$\Delta f = \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial f}{\partial r} \right) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial f}{\partial \theta} \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 f}{\partial \varphi^2}.$$

(iii) Euler's Angles (Fig. 10). Let (x, y, z) and (ξ, η, ζ) be two linear orthogonal coordinate systems with common origin O . Denote the angle between the z -axis and the ζ -axis by θ ; the angle between the zx -plane and the $z\zeta$ -plane by φ ; and the angle between the η -axis and the intersection OK of the xy -plane and the $\xi\eta$ -plane (or the angle between the ξ -axis and the intersection OL of the $z\zeta$ -plane and the $\xi\eta$ -plane) by ψ . The angles θ , φ , and ψ are called Euler's angles. The direction cosines of one coordinate axis with respect to the other coordinate system are as follows:

	x	y	z
ξ	$\cos \varphi \cos \theta \cos \psi - \sin \varphi \sin \psi$	$\sin \varphi \cos \theta \cos \psi + \cos \varphi \sin \psi$	$-\sin \theta \cos \psi$
η	$-\cos \varphi \cos \theta \sin \psi - \sin \varphi \cos \psi$	$-\sin \varphi \cos \theta \sin \psi + \cos \varphi \cos \psi$	$\sin \theta \sin \psi$
ζ	$\cos \varphi \sin \theta$	$\sin \varphi \sin \theta$	$\cos \theta$

(iv) Rotational (or Revolutinal) Coordinates (u, v, ρ) . Let (u, v) be curvilinear coordinates (Section (1)) on the $z\rho$ -plane. The rotational coordinates (u, v, ρ) are given by the combination of $x = \rho \cos \varphi$, $y = \rho \sin \varphi$ with the coordinates on the $z\rho$ -plane. We have

$$ds^2 = \rho^2 du^2 + q^2 dv^2 + \rho^2 d\varphi^2,$$

where p, q are the corresponding values for the coordinates (u, v) .

(v) Generalized Cylindrical Coordinates (u, v, z) . These are a combination of curvilinear coordinates (u, v) on the xy -plane with z . We have

$$ds^2 = p^2 du^2 + q^2 dv^2 + dz^2.$$

For various selections of (u, v) we have coordinates as follows:

(u, v)	Rotational Coordinate System	Generalized Cylindrical Coordinate System
Linear rectangular coordinates	Circular cylindrical coordinates	Linear rectangular coordinates
Polar coordinates ((1)(i))	Spherical coordinates	Circular cylindrical coordinates
Elliptic coordinates ((1)(ii))	Spheroidal coordinates ⁽¹⁾	Elliptic cylindrical coordinates
Parabolic coordinates ((1)(iii))	Rotational parabolic coordinates ⁽²⁾	Parabolic cylindrical coordinates
Equilateral hyperbolic coordinates ((1)(iv))	Rotational hyperbolic coordinates	Hyperbolic cylindrical coordinates
Bipolar coordinates ((1)(v))	Toroidal coordinates ⁽³⁾ Bipolar coordinates ⁽⁴⁾	Bipolar cylindrical coordinates

Notes

- (1) When the ρ -axis is a minor or major axis, we have prolate or oblate spheroidal coordinates, respectively.
- (2) We take the z -axis as the common principal axis of the parabolas.
- (3) Where the line passing through two fixed points is the ρ -axis.
- (4) Where the line passing through two fixed points is the z -axis.

(vi) Ellipsoidal Coordinates (λ, μ, ν) (Fig. 11). Among the family of confocal quadrics

$$\frac{x^2}{a^2 + \rho} + \frac{y^2}{b^2 + \rho} + \frac{z^2}{c^2 + \rho} = 1 \quad (a > b > c > 0),$$

there are three values of ρ for which the surface passes through a given point $P(x,y,z)$. Denote the three values of ρ by λ, μ, ν , where $\lambda > -c^2 > \mu > -b^2 > \nu > -a^2$. The surfaces corresponding to $\rho=\lambda, \rho=\mu$, and $\rho=\nu$ are an ellipsoid, a hyperboloid of one sheet, and a hyperboloid of two sheets, respectively. We have

$$x^2 = \frac{h(a)}{(a^2 - b^2)(a^2 - c^2)}, \quad y^2 = \frac{h(b)}{(b^2 - c^2)(b^2 - a^2)},$$
$$z^2 = \frac{h(c)}{(c^2 - a^2)(c^2 - b^2)}; \quad h(\alpha) \equiv (\lambda + \alpha^2)(\mu + \alpha^2)(\nu + \alpha^2).$$
$$g_1 = \frac{\sqrt{(\lambda - \mu)(\lambda - \nu)}}{2\rho(\lambda)}, \quad g_2 = \frac{\sqrt{(\mu - \nu)(\mu - \lambda)}}{2\rho(\mu)},$$
$$g_3 = \frac{\sqrt{(\nu - \lambda)(\nu - \mu)}}{2\rho(\nu)}; \quad \rho(t) \equiv \sqrt{(t + a^2)(t + b^2)(t + c^2)}.$$

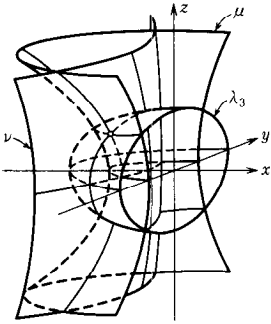


Fig. 11

4. Differential Geometry

(I) Classical Differential Geometry (→ 111 Differential Geometry of Curves and Surfaces)

(1) Plane Curves (Fig. 12). At a point $P(x_0,y_0)$ on a curve $y=f(x)$, the equation of the tangent line is $y-y_0=f'(x_0)(x-x_0)$,

$PT = |y_0\sqrt{1+y_0'^2} / y_0'|,$

and the tangential shadow $TM=y_0/y_0'$. The equation of the normal line is $f'(x_0)(y-y_0)+(x-x_0)=0$,

$PN = |y_0\sqrt{1+y_0'^2} |,$

and the normal shadow $MN=y_0y_0'$. The slope of the tangent is $\tan \alpha=f'(x_0)=y_0'$. The curvature at P is

$\kappa = 1/PQ = f''(x_0)/[1+f'(x_0)^2]^{3/2}$

The coordinates of the center of curvature Q are

$(x_0-f'(x_0)[1+f'(x_0)^2]/f''(x_0), \quad f(x_0)+[1+f'(x_0)^2]/f''(x_0)).$

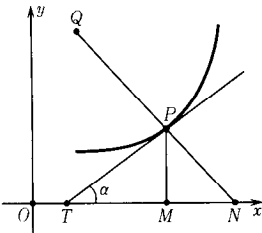


Fig. 12

(2) Space Curves $x_i = x_i(t)$ ($i = 1, 2, 3$), or $\mathbf{x} = \mathbf{x}(t)$. The line element of a curve $\mathbf{x} = \mathbf{x}(t)$ is

$ds = \sqrt{(dx_1)^2 + (dx_2)^2 + (dx_3)^2} = \sqrt{\sum_{\alpha=1}^3 \dot{x}_\alpha^2} \, dt \quad \left(\cdot = \frac{d}{dt} \right).$

The curvature is

$\kappa = \sqrt{\sum \ddot{x}_\alpha^2 - \dot{s}^2 / s^2}$

For $t=s$ (arc length), the curvature is $\kappa = \sqrt{(\sum x_\alpha''^2)}$, and the torsion is $\tau = [\det(x'_\alpha, x''_\alpha, x'''_\alpha)]_{\alpha=1,2,3} / \kappa^2$, where $' = d/ds$. When we denote Frenet's frame by (ξ, η, ζ) , we have $\xi = \mathbf{x}'$, $\eta = \xi' / \kappa$, $\zeta = \xi \times \eta$ (vector product).

The Frenet-Serret formulas are

$$\xi' = \kappa\eta, \quad \eta' = -\kappa\xi + \tau\xi, \quad \xi' = -\tau\eta.$$

(3) Surface in 3-Dimensional Space $x_\alpha = x_\alpha(u_1, u_2)$ ($\alpha = 1, 2, 3$). The first fundamental form of the surface is

$$g_{jk} = \sum_{\alpha=1}^3 \frac{\partial x_\alpha}{\partial u_j} \frac{\partial x_\alpha}{\partial u_k} \quad (j, k = 1, 2). \quad g = \det(g_{jk}) > 0.$$

Let (g^{jk}) be the inverse matrix of (g_{jk}) . The tangent plane at the point $x_\alpha^{(0)}$ is given by

$$\det(x_\alpha - x_\alpha^{(0)}, (\partial x_\alpha / \partial u_1)^{(0)}, (\partial x_\alpha / \partial u_2)^{(0)}) = 0.$$

The normal line at the point $x_\alpha^{(0)}$ is given by $x_\alpha - x_\alpha^{(0)} = t\nu_\alpha^{(0)}$, where t is a parameter, and ν_α is the unit normal vector, given by

$$\nu_\alpha = \frac{1}{\sqrt{g}} \frac{\partial(x_\beta, x_\gamma)}{\partial(u_1, u_2)} \quad (\delta_{\alpha\beta\gamma}^{123} = +1).$$

The second fundamental form is

$$h_{jk} \equiv \sum_{\alpha=1}^3 \nu_\alpha \frac{\partial^2 x_\alpha}{\partial u_j \partial u_k} = - \sum_{\alpha=1}^3 \frac{\partial \nu_\alpha}{\partial u_j} \frac{\partial x_\alpha}{\partial u_k}. \quad h \equiv \det(h_{jk}).$$

The principal radii of curvature R_1, R_2 are the roots of the quadratic equation

$$\frac{1}{R^2} - \sum_{j,k} g^{jk} h_{jk} \frac{1}{R} + \frac{h}{g} = 0.$$

The mean curvature (or Germain's curvature) is

$$H \equiv \frac{1}{2} \left(\frac{1}{R_1} + \frac{1}{R_2} \right) = \frac{1}{2} \sum_{j,k} g^{jk} h_{jk},$$

and $H=0$ is the condition for the given surface to be a minimal surface. The Gaussian curvature (or total curvature) is

$$K \equiv \frac{1}{R_1 R_2} = \frac{h}{g},$$

and $K=0$ is the condition for the surface to be developable.

We use the notations of Riemannian geometry, with g_{jk} the fundamental tensor:

$$\frac{\partial^2 x_\alpha}{\partial u_j \partial u_k} = \sum_{a=1}^3 \left\{ \begin{matrix} a \\ jk \end{matrix} \right\} \frac{\partial x_\alpha}{\partial u_a} + h_{jk} \nu_\alpha \quad (\text{Gauss's formula}).$$

$$R_{ijkl} = h_{jl} h_{ik} - h_{jk} h_{il} \quad (\text{Gauss's equation}). \quad h_{jk;l} = h_{jl;k} \quad (\text{Codazzi-Mainardi equation}).$$

$$K = \frac{R_{1212}}{g} = \frac{R}{2} = \frac{1}{\sqrt{g}} \left[\frac{\partial}{\partial u_2} \left(\frac{\sqrt{g}}{g_{11}} \left\{ \begin{matrix} 2 \\ 11 \end{matrix} \right\} \right) - \frac{\partial}{\partial u_1} \left(\frac{\sqrt{g}}{g_{11}} \left\{ \begin{matrix} 2 \\ 12 \end{matrix} \right\} \right) \right]$$

$$= \frac{1}{\sqrt{g}} \left[\frac{\partial}{\partial u_1} \left(\frac{\sqrt{g}}{g_{22}} \left\{ \begin{matrix} 1 \\ 22 \end{matrix} \right\} \right) - \frac{\partial}{\partial u_2} \left(\frac{\sqrt{g}}{g_{22}} \left\{ \begin{matrix} 1 \\ 12 \end{matrix} \right\} \right) \right].$$

$$\frac{\partial \nu_\alpha}{\partial u_j} = - \sum_{k,l} h_{jk} g^{kl} \frac{\partial x_\alpha}{\partial u_l} \quad (\text{Weingarten's formula}).$$

The third fundamental form is given by

$$l_{jk} \equiv \sum_{\alpha=1}^3 \frac{\partial \nu_\alpha}{\partial u_j} \frac{\partial \nu_\alpha}{\partial u_k} = \sum_{s,t} g^{st} h_{js} h_{kt} = 2Hh_{jk} - Kg_{jk}. \quad \det(l_{jk}) = K^2 g = Kh.$$

(4) Geodesic Curvature. Let $C: u_i = u_i(s)$ be a curve on a surface S and ρ be the curvature of C at a point P . Let θ be the angle between the osculating plane of C and the plane tangent to S . The geodesic curvature ρ_g of C at P is given by

$$\rho_g = \rho \cos \theta = \sqrt{g} \det \left(\frac{du_i}{ds}, \frac{d^2 u_i}{ds^2} + \sum_{j,k} \left\{ \begin{matrix} i \\ jk \end{matrix} \right\} \frac{du_j}{ds} \frac{du_k}{ds} \right)_{i=1,2}$$

$\rho_g = 0$ is the condition for C to be a geodesic. Let D be a simply connected domain on the surface S , whose boundary Γ consists of n smooth curves. Let θ_α be the outer angle at the intersection of two consecutive curves ($\alpha = 1, \dots, n$). Then we have the Gauss-Bonnet formula:

$$\int_{\Gamma} \rho_g ds + \iint_D K dS = 2\pi - \sum_{\alpha=1}^n \theta_\alpha.$$

(II) Riemannian Geometry, Tensor Calculus (→ 417 Tensor Calculus)

In the present section, we use Einstein's convention (omission of the summation symbol).

(1) Numerical Tensor.

$$\text{Kronecker's } \delta \quad \delta_{jk}, \quad \delta^{jk}, \quad \delta_k^j = \begin{cases} 1 & (j=k) \\ 0 & (j \neq k). \end{cases}$$

$$\delta_{k_1 \dots k_p}^{j_1 \dots j_p} = \det(\delta_{k_\nu}^{j_\mu})_{\mu, \nu=1, \dots, p} = \begin{cases} 0 & (\{j_\mu\} \neq \{k_\nu\}), \\ +1 & (\{j_\mu\} = \{k_\nu\} \text{ and } (j_\mu) \text{ is an even permutation of } (k_\nu)), \\ -1 & (\{j_\mu\} = \{k_\nu\} \text{ and } (j_\mu) \text{ is an odd permutation of } (k_\nu)). \end{cases}$$

$$\text{Eddington's } \varepsilon \quad \varepsilon_{j_1 \dots j_n} = \delta_{j_1 \dots j_n}^{1 \dots n}, \quad \varepsilon^{j_1 \dots j_n} = \delta_{1 \dots n}^{j_1 \dots j_n}.$$

$$\delta_{k_1 \dots k_p j_{p+1} \dots j_n}^{j_1 \dots j_p j_{p+1} \dots j_n} = (n-p)! \delta_{k_1 \dots k_p}^{j_1 \dots j_p}, \quad \det(a_\nu^\mu)_{\mu, \nu=1, \dots, n} = \varepsilon^{j_1 \dots j_n} a_{j_1}^1 a_{j_2}^2 \dots a_{j_n}^n = \varepsilon_{j_1 \dots j_n} a_1^j a_2^j \dots a_n^j.$$

(2) Fundamental Objects in Riemannian Geometry. Let g_{jk} be the fundamental tensor, and (g^{jk}) be the inverse matrix of (g_{jk}) . We put $g \equiv \det(g_{jk})$.

The Christoffel symbol is

$$\left\{ \begin{matrix} i \\ jk \end{matrix} \right\} = \frac{1}{2} g^{ia} \left[\frac{\partial g_{aj}}{\partial x^k} + \frac{\partial g_{ak}}{\partial x^j} - \frac{\partial g_{jk}}{\partial x^a} \right] = \left\{ \begin{matrix} i \\ kj \end{matrix} \right\}, \quad \left\{ \begin{matrix} a \\ ak \end{matrix} \right\} = \frac{\partial \log \sqrt{g}}{\partial x^k},$$

which has the transformation rule

$$\left\{ \begin{matrix} \bar{i} \\ \bar{j} \bar{k} \end{matrix} \right\} = \frac{\partial \bar{x}^i}{\partial x^j} \left(\frac{\partial x^j}{\partial \bar{x}^{\bar{j}}} \frac{\partial x^k}{\partial \bar{x}^{\bar{k}}} \left\{ \begin{matrix} i \\ jk \end{matrix} \right\} + \frac{\partial^2 x^i}{\partial \bar{x}^{\bar{j}} \partial \bar{x}^{\bar{k}}} \right)$$

under a coordinate transformation.

A geometrical object Γ_{jk}^i with a similar transformation rule is called the coefficient of the affine connection. The torsion tensor is

$$S_{jk}^i \equiv \Gamma_{jk}^i - \Gamma_{kj}^i.$$

The equation of a geodesic is

$$\frac{d^2 x^i}{ds^2} + \left\{ \begin{matrix} i \\ jk \end{matrix} \right\} \frac{dx^j}{ds} \frac{dx^k}{ds} = 0.$$

The covariant derivative of a tensor of weight W with respect to a coefficient of affine connection Γ_{jk}^i is given by

$$T_{k_1 \dots k_q l}^{j_1 \dots j_p} \equiv \partial T_{k_1 \dots k_q}^{j_1 \dots j_p} / \partial x^l + \sum_{\nu=1}^p T_{k_1 \dots k_q}^{j_1 \dots j_{\nu-1} a j_{\nu+1} \dots j_p} \Gamma_{al}^{j_\nu} - \sum_{\mu=1}^q T_{k_1 \dots k_{\mu-1} a k_{\mu+1} \dots k_q}^{j_1 \dots j_p} \Gamma_{lk_\mu}^a - W T_{k_1 \dots k_q}^{j_1 \dots j_p} \Gamma_{al}^a.$$

For the Christoffel symbol, we denote the covariant derivative by $;/l$. Then we have the following formulas:

$$g_{jk;l} = 0, \quad g^{jk};_l = 0, \quad \delta_{k;l}^j = 0, \quad \sqrt{g} \varepsilon_{j_1 \dots j_n l} = 0, \quad (1/\sqrt{g}) \varepsilon^{j_1 \dots j_n}_l = 0.$$

For a scalar f $\text{grad } f = (f_{;j})$,

for a covariant vector v_j $\text{rot } v = (v_{j;k} - v_{k;j}) = (\partial v_j / \partial x^k - \partial v_k / \partial x^j)$,

and for a contravariant vector v^j $\text{div } v = v^j_{;j} = \frac{1}{\sqrt{g}} \frac{\partial (\sqrt{g} v^j)}{\partial x^j}$.

Beltrami's differential operator of the first kind is

$$\Delta_1 f \equiv g^{jk} f_{;j} f_{;k}.$$

Beltrami's differential operator of the second kind is

$$\Delta_2 f = \operatorname{div} \operatorname{grad} f = \frac{1}{\sqrt{g}} \frac{\partial (\sqrt{g} g^{jk} (\partial f / \partial x^k))}{\partial x^j}.$$

For a domain D with sufficiently smooth boundary Γ , we denote the directional derivative along the inner normal by $\partial / \partial n$, the volume element by dV , and the surface element on Γ by dS . Then we have Green's formulas,

$$\int_D (\Delta_1(\varphi\psi) + \psi\Delta_2\varphi) dV = - \int_{\Gamma} \psi \frac{\partial \varphi}{\partial n} dS, \quad \int_D (\varphi\Delta_2\psi - \psi\Delta_2\varphi) dV = \int_{\Gamma} \left(\psi \frac{\partial \varphi}{\partial n} - \varphi \frac{\partial \psi}{\partial n} \right) dS.$$

We denote the curvature tensor with respect to the coefficients of a general affine connection Γ_{jk}^i by B_{jkl}^i , and by R_{jkl}^i when $\Gamma_{jk}^i = \left\{ \begin{smallmatrix} i \\ jk \end{smallmatrix} \right\}$. We have the following formulas:

$$B_{jkl}^i \equiv \frac{\partial \Gamma_{jl}^i}{\partial x^k} - \frac{\partial \Gamma_{jk}^i}{\partial x^l} + \Gamma_{jl}^a \Gamma_{ak}^i - \Gamma_{jk}^a \Gamma_{al}^i;$$

$$R_{ijkl} \equiv g_{ai} R_{jkl}^a = -R_{ijlk} = R_{klij} = \frac{1}{2} \left[\frac{\partial^2 g_{ik}}{\partial x^j \partial x^l} + \frac{\partial^2 g_{jl}}{\partial x^i \partial x^k} - \frac{\partial^2 g_{jk}}{\partial x^i \partial x^l} - \frac{\partial^2 g_{il}}{\partial x^j \partial x^k} \right]$$

$$+ g_{ab} \left(\left\{ \begin{smallmatrix} b \\ ik \end{smallmatrix} \right\} \left\{ \begin{smallmatrix} a \\ jl \end{smallmatrix} \right\} - \left\{ \begin{smallmatrix} b \\ il \end{smallmatrix} \right\} \left\{ \begin{smallmatrix} a \\ jk \end{smallmatrix} \right\} \right);$$

$$\text{Bianchi's first identity} \quad R_{jkl}^i + R_{klj}^i + R_{ljk}^i = 0,$$

$$- (B_{jkl}^i + B_{klj}^i + B_{ljk}^i) = 2(S_{jk|l}^i + S_{kl|j}^i + S_{lj|k}^i) + 4(S_{ja}^i S_{kl}^a + S_{ka}^i S_{lj}^a + S_{la}^i S_{jk}^a);$$

$$\text{Bianchi's second identity} \quad R_{jkl;m}^i + R_{jlm;k}^i + R_{jmk;l}^i = 0,$$

$$B_{jkl|m}^i + B_{jlm|k}^i + B_{jmk|l}^i = -2(B_{jma}^i S_{kl}^a + B_{jka}^i S_{lm}^a + B_{jla}^i S_{mk}^a);$$

$$\text{Ricci's tensor} \quad R_{jk} \equiv -R_{jki}^i = R_{kj};$$

$$\text{scalar curvature} \quad R \equiv g^{jk} R_{jk};$$

$$\text{Ricci's formula} \quad T_{k_1 \dots k_q | s | t}^{j_1 \dots j_p} - T_{k_1 \dots k_q | t | s}^{j_1 \dots j_p}$$

$$= - \sum_{\nu=1}^p T_{k_1 \dots k_{\nu-1} a_{\nu}^{j_{\nu}} \dots j_p}^{j_1 \dots j_{\nu-1} a_{\nu}^{j_{\nu}} \dots j_p} B_{ast}^{j_{\nu}} + \sum_{\mu=1}^q T_{k_1 \dots k_{\mu-1} a_{\mu}^{j_{\mu}} \dots j_p}^{j_1 \dots j_{\mu-1} a_{\mu}^{j_{\mu}} \dots j_p} B_{k_{\mu} st}^{j_{\mu}} + 2 T_{k_1 \dots k_q | l}^{j_1 \dots j_p} S_{st}^l + W T_{k_1 \dots k_q}^{j_1 \dots j_p} B_{ast}^a,$$

where S and B are the torsion and curvature tensors given above, respectively, and W is the weight of the tensor T .

(3) Special Riemannian Spaces (\rightarrow 364 Riemannian Manifolds). In the present Section (3), n means the dimension of the space.

(i) Space of Constant Curvature $R_{jkl}^i = \rho(g_{jl}\delta_k^i - g_{jk}\delta_l^i)$; $\rho = R/n(n-1)$,

(ii) Einstein Space $R_{jk} = \rho g_{jk}$, $\rho = R/n$, for $n \geq 3$, where R is a constant.

(iii) Locally Symmetric Riemannian Space $R_{jkl;m}^i = 0$.

(iv) Projectively Flat Space. Weyl's projective curvature tensor is defined by

$$W_{jkl}^i \equiv R_{jkl}^i + \frac{1}{n-1} (R_{jk}\delta_l^i - R_{jl}\delta_k^i).$$

The condition for the space to be projectively flat is given by $W_{jkl}^i = 0$, $R_{jk;l} = R_{jl;k}$.

If $n \geq 3$, the latter condition follows from the former condition, and the space reduces to a space of constant curvature. If $n = 2$, the former condition $W = 0$ always holds.

(v) Conircularly Flat Space $Z_{jkl}^i \equiv R_{jkl}^i + \frac{R}{n(n-1)} (g_{jk}\delta_l^i - g_{jl}\delta_k^i) = 0$. This space reduces to a space of constant curvature.

(vi) Conformally Flat Space. Weyl's conformal curvature tensor is defined by

$$C_{jkl}^i \equiv R_{jkl}^i + \frac{1}{n-2} (R_{jk}\delta_l^i - R_{jl}\delta_k^i + g_{jk}R_l^i - g_{jl}R_k^i) - \frac{R(g_{jk}\delta_l^i - g_{jl}\delta_k^i)}{(n-1)(n-2)},$$

$$\Pi_{jk} \equiv -\frac{R_{jk}}{(n-2)} + \frac{Rg_{jk}}{2(n-1)(n-2)}.$$

The condition for the space to be conformally flat is given by $C_{jkl}^i = 0$, $\Pi_{jk;l} = \Pi_{jl;k}$.

If $n \geq 4$, the latter condition follows from the former condition, and if $n = 3$, the former condition $C = 0$ always holds.

5. Lie Algebras, Symmetric Riemannian Spaces, and Singularities

(I) The Classification of Complex Simple Lie Algebras and Compact Real Simple Lie Algebras
(→ 248 Lie Algebras)

(1) Lie Algebra. The unitary restriction of a noncommutative finite-dimensional complex simple Lie algebra \mathfrak{g} is a compact real simple Lie algebra \mathfrak{g}_u , and \mathfrak{g} is given by the complexification $\mathfrak{g}_u^{\mathbb{C}}$ of \mathfrak{g}_u . There exists a bijective correspondence between the classifications of these two kinds of Lie algebras. Using Dynkin diagrams, the classification is done as in Fig. 14 (→ 248 Lie Algebras). The system of fundamental roots $\{\alpha_1, \dots, \alpha_l\}$ of a simple Lie algebra \mathfrak{g} is in one-to-one correspondence with the vertices of a Dynkin diagram shown by simple circles in Fig. 14. The number of simple circles coincides with the rank l of \mathfrak{g} . The double circle in Fig. 14 means -1 times the highest root θ . Sometimes we mean by the term “Dynkin diagram” the diagram without the double circle and the lines issuing from it. Here we call the diagram with double circle representing $-\theta$ the extended Dynkin diagram. Corresponding to the value of the inner product with respect to the Killing form $-2(\alpha_i, \alpha_j)/(\alpha_j, \alpha_j)$ ($i \neq j$) (which must be 0, 1, 2, or 3), we connect two vertices representing α_i and α_j as in Fig. 13. When the value is 0, we do not connect α_i and α_j . In Fig. 13, the left circle corresponds to α_i and the right circle to α_j .

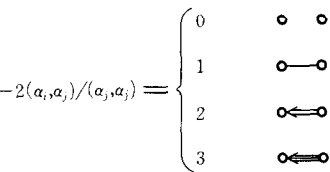


Fig. 13

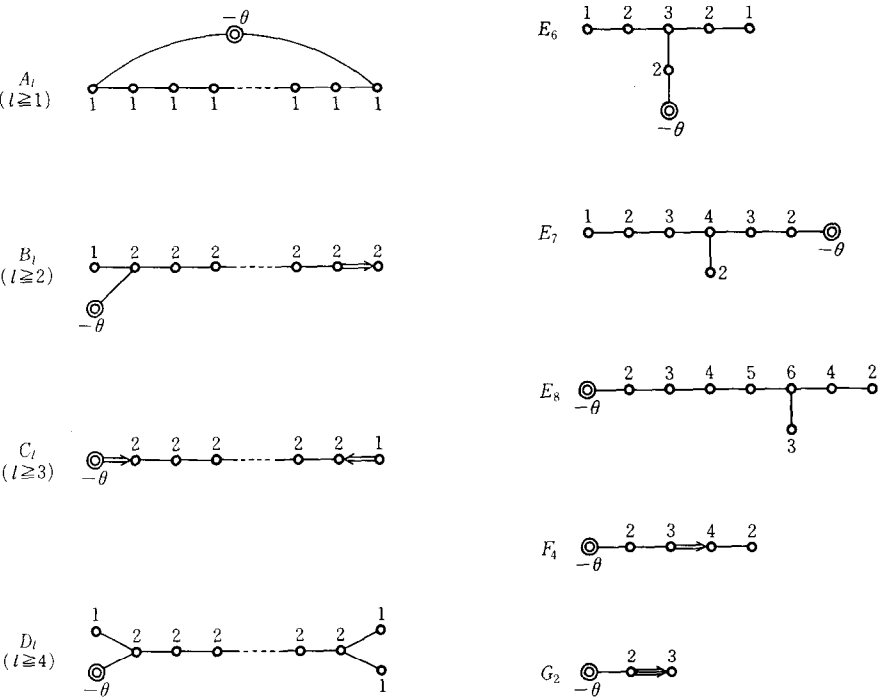


Fig. 14 We have relations $B_1 = C_1 = A_1$, $C_2 = B_2$, and $D_3 = A_3$. ($D_2 = A_1 + A_1$, which is not simple.) In this figure, the number at each vertex means the coefficient m_i in $\theta = \sum m_i \alpha_i$.

From Fig. 14, we have the following information.
(i) The quotient group of the automorphism group $A(\mathfrak{g})$ of \mathfrak{g} with respect to the inner automorphism group $I(\mathfrak{g})$ is isomorphic to the automorphism group of the corresponding Dynkin diagram. The order of the latter group is 2 for A_l ($l \geq 2$) since the diagram is symmetric. It is also 2 for D_l ($l \geq 5$) and for E_6 , and it is 6 ($=3!$) for D_4 . For all other cases, the order is 1.

- (ii) The order of the center of the simply connected Lie group associated with \mathfrak{g} is equal to the index of the subgroup consisting of elements stabilizing $-\theta$ in the group of automorphisms of the extended Dynkin diagram of \mathfrak{g} (S. Murakami). This index is equal to the order of the fundamental group of the adjoint group of \mathfrak{g} and the number of connected Lie groups, whose Lie algebra is \mathfrak{g} .
- (iii) Any parabolic Lie subalgebra of \mathfrak{g} is isomorphic to a subalgebra generated by the root vector X_α (and elements of the Cartan subalgebra) such that $\alpha = \sum n_i \alpha_i$, where $\{\alpha_1, \dots, \alpha_l\}$ is a system of fundamental roots, $n_i \geq 0$ ($i = 1, \dots, l$) or $n_i \leq 0$ ($i = 1, \dots, l$), and $n_j = 0$ for α_j belonging to a fixed subset S of $\{\alpha_1, \dots, \alpha_l\}$.

Hence, isomorphism classes of parabolic Lie subalgebras are in one-to-one correspondence with the set of subsets S of $\{\alpha_1, \dots, \alpha_l\}$.

- (iv) Maximal Lie subalgebra \mathfrak{f} of \mathfrak{g} with the same rank l as \mathfrak{g} . The Lie subalgebra \mathfrak{f} is classified by the following rule. First we remove a vertex α_i from the Dynkin diagram. If the number m_i attached to the vertex is 1, \mathfrak{f} is given by the product of the simple Lie algebra corresponding to the Dynkin diagram after removing the vertex α_i and a one-dimensional Lie subalgebra. If $m_i > 1$, \mathfrak{f} is given by the diagram after removing α_i from the extended Dynkin diagram.

(2) Lie Groups. The classical complex simple Lie groups of rank n represented by A, B, C, D (in Cartan's symbolism) are the complex special linear group $SL(n+1, \mathbb{C})$, the complex special orthogonal group $SO(2n+1, \mathbb{C})$, the complex symplectic group $Sp(n, \mathbb{C})$, and the complex special orthogonal group $SO(2n, \mathbb{C})$, respectively. The classical compact simple Lie groups of rank n represented by A, B, C, D are the special unitary group $SU(n+1)$, the special orthogonal group $SO(2n+1)$, the unitary-symplectic group $Sp(n)$, and the special orthogonal group $SO(2n)$, respectively (\rightarrow 60 Classical Groups).

Cartan's Symbol	Complex Form	Compact Form	Dimension	Rank
A_n	$SL(n+1, \mathbb{C})$	$SU(n+1)$	$(n+1)^2 - 1$	n
B_n	$SO(2n+1, \mathbb{C})$	$SO(2n+1)$	$2n^2 + n$	n
C_n	$Sp(n, \mathbb{C})$	$Sp(n)$	$2n^2 + n$	n
D_n	$SO(2n, \mathbb{C})$	$SO(2n)$	$2n^2 - n$	n
G_2	$\text{Aut } \mathbb{C}^c$	$\text{Aut } \mathbb{C}$	14	2
F_4	$\text{Aut } \mathfrak{H}^c$	$\text{Aut } \mathfrak{H}$	52	4
E_6			78	6
E_7			133	7
E_8			248	8

Here \mathbb{C} is the Cayley algebra over \mathbb{R} , \mathbb{C}^c is the complexification of \mathbb{C} , \mathfrak{H} is the Jordan algebra of Hermitian matrices of order 3 over \mathbb{C} , \mathfrak{H}^c is the complexification of \mathfrak{H} , and $\text{Aut } A$ is the automorphism group of A .

(II) Classification of Noncompact Real Simple Lie Algebras

Classical Cases

Cartan's Symbol	Noncompact Real Simple Lie Algebra \mathfrak{g}	Maximal Compact Lie Algebra of \mathfrak{g}
AI	$\mathfrak{sl}(p+1; \mathbb{R})$	$\mathfrak{su}(p+1)$
AII	$\mathfrak{sl}(n; \mathbb{H})$	$\mathfrak{sp}(n)$
AIII	$\mathfrak{su}(p, q; \mathbb{C})$	$\mathfrak{su}(p) + \mathfrak{su}(q)$
BI	$\mathfrak{so}(p, q; \mathbb{R})$	$\mathfrak{so}(p) + \mathfrak{so}(q) \quad (p+q=2m+1)$
BII	$\mathfrak{so}(1, n-1; \mathbb{R})$	$\mathfrak{so}(n-1) \quad (n=2m+1)$
CI	$\mathfrak{sp}(p; \mathbb{R})$	$\mathfrak{u}(p)$
CII	$\mathfrak{u}(p, q; \mathbb{H})$	$\mathfrak{sp}(p) + \mathfrak{sp}(q)$
DI	$\mathfrak{so}(p, q; \mathbb{R})$	$\mathfrak{so}(p) + \mathfrak{so}(q) \quad (p+q=2m)$
DII	$\mathfrak{so}(1, n-1; \mathbb{R})$	$\mathfrak{so}(n-1) \quad (n=2m)$
DIII	$\mathfrak{so}(p; \mathbb{H})$	$\mathfrak{u}(2p)$

Here the field F is the real field \mathbf{R} , the complex field \mathbf{C} , or the quaternion field \mathbf{H} ($\mathbf{R} \subset \mathbf{C} \subset \mathbf{H}$). \mathbf{H} is an algebra over \mathbf{R} . For a quaternion $x = x_0 + x_1i + x_2j + x_3k$ ($x_0, x_1, x_2, x_3 \in \mathbf{R}$), we put

$$\bar{x} = x_0 - x_1i - x_2j - x_3k,$$
$$x^* = x_0 + x_1i - x_2j + x_3k.$$

Then $\mathfrak{gl}(n; F) = \{\text{set of all square matrices over } F \text{ of order } n\}$,

$$\mathfrak{sl}(n; F) = \{A \in \mathfrak{gl}(n; F) \mid \text{tr } A = 0\},$$
$$\mathfrak{so}(p, q; F) = \{A \in \mathfrak{gl}(p + q; F) \mid A^* I_{p, q} + I_{p, q} A = 0\},$$

where $I_{p, q}$ is the symmetric transformation of the Euclidean space \mathbf{R}^{p+q} with respect to \mathbf{R}^p , i.e.,

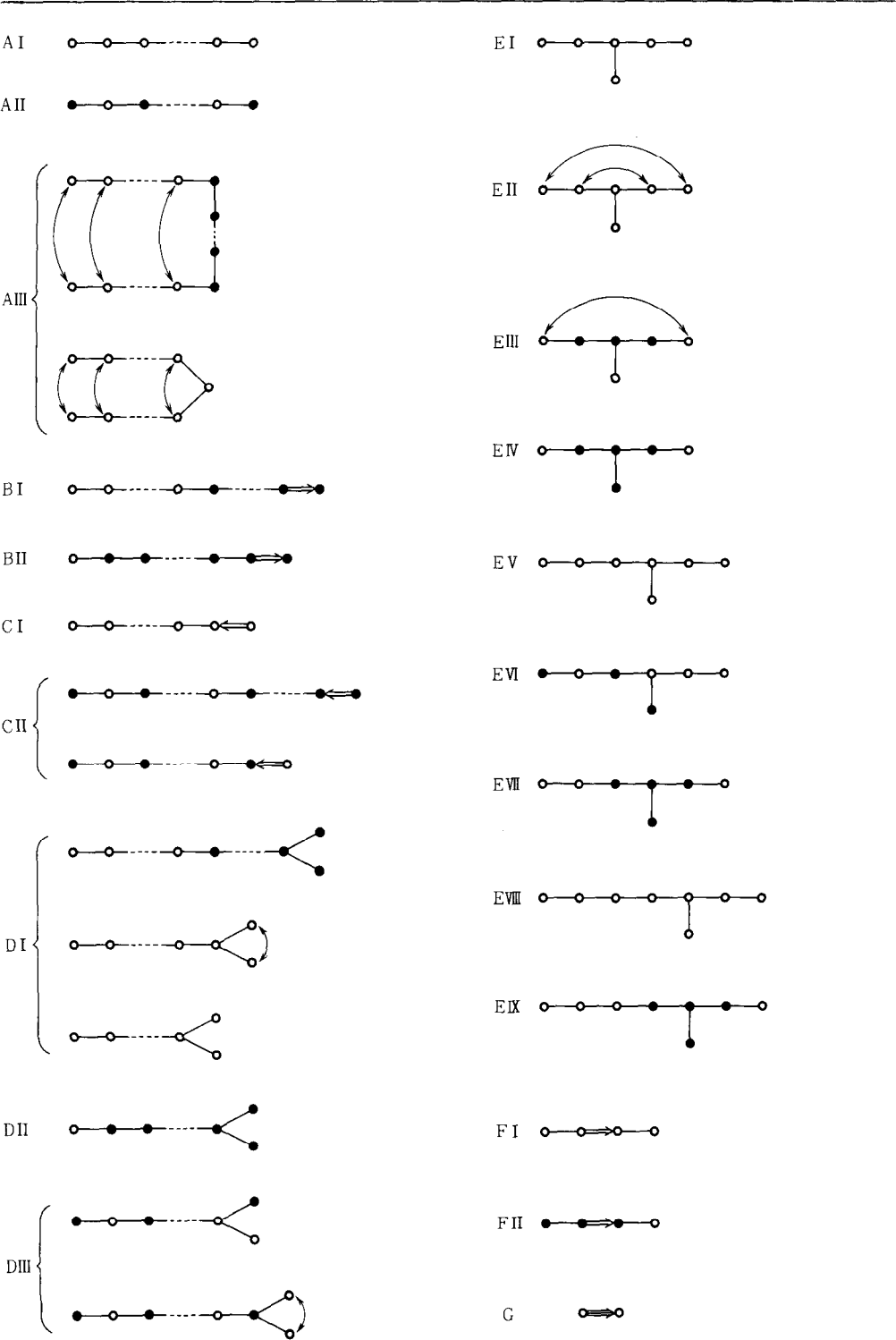


Fig. 15

$I_{p,q}$ is the diagonal sum of the unit matrix I_p of order p and $-I_q$. We have

$$\begin{aligned}\mathfrak{so}(n; F) &= \mathfrak{so}(n, 0; F), \\ \mathfrak{u}(p, q; F) &= \{A \in \mathfrak{gl}(p+q; F) \mid {}^t \bar{A} I_{p,q} + I_{p,q} A = 0\}, \\ \mathfrak{u}(n; F) &= \mathfrak{u}(n, 0; F), \\ \mathfrak{sp}(n; F) &= \{A \in \mathfrak{gl}(2n; F) \mid {}^t \bar{A} J + J A = 0\},\end{aligned}$$

where J is the matrix of an alternating form $\sum_{i=1}^n (x_i y_{i+n} - x_{i+n} y_i)$ of order $2n$.

A noncompact real simple Lie algebra \mathfrak{g} is classified by the relation of the complex conjugation operator σ with respect to the complexification \mathfrak{g}^c of \mathfrak{g} . The results are given by Satake's diagram (Fig. 15).

In the diagram, the fundamental root corresponding to a black circle is multiplied by -1 under σ for a suitable choice of Cartan subalgebra, and the arc with an arrow means that two elements corresponding to both ends of the arc are mutually transformed by a special transformation p such that $\sigma = pw$ ($w \in W$).

(III) Classification of Irreducible Symmetric Riemannian Spaces (\rightarrow 412 Symmetric Riemannian Spaces and Real Forms)

A simply connected irreducible symmetric Riemannian space $M = G/K$ is either a space in the following table or a simply connected compact simple Lie group mentioned in (I). The noncompact forms uniquely corresponding to the compact symmetric Riemannian space are in one-to-one correspondence with the noncompact real simple Lie algebras mentioned in (II).

Cartan's Symbol	$G/K = M$	Dimension	Rank
AI	$SU(n)/SO(n) \quad (n \geq 2)$	$(n-1)(n+2)/2$	$n-1$
AII	$SU(2n)/Sp(n) \quad (n \geq 1)$	$(n-1)(2n+1)$	$n-1$
AIII	$U(p+q)/U(p) \times U(q) \quad (p \geq q \geq 1)$	$2pq$	q
BDI	$SO(p+q)/SO(p) \times SO(q) \quad (p \geq q \geq 2, p+q \neq 4)$	pq	q
BDII	$SO(n+1)/SO(n) \quad (n \geq 2)$	n	1
DIII	$SO(2l)/U(l) \quad (l \geq 4)$	$l(l-1)$	$[l/2]$
CI	$Sp(n)/U(n) \quad (n \geq 3)$	$n(n+1)$	n
CII	$Sp(p+q)/Sp(p) \times Sp(q) \quad (p \geq q \geq 1)$	$4pq$	q
EI	$E_6/Sp(4)$	42	6
EII	$E_6/SU(2) \cdot SU(6)$	40	4
EIII	$E_6/Spin(10) \cdot SO(2)$	32	2
EIV	E_6/F_4	26	2
EV	$E_7/SU(8)$	70	7
EVI	$E_7/Spin(12) \cdot SU(2)$	64	4
EVII	$E_7/E_6 \cdot SO(2)$	54	3
EVIII	$E_8/Spin(16)$	128	8
EIX	$E_8/E_7 \cdot SU(2)$	112	4
FI	$F_4/Sp(3) \cdot SU(2)$	28	4
FII	$F_4/Spin(9)$	16	1
G	$G_2/SO(4)$	8	2

Notes

The group $G = U(p+q)$ in AIII is not effective, unless it is replaced by $SU(p+q)$. To be precise, $K = Sp(4)$ in EI should be replaced by its quotient group factored by a subgroup of order 2 of its center. K in EII is not a direct product of simple groups; the order of its fundamental group $\pi_1(K)$ is 2. To be precise, K in EV or EVIII should be replaced by its quotient group factored by a subgroup of order 2 of its center. The K 's in EII, EIII, EVI, EVII, EIX, and FI are not direct products. The fundamental group $\pi_1(K)$ of K is the infinite cyclic group \mathbf{Z} for EIII, EVII; for all other cases, the order of $\pi_1(K)$ is 2.

In EIII, EVII, the groups E_6, E_7 are adjoint groups of compact simple Lie algebras. In other cases, E_6 and E_7 (E_8, F_4 and G_2 also) are simply connected Lie groups.

The compact symmetric Riemannian space M is a complex Grassmann manifold for AIII, a real Grassmann manifold for BDI, a sphere for BDII, a quaternion Grassmann manifold for CII, and a Cayley projective plane for FII.

(IV) Isomorphic Relations among Classical Lie Algebras

The isomorphic relations among the classical Lie algebras over \mathbf{R} or \mathbf{C} are all given in the following table. In the table, we denote, for example, the real form of type AI of the complex Lie algebra with rank 3 by $A_3\mathbf{I}$ in Cartan's symbolism. When there are nonisomorphic real forms of the same type and same rank (e.g., in the case of $D_3\mathbf{I}$) we distinguish them by the rank of the corresponding symmetric Riemannian space and denote them by, e.g., $D_3\mathbf{I}_p$, where p is the index of total isotropy of the sesquilinear form which is invariant under the corresponding Lie algebra.

Cartan's Symbol	Isomorphisms among Classical Lie Algebras
$A_1 = B_1 = C_1$ $B_2 = C_2$ $A_3 = D_3$ $A_1\mathbf{I} = A_1\mathbf{III} = B_1\mathbf{I} = C_1\mathbf{I}$ $B_2\mathbf{I}_2 = C_2\mathbf{I}$ $B_2\mathbf{I}_1 = C_2\mathbf{II}$ $A_3\mathbf{I} = D_3\mathbf{I}_3$ $A_3\mathbf{II} = D_3\mathbf{I}_1$ $A_3\mathbf{III}_2 = D_3\mathbf{I}_2$ $A_3\mathbf{III}_1 = D_3\mathbf{III}$ $D_4\mathbf{I}_2 = D_4\mathbf{III}$ $D_2 = A_1 \times A_1^*$ $D_2\mathbf{I}_2 = A_1\mathbf{I} \times A_1\mathbf{I}$ $D_2\mathbf{III} = A_1 \times A_1\mathbf{I}^*$ $D_2\mathbf{I}_1 = A_1^*$	$\mathfrak{sl}(2, \mathbf{C}) \cong \mathfrak{so}(3, \mathbf{C}) \cong \mathfrak{sp}(1, \mathbf{C}); \mathfrak{su}(2) \cong \mathfrak{so}(3) \cong \mathfrak{sp}(1)$ $\mathfrak{so}(5, \mathbf{C}) \cong \mathfrak{sp}(2, \mathbf{C}); \mathfrak{so}(5) \cong \mathfrak{sp}(2)$ $\mathfrak{sl}(4, \mathbf{C}) \cong \mathfrak{so}(6, \mathbf{C}); \mathfrak{su}(4) \cong \mathfrak{so}(6)$ $\mathfrak{sl}(2, \mathbf{R}) \cong \mathfrak{su}(1, 1; \mathbf{C}) \cong \mathfrak{so}(2, 1; \mathbf{R}) \cong \mathfrak{sp}(1; \mathbf{R})$ $\mathfrak{so}(3, 2; \mathbf{R}) \cong \mathfrak{sp}(2, \mathbf{R})$ $\mathfrak{so}(4, 1; \mathbf{R}) \cong \mathfrak{u}(1, 1; \mathbf{H})$ $\mathfrak{sl}(4, \mathbf{R}) \cong \mathfrak{so}(3, 3; \mathbf{R})$ $\mathfrak{sl}(2, \mathbf{H}) \cong \mathfrak{so}(5, 1; \mathbf{R})$ $\mathfrak{su}(2, 2; \mathbf{C}) \cong \mathfrak{so}(4, 2; \mathbf{R})$ $\mathfrak{su}(3, 1; \mathbf{C}) \cong \mathfrak{so}(3; \mathbf{H})$ $\mathfrak{so}(6, 2; \mathbf{R}) \cong \mathfrak{so}(4, \mathbf{H})$ $\mathfrak{so}(4, \mathbf{C}) \cong \mathfrak{sl}(2, \mathbf{C}) \times \mathfrak{sl}(2, \mathbf{C}); \mathfrak{so}(4) \cong \mathfrak{su}(2) \times \mathfrak{su}(2)$ $\mathfrak{so}(2, 2; \mathbf{R}) \cong \mathfrak{sl}(2, \mathbf{R}) \times \mathfrak{sl}(2; \mathbf{R})$ $\mathfrak{so}(2; \mathbf{H}) \cong \mathfrak{su}(2) \times \mathfrak{sl}(2; \mathbf{R})$ $\mathfrak{so}(3, 1; \mathbf{R}) \cong \mathfrak{sl}(2, \mathbf{C})$

Note

(*) In these 3 cases, there are isomorphisms given by the replacement of $\mathfrak{sl}(2, \mathbf{C})$ or $\mathfrak{su}(2)$ by isomorphic Lie algebras of type B_1 or type C_1 due to the isomorphism $A_1 \cong B_1 \cong C_1$.

(V) Lists of Normal Forms of Singularities with Modulus Number $m=0, 1$, and 2 (→ 418 Theory of Singularities)

Letters A, \dots, Z stand here for stable equivalence classes of function germs (or families of function germs).

(1) Simple Singularities ($m=0$). There are 2 infinite series A, D , and 3 “exceptional” singularities E_6, E_7, E_8 :

Notation	Normal form	Restrictions
A_n	$x^{n+1} + y^2 + z^2$	$n \geq 1$
D_n	$x^{n-1} + xy^2 + z^2$	$n \geq 4$
E_6	$x^4 + y^3 + z^2$	
E_7	$x^3y + y^3 + z^2$	
E_8	$x^5 + y^3 + z^2$	

(2) Unimodular Singularities ($m=1$). There are 3 families of parabolic singularities, one series of hyperbolic singularities (with 3 subscripts), and 14 families of exceptional singularities.

The parabolic singularities

Notation	Normal form	Restrictions
$P_8 = \tilde{E}_6$	$x^3 + y^3 + z^3 + axyz$	$a^3 + 27 \neq 0$
$X_9 = \tilde{E}_7$	$x^4 + y^4 + z^2 + axyz$	$a^4 - 64 \neq 0$
$J_{10} = \tilde{E}_8$	$x^6 + y^3 + z^2 + axyz$	$a^6 - 432 \neq 0$

The hyperbolic singularities

Notation	Normal form	Restrictions
T_{pqr}	$x^p + y^q + z^r + axyz$	$a \neq 0, \frac{1}{p} + \frac{1}{q} + \frac{1}{r} < 1$

The 14 exceptional families

Notation	Normal form	Gabrielov numbers	Dolgachëv numbers	Notation	Normal form	Gabrielov numbers	Dolgachëv numbers
K_{12}	$x^3 + y^7 + z^2 + axy^5$	2 3 7	2 3 7	W_{13}	$x^4 + xy^4 + z^2 + ay^6$	2 5 6	3 4 4
K_{13}	$x^3 + xy^5 + z^2 + ay^8$	2 3 8	2 4 5	Q_{10}	$x^3 + y^4 + yz^2 + axy^3$	3 3 4	2 3 9
K_{14}	$x^3 + y^8 + z^2 + axy^6$	2 3 9	3 3 4	Q_{11}	$x^3 + y^2z + xz^3 + az^5$	3 3 5	2 4 7
Z_{11}	$x^3y + y^5 + z^2 + axy^4$	2 4 5	2 3 8	Q_{12}	$x^3 + y^5 + yz^2 + axy^4$	3 3 6	3 3 6
Z_{12}	$x^3y + xy^4 + z^2 + ax^2y^3$	2 4 6	2 4 6	S_{11}	$x^4 + y^2z + xz^2 + ax^3z$	3 4 4	2 5 6
Z_{13}	$x^3y + y^6 + z^2 + axy^5$	2 4 7	3 3 5	S_{12}	$x^2y + y^2z + xz^3 + az^5$	3 4 5	3 4 5
W_{12}	$x^4 + y^5 + z^2 + ax^2y^3$	2 5 5	2 5 5	U_{12}	$x^3 + y^3 + z^4 + axyz^2$	4 4 4	4 4 4

(3) Bimodular Singularities ($m = 2$). There are 8 infinite series and 14 exceptional families. In all the formulas, $\mathfrak{a} = a_0 + a_1y$.

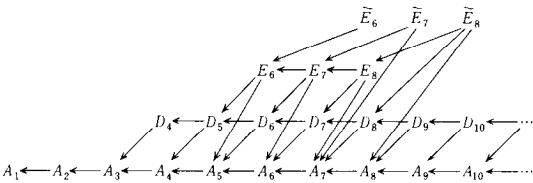
The 8 infinite series of bimodular singularities

Notation	Normal form	Restriction	Milnor number
$J_{3,0}$	$x^3 + bx^2y^3 + y^9 + z^2 + cxy^7$	$4b^3 + 27 \neq 0$	16
$J_{3,p}$	$x^3 + x^2y^3 + z^2 + ay^{9+p}$	$p > 0, a_0 \neq 0$	$16 + p$
$Z_{1,0}$	$y(x^3 + dx^2y^2 + cxy^5 + y^6) + z^2$	$4d^3 + 27 \neq 0$	15
$Z_{1,p}$	$y(x^3 + x^2y^2 + ay^{6+p}) + z^2$	$p > 0, a_0 \neq 0$	$15 + p$
$W_{1,0}$	$x^4 + ax^2y^3 + y^6 + z^2$	$a_0^2 \neq 4$	15
$W_{1,p}$	$x^4 + x^2y^3 + ay^{6+p} + z^2$	$p > 0, a_0 \neq 0$	$15 + p$
$W_{1,2q-1}^\#$	$(x^2 + y^3)^2 + axy^{4+q} + z^2$	$q > 0, a_0 \neq 0$	$15 + 2q - 1$
$W_{1,2q}^\#$	$(x^2 + y^3)^2 + ax^2y^{3+q} + z^2$	$q > 0, a_0 \neq 0$	$15 + 2q$
$Q_{2,0}$	$x^3 + yz^2 + ax^2y^2 + xy^4$	$a_0^2 \neq 4$	14
$Q_{2,p}$	$x^3 + yz^2 + x^2y^2 + az^{6+p}$	$p > 0, a_0 \neq 0$	$14 + p$
$S_{1,0}$	$x^2z + yz^2 + y^5 + azy^3$	$a_0^2 \neq 4$	14
$S_{1,p}$	$x^2z + yz^2 + x^2y^2 + ay^{5+p}$	$p > 0, a_0 \neq 0$	$14 + p$
$S_{1,2q-1}^\#$	$x^2z + yz^2 + zy^3 + axy^{2+q}$	$q > 0, a_0 \neq 0$	$14 + 2q - 1$
$S_{1,2q}^\#$	$x^2z + yz^2 + zy^3 + ax^2y^{2+q}$	$q > 0, a_0 \neq 0$	$14 + 2q$
$U_{1,0}$	$x^3 + xz^2 + xy^3 + ay^3z$	$a_0(a_0^2 + 1) \neq 0$	14
$U_{1,2q-1}$	$x^3 + xz^2 + xy^3 + ay^{1+q}z^2$	$q > 0, a_0 \neq 0$	$14 + 2q - 1$
$U_{1,2q}$	$x^3 + xz^2 + xy^3 + ay^{3+q}z$	$q > 0, a_0 \neq 0$	$14 + 2q$

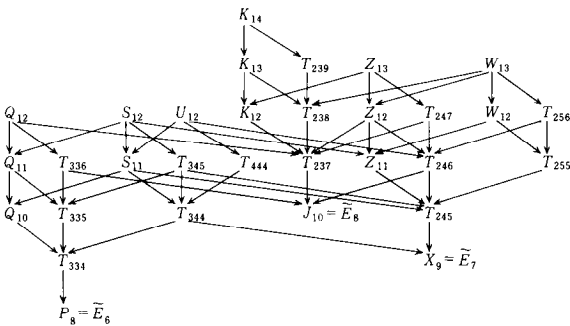
The 14 exceptional families

Notation	Normal form	Notation	Normal form
E_{18}	$x^3 + y^{10} + z^2 + axy^7$	W_{18}	$x^4 + y^7 + z^2 + ax^2y^4$
E_{19}	$x^3 + xy^7 + z^2 + ay^{11}$	Q_{16}	$x^3 + yz^2 + y^7 + z^2 + axy^5$
E_{20}	$x^3 + y^{11} + z^2 + axy^8$	Q_{17}	$x^3 + yz^2 + xy^5 + z^2 + ay^8$
Z_{17}	$x^3y + y^8 + z^2 + axy^6$	Q_{18}	$x^3 + yz^2 + y^8 + z^2 + axy^6$
Z_{18}	$x^3y + xy^6 + z^2 + ay^9$	S_{16}	$x^2z + yz^2 + xy^4 + z^2 + ay^6$
Z_{19}	$x^3y + y^9 + z^2 + axy^7$	S_{17}	$x^2z + yz^2 + y^6 + z^2 + azy^4$
W_{17}	$x^4 + xy^5 + z^2 + ay^7$	U_{16}	$x^3 + xz^2 + y^5 + z^2 + ax^2y^2$

Adjacency relations between simple and simply elliptic singularities



Adjacency relations among unimodular singularities



References

[1] V. I. Arnol'd, Singularity theory, Lecture note ser. 53, London Math. Soc., 1981.
[2] E. Breiskorn, Die Hierarchie der 1-modularen Singularitäten, Manuscripta Math., 27 (1979), 183–219.
[3] K. Saito, Einfach elliptische Singularitäten, Inventiones Math., 23 (1974), 289–325.

6. Topology

(I) *h*-Cobordism Groups of Homotopy Spheres and Groups of Differentiable Structures on Combinatorial Spheres

(1) The Structure of the *h*-Cobordism Group θ_n of *n*-Dimensional Homotopy Spheres. In the following table, values of θ_n have the following meanings: 0 means that the group consists only of the identity element, an integer *l* means that the group is isomorphic to the cyclic group of order *l*, 2^{*l*} means that the group is the direct sum of *l* groups of order 2, + means the direct sum, and ? means that the structure of the group is unknown.

<i>n</i>	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
$\theta_n \cong$	0	0	?	0	0	0	28	2	2 ³ or 4+2	6	992	0	3	2	8128+2	2	2 ⁴ or 4+2 ²	8+2

(2) The Group Γ_n of Differentiable Structures on the *n*-Dimensional Combinatorial Sphere.
 $\Gamma_n \cong \theta_n \quad (n \neq 3), \quad \Gamma_3 = 0.$

(II) Adem's Formula Concerning Steenrod Operators *Sq* and \mathcal{P} (→ 64 Cohomology Operations)

For the cohomology operators *Sq* and \mathcal{P} , we have

$$Sq^a Sq^b = \sum_{c=0}^{\lfloor a/2 \rfloor} \binom{b-c-1}{a-2c} Sq^{a+b-c} Sq^c \quad (a < 2b).$$

$$\begin{aligned}\mathcal{P}^a \mathcal{P}^b &= \sum_{c=0}^{\lfloor a/p \rfloor} (-1)^{a+c} \binom{(b-c)(p-1)-1}{a-pc} \mathcal{P}^{a+b-c} \mathcal{P}^c \quad (a < pb), \\ \mathcal{P}^a \delta \mathcal{P}^b &= \sum_{c=0}^{\lfloor a/p \rfloor} (-1)^{a+c} \binom{(b-c)(p-1)}{a-pc} \delta \mathcal{P}^{a+b-c} \mathcal{P}^c \\ &\quad + \sum_{c=0}^{\lfloor (a-1)/p \rfloor} (-1)^{a+c-1} \binom{(b-c)(p-1)-1}{a-pc-1} \mathcal{P}^{a+b-c} \delta \mathcal{P}^c \quad (a \leq pb).\end{aligned}$$

Several simple cases of the formula above are as follows.

$$\begin{aligned}Sq^1 Sq^{2n} &= Sq^{2n+1}, & Sq^1 Sq^{2n+1} &= 0, \\ Sq^2 Sq^{4n} &= Sq^{4n+2} + Sq^{4n+1} Sq^1, & Sq^2 Sq^{4n+1} &= Sq^{4n+2} Sq^1, \\ Sq^2 Sq^{4n+2} &= Sq^{4n+3} Sq^1, & Sq^2 Sq^{4n+3} &= Sq^{4n+5} + Sq^{4n+4} Sq^1, \\ Sq^4 Sq^{8n} &= Sq^{8n+4} + Sq^{8n+3} Sq^1 + Sq^{8n+2} Sq^2, & Sq^4 Sq^{8n+1} &= Sq^{8n+4} Sq^1 + Sq^{8n+3} Sq^2, \\ Sq^4 Sq^{8n+2} &= Sq^{8n+4} Sq^2, & Sq^4 Sq^{8n+3} &= Sq^{8n+5} Sq^2, \\ Sq^4 Sq^{8n+4} &= Sq^{8n+7} Sq^1 + Sq^{8n+6} Sq^2, & Sq^4 Sq^{8n+5} &= Sq^{8n+9} + Sq^{8n+8} Sq^1 + Sq^{8n+7} Sq^2, \\ Sq^4 Sq^{8n+6} &= Sq^{8n+10} + Sq^{8n+8} Sq^2, & Sq^4 Sq^{8n+7} &= Sq^{8n+11} + Sq^{8n+9} Sq^2, \\ \mathcal{P}^1 \mathcal{P}^n &= (n+1) \mathcal{P}^{n+1}, \\ \mathcal{P}^1 \delta \mathcal{P}^n &= n \cdot \delta \mathcal{P}^{n+1} + \mathcal{P}^{n+1} \delta.\end{aligned}$$

(III) Cohomology Ring $H^*(\pi, n; \mathbf{Z}_p)$ of Eilenberg-MacLane Complex ($\rightarrow 70$ Complexes)

\mathbf{Z} means the set of integers, and $\mathbf{Z}_p = \mathbf{Z}/p\mathbf{Z}$, where p is a prime number.

(1) The case $p=2$, $\pi = \mathbf{Z}$ or \mathbf{Z}_{2^f} ($f \geq 1$). The degree of a finite sequence $I = (i_r, i_{r-1}, \dots, i_1)$ of positive integers is defined by $d(I) = i_1 + i_2 + \dots + i_r$. If such a sequence satisfies $i_{k+1} \geq 2i_k$ ($k = 1, \dots, r-1$), it is called admissible, and we define its excess by

$$e(I) = (i_r - 2i_{r-1}) + \dots + (i_2 - 2i_1) + i_1 = 2i_r - d(I).$$

Further, we put $Sq^I = Sq^{i_r} Sq^{i_{r-1}} \dots Sq^{i_1}$. Then we have $H^*(\mathbf{Z}_{2^f}, n; \mathbf{Z}_2) = \mathbf{Z}_2[Sq^I u_n | I \text{ is admissible, } e(I) < n]$, $H^*(\mathbf{Z}, n; \mathbf{Z}_2) = \mathbf{Z}_2[Sq^I u_n | I \text{ is admissible, } e(I) < n, i_1 > 1]$.

Here, $u_n \in H^n(\pi, n; \mathbf{Z}_2)$ is the fundamental cohomology class. $I = \emptyset$ (empty) is also admissible, and for this case we put $n(I) = e(I) = 0$, $Sq^I = 1$. Due to the Künneth theorem, we have

$$H^*(\pi + \pi', n; \mathbf{Z}_p) = H^*(\pi, n; \mathbf{Z}_p) \otimes H^*(\pi', n; \mathbf{Z}_p)$$

if π is finitely generated. In particular, we have

$$\begin{aligned}H^*(\mathbf{Z}_2, 1; \mathbf{Z}_2) &= \mathbf{Z}_2[u_1], \\ H^*(\mathbf{Z}_2, 2; \mathbf{Z}_2) &= \mathbf{Z}_2[u_2, Sq^1 u_2, Sq^2 Sq^1 u_2, \dots, Sq^{2^r} Sq^{2^r-1} \dots Sq^1 u_2, \dots], \\ H^*(\mathbf{Z}_2, 3; \mathbf{Z}_2) &= \mathbf{Z}_2[u_3, Sq^{2^r} Sq^{2^r-1} \dots Sq^1 u_3, Sq^{(2^r+1)2^r} Sq^{(2^r+1)2^r-1} \dots \\ &\quad Sq^{2^r+1} Sq^{2^r-1} \dots Sq^1 u_3 | r \geq 0, \quad s \geq 0].\end{aligned}$$

(2) The case $p \neq 2$, $\pi = \mathbf{Z}$ or \mathbf{Z}_{p^f} ($f \geq 1$). We define the degree of a finite sequence $I = (i_r, i_{r-1}, \dots, i_1, i_0)$ of nonnegative integers by $d(I) = i_r + \dots + i_1 + i_0$. The sequence I is called admissible if it satisfies the following conditions:

$$\begin{aligned}i_k &= 2\lambda_k(p-1) + \epsilon_k \quad (\lambda_k \text{ is a nonnegative integer, } \epsilon_k = 0 \text{ or } 1 \ (0 \leq k \leq r)), \text{ and} \\ i_0 &= 0 \text{ or } 1, \quad i_1 \geq 2p-2, \quad i_{k+1} \geq pi_k \quad (1 \leq k \leq r-1).\end{aligned}$$

We define its excess by $e(I) = pi_r - (p-1)d(I)$. Further, we put $\mathcal{P}^I = \delta^{\epsilon_r} \mathcal{P}^{\lambda_r} \dots \delta^{\epsilon_1} \mathcal{P}^{\lambda_1} \delta^{\epsilon_0}$, and assume that $u_n \in H^n(\pi, n; \mathbf{Z})$ is the fundamental cohomology class. Then we have

$$\begin{aligned}H^*(\mathbf{Z}_{p^f}, n; \mathbf{Z}_p) &= \mathbf{Z}_p[\mathcal{P}^I u_n | I \text{ is admissible, } e(I) < n(p-1), n+d(I) \text{ is even}] \\ &\quad \otimes \wedge_{\mathbf{Z}_p} (\mathcal{P}^I u_n | I \text{ is admissible, } e(I) < n(p-1), n+d(I) \text{ is odd}).\end{aligned}$$

$H^*(\mathbf{Z}, n; \mathbf{Z})$ is given by the above formula when the admissible sequence is I with $i_0 = 0$.

(IV) Cohomology Ring of Compact Connected Lie Groups (\rightarrow 427 Topology of Lie Groups and Homogeneous Spaces)

(1) General Remarks. Let G be a compact connected Lie group with rank l and dimension n . We have $H^*(G; \mathbf{R}) \cong \bigwedge_{\mathbf{R}}(x_1, \dots, x_l)$, where $\bigwedge_K(x_1, \dots, x_l)$ means the exterior algebra over K of a linear space $V = Kx_1 + \dots + Kx_l$ with the basis $\{x_1, \dots, x_l\}$ over K . We define a new degree in $\bigwedge_K(x_1, \dots, x_l)$ by putting $\deg x_i = m_i$ (m_i is odd) ($1 \leq i \leq l$), where $m_1 + \dots + m_l = n$. The \cong means isomorphism as graded rings.

(2) Classical Compact Simple Lie Groups. We set $\deg x_i = i$.

$$H^*(U(n); \mathbf{R}) \cong \bigwedge_{\mathbf{R}}(x_1, x_3, \dots, x_{2n-1}),$$

$$H^*(SU(n); \mathbf{R}) \cong \bigwedge_{\mathbf{R}}(x_3, x_5, \dots, x_{2n-1}),$$

$$H^*(Sp(n); \mathbf{R}) \cong \bigwedge_{\mathbf{R}}(x_3, x_7, \dots, x_{4n-1}),$$

$$H^*(SO(n); \mathbf{Z}_2) \cong (\text{Having } x_1, x_2, \dots, x_{n-1} \text{ as a simple system of generators})$$

$$\cong \mathbf{Z}_2[x_1, x_3, \dots, x_{2n'-1}] / (x_i^{2^{s(i)}} \mid i = 1, \dots, n')$$

$$(n' = \lfloor n/2 \rfloor, s(i) \text{ is the least integer satisfying } 2^{s(i)}(2i-1) \geq n)$$

$$H^*(SO(2n); K) \cong \bigwedge_K(x_3, x_7, \dots, x_{4n-5}, x_{2n-1}),$$

$$H^*(SO(2n-1); K) \cong \bigwedge_K(x_3, x_7, \dots, x_{4n-5}), \text{ where } K \text{ is a commutative field whose characteristic is not 2.}$$

$$\text{For } SO(n), Sq^a(x_i) = \binom{i}{a} x_{i+a}. \text{ For } SU(n), p^a(x_{2i-1}) = \binom{i-1}{a} x_{2i-1+2a(p-1)}.$$

$$\text{For } Sp(n), \mathcal{P}^a(x_{4i-1}) = (-1)^{a(p-1)/2} \binom{2i-1}{a} x_{4i-1+2a(p-1)}.$$

(3) Exceptional Compact Simple Lie Groups. n and m_i ($1 \leq i \leq l$) given in (1) are as follows.

$$G_2: \quad n = 14, \quad m_i = 3, \quad 11.$$

$$F_4: \quad n = 52, \quad m_i = 3, \quad 11, \quad 15, \quad 23.$$

$$E_6: \quad n = 78, \quad m_i = 3, \quad 9, \quad 11, \quad 15, \quad 17, \quad 23.$$

$$E_7: \quad n = 133, \quad m_i = 3, \quad 11, \quad 15, \quad 19, \quad 23, \quad 27, \quad 35.$$

$$E_8: \quad n = 248, \quad m_i = 3, \quad 15, \quad 23, \quad 27, \quad 35, \quad 39, \quad 47, \quad 59.$$

(4) p -Torsion Groups of Exceptional Groups. The p -torsion groups of exceptional Lie groups are unit groups except when $p=2$ for G_2 ; $p=2, 3$ for F_4, E_6, E_7 ; and $p=2, 3, 5$ for E_8 . The cohomology ring of \mathbf{Z}_p as a coefficient group in these exceptional cases is given as follows. Here we put $\deg x_i = i$.

$$H^*(G_2; \mathbf{Z}_2) = \mathbf{Z}_2[x_3] / (x_3^4) \otimes \bigwedge_{\mathbf{Z}_2}(Sq^2x_3);$$

$$H^*(F_4; \mathbf{Z}_2) = \mathbf{Z}_2[x_3] / (x_3^4) \otimes \bigwedge_{\mathbf{Z}_2}(Sq^2x_3, x_{15}, Sq^8x_{15}),$$

$$H^*(F_4; \mathbf{Z}_3) = \mathbf{Z}_3[\delta \mathcal{P}^1x_3] / ((\delta \mathcal{P}^1x_3)^3) \otimes \bigwedge_{\mathbf{Z}_3}(x_3, \mathcal{P}^1x_3, x_{11}, \mathcal{P}^1x_{11});$$

$$H^*(E_6; \mathbf{Z}_2) = \mathbf{Z}_2[x_3] / (x_3^4) \otimes \bigwedge_{\mathbf{Z}_2}(Sq^2x_3, Sq^4Sq^2x_3, x_{15}, Sq^8Sq^4Sq^2x_3, Sq^8x_{15}),$$

$$H^*(E_6; \mathbf{Z}_3) = \mathbf{Z}_3[\delta \mathcal{P}^1x_3] / ((\delta \mathcal{P}^1x_3)^3) \otimes \bigwedge_{\mathbf{Z}_3}(x_3, \mathcal{P}^1x_3, x_9, x_{11}, \mathcal{P}^1x_{11}, x_{17});$$

$$H^*(E_7; \mathbf{Z}_2) = \mathbf{Z}_2[x_3, Sq^2x_3, Sq^4Sq^2x_3] / (x_3^4, (Sq^2x_3)^4, (Sq^4Sq^2x_3)^4)$$

$$\otimes \bigwedge_{\mathbf{Z}_2}(x_{15}, Sq^8Sq^4Sq^2x_3, Sq^8x_{15}, Sq^4Sq^8x_{15}),$$

$$H^*(E_7; \mathbf{Z}_3) = \mathbf{Z}_3[\delta \mathcal{P}^1x_3] / ((\delta \mathcal{P}^1x_3)^3) \otimes \bigwedge_{\mathbf{Z}_3}(x_3, \mathcal{P}^1x_3, x_{11}, \mathcal{P}^1x_{11}, \mathcal{P}^3\mathcal{P}^1x_3, x_{27}, x_{35});$$

$$H^*(E_8; \mathbf{Z}_2) = \mathbf{Z}_2[x_3, Sq^2x_3, Sq^4Sq^2x_3, x_{15}] / (x_3^{16}, (Sq^2x_3)^8, (Sq^4Sq^2x_3)^4, x_{15}^4)$$

$$\otimes \bigwedge_{\mathbf{Z}_2}(Sq^8Sq^4Sq^2x_3, Sq^8x_{15}, Sq^4Sq^8x_{15}, Sq^2Sq^4Sq^8x_{15}),$$

$$H^*(E_8; \mathbf{Z}_3) = \mathbf{Z}_3[\delta \mathcal{P}^1x_3, \delta \mathcal{P}^3\mathcal{P}^1x_3] / ((\delta \mathcal{P}^1x_3)^3, (\delta \mathcal{P}^3\mathcal{P}^1x_3)^3)$$

$$\otimes \bigwedge_{\mathbf{Z}_3}(x_3, \mathcal{P}^1x_3, x_{15}, \mathcal{P}^3\mathcal{P}^1x_3, \mathcal{P}^3x_{15}, x_{35}, x_{39}, x_{47}),$$

$$H^*(E_8; \mathbf{Z}_5) = \mathbf{Z}_5[\delta \mathcal{P}^1x_3] / ((\delta \mathcal{P}^1x_3)^5) \otimes \bigwedge_{\mathbf{Z}_5}(x_3, \mathcal{P}^1x_3, x_{15}, \mathcal{P}^1x_{15}, x_{27}, x_{35}, x_{39}, x_{47}).$$

(V) Cohomology Rings of Classifying Spaces (→ 56 Characteristic Classes, 427 Topology of Lie Groups and Homogeneous Spaces C)

(1) Let $H^*(G; K) = \bigwedge_K(x_1, x_2, \dots, x_n)$. Then the $\deg x_i$ are odd and the x_i may be assumed to be transgressive. y_i being its image, the following formula holds:

$$H^*(BG; K) = K[y_1, y_2, \dots, y_n] \quad (\text{Borel's theorem}).$$

(2) $H^*(BU(n)) = H^*(BGL(n, \mathbb{C})) = \mathbb{Z}[c_1, c_2, \dots, c_n],$

$$H^*(BSU(n)) = H^*(BSL(n, \mathbb{C})) = \mathbb{Z}[c_2, \dots, c_n],$$

$$H^*(BSp(n)) = \mathbb{Z}[q_1, q_2, \dots, q_n],$$

$$H^*(BO(n); K_2) = H^*(BGL(n, \mathbb{R}); \mathbb{Z}_2) = K_2[w_1, w_2, \dots, w_n],$$

$$H^*(BSO(n); K_2) = H^*(BSL(n, \mathbb{R}); \mathbb{Z}_2) = K_2[w_2, \dots, w_n],$$

$$H^*(BSO(2m+1); K) = K[p_1, p_2, \dots, p_m],$$

$$H^*(BSO(2m); K) = K[p_1, p_2, \dots, p_{m-1}, \chi].$$

Here, K denotes a field of characteristic $\neq 2$, and K_2 is the field of characteristic 2. The c_i denote the i th Chern classes and the q_i the i th symplectic Pontryagin classes, the w_i the i th Stiefel-Whitney classes. Moreover, the p_i denote the i th Pontryagin classes, and χ the Euler class. Their degrees are given as follows: $\deg c_i = 2i$, $\deg q_i = \deg p_i = 4i$, $\deg w_i = i$, and $\deg \chi = 2m$.

(3) Wu's Formula. Let $H^2(BSO(n); \mathbb{Z}_2) = \mathbb{Z}_2[w_2, \dots, w_n]$ and $H^*(BU(n), \mathbb{Z}_2) = \mathbb{Z}_2[c_1, c_2, \dots, c_n]$. We have

$$Sq^j w_i = \sum_{0 \leq t \leq j} \binom{i-j+t-1}{t} w_{j-t} w_{i+t} \quad (w_0 = 1),$$

$$Sq^{2j} c_i = \sum_{0 \leq t \leq j} \binom{i-j+t-1}{t} c_{j-t} c_{i+t} \quad (c_0 = 1).$$

Here the symbol $\binom{a}{b}$ denotes the binomial coefficient for $a \geq b$; $\binom{a}{b} = 1$, and $\binom{a}{b} = 0$ otherwise.

(VI) Homotopy Groups of Spheres (→ 202 Homotopy Theory)

Table of the $(n+k)$ th Homotopy Group $\pi_{n+k}(S^n)$ of the n -Dimensional Sphere S^n . The table represents Abelian groups. 0 stands for the unit group; integer l the cyclic group of order l ; ∞ the infinite cyclic group; 2^l the direct sum of l groups of order 2; and $+$ means the direct sum.

$n \backslash k$	<0	0	1	2	3	4	5	6	7	8	9	10	11	12	13
1	0	∞	0	0	0	0	0	0	0	0	0	0	0	0	0
2	0	∞	∞	2	2	12	2	2	3	15	2	2^2	$12+2$	$84+2^2$	2^2
3	0	∞	∞	2	12	2	2	3	15	2	2^2	$12+2$	$84+2^2$	2^2	6
4	0	∞	2	2	$\infty+12$	2^2	2^2	$24+3$	15	2	2^3	$120+12+2$	$84+2^5$	2^6	$24+6+2$
5	0	∞	2	2	24	2	2	2	30	2	2^3	$72+2$	$504+2^2$	2^3	$6+2$
6	0	∞	2	2	24	0	∞	2	60	$24+2$	2^3	$72+2$	$504+4$	240	6
7	0	∞	2	2	24	0	0	2	120	2^3	2^4	$24+2$	$504+2$	0	6
8	0	∞	2	2	24	0	0	2	$\infty+120$	2^4	2^5	$24+24+2$	$504+2$	0	$6+2$
9	0	∞	2	2	24	0	0	2	240	2^3	2^4	$24+2$	$504+2$	0	6
10	0	∞	2	2	24	0	0	2	240	2^2	$\infty+2^3$	$12+2$	504	12	6
11	0	∞	2	2	24	0	0	2	240	2^2	2^3	$6+2$	504	2	$6+2$
12	0	∞	2	2	24	0	0	2	240	2^2	2^3	6	$\infty+504$	2^2	$6+2$
13	0	∞	2	2	24	0	0	2	240	2^2	2^3	6	504	2	6
14	0	∞	2	2	24	0	0	2	240	2^2	2^3	6	504	0	$\infty+3$
>15	0	∞	2	2	24	0	0	2	240	2^2	2^3	6	504	0	3

Table of the $(n+k)$ th Homotopy Group $\pi_{n+k}(S^n)$ of the n -Dimensional Sphere S^n (Continued)

$n \backslash k$	14	15	16	17	18	19	20	21	22
1	0	0	0	0	0	0	0	0	0
2	6	30	30	$6+2$	$12+2^2$	$12+2^2$	$132+2$	2^2	2
3	30	30	$6+2$	$12+2^2$	$12+2^2$	$132+2$	2^2	2	210
4	$2520+6+2$	30	$6+6+2$	$24+12+4+2^2$	$120+12+2^5$	$132+2^5$	2^6	$24+2^2$	$9240+6+2$
5	$6+2$	$30+2$	2^2	$4+2^2$	$24+2^2$	$264+2$	$6+2^2$	$6+2$	$90+2^2$
6	$12+2$	$60+2$	$504+2^2$	2^4	$24+6+2$	$1056+8$	$480+12$	6	$180+2^2$
7	$24+4$	$120+2^3$	2^4	2^4	$24+2$	$264+2$	24	$6+2$	$72+2^3$
8	$240+24+4$	$120+2^5$	2^7	$6+2^4$	$504+24+2$	$264+2$	$24+3$	$12+2^3$	$1440+24+2^4$
9	$16+4$	$240+2^3$	2^4	2^4	$24+2$	$264+2$	24	$6+2^2$	$144+2^3$
10	$16+2$	$240+2^2$	$240+2$	2^3	$24+2^2$	$264+6$	$504+24$	$6+2^2$	$144+6+2$
11	$16+2$	$240+2$	2	2^3	$8+4+2$	$264+2^3$	$24+2^2$	2^4	$48+2^2$
12	$48+4+2$	$240+2$	2	2^4	$480+4+4+2$	$264+2^5$	$24+2^5$	$6+2^4$	$2016+12+2^2$
13	$16+2$	$480+2$	2	2^4	$8+8+2$	$264+2^3$	$24+2^3$	$4+2^3$	$16+2^2$
14	$8+2$	$480+2$	$24+2$	2^4	$8+8+2$	$264+4+2$	$240+24$	$4+2^2$	$16+2^2$
15	$4+2$	$480+2$	2^3	2^5	$8+8+2$	$264+2^2$	2^4	2^3	$16+2^3$
16	$2+2$	$\infty+480+2$	2^4	2^6	$24+8+8+2$	$264+2^2$	2^4	2^4	$240+16+2^3$
17	$2+2$	$480+2$	2^3	2^5	$8+8+2$	$264+2^2$	24	2^3	$16+2^3$
18	$2+2$	$480+2$	2^2	$\infty+2^4$	$8+4+2$	$264+2$	$24+12$	2^3	$16+2^2$
19	$2+2$	$480+2$	2^2	2^4	$8+2^2$	$264+2$	$24+2$	2^4	$16+2^2$
20	$2+2$	$480+2$	2^2	2^4	$8+2$	$\infty+264+2$	$24+2^2$	2^4	$16+2^2$
21	$2+2$	$480+2$	2^2	2^4	$8+2$	$264+2$	$24+2$	2^3	$8+2^2$
22	$2+2$	$480+2$	2^2	2^4	$8+2$	$264+2$	24	$\infty+2^2$	$4+2^2$
23	$2+2$	$480+2$	2^2	2^4	$8+2$	$264+2$	24	2^2	2^3
>24	$2+2$	$480+2$	2^2	2^4	$8+2$	$264+2$	24	2^2	2^2

Remarks

(1) When $n > k+1$ (below the broken line in the table), $\pi_{n+k}(S^n)$ is independent of n and is isomorphic with the k th stable homotopy group G_k .

(2) Let $\iota_n \in \pi_n(S^n)$ be the identity on S^n ; $\eta_2 \in \pi_3(S^2)$, $\nu_4 \in \pi_7(S^4)$, $\sigma_8 \in \pi_{15}(S^8)$ be the Hopf mapping $S^3 \rightarrow S^2$, $S^7 \rightarrow S^4$, $S^{15} \rightarrow S^8$ (induced mapping in the homotopy class), respectively; and $[\iota_{2m}, \iota_{2m}] \in \pi_{4m-1}(S^{2m})$ ($m \neq 1, 2, 4$) be the Whitehead product of ι_{2m} . These objects generate infinite cyclic groups which are direct factors of $\pi_{n+k}(S^n)$ corresponding to the original mappings.

(3) $\eta_{n+2} = E^n \eta_2$, $\nu_{n+4} = E^n \nu_4$, $\sigma_{n+8} = E^n \sigma_8$ ($n \geq 1$) (E is the suspension) are the generator for $\pi_{n+k}(S^n)$, which contains the mappings.

(4) The orders of the following compositions are $2:gs_{n+7}$

$$\eta_n \circ \eta_{n+1} (n \geq 2), \quad \nu_n \circ \nu_{n+3} (n \geq 5), \quad \sigma_n \circ \sigma_{n+7} (n \geq 16), \quad \eta_n \circ \nu_{n+1} (n = 3, 4),$$

$$\nu_n \circ \eta_{n+3} (n = 4, 5), \quad \eta_n \circ \sigma_{n+1} (n \geq 7), \quad \sigma_n \circ \eta_{n+7} (n \geq 8), \quad \nu_{10} \circ \sigma_{13}, \quad \sigma_{11} \circ \nu_{18},$$

$$\eta_n \circ \eta_{n+1} \circ \eta_{n+2} (n \geq 2), \quad \nu_n \circ \nu_{n+3} \circ \nu_{n+6} (n \geq 4), \quad \sigma_n \circ \sigma_{n+7} \circ \sigma_{n+14} (n \geq 9).$$

(VII) The Homotopy Groups $\pi_k(G)$ of Compact Connected Lie Groups G

Here the group G is one of the following:

$$SO(n) (n \geq 2), \quad Spin(n) (n \geq 3), \quad U(n) (n \geq 1), \quad SU(n) (n \geq 2),$$

$$Sp(n) (n \geq 1), \quad G_2, \quad F_4, \quad E_6, \quad E_7, \quad E_8.$$

(1) The Fundamental Group $\pi_1(G)$.

$$\pi_1(G) \cong \begin{cases} \infty & (G = U(n) (n \geq 1), \quad SO(2)), \\ 2 & (G = SO(n) (n \geq 3)), \\ 0 & (\text{for all other groups } G). \end{cases}$$

(2) Isomorphic Relations ($k \geq 2$).

$$\pi_k(U(n)) \cong \pi_k(SU(n)) (n \geq 2),$$

(4) Stable Homotopy Groups. For sufficiently large n for fixed k , the homotopy groups for classical compact simple Lie groups $G = Sp(n)$, $SU(n)$, $SO(n)$ become stable. We denote them by the following notations. Here we assume $k \geq 2$.

$$\begin{aligned}\pi_k(Sp) &= \pi_k(Sp(n)) & (n \geq (k-1)/4), \\ \pi_k(U) &= \pi_k(U(n)) \cong \pi_k(SU(n)) & (n \geq (k+1)/2), \\ \pi_k(O) &= \pi_k(SO(n)) & (n \geq k+2).\end{aligned}$$

Bott periodicity theorem

$$\pi_k(Sp) \cong \begin{cases} \infty & (k \equiv 3, 7 \pmod{8}), \\ 2 & (k \equiv 4, 5 \pmod{8}), \\ 0 & (k \equiv 0, 1, 2, 6 \pmod{8}). \end{cases}$$

$$\pi_k(O) \cong \begin{cases} \infty & (k \equiv 3, 7 \pmod{8}), \\ 2 & (k \equiv 0, 1 \pmod{8}), \\ 0 & (k \equiv 2, 4, 5, 6 \pmod{8}). \end{cases}$$

$$\pi_k(U) \cong \begin{cases} \infty & (k \equiv 1 \pmod{2}), \\ 0 & (k \equiv 0 \pmod{2}). \end{cases}$$

(5) Metastable Homotopy Groups.

(a, b) means the greatest common divisor of two integers a and b .

$$\pi_{2n}(SU(n)) \cong n!.$$

$$\pi_{2n+1}(SU(n)) \cong \begin{cases} 2 & (n \text{ even}), \\ 0 & (n \text{ odd}). \end{cases}$$

$$\pi_{2n+2}(SU(n)) \cong \begin{cases} (n+1)! + 2 & (n \text{ even}, \geq 4), \\ (n+1)!/2 & (n \text{ odd}). \end{cases}$$

$$\pi_{2n+3}(SU(n)) \cong \begin{cases} (24, n) & (n \text{ even}), \\ (24, n+3)/2 & (n \text{ odd}). \end{cases}$$

$$\pi_{2n+4}(SU(n)) \cong \begin{cases} (n+2)!(24, n)/48 & (n \text{ even}, \geq 4), \\ (n+2)!(24, n+3)/24 & (n \text{ odd}). \end{cases}$$

$$\pi_{2n+5}(SU(n)) \cong \pi_{2n+5}(U(n+1)).$$

$$\pi_{2n+6}(SU(n)) \cong \begin{cases} \pi_{2n+6}(U(n+1)) & (n \equiv 2, 3 \pmod{4}, n \geq 3), \\ \pi_{2n+6}(U(n+1)) + 2 & (n \equiv 0, 1 \pmod{4}). \end{cases}$$

$$\pi_{4n+2}(Sp(n)) \cong \begin{cases} (2n+1)! & (n \text{ even}), \\ 2(2n+1)! & (n \text{ odd}). \end{cases}$$

$$\pi_{4n+3}(Sp(n)) \cong 2.$$

$$\pi_{4n+4}(Sp(n)) \cong \begin{cases} 2+2 & (n \text{ even}), \\ 2 & (n \text{ odd}). \end{cases}$$

$$\pi_{4n+5}(Sp(n)) \cong \begin{cases} (24, n+2) + 2 & (n \text{ even}), \\ (24, n+2) & (n \text{ odd}). \end{cases}$$

$$\pi_{4n+6}(Sp(n)) \cong \begin{cases} (2n+3)!(24, n+2)/12 & (n \text{ even}), \\ (2n+3)!(24, n+2)/24 & (n \text{ odd}). \end{cases}$$

$$\pi_{4n+7}(Sp(n)) \cong 2.$$

$$\pi_{4n+8}(Sp(n)) \cong 2+2.$$

The homotopy groups $\pi_{n+i}(SO(n))$ for $n \geq 16$, $3 \geq i \geq -1$ are determined by the isomorphism

$$\pi_{n+i}(SO(n)) \cong \pi_{n+i}(O) + \pi_{n+i+1}(V_{i+3+n, i+3}(\mathbf{R}))$$

and the homotopy groups of $V_{m+n, m}(\mathbf{R})$ given below.

(6) Homotopy Groups of Real Stiefel Manifolds $V_{m+n, m}(\mathbf{R}) = O(m+n)/I_m \times O(n)$.

$$\pi_{n+k}(V_{n+1, 1}) \cong \pi_{n+k}(S^n).$$

$$\pi_{n-k}(V_{m+n, m}) \cong 0 \quad (k \geq 1).$$

$$\pi_n(V_{m+n, m}) \cong \begin{cases} 2 & (n=2s-1, m \geq 2), \\ \infty & (n=2s). \end{cases}$$

$\pi_{n+k}(V_{m+n, m})$ ($k=1, 2, 3, 4, 5$) are given in the following table.

	m	1	2	3	4	5	6	$8s-1$	$8s$	$8s+1$	$8s+2$	$8s+3$	$8s+4$	$8s+5$	$8s+6$
π_{n+1}	2	0	∞^2	2	$2+\infty$	2	$2+\infty$	2	$2+\infty$	2	$2+\infty$	2	$2+\infty$	2	$2+\infty$
	≥ 3	0	∞	0	2^2	2	4	0	2^2	2	4	0	2^2	2	4
π_{n+2}	2	∞	2^2	4	2^2	4	2^2	4	2^2	4	2^2	4	2^2	4	2^2
	3	∞^2	2	$2+\infty$	2^2	$4+\infty$	2	$2+\infty$	2^2	$4+\infty$	2	$2+\infty$	2^2	$4+\infty$	2
	≥ 4	∞	0	2	2^2	8	0	2	2^2	8	0	2	2^2	8	0
π_{n+3}	2	2	2^2	2	$\infty+12+2$	2^2	$24+2$	2^2	$24+2$	2^2	$24+4$	2^2	$24+2$	2^2	$24+2$
	3	2^2	2	2	$\infty+12+4$	2^3	$12+2$	2^2	$24+4$	2^3	$12+2$	2^2	$24+4$	2^3	$12+2$
	4	2	∞	2	∞^2+12+4	2^2	$12+\infty$	2^2	$24+4+\infty$	2^2	$12+\infty$	2^2	$24+4+\infty$	2^2	$12+\infty$
	≥ 5	0	0	2	$12+4+\infty$	2	12	2	$24+8$	2	12	2^2	$4+48$	2	12
π_{n+4}	2	2	12^2	$\infty+2$	2^2+24	2^2	24	2	24	2	24	2	24	2	24
	3	2^2	0	$\infty+4$	2^4	2^3	2	4	2^2	2^2	2	4	2^2	2^2	2
	4	2	0	$4+\infty$	2^5	2^2	2	8	2^3	2	2	8	2^3	2	2
	5	∞	0	$4+\infty^2$	2^4	$2+\infty$	2	$8+\infty$	2^2	∞	2	$8+\infty$	2^2	∞	2
	≥ 6	0	0	$4+\infty$	2^3	2	0	8	2	0	2	16	2	0	0
π_{n+5}	2	12	2^2	2	2^3	0	∞	0	0	0	0	0	0	0	0
	3	12^2	∞	$2+24$	2^4	24	$\infty+2$	24	2	24	2	24	2	24	2
	4	0	∞	2^3	2^5	2	$\infty+4$	2^2	2^2	2	4	2^2	2^2	2	4

(VIII) Immersion and Embedding of Projective Spaces (\rightarrow 114 Differential Topology)

We denote immersion by \subset , and embedding by \subseteq . $\mathbf{P}^n(A)$ is an n -dimensional real or complex projective space where $A = \mathbf{R}$ or \mathbf{C} , $k\{\mathbf{P}^n(A)\}$ is the integer k such that $\mathbf{P}^n(A) \subset \mathbf{R}^k$ and $\mathbf{P}^n(A) \not\subset \mathbf{R}^{k-1}$, and $\tilde{k}\{\mathbf{P}^n(A)\}$ is the integer k such that $\mathbf{P}^n(A) \subseteq \mathbf{R}^k$ and $\mathbf{P}^n(A) \not\subseteq \mathbf{R}^{k-1}$.

In the table, for example, numbers 9–11 in the row $k\{\mathbf{P}^n(\mathbf{R})\}$ for $n=6$ mean $\mathbf{P}^6(\mathbf{R}) \not\subset \mathbf{R}^8$, $\mathbf{P}^6(\mathbf{R}) \subset \mathbf{R}^{11}$.

n	1	2	3	4	5	6	7	8	9	10	11	12
$k\{\mathbf{P}^n(\mathbf{R})\}$	2	4	5	8	9	9~11	9~12	16	17	17~19
$\tilde{k}\{\mathbf{P}^n(\mathbf{R})\}$	2	3	4	7	7	7	8	15	15	16	16	17~19
$k\{\mathbf{P}^n(\mathbf{C})\}$	3	7	9	15	17	22	22~25	31	33	38	38~41	...
$\tilde{k}\{\mathbf{P}^n(\mathbf{C})\}$	3	7	8~9	15	16~17	22	22~25	31	32~33	38	38~41	...

	2^r	2^r+1	2^r+2	2^r+3	2^r+2^s ($r > s > 0$)
$k\{\mathbf{P}^n(\mathbf{R})\}$	$2n$	$2n-1$	$2n-3 \sim 2n-1$
$\tilde{k}\{\mathbf{P}^n(\mathbf{R})\}$	$2n-1$	$2n-3$	$2n-4$	$2n-6$...
$k\{\mathbf{P}^n(\mathbf{C})\}$	$4n-1$	$4n-3$	$4n-2$...	$4n-2$
$\tilde{k}\{\mathbf{P}^n(\mathbf{C})\}$	$4n-1$	$4n-4 \sim 4n-3$	$4n-2$...	$4n-2$

7. Knot Theory (→ 235 Knot Theory)

Let k be a projection on a plane of a knot K . We color the domains separated by k , white and black alternatively. The outermost (unbounded) domain determined by k is colored white. In Fig. 16, hatching means black. Take a point (a black point in Fig. 16) in each black domain. The self-intersections of k are represented by white points (Fig. 16). Through each white point we draw a line segment connecting the black points in the black regions meeting at the white point. In Fig. 16, we show this as a broken line. We assign the signature $+$ if the torsion of K at the intersection of k has the orientation of a right-hand screw (as in Fig. 17, left), and the signature $-$ if the orientation is opposite (as in Fig. 17, right). The picture of the line segments with signatures is called the graph corresponding to the projection k of the knot K . Given such a graph, we can reconstruct the original knot K .

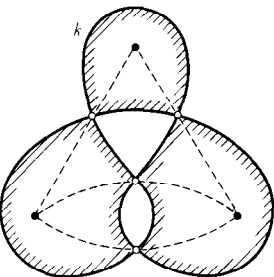


Fig. 16

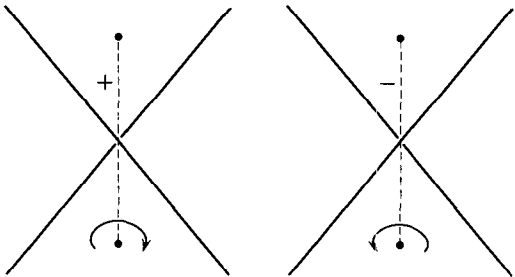


Fig. 17

Fig. 18 shows the classification table of knots for which the numbers of intersections of k are 3 to 8 when we minimize the intersections. The projection of k is described by a solid line, and the graph by broken lines. We omit the signatures since for each graph from 3_1 to 8_{18} they are all $+$ or all $-$. Such knots are called alternating knots.

8. Inequalities (→ 88 Convex Analysis, 211 Inequalities)

(1) $|a + b| \leq |a| + |b|,$

$$|a - b| \geq ||a| - |b||.$$

For real a_v , we have $\sum a_v^2 \geq 0$, and the equality holds only if all $a_v = 0$.

(2) $n! < n^n < (n!)^2 \quad (n \geq 3).$

$$e^n \geq n^n / n!.$$

$$n^{1/n} < 3^{1/3} \quad (n \neq 3).$$

(3) $2/\pi < (\sin x)/x < 1 \quad (0 < x < \pi/2) \quad (\text{Jordan's inequality}).$

(4) Denote the elementary symmetric polynomials of positive numbers $a_1, \dots, a_n > 0$ by S_r ($r = 1, \dots, n$). Then

$$S_1 / \binom{n}{1} \geq \left[S_2 / \binom{n}{2} \right]^{1/2} \geq \dots \geq \left[S_r / \binom{n}{r} \right]^{1/r} \geq \dots \geq \left[S_n / \binom{n}{n} \right]^{1/n}.$$

If at least one equality holds, then $a_1 = \dots = a_n$. In particular, from the two external terms, we have the following inequalities concerning mean values:

$$\frac{1}{n} \sum_{v=1}^n a_v \geq \left(\prod_{v=1}^n a_v \right)^{1/n} \geq n / \sum_{v=1}^n \frac{1}{a_v}.$$

For weighted means, we have

$$\sum_{v=1}^n \lambda_v a_v \geq \prod_{v=1}^n a_v^{\lambda_v} \quad \left(\sum \lambda_v = 1, \quad \lambda_v > 0 \right).$$

(5) When $a_v > 0, b_v > 0, p > 1, q > 1, (1/p) + (1/q) = 1$,

$$\left[\sum_{v=1}^n (a_v)^p \right]^{1/p} \left[\sum_{v=1}^n (b_v)^q \right]^{1/q} \geq \sum_{v=1}^n a_v b_v \quad (\text{H\"older's inequality}).$$

The equality holds only if $(a_v)^p = c(b_v)^q$ (c is a constant).

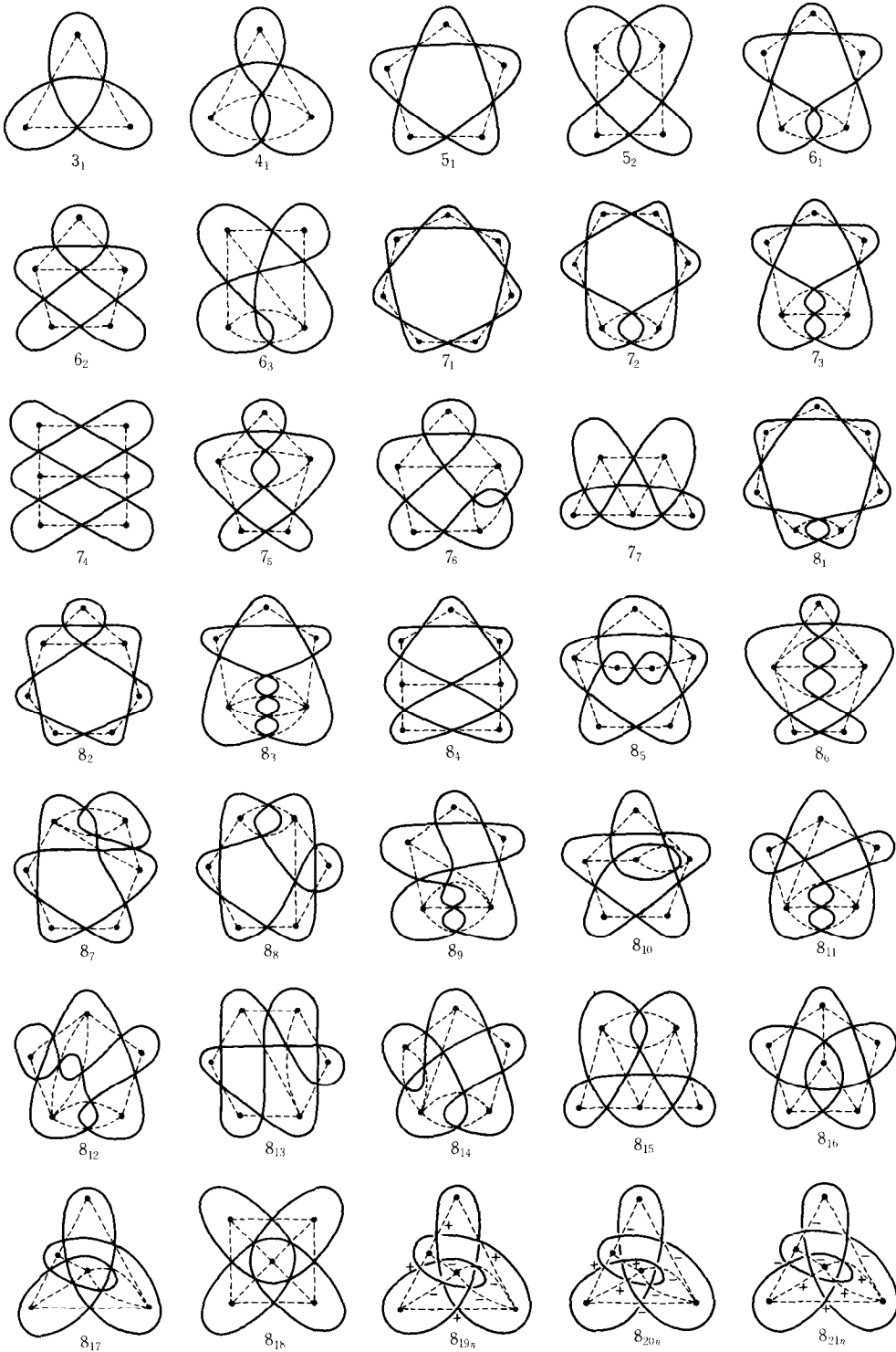


Fig. 18
Classification table of knots. The signatures from 3_1 to 8_{18} are all + or all -.

When $p=q=2$, the inequality is called Cauchy's inequality, the Cauchy-Schwarz inequality, or Bunyakovskii's inequality. As special cases, we have

$$\left(\sum_{\nu=1}^n a_{\nu} \right) \left(\sum_{\nu=1}^n \frac{1}{a_{\nu}} \right) \geq n^2 \quad (a_{\nu} > 0),$$

$$\left(\sum_{\nu=1}^n a_{\nu} \right)^2 \leq n \left(\sum_{\nu=1}^n a_{\nu}^2 \right) \quad (a_{\nu} > 0).$$

When $0 < p < 1$, we have an inequality by reversing the inequality sign in Hölder's inequality.

(6) When $a_\nu > 0$, $b_\nu > 0$, $p > 0$, and $\{a_\nu\}$ and $\{b_\nu\}$ are not proportional, we have

$$\left[\sum_{\nu=1}^n (a_\nu + b_\nu)^p \right]^{1/p} \leq \left[\sum_{\nu=1}^n (a_\nu)^p \right]^{1/p} + \left[\sum_{\nu=1}^n (b_\nu)^p \right]^{1/p} \quad (p \geq 1) \quad (\text{Minkowski's inequality}).$$

The integral inequality corresponding to (5) or (6) has the same name.

(7) If $a_{\mu\nu} > 0$, $\sum_{\mu=1}^n a_{\mu\nu} = \sum_{\nu=1}^n a_{\mu\nu} = 1$, $b_\nu > 0$,

$$\prod_{\nu=1}^n b_\nu \leq \prod_{\mu=1}^n \left(\sum_{\nu=1}^n a_{\mu\nu} b_\nu \right).$$

In particular, for the determinant $\Delta = \det(a_{\mu\nu})$,

$$|\Delta|^2 \leq \prod_{\nu=1}^n \left(\sum_{\mu=1}^n |a_{\mu\nu}|^2 \right).$$

The equality in this holds only if all rows are mutually orthogonal. If all $|a_{\mu\nu}| \leq M$, we have

$$|\Delta| \leq n^{n/2} M^n \quad (\text{Hadamard's estimation}).$$

(8) Suppose that a function $f(x)$ is continuous, strictly monotone increasing in $x \geq 0$, and $f(0) = 0$. Denote the inverse function of f by f^{-1} . For $a, b > 0$, we have

$$ab \leq \int_0^a f(x) dx + \int_0^b f^{-1}(x) dx \quad (\text{Young's inequality}),$$

and the equality holds only if $b = f(a)$.

In particular, for $f(x) = x^{p-1}$ ($p > 1$), we have

$$\frac{a^p}{p} + \frac{b^q}{q} \geq ab,$$

where $(1/p) + (1/q) = 1$.

(9) If $p, q > 1$, $(1/p) + (1/q) = 1$, $a_\mu \geq 0$, $b_\nu \geq 0$,

$$\sum_{\mu, \nu=0}^{\infty} \frac{a_\mu b_\nu}{\mu + \nu + 1} \leq \frac{\pi}{\sin(\pi/p)} \left[\sum_{\mu=0}^{\infty} (a_\mu)^p \right]^{1/p} \left[\sum_{\nu=0}^{\infty} (b_\nu)^q \right]^{1/q} \quad (\text{Hilbert's inequality}),$$

and the equality holds only when the right-hand side vanishes.

(10) For a continuous function $f(x) \geq 0$ ($0 \leq x < \infty$), we put

$$F(x) = \int_0^x f(t) dt,$$

and assume that $p > 1$. Then

$$\int_0^\infty \left[\frac{F(x)}{x} \right]^p dx \leq \left(\frac{p}{p-1} \right)^p \int_0^\infty [f(x)]^p dx \quad (\text{Hardy's inequality}),$$

and the equality holds only if $f(x)$ is identically 0.

Further, if $f(x) > 0$,

$$\int_0^\infty \exp \left[\frac{1}{x} \int_0^x \log f(t) dt \right] dx < e \int_0^\infty f(x) dx \quad (\text{Carleman's inequality}).$$

(11) Let $a \leq x < \xi \leq b$, $p > 1$, and

$$\sup_{\xi} \frac{1}{\xi - x} \int_x^{\xi} f(t) dt = \theta(x).$$

Then

$$\int_a^b [\theta(x)]^p dx \leq 2 \left(\frac{p}{p-1} \right)^p \int_a^b |f(x)|^p dx \quad (\text{Hardy-Littlewood supremum theorem}).$$

(12) If $f(x)$ is piecewise smooth in $0 \leq x \leq \pi$ and $f(0) = f(\pi) = 0$,

$$\int_0^\pi [f'(x)]^2 dx \geq \int_0^\pi [f(x)]^2 dx \quad (\text{Wirtinger's inequality}),$$

and the equality holds only if $f(x)$ is a constant multiple of $\sin x$.

9. Differential and Integral Calculus

(I) Derivatives and Primitive Functions (→ 106 Differential Calculus, 216 Integral Calculus)

$F(x) \equiv \int f(x) dx$	$f(x) \equiv F'(x)$
$\alpha\varphi + \beta\psi$ (α, β constants)	$\alpha\varphi' + \beta\psi'$
$\varphi \cdot \psi$	$\varphi' \cdot \psi + \varphi \cdot \psi'$
φ / ψ ($\psi \neq 0$)	$(\varphi' \cdot \psi - \varphi \cdot \psi') / \psi^2$
$\log \varphi $ ($\varphi \neq 0$)	φ' / φ (logarithmic differentiation)
$\Phi(\varphi)$ (composite)	$(d\Phi/d\varphi) \cdot \varphi'$
c (constant)	0
x^n	nx^{n-1}
$x^{n+1}/(n+1)$	x^n ($n \neq -1$)
$\log x $	$1/x$
$\log_a x $	$(\log_a e)/x$
$x(\log x - 1)$	$\log x$
$\exp x = e^x$	$\exp x = e^x$
a^x ($a > 0$)	$a^x \log a$
x^x	$x^x(1 + \log x)$
$(x-1)e^x$	xe^x
$\sin x$	$\cos x$
$\cos x$	$-\sin x$
$\tan x$	$\sec^2 x$
$\cot x$	$-\operatorname{cosec}^2 x$
$\sec x$	$\sec x \tan x$
$\operatorname{cosec} x$	$-\operatorname{cosec} x \cot x$
$\sinh x = (e^x - e^{-x})/2$	$\cosh x$
$\cosh x = (e^x + e^{-x})/2$	$\sinh x$
$\tanh x = \sinh x / \cosh x$	$\operatorname{sech}^2 x$
$\coth x = \cosh x / \sinh x$	$-\operatorname{cosech}^2 x$
$\operatorname{sech} x = 1 / \cosh x$	$-\operatorname{sech} x \tanh x$
$\operatorname{cosech} x = 1 / \sinh x$	$-\operatorname{cosech} x \coth x$
$\arcsin x$ ($ F < \pi/2$)	$1/\sqrt{1-x^2}$
$\arccos x$ ($0 < F < \pi$)	$-1/\sqrt{1-x^2}$
$\arctan x$ ($ F < \pi/2$)	$1/(1+x^2)$
$\operatorname{arccot} x$ ($ F < \pi/2$)	$-1/(1+x^2)$
$\operatorname{arcsec} x$ ($0 < F < \pi$)	$1/ x \sqrt{x^2-1}$
$\operatorname{arc cosec} x$ ($ F < \pi/2$)	$-1/ x \sqrt{x^2-1}$
$\operatorname{arsinh} x = \log(x + \sqrt{x^2+1})$	$1/\sqrt{x^2+1}$
$\operatorname{arcosh} x = \log(x + \sqrt{x^2-1})$	$1/\sqrt{x^2-1}$
$\frac{1}{2} \log \left \frac{1+x}{1-x} \right = \begin{cases} \operatorname{artanh} x & (x < 1) \\ \operatorname{arcoth} x & (x > 1) \end{cases}$	$\frac{1}{1-x^2}$
$\operatorname{arsech} x$	$-1/x\sqrt{1-x^2}$
$\operatorname{arc cosech} x$	$-1/ x \sqrt{1+x^2}$

$F(x) \equiv \int f(x) dx$	$f(x) \equiv F'(x)$
$\frac{1}{2a} \log \left \frac{a-x}{a+x} \right \quad (a > 0)$	$\frac{1}{x^2 - a^2}$
$(1/a) \arctan(x/a)$	$1/(x^2 + a^2)$
$(x\sqrt{1-x^2} + \arcsin x)/2$	$\sqrt{1-x^2}$
$[x\sqrt{x^2 \pm 1} \pm \log(x + \sqrt{x^2 \pm 1})]/2$	$\sqrt{x^2 \pm 1}$
$-\log \cos x $	$\tan x$
$\log \sin x $	$\cot x$
$\log \tan x $	$1/\sin x \cos x$
$\log \tan[(\pi/4) + (x/2)] $	$\sec x$
$\log \tan(x/2) $	$\operatorname{cosec} x$
$(x/2) - (1/4)\sin 2x$	$\sin^2 x$
$\sin x - x \cos x$	$x \sin x$
$\cos x + x \sin x$	$x \cos x$
$\frac{n \sin mx \sin nx + m \cos mx \cos nx}{n^2 - m^2} \quad (n^2 \neq m^2)$	$\sin mx \cos nx$
$e^{bx} \frac{b \sin ax - a \cos ax}{a^2 + b^2}$	$e^{bx} \sin ax$
$e^{bx} \frac{b \cos ax + a \sin ax}{a^2 + b^2}$	$e^{bx} \cos ax$
$x \arcsin x + \sqrt{1-x^2}$	$\arcsin x$
$x \arctan x - (1/2)\log(1+x^2)$	$\arctan x$
$\det(\varphi_{jk})_{j,k=1,\dots,n}$	$\sum \det(\varphi_{j1} \dots \varphi_{j\nu-1} \varphi'_{j\nu} \varphi_{j\nu+1} \dots \varphi_{jn})_{j=1,\dots,n}$

(II) Recurrence Formulas for Indefinite Integrals

(1) $I_m \equiv \int \frac{dx}{(1+x^2)^m} \quad (m \text{ is a positive integer}).$

$$I_m = \frac{1}{2m-2} \frac{x}{(1+x^2)^{m-1}} + \frac{2m-3}{2m-2} I_{m-1} \quad (m \geq 2); \quad I_1 = \arctan x.$$

(2) $I_m \equiv \int \frac{x^m}{\sqrt{ax^2+bx+c}} dx \quad (m \text{ is an integer}, a \neq 0).$

The case $m < 0$ is reduced to the case $m \geq 0$ by the change of variable $1/x = t$.

$$I_m = \frac{1}{ma} x^{m-1} \sqrt{ax^2+bx+c} - \frac{(2m-1)b}{2ma} I_{m-1} - \frac{(m-1)c}{ma} I_{m-2} \quad (m \geq 1);$$

$$I_0 = \begin{cases} (1/\sqrt{a}) \log |2ax+b+2\sqrt{a} \sqrt{ax^2+bx+c}| & (a > 0), \\ \frac{-1}{\sqrt{-a}} \arcsin \frac{2ax+b}{\sqrt{b^2-4ac}} & (a < 0); \end{cases}$$

In this case, for the integrand to be a real function it is necessary that $b^2 - 4ac > 0$.

(3) $I_m \equiv \int x^m e^x dx \quad (m \text{ is an integer}).$

$$I_m = x^m e^x - m I_{m-1}; \quad I_0 = e^x, \quad I_{-1} = \operatorname{Ei} x,$$

where Ei is the exponential integral function (\rightarrow Table 19.II.3, this Appendix).

$$\begin{aligned}
 (4) \quad I_{m,n} &\equiv \int x^m (\log x)^n dx \quad (m, n \text{ are integers, } n \geq 0). \\
 I_{m,n} &= \frac{x^{m+1}}{m+1} (\log x)^n - \frac{n}{m+1} I_{m,n-1}; \quad I_{m,0} = \frac{x^{m+1}}{m+1} \quad (m \neq -1), \quad I_{-1,n} = (\log x)^{n+1} / (n+1). \\
 (5) \quad I_m &\equiv \int x^m \sin x dx, \quad J_m \equiv \int x^m \cos x dx \quad (m \text{ is a nonnegative integer}). \\
 I_m &= -x^m \cos x + m J_{m-1} = x^{m-1} (m \sin x - x \cos x) - m(m-1) I_{m-2}, \\
 J_m &= x^m \sin x - m I_{m-1} = x^{m-1} (x \sin x + m \cos x) - m(m-1) J_{m-2}; \\
 I_0 &= -\cos x, \\
 J_0 &= \sin x, \\
 (6) \quad I_{m,n} &\equiv \int \sin^m x \cdot \cos^n x dx \quad (m, n \text{ are integers}). \\
 I_{m,n} &= \frac{\sin^{m+1} x \cos^{n-1} x}{m+n} + \frac{n-1}{m+n} I_{m,n-2} \quad \left. \begin{aligned} & \\ & \end{aligned} \right\} (m+n \neq 0), \\
 I_{m,n} &= \frac{-\sin^{m-1} x \cos^{n+1} x}{m+n} + \frac{m-1}{m+n} I_{m-2,n} \quad \left. \begin{aligned} & \\ & \end{aligned} \right\} \\
 I_{m,n} &= \frac{-\sin^{m+1} x \cos^{n+1} x}{n+1} + \frac{m+n+2}{n+1} I_{m,n+2} \quad (n \neq -1), \\
 I_{m,n} &= \frac{\sin^{m+1} x \cos^{n+1} x}{m+1} + \frac{m+n+2}{m+1} I_{m+2,n} \quad (m \neq -1); \\
 I_{1,1} &= (\sin^2 x)/2, \quad I_{1,0} = -\cos x, \quad I_{1,-1} = -\log |\cos x|, \quad I_{0,1} = \sin x, \quad I_{0,0} = x, \\
 I_{0,-1} &= \log |\tan[(x/2) + (\pi/4)]|, \quad I_{-1,1} = \log |\sin x|, \\
 I_{-1,0} &= \log |\tan(x/2)|, \quad I_{-1,-1} = \log |\tan x|. \\
 I_{m,-m} &\equiv \int \tan^m x dx = \frac{\tan^{m-1} x}{m-1} - I_{m-2,-(m-2)} \quad (m \neq 1).
 \end{aligned}$$

(III) Derivatives of Higher Order

$f(x)$	$f^{(n)}(x)$
$\varphi \cdot \psi$	$\sum_{\nu=0}^n \binom{n}{\nu} \varphi^{(\nu)} \psi^{(n-\nu)} \quad (\text{Leibniz's formula})$
x^k	$\prod_{\nu=0}^{n-1} (k-\nu) x^{k-n}$
$(x+a)^n$	$n!$
$\exp x$	$\exp x$
$a^x \quad (a > 0)$	$a^x (\log a)^n$
$\log x$	$(-1)^{n-1} (n-1)! / x^n$
$\sin x$	$\sin[x + (n\pi/2)]$
$\cos x$	$\cos[x + (n\pi/2)]$
$e^{ax} \cos bx$	$r^n e^{ax} \cos(bx + n\theta) \quad (\text{where } a = r \cos \theta, b = r \sin \theta)$
$\arcsin x$	$\frac{1}{2^{n-1}} \sum_{\nu=0}^{n-1} (-1)^\nu \binom{n-1}{\nu} (2\nu-1)!! (2n-2\nu-3)!! (1+x)^{-(1/2)-\nu} (1-x)^{(1/2)-n+\nu}$ (where $(2\nu-1)!! \equiv 1 \cdot 3 \cdot 5 \cdot \dots \cdot (2\nu-1)$, $(-1)!! \equiv 1$)
$\arctan x$	$(-1)^{n-1} (n-1)! \sin^n \theta \sin n\theta \quad (\text{where } x = \cot \theta)$

$$\left(\frac{1}{f}\right)' = -\frac{f'}{f^2}, \quad \left(\frac{1}{f}\right)'' = \frac{2f'^2 - ff''}{f^3}, \quad \left(\frac{1}{f}\right)''' = \frac{6ff'f'' - 6f'^3 - f^2 f'''}{f^4}.$$

Higher-order derivatives of a composite function $g(t) \equiv f(x_1(t), \dots, x_n(t))$

$$\frac{dg}{dt} = \sum_{i=1}^n \frac{\partial f}{\partial x_i} \frac{dx_i}{dt}, \quad \frac{d^2g}{dt^2} = \sum_{i=1}^n \frac{\partial f}{\partial x_i} \frac{d^2x_i}{dt^2} + \sum_{i,j=1}^n \frac{\partial^2 f}{\partial x_i \partial x_j} \frac{dx_i}{dt} \frac{dx_j}{dt},$$

$$\frac{d^3g}{dt^3} = \sum_{i=1}^n \frac{\partial f}{\partial x_i} \frac{d^3x_i}{dt^3} + 2 \sum_{i,j=1}^n \frac{\partial^2 f}{\partial x_i \partial x_j} \frac{d^2x_i}{dt^2} \frac{dx_j}{dt} + \sum_{i,j,k=1}^n \frac{\partial^3 f}{\partial x_i \partial x_j \partial x_k} \frac{dx_i}{dt} \frac{dx_j}{dt} \frac{dx_k}{dt}.$$

For a function $z = z(x_1, \dots, x_n)$ determined implicitly by $F(z; x_1, \dots, x_n) = 0$, we have

$$\frac{\partial z}{\partial x_i} = -\frac{F_{x_i}}{F_z}, \quad \frac{\partial^2 z}{\partial x_i \partial x_j} = -\frac{F_{x_i x_j}}{F_z} + \frac{F_{x_i} F_{x_j z} + F_{x_j} F_{x_i z}}{F_z^2} - \frac{F_{x_i} F_{x_j} F_{zz}}{F_z^3}.$$

Schwarzian derivative:

$$\{y; x\} \equiv \left(\frac{d^3y}{dx^3} \right) / \left(\frac{dy}{dx} \right) - \frac{3}{2} \left[\left(\frac{d^2y}{dx^2} \right) / \left(\frac{dy}{dx} \right) \right]^2, \quad \{y; x\} = 0 \Leftrightarrow y = (ax + b)/(cx + d),$$

$$\{y; x\} = \left(\frac{dz}{dx} \right)^2 [\{y; z\} - \{x; z\}] = - \left(\frac{dy}{dx} \right)^2 \{x; y\}, \quad \{(ay + b)/(cy + d); x\} = \{y; x\}.$$

(IV) The Taylor Expansion and Remainder

If $f(x)$ is n times continuously differentiable in the interval $[a, b]$ (i.e., of class C^n),

$$f(b) = \sum_{\nu=0}^{n-1} \frac{(b-a)^\nu}{\nu!} f^{(\nu)}(a) + R_n \quad (\text{Taylor's formula}).$$

R_n is called the remainder, and is represented as follows:

$$R_n = \frac{1}{(n-1)!} \int_a^b (b-x)^{n-1} f^{(n)}(x) dx = \frac{(b-a)^p (b-\xi)^{n-p}}{(n-1)!p} f^{(n)}(\xi)$$

$$= \frac{(b-a)^n}{(n-1)!p} (1-\theta)^{n-p} f^{(n)}(a+\theta(b-a)) \quad (n \geq p > 0, \quad 0 < \theta < 1, \quad a < \xi < b, \quad \xi = a + \theta(b-a))$$

(Roche-Schlömilch remainder);

$$= \frac{1}{n!} (b-a)^n f^{(n)}(\xi) \quad (\text{Lagrange's remainder});$$

$$= \frac{1}{(n-1)!} (b-a)(b-\xi)^{n-1} f^{(n)}(\xi) \quad (\text{Cauchy's remainder}).$$

If $f(x, y)$ is m times continuously differentiable in a neighborhood of a point (x_0, y_0) ,

$$\begin{aligned} f(x_0 + h, y_0 + k) &= \sum_{\lambda=0}^{m-1} \frac{1}{\lambda!} \left(h \frac{\partial}{\partial x} + k \frac{\partial}{\partial y} \right)^\lambda f(x_0, y_0) + R_m \\ &= \sum_{0 \leq \mu + \nu \leq m-1; \mu, \nu \geq 0} \frac{1}{\mu! \nu!} h^\mu k^\nu \frac{\partial^{\mu+\nu} f(x_0, y_0)}{\partial x^\mu \partial y^\nu} + R_m, \end{aligned}$$

$$R_m = \frac{1}{m!} \left(h \frac{\partial}{\partial x} + k \frac{\partial}{\partial y} \right)^m f(x_0 + \theta h, y_0 + \theta k) \quad (0 < \theta < 1).$$

If all partial derivatives up to order $m-1$ are totally differentiable,

$$\begin{aligned} f(x_1 + h_1, \dots, x_n + h_n) &= \sum_{\nu=0}^{m-1} \frac{1}{\nu!} \left(\sum_{\mu=1}^n h_\mu \frac{\partial}{\partial x_\mu} \right)^\nu f(x_1, \dots, x_n) + R_m \\ &= \sum \frac{1}{\nu_1! \dots \nu_n!} h_1^{\nu_1} \dots h_n^{\nu_n} \frac{\partial^{\nu_1 + \dots + \nu_n} f(x_1, \dots, x_n)}{\partial x_1^{\nu_1} \dots \partial x_n^{\nu_n}} + R_m, \end{aligned}$$

where the Σ means the sum for ν_1, \dots, ν_n in the domain $0 \leq \nu_1 + \dots + \nu_n \leq m-1; \nu_1, \dots, \nu_n \geq 0$.

The remainder R_m is expressed as

$$R_m = \frac{1}{m!} \left(\sum_{\mu=1}^n h_\mu \frac{\partial}{\partial x_\mu} \right)^m f(x_1 + \theta h_1, \dots, x_n + \theta h_n) \quad (0 < \theta < 1).$$

(V) Definite Integrals [4]

In the following formulas, we assume that m, n are positive integers. δ_{mn} is Kronecker's delta ($\delta_{mn} = 0$ or 1 for $m \neq n$ or $m = n$), Γ is the gamma function, B is the beta function, and C is the Euler constant.

For simplicity, we put

$$m!! \equiv \begin{cases} 1 \cdot 3 \cdot 5 \cdot \dots \cdot (m-2) \cdot m = 2^{(m+1)/2} \Gamma[(m/2) + 1] / \sqrt{\pi} = m! / 2^{(m-1)/2} [(m-1)/2]! & (m \text{ is odd}), \\ 2 \cdot 4 \cdot 6 \cdot \dots \cdot (m-2) \cdot m = 2^{m/2} \Gamma[(m/2) + 1] = 2^{m/2} (m/2)! & (m \text{ is even}). \end{cases}$$

In an n -dimensional real space, the volume of the domain

$$|x_1|^p + \dots + |x_n|^p \leq 1 \quad (p > 0) \quad \text{is} \quad \frac{2^n [\Gamma(1/p)]^n}{p^{n-1} n \Gamma(n/p)}.$$

For $p=2$, this is the volume of the unit hypersphere, which is

$$\frac{\pi^{n/2}}{\Gamma[(n/2) + 1]} = \begin{cases} (2\pi)^{n/2} / n!! & (n \text{ is even}), \\ 2(2\pi)^{(n-1)/2} / n!! & (n \text{ is odd}). \end{cases}$$

The surface area of the $(n-1)$ -dimensional unit hypersphere

$$|x_1|^2 + \dots + |x_n|^2 = 1 \quad \text{is} \quad \frac{2\pi^{n/2}}{\Gamma(n/2)} = \begin{cases} (2\pi)^{n/2} / (n-2)!! & (n \text{ is even}), \\ 2(2\pi)^{(n-1)/2} / (n-2)!! & (n \text{ is odd}). \end{cases}$$

$$\int_0^\infty x^{p-1} e^{-x} dx = \int_0^1 \left(\log \frac{1}{x} \right)^{p-1} dx \equiv \Gamma(p).$$

$$\int_0^1 x^p \left(\log \frac{1}{x} \right)^q dx = \frac{\Gamma(q+1)}{(p+1)^{q+1}} \quad (\operatorname{Re} p, \operatorname{Re} q > -1).$$

$$\int_0^1 x^{p-1} (1-x)^{q-1} dx = \int_0^\infty \frac{x^{p-1}}{(1+x)^{p+q}} dx \equiv B(p, q) = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}.$$

$$\int_0^\infty \frac{x^a}{(1+x^c)^{1+b}} dx = \frac{1}{c} \frac{\Gamma[(a+1)/c] \Gamma[b - \{(a-c+1)/c\}]}{\Gamma(1+b)} \quad \left(\operatorname{Re} c > 0; \operatorname{Re} a, \operatorname{Re} b > -1; \operatorname{Re} b > \operatorname{Re} \frac{a-c+1}{c} \right).$$

$$\int_0^a \frac{(a-x)^{p-1} (x-b)^{q-1}}{|x-c|^{p+q}} dx = \frac{(a-b)^{p+q-1}}{|a-c|^q |b-c|^p} B(p, q) \quad (0 < c < b < a \text{ or } 0 < b < a < c; \operatorname{Re} p, \operatorname{Re} q > 0).$$

$$\int_{-\infty}^\infty \frac{1}{(1+x^2)^{n+1}} dx = \frac{\pi(2n)!}{2^{2n}(n!)^2} = \pi \frac{(2n-1)!!}{(2n)!!}.$$

$$\int_{-\infty}^\infty \frac{x^{2m}}{1+x^{2n}} dx = \frac{\pi}{n \sin[(2m+1)\pi/2n]} \quad (2m+1 < 2n).$$

$$\int_0^\infty \frac{x^{a-1}}{1+x} dx = \Gamma(a)\Gamma(1-a) = \frac{\pi}{\sin a\pi} \quad (0 < a < 1).$$

$$\int_0^\infty e^{-a^2 x^2} dx = \frac{\sqrt{\pi}}{2|a|}. \quad \int_0^\infty x^p e^{-x^2} dx = \frac{1}{2} \Gamma\left(\frac{p+1}{2}\right) = \begin{cases} (2n-1)!! \sqrt{\pi} / 2^{n+1} & (p=2n), \\ n! / 2 & (p=2n+1). \end{cases}$$

$$\int_0^\infty (e^{-a^2/x^2} - e^{-b^2/x^2}) dx = (b-a)\sqrt{\pi} \quad (a, b \geq 0). \quad \int_0^\infty e^{-[x-(1/x)]^2} dx = \frac{\sqrt{\pi}}{2}.$$

$$\int_0^\infty e^{-x^2 - (a^2/x^2)} dx = \frac{e^{-2a\sqrt{\pi}}}{2} \quad (a \geq 0). \quad \int_0^\infty \frac{1}{e^{ax} + e^{-ax}} dx = \frac{\pi}{4a}.$$

$$\int_0^\infty \frac{x}{e^{ax} - e^{-ax}} dx = \frac{\pi^2}{8a^2} \quad (a > 0).$$

$$\int_0^\infty \frac{x}{e^x - e^{-x}} dx = \int_0^1 \frac{\log(1/x)}{1-x^2} dx = \frac{\pi^2}{8}.$$

$$\begin{aligned}
 \int_0^\infty \frac{x}{e^x+1} dx &= \int_0^1 \frac{\log(1/x)}{1+x} dx = \frac{\pi^2}{12}, \quad \int_0^\infty \frac{x}{e^x-1} dx = \int_0^1 \frac{\log(1/x)}{1-x} dx = \frac{\pi^2}{6}. \\
 \int_0^\infty \log\left(\frac{e^x+1}{e^x-1}\right) dx &= \int_0^1 \log\left(\frac{1+x}{1-x}\right) \frac{1}{x} dx = \frac{\pi^2}{4}. \\
 \int_0^1 \frac{\log x}{\sqrt{1-x^2}} dx &= \int_0^{\pi/2} \log \sin x dx = -\frac{\pi}{2} \log 2. \\
 \int_0^1 \frac{\log(1+x)}{1+x^2} dx &= \frac{\pi}{8} \log 2, \quad \int_0^1 \frac{\log x}{1+x^2} dx = -\int_1^\infty \frac{\log x}{1+x^2} dx = \sum_{n=0}^\infty \frac{(-1)^{n-1}}{(2n+1)^2} = -0.91596\dots \\
 \int_0^\infty \frac{(\log x)^2}{1+x+x^2} dx &= \frac{16\pi^3}{81\sqrt{3}}, \quad \int_0^1 \frac{x^p-x^q}{\log x} dx = \log \frac{p+1}{q+1} \quad (p, q > -1). \\
 \int_0^1 \log|\log x| dx &= \int_0^\infty e^{-t} \log t dt = -C = -0.57721\dots \\
 \int_0^\pi \sin mx \sin nx dx &= \int_0^\pi \cos mx \cos nx dx = \delta_{mn} \frac{\pi}{2}. \\
 \int_0^\pi \sin mx \cos nx dx &= \begin{cases} [1-(-1)^{m+n}] \frac{m}{m^2-n^2} & (m \neq n); \\ 0 & (m = n). \end{cases} \\
 \int_0^{\pi/2} \sin^p x \cos^q x dx &= \frac{1}{2} B\left(\frac{p+1}{2}, \frac{q+1}{2}\right) \quad (\operatorname{Re} p, \operatorname{Re} q > -\frac{1}{2}); \\
 &= \begin{cases} (\pi/2)(p-1)!!(q-1)!!/(p+q)!! & (p, q \text{ are even positive integers}), \\ (p-1)!!(q-1)!!/(p+q)!! & (p, q \text{ are positive integers not both even}). \end{cases} \\
 \int_0^{\pi/2} \sin^p x dx = \int_0^{\pi/2} \cos^p x dx &= \frac{\sqrt{\pi}}{2} \frac{\Gamma[(p+1)/2]}{\Gamma[(p/2)+1]} \quad (\operatorname{Re} p > -1); \\
 &= \begin{cases} (\pi/2)(2n-1)!!/(2n)!! & (p=2n), \\ (2n)!!/(2n+1)!! & (p=2n+1). \end{cases} \\
 \int_{-\infty}^\infty \sin(x^2) dx = \int_{-\infty}^\infty \cos(x^2) dx &= \int_0^\infty \frac{\sin x}{\sqrt{x}} dx = \int_0^\infty \frac{\cos x}{\sqrt{x}} dx = \sqrt{\frac{\pi}{2}} \quad (\text{Fresnel integral}). \\
 \int_0^\infty \frac{\sin ax}{x} dx = \frac{\pi}{2} \quad (a > 0), \quad \int_0^\infty \frac{\tan x}{x} dx &= \frac{\pi}{2} \\
 &\quad \left(\text{take Cauchy's principal value at } x = \left(n + \frac{1}{2}\right)\pi\right). \\
 \int_0^\infty \frac{\sin^{2n+1} x}{x} dx &= \frac{\pi}{2} \frac{(2n-1)!!}{(2n)!!}, \quad \int_0^\infty \frac{\sin^2 x}{x^2} dx = \frac{\pi}{2}, \quad \int_0^\infty \frac{\sin(x^2)}{x} dx = \frac{\pi}{4}. \\
 \int_0^\infty \frac{\sin qx}{x^p} dx &= \frac{\pi q^{p-1}}{2\Gamma(p)\sin(p\pi/2)} \quad (0 < p < 2). \\
 \int_0^\infty \frac{\cos px}{1+x^2} dx &= \frac{\pi}{2} e^{-|p|}, \quad \int_0^\infty \frac{\cos^2 ax}{1+x^2} dx = \frac{\pi}{4} (1+e^{-2a}) \quad (a > 0). \\
 \int_0^\infty \frac{\sin ax}{x(1+x^2)} dx &= \frac{\pi}{2} (1-e^{-a}) \quad (a > 0), \quad \int_0^\infty \frac{x \sin ax}{1+x^2} dx = \frac{\pi}{2} e^{-a} \quad (a > 0). \\
 \int_0^\infty \frac{\sin^{2m+1} x \cos^{2n} x}{x} dx &= \int_0^\infty \frac{\sin^{2m+1} x \cos^{2n-1} x}{x} dx = \frac{\pi}{2} \frac{(2m-1)!!(2n-1)!!}{(2m+2n)!!}. \\
 \int_0^\infty \frac{\sin ax \cos bx}{x} dx &= \begin{cases} \pi/2 & (a > b > 0), \\ \pi/4 & (a = b > 0), \\ 0 & (b > a > 0) \end{cases} \quad (\text{Dirichlet's discontinuous factor}). \\
 \int_0^{2\pi} \frac{1}{1+a \cos x} dx &= \frac{2\pi}{\sqrt{1-a^2}} \quad (|a| < 1), \quad \int_0^{\pi/2} \frac{1}{a^2 \cos^2 x + b^2 \sin^2 x} dx = \frac{\pi}{2ab} \quad (ab \neq 0). \\
 \int_0^\pi \frac{x \sin x}{1+\cos^2 x} dx &= \frac{\pi^2}{4}.
 \end{aligned}$$

$$\int_0^\pi \frac{\cos nx}{1-2a \cos x + a^2} dx = \begin{cases} \pi a^n / (1-a^2) & (|a| < 1), \\ \pi / a^n (a^2-1) & (|a| > 1). \end{cases}$$

References

- [1] B. O. Peirce, A short table of integrals, Ginn, Boston, second revised edition, 1910.
 [2] D. Bierens de Haan, Nouvelles tables d'intégrales définies, Leiden, 1867.
 There are several mistakes in this table. For the errata, see
 [3] C. F. Lindmann, Examen des nouvelles tables de M. Bierens de Haan, Handlingar Svenska Vetenskaps-Akad., 1891.
 [4] E. W. Sheldon, Critical revision of de Haan's tables of definite integrals, Amer. J. Math., 34 (1912), 39-114.

10. Series (→ 379 Series)

(I) Finite Series

- (1) $S_k \equiv 1^k + 2^k + \dots + n^k$ (k is an integer). For $k \geq 0$, we have

$$S_k = \frac{B_{k+1}(n+1) - B_{k+1}(1)}{k+1} = \sum_{i=0}^k (-1)^i \binom{k+1}{i} \frac{B_{2i}(n+1)^{k+1-i}}{k+1},$$

where B_l is a Bernoulli number and $B_l(x)$ is a Bernoulli polynomial. In particular,

$$S_0 = n, \quad S_1 = n(n+1)/2, \quad S_2 = n(n+1)(2n+1)/6, \quad S_3 = n^2(n+1)^2/4,$$

$$S_4 = n(n+1)(2n+1)(3n^2+3n-1)/30.$$

For $k < 0$ and $k = -l$,

$$\begin{aligned} S_{-l} &= c_l - \left[(-1)^l / (l-1)! \right] \left[d^l \log \Gamma(x) / dx^l \right]_{x=n+1} \\ &= c_l - \frac{1}{(l-1)(n+1)^{l-1}} - \frac{1}{2(n+1)^l} + \sum_{i=1}^{\infty} (-1)^i \frac{B_{2(i+1)}}{(i+1)!} \frac{(l+i-1)!}{(l-1)!} \frac{1}{(n+1)^{l+i}}. \end{aligned}$$

For $l=1$, the second term in the latter formula is replaced by $\log[1/(n+1)]$. Here Γ is the gamma function, and the constants c_l are

$$c_l = \begin{cases} C \quad (\text{Euler constant}) & (l=1), \\ \zeta(l) \quad (\zeta \text{ is the Riemann zeta function}) & (l \geq 2). \end{cases}$$

$$(2) \quad \sum_{i=1}^n i(i+1)\dots(i+m-1) \equiv \sum_{i=1}^n \frac{(i+m-1)!}{(i-1)!} = \frac{1}{m+1} \frac{(n+m)!}{(n-1)!},$$

$$\sum_{i=1}^n \frac{(i-1)!}{(i+m-1)!} = \frac{1}{m-1} \left[\frac{1}{(m-1)!} - \frac{n!}{(n+m-1)!} \right] \quad (m \geq 2),$$

$$\sum_{i=1}^n i! i = (n+1)! - 1, \quad \sum_{i=1}^n i \binom{n}{i} = n2^{n-1},$$

$$\sum_{i=m}^n \binom{i}{m} \binom{n+s-i-1}{n-i} = \binom{n+s}{m+s} \quad (m \leq n),$$

$$\sum_{i=0}^n \binom{n}{i} \binom{m}{r-i} = \binom{n+m}{r}.$$

$$\sum_{i=1}^n a^i = \begin{cases} a(a^n-1)/(a-1) & (a \neq 1) \\ n & (a = 1) \end{cases} \quad (\text{geometric progression})$$

$$\sum_{j=0}^n (a+jd) = (n+1)a + \frac{n(n+1)}{2} d = \frac{n+1}{2} (a+a+nd) \quad (\text{arithmetic progression})$$

$$\sum_{j=0}^n \sin(\alpha+j\beta) = \sin\left(\alpha + \frac{n}{2}\beta\right) \sin \frac{(n+1)\beta}{2} \bigg/ \sin \frac{\beta}{2},$$

$$\sum_{j=0}^n \cos(\alpha + j\beta) = \cos\left(\alpha + \frac{n}{2}\beta\right) \sin \frac{(n+1)\beta}{2} / \sin \frac{\beta}{2},$$

$$\sum_{j=0}^n \operatorname{cosec} 2^j \alpha = \cot(\alpha/2) - \cot 2^n \alpha.$$

(II) Convergence Criteria for Positive Series $\sum a_n$

In the present Section II, we assume that $a_n \geq 0$.

Cauchy's criterion: The series converges when $\limsup \sqrt[n]{a_n} < 1$ and it diverges when $\limsup \sqrt[n]{a_n} > 1$.

d'Alembert's criterion: The series converges when $\limsup a_{n+1}/a_n < 1$ and diverges when $\liminf a_{n+1}/a_n > 1$.

Raabe's criterion: The series converges when $\liminf n[(a_n/a_{n+1}) - 1] > 1$ and diverges when $\limsup n[(a_n/a_{n+1}) - 1] < 1$.

Kummer's criterion: For a positive divergent series $\sum(1/b_n)$, the series $\sum a_n$ converges when $\liminf [(b_n a_n/a_{n+1}) - b_{n+1}] > 0$ and diverges when $\limsup [(b_n a_n/a_{n+1}) - b_{n+1}] < 0$ diverges.

Gauss's criterion: Suppose $a_n/a_{n+1} = 1 + (k/n) + (\theta_n/n^2)$, where $\lambda > 1$ and $\{\theta_n\}$ is bounded. Then the series $\sum a_n$ converges when $k > 1$; and diverges when $k \leq 1$.

Schlömilch's criterion: For a decreasing positive sequence $a_n \downarrow 0$, let n_v be an increasing sequence of positive integers and suppose that $(n_{v+2} - n_{v+1})/(n_{v+1} - n_v)$ is bounded. Then the two series $\sum a_n$ and $\sum (n_{v+1} - n_v) a_{n_v}$ converge or diverge simultaneously.

Logarithmic criterion: For a positive integer k , we put

$$\log_k x \equiv \log(\log_{k-1} x), \quad \log_1 x = \log x.$$

Then for sufficiently large n we have

The first logarithmic criterion: If

$$a_n - 1/(n \log_1 n \dots \log_{k-1} n (\log_k n)^p) \begin{cases} \leq 0, & p > 1 \\ \geq 0, & p \leq 1 \end{cases} \begin{cases} \text{then } \sum a_n \text{ converges,} \\ \text{then } \sum a_n \text{ diverges.} \end{cases}$$

The second logarithmic criterion: If

$$\frac{a_{n+1}}{a_n} - \frac{n}{n+1} \frac{\log_1 n}{\log_1(n+1)} \dots \frac{\log_{k-1} n}{\log_{k-1}(n+1)} \left(\frac{\log_k n}{\log_k(n+1)} \right)^p \begin{cases} \leq 0, & p > 1 \\ \geq 0, & p \leq 1 \end{cases} \begin{cases} \text{then } \sum a_n \text{ converges,} \\ \text{then } \sum a_n \text{ diverges.} \end{cases}$$

(III) Infinite Series

$$\sum_{i=1}^{\infty} \frac{(-1)^{i-1}}{i} = \log 2, \quad \sum_{i=1}^{\infty} \frac{(-1)^{i-1}}{2i-1} = \frac{\pi}{4} \quad (\text{Leibniz's formula}),$$

$$\sum_{i=1}^{\infty} \frac{(-1)^{i-1}}{2i(2i+1)(2i+2)} = \frac{\pi-3}{4},$$

$$\sum_{i=0}^{\infty} \frac{(2i)!}{2^{2i}(i!)^2} \frac{1}{2i+1} = \frac{\pi}{2}, \quad \sum_{i=1}^{\infty} \left(\frac{1}{i} - \log\left(1 + \frac{1}{i}\right) \right) = C \quad (C \text{ is Euler's constant}).$$

Putting

$$\zeta(n) = \sum_{i=1}^{\infty} \frac{1}{i^n}, \quad \alpha(n) = \sum_{i=1}^{\infty} \frac{1}{(2i-1)^n}, \quad \beta(n) = \sum_{i=1}^{\infty} \frac{(-1)^{i-1}}{i^n}, \quad \varepsilon(n) = \sum_{i=1}^{\infty} \frac{(-1)^{i-1}}{(2i-1)^n},$$

we have

$$\zeta(2n) = \frac{(2\pi)^{2n}}{2(2n)!} B_{2n}, \quad \alpha(2n) = \frac{(2^{2n}-1)\pi^{2n}}{2(2n)!} B_{2n},$$

$$\beta(2n) = \frac{(2^{2n-1}-1)\pi^{2n}}{(2n)!} B_{2n}, \quad \varepsilon(2n+1) = \frac{\pi^{2n+1}}{2^{2n+2}(2n)!} E_{2n}.$$

where B_n is a Bernoulli number, and E_n is an Euler number.

$$\zeta(2) = \pi^2/6, \quad \zeta(4) = \pi^4/90, \quad \zeta(6) = \pi^6/945.$$

$$\alpha(2) = \pi^2/8, \quad \alpha(4) = \pi^4/96, \quad \alpha(6) = \pi^6/960.$$

$$\beta(2) = \pi^2/12, \quad \beta(4) = 7\pi^4/720, \quad \beta(6) = 31\pi^6/30240.$$

$$\varepsilon(1) = \pi/4, \quad \varepsilon(2) = 0.915965594177219 \ 015054603514932\dots \quad (\text{Catalan's constant}),$$

$$\varepsilon(3) = \pi^3/32, \quad \varepsilon(5) = 5\pi^5/1536, \quad \varepsilon(7) = 61\pi^7/92160.$$

(IV) Power Series (\rightarrow 339 Power Series)

(1) Binomial Series $(1+x)^\alpha = \sum_{i=0}^{\infty} \binom{\alpha}{i} x^i$. This converges always in $|x| < 1$. If $\alpha > 0$, it converges in $-1 \leq x \leq 1$, and if $-1 < \alpha < 0$, it converges in $-1 < x \leq 1$. When α is 0 or a positive integer, it reduces to a polynomial and converges in $|x| < \infty$.

$$\frac{1}{1+x} = \sum_{i=0}^{\infty} (-1)^i x^i \quad (|x| < 1),$$

$$\sqrt{1+x} = \sum_{i=0}^{\infty} \frac{(-1)^{i-1}(2i)!}{(2i-1)2^{2i}(i!)^2} x^i \quad (|x| < 1), \quad \frac{1}{\sqrt{1+x}} = \sum_{i=0}^{\infty} \frac{(-1)^i(2i)!}{2^{2i}(i!)^2} x^i \quad (|x| < 1).$$

(2) Elementary Transcendental Functions (\rightarrow 131 Elementary Functions).

$$e^x \equiv \exp x = \sum_{i=0}^{\infty} \frac{x^i}{i!} = \lim_{n \rightarrow \infty} \left(1 + \frac{x}{n}\right)^n, \quad a^x = \exp(x \log a) \quad (|x| < \infty).$$

$$\log(1+x) = \sum_{i=1}^{\infty} \frac{(-1)^{i-1}}{i} x^i \quad (-1 < x \leq 1), \quad \log x = 2 \sum_{i=0}^{\infty} \frac{1}{2i+1} \left(\frac{x-1}{x+1}\right)^{2i+1} \quad (0 < x < \infty).$$

$$\sin x = \sum_{i=0}^{\infty} \frac{(-1)^i}{(2i+1)!} x^{2i+1}, \quad \cos x = \sum_{i=0}^{\infty} \frac{(-1)^i}{(2i)!} x^{2i} \quad (|x| < \infty).$$

$$\tan x = \sum_{i=1}^{\infty} \frac{2^{2i}(2^{2i}-1)B_{2i}}{(2i)!} x^{2i-1} \quad \left(|x| < \frac{\pi}{2}\right) \quad (B_i \text{ is a Bernoulli number}),$$

$$\cot x = \frac{1}{x} - \sum_{i=1}^{\infty} \frac{2^{2i}B_{2i}}{(2i)!} x^{2i-1} \quad \left(0 < |x| < \frac{\pi}{2}\right),$$

$$\sec x = \sum_{i=0}^{\infty} \frac{E_{2i}}{(2i)!} x^{2i} \quad \left(|x| < \frac{\pi}{2}\right) \quad (E_i \text{ is an Euler number}),$$

$$\operatorname{cosec} x = \frac{1}{x} + \sum_{i=1}^{\infty} \frac{(2^{2i}-2)B_{2i}}{(2i)!} x^{2i-1} \quad (0 < |x| < \pi).$$

$$\arcsin x = \sum_{i=0}^{\infty} \frac{(2i)!}{2^{2i}(i!)^2} \frac{x^{2i+1}}{2i+1} \quad (|x| \leq 1), \quad \arctan x = \sum_{i=0}^{\infty} \frac{(-1)^i}{2i+1} x^{2i+1} \quad (|x| \leq 1).$$

(V) Partial Fractions for Elementary Functions

$$\begin{aligned}\tan x &= \sum_{n=0}^{\infty} \frac{8x}{(2n+1)^2\pi^2 - 4x^2}, \quad \cot x = \frac{1}{x} + 2x \sum_{n=1}^{\infty} \frac{1}{x^2 - n^2\pi^2}, \\ \sec x &= 4 \sum_{n=1}^{\infty} \frac{(-1)^n(2n-1)\pi}{(2n-1)^2\pi^2 - 4x^2}, \quad \operatorname{cosec} x = \frac{1}{x} + 2x \sum_{n=1}^{\infty} \frac{(-1)^n}{x^2 - n^2\pi^2}, \\ \sec^2 x &= \sum_{n=-\infty}^{\infty} \frac{1}{[x + \{(2n+1)\pi/2\}]^2}, \quad \operatorname{cosec}^2 x = \sum_{n=-\infty}^{\infty} \frac{1}{(x + n\pi)^2}.\end{aligned}$$

(VI) Infinite Products (\rightarrow 379 Series F)

$$\prod_{n=1}^{\infty} \frac{4n^2}{4n^2 - 1} = \frac{\pi}{2} \quad (\text{Wallis formula}), \quad \prod_{n=1}^{\infty} \left(1 + \frac{x}{a+n}\right) e^{-x/n} = e^{-Cx} \frac{\Gamma(1+a)}{\Gamma(1+a+x)}$$

(C is Euler's constant).

$$\prod_{n=1}^{\infty} \left(1 - \frac{x}{2n-1}\right) \left(1 + \frac{x}{2n}\right) = \sqrt{\pi} / \Gamma\left(1 + \frac{x}{2}\right) \Gamma\left(\frac{1}{2} - \frac{x}{2}\right).$$

$$\prod_p 1/(1-p^{-s}) = \zeta(s) \quad (p \text{ ranges over all prime numbers, } s > 1),$$

$$\prod_{n=1}^{\infty} \left(1 - \frac{x^2}{n^2\pi^2}\right) = \frac{\sin x}{x}, \quad \prod_{n=1}^{\infty} \cos \frac{x}{2^n} = \frac{\sin x}{x}, \quad \prod_{n=1}^{\infty} \left(1 - \frac{4x^2}{(2n-1)^2\pi^2}\right) = \cos x.$$

$$\text{For } |q| < 1, \text{ putting } q_1 \equiv \prod_{n=1}^{\infty} (1 + q^{2n}), \quad q_2 \equiv \prod_{n=1}^{\infty} (1 + q^{2n-1}), \quad q_3 \equiv \prod_{n=1}^{\infty} (1 - q^{2n-1}),$$

$$q_4 \equiv \prod_{n=1}^{\infty} (1 - q^{2n}) \quad \text{we have} \quad q_1 q_2 q_3 = 1.$$

Further, putting $q = e^{i\pi\tau}$, we have the following formulas concerning ϑ -functions (\rightarrow 134 Elliptic Functions):

$$\vartheta_4(0, \tau) = q_4 q_3^2, \quad \vartheta_2(0, \tau) = 2q^{1/4} q_4 q_1^2, \quad \vartheta_3(0, \tau) = q_4 q_2^2, \quad \vartheta_1'(0, \tau) = 2\pi q^{1/4} q_4^3.$$

11. Fourier Analysis

(I) Fourier Series (\rightarrow 159 Fourier Series)

$$\begin{aligned}(1) \text{ Fourier coefficients} \quad a_0 &= \frac{1}{a} \int_0^a f(x) dx, \quad a_n = \frac{2}{a} \int_0^a f(x) \cos \frac{n\pi x}{a} dx \\ b_n &= \frac{2}{a} \int_0^a f(x) \sin \frac{n\pi x}{a} dx.\end{aligned}$$

$$\text{Fourier cosine series} \quad a_0 + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{a} = \begin{cases} f(x) & (0 < x < a), \\ f(-x) & (-a < x < 0). \end{cases}$$

$$\text{Fourier sine series} \quad \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{a} = \begin{cases} f(x) & (0 < x < a), \\ -f(-x) & (-a < x < 0). \end{cases}$$

The next table shows the Fourier coefficients of the functions $F(x)$ directly in the following manner from a given function $f(x)$ on the interval $[0, a]$. For x in $[-a, 0]$ and when the cosine series $\{a_n\}$ is in question, we set $f(x) = f(-x)$, and when the sine series $\{b_n\}$ is in question we set $f(x) = -f(-x)$. Thus $f(x)$ is extended in two ways to functions on $[-a, a]$. The functions $F(x)$ are the periodic continuations of such functions. We remark that the sum of the Fourier series given by the Fourier coefficients in the right hand side has, in general, some singularities (discontinuity of the function or its higher derivatives, for example) at the points given by the integral multiples of a . We assume that μ is not an integer.

$f(x)$	a_0	$a_n \quad (n=1, 2, \dots)$	$b_n \quad (n=1, 2, \dots)$
1	1	0	$[1 + (-1)^{n+1}]2a/n\pi$
x	$\frac{a}{2}$	$[1 + (-1)^{n+1}]\frac{-2a}{n^2\pi^2}$	$(-1)^{n+1}\frac{2a}{n\pi}$
x^2	$\frac{a^2}{3}$	$(-1)^n\frac{4a^2}{n^2\pi^2}$	$(-1)^{n-1}\frac{2a^2}{n\pi} - [1 + (-1)^{n+1}]\frac{4a^2}{n^3\pi^3}$
e^{kx}	$\frac{e^{ka} - 1}{ka}$	$\frac{2ka[(-1)^n e^{ka} - 1]}{k^2a^2 + n^2\pi^2}$	$\frac{2n\pi[1 - (-1)^n]e^{ka}}{k^2a^2 + n^2\pi^2}$
$\cos \frac{\mu\pi x}{a}$	$\frac{\sin \mu\pi}{\mu\pi}$	$(-1)^n \frac{2}{\pi} \frac{\mu \sin \mu\pi}{\mu^2 - n^2}$	$\frac{2}{\pi} \frac{[(-1)^n \cos \mu\pi - 1]}{\mu^2 - n^2}$
$\sin \frac{\mu\pi x}{a}$	$\frac{1 - \cos \mu\pi}{\mu\pi}$	$\frac{2}{\pi} \frac{\mu[1 - (-1)^n \cos \mu\pi]}{\mu^2 - n^2}$	$(-1)^n \frac{2}{\pi} \frac{n \sin \mu\pi}{\mu^2 - n^2}$
$\frac{1 - \lambda^2}{1 - 2\lambda \cos(\pi x/a) + \lambda^2}$	1	$2\lambda^n \quad (\lambda < 1)$	
$\frac{\lambda \sin(\pi x/a)}{1 - 2\lambda \cos(\pi x/a) + \lambda^2}$			$\lambda^n \quad (\lambda < 1)$
$B_{2m}(x/2a)$	0	$(-1)^{m+1}2(2m)!/(2n\pi)^{2m}$	
$B_{2m+1}(x/a)$			$(-1)^{m+1}2(2m+1)!/(2n\pi)^{2m+1}$
$\log \sin(\pi x/2a)$	$-\log 2$	$-1/n$	
$(1/2)\cot(\pi x/2a)$			$1^{(1)}$

Note

(1)The Fourier series does not converge in the sense of Cauchy, but it is summable, for example, by the Cesàro summation of the first order.

$$\begin{aligned}
 (2) \quad & \sum_{n=1}^{\infty} (-1)^{n-1} \frac{\cos nx}{n} = \log\left(2 \cos \frac{x}{2}\right) \quad (-\pi < x < \pi), \quad \sum_{n=1}^{\infty} \frac{\sin nx}{n} = \frac{1}{2}(\pi - x) \quad (0 < x < 2\pi). \\
 & \sum_{n=1}^{\infty} \frac{\cos(2n-1)x}{2n-1} = \frac{1}{2} \log \left| \cot \frac{x}{2} \right| \quad (0 < x < 2\pi, \quad x \neq \pi), \\
 & \sum_{n=1}^{\infty} \frac{\sin(2n-1)x}{2n-1} = \begin{cases} \pi/4 & (0 < x < \pi), \\ -\pi/4 & (\pi < x < 2\pi). \end{cases} \\
 & \sum_{n=1}^{\infty} \frac{\cos nx}{n^2} = \frac{1}{4}(x - \pi)^2 - \frac{\pi^2}{12} \quad (0 \leq x \leq 2\pi), \\
 & \sum_{n=1}^{\infty} \frac{\sin nx}{n^2} = -x \log 2 - \int_0^x \log\left(\sin \frac{t}{2}\right) dt \quad (0 \leq x < 2\pi). \\
 & \sum_{n=1}^{\infty} \frac{a^n}{n!} \cos nx = e^{a \cos x} \cos(a \sin x) - 1, \quad \sum_{n=1}^{\infty} \frac{a^n}{n!} \sin nx = e^{a \cos x} \sin(a \sin x). \\
 & \sum_{n=1}^{\infty} (-1)^{n-1} \frac{\cos nx}{n^2 - a^2} = \frac{\pi \cos ax}{2a \sin a\pi} - \frac{1}{2a^2} \quad (-\pi \leq x \leq \pi), \\
 & \sum_{n=1}^{\infty} (-1)^{n-1} \frac{n \sin nx}{n^2 - a^2} = \frac{\pi \sin ax}{2 \sin a\pi} \quad (-\pi < x < \pi).
 \end{aligned}$$

In the final two formulas, we assume that a is not an integer.

(II) Fourier Transforms (\rightarrow 160 Fourier Transform)

The Fourier transform $\mathcal{F}[f]$ and the inverse Fourier transform $\mathcal{F}^{-1}[g]$ for integrable functions f and g are defined as

$$\mathcal{F}[f(x)] = \mathcal{F}[f](\xi) = (2\pi)^{-n/2} \int_{\mathbf{R}^n} f(x) e^{-ix\xi} dx,$$
$$\mathcal{F}[g(\xi)] = \mathcal{F}[g](x) = (2\pi)^{-n/2} \int_{\mathbf{R}^n} g(\xi) e^{ix\xi} d\xi, \quad x\xi = x_1\xi_1 + x_2\xi_2 + \dots + x_n\xi_n.$$

In some textbooks the factor $(2\pi)^{-n/2}$ is deleted or the symbols i and $-i$ are switched when defining \mathcal{F} and \mathcal{F} . However, conversion of the formulas above to ones due to other definitions is straightforward. These transforms are also defined for some nonintegrable functions, or even more generally for tempered distributions. The Fourier transform \mathcal{F} and the inverse Fourier transform \mathcal{F} defined on the space of tempered distributions $\mathcal{S}' = \mathcal{S}'(\mathbf{R}^n)$ are linear homeomorphic mappings from \mathcal{S}' to itself. Useful formulas of these transforms are given in the table below, where $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n)$ ($\alpha_j = 0, 1, 2, \dots$), $|\alpha| = \alpha_1 + \alpha_2 + \dots + \alpha_n$, C is Euler's constant, $\lambda \in \mathbf{C}$, and $\mathbf{Z}_+ = \{m \in \mathbf{Z} | m \geq 0\}$.

Case 1. $n = 1$. First we explain the meaning of the symbols appearing in the table:

$$x_+ = \max(x, 0) \quad (\text{the positive part of } x),$$
$$x_- = \max(-x, 0) \quad (\text{the negative part of } x),$$
$$x_+^\lambda \text{ and } x_-^\lambda \text{ are understood in the sense of finite parts } (\rightarrow 125 \text{ Distributions and Hyperfunctions}),$$
$$(x + i\varepsilon)^\lambda = \exp[\lambda \operatorname{Log}(x + i\varepsilon)] \quad (\varepsilon \neq 0; \operatorname{Log} \text{ is the principal value of } \log)$$
$$= (x^2 + \varepsilon^2)^{\lambda/2} \exp[i\lambda \operatorname{Arg}(x + i\varepsilon)] \quad (-\pi < \operatorname{Arg} z \leq \pi),$$
$$(x \pm i0)^\lambda = \lim_{\varepsilon \downarrow 0} (x \pm i\varepsilon)^\lambda \quad (\text{limit in the sense of distributions}).$$

Then the following formula holds:

$$(x \pm i0)^\lambda = x_+^\lambda + e^{\pm i\lambda\pi} x_-^\lambda$$
$$\operatorname{Pf} x^m = x_+^m + (-1)^m x_-^m \quad (m \in \mathbf{Z}) \quad (\operatorname{Pf} \text{ is the finite part}).$$

In the special case $m = -1$, $\operatorname{Pf} x^{-1}$ coincides with Cauchy's principal value p.v. x^{-1} .

$T \in \mathcal{S}'$	$\mathcal{F}[T] \in \mathcal{S}'$
$\delta(x)$	$\sqrt{2\pi}$
$P(x)$ (polynomial)	$\sqrt{2\pi} P(i d/d\xi) \delta(\xi)$
p.v. $1/x$	$\sqrt{\pi/2} i \operatorname{sgn} \xi$
$\operatorname{Pf} x^{-m}$	$\sqrt{\pi/2} [(-i)^m / (m-1)!] \xi^{m-1} \operatorname{sgn} \xi \quad (m \in \mathbf{N}, \mathbf{Z})$
x_+^λ	$\frac{\Gamma(\lambda+1)}{\sqrt{2\pi}} [e^{-i\pi(\lambda+1)/2} \xi_+^{-\lambda-1} + e^{i\pi(\lambda+1)/2} \xi_-^{-\lambda-1}]$ $\left[\frac{\Gamma(\lambda+1)}{\sqrt{2\pi}} e^{-i\pi(\lambda+1)/2} (\xi + i0)^{-\lambda-1} \right] \quad (\lambda \notin \mathbf{Z})$
x_+^m	$(i^m / \sqrt{2\pi}) [\pi \delta^{(m)} - i(-1)^m m! \operatorname{Pf} \xi^{-m-1}] \quad (m \in \mathbf{Z}_+)$
x_+^{-m}	$\frac{(-i)^{m-1}}{\sqrt{2\pi(m-1)!}} \left[\left(\sum_{j=1}^{m-1} \frac{1}{j} - C \right) \xi^{m-1} - \frac{i\pi}{2} \xi^{m-1} \operatorname{sgn} \xi - \xi^{m-1} \log \xi \right] \quad (m \in \mathbf{N}, \mathbf{Z})$
x_-^λ	$\frac{\Gamma(\lambda+1)}{\sqrt{2\pi}} [e^{i\pi(\lambda+1)/2} \xi_+^{-\lambda-1} + e^{-i\pi(\lambda+1)/2} \xi_-^{-\lambda-1}]$ $\left[\frac{\Gamma(\lambda+1)}{\sqrt{2\pi}} e^{i\pi(\lambda+1)/2} (\xi - i0)^{-\lambda-1} \right] \quad (\lambda \notin \mathbf{Z})$
x_-^m	$\frac{(-i)^m}{\sqrt{2\pi}} [\pi \delta^{(m)} + i(-1)^m m! \operatorname{Pf} \xi^{-m-1}] \quad (m \in \mathbf{Z}_+)$
x_-^{-m}	$\frac{i^{m-1}}{\sqrt{2\pi(m-1)!}} \left[\left(\sum_{j=1}^{m-1} \frac{1}{j} - C \right) \xi^{m-1} + \frac{i\pi}{2} \xi^{m-1} \operatorname{sgn} \xi - \xi^{m-1} \log \xi \right] \quad (m \in \mathbf{N}, \mathbf{Z})$
$(x + i0)^\lambda$	$[\sqrt{2\pi} e^{i\pi\lambda/2} / \Gamma(-\lambda)] \xi_+^{-\lambda-1} \quad (\lambda \notin \mathbf{Z}_+)$
$(x - i0)^\lambda$	$[\sqrt{2\pi} e^{-i\pi\lambda/2} / \Gamma(-\lambda)] \xi_-^{-\lambda-1} \quad (\lambda \notin \mathbf{Z}_+)$
$(x \pm i0)^m = x^m$	$\sqrt{2\pi} i^m \delta^{(m)} \quad (m \in \mathbf{Z}_+)$
$x^{-1} \log x $	$\sqrt{\pi/2} i \operatorname{sgn} \xi \cdot (C + \log \xi)$
$e^{-x^2/\alpha}$	$\sqrt{a/2} e^{-a\xi^2/4} \quad (a > 0)$
$\begin{cases} e^{-ax} & (x > 0) \\ 0 & (x \leq 0) \end{cases}$	$\frac{1}{\sqrt{2\pi}} \frac{1}{a + i\xi} \quad (a > 0)$

$T (\in \mathcal{S}')$	$\mathcal{F}[T] (\in \mathcal{S}')$
$e^{-a x }$	$\frac{\sqrt{a + \sqrt{a^2 + \xi^2}}}{\sqrt{a^2 + \xi^2}} \quad (a > 0)$
$\sqrt{ x }$	$\frac{\sqrt{2}}{\pi} \frac{ab}{\sqrt{\xi^2 + b^2}} K_1(a\sqrt{\xi^2 + b^2}) \quad (a > 0, b > 0)$
$e^{-b\sqrt{x^2 + a^2}}$	$-\frac{\sqrt{2\pi}}{ \xi } (e^{-a \xi } - e^{-b \xi }) \quad (a \geq 0, b \geq 0)$
$\log \frac{x^2 + a^2}{x^2 + b^2}$	$-\sqrt{\pi/2} \operatorname{sgn} a \cdot (e^{- a \xi /\sqrt{2}} - e^{- b \xi /\sqrt{2}}) \quad (a \in \mathbf{R}, a \neq 0)$
$\arctan(x/a)$	$-\frac{i}{\sqrt{2\pi}} \Gamma(v) \cos \frac{v\pi}{2} \left(\frac{1}{ \xi - a ^v} - \frac{1}{ \xi + a ^v} \right) \quad (v \notin \mathbf{Z})$
$\frac{\sin ax}{ x ^{1-v}}$	$\frac{1}{\sqrt{2\pi}} \Gamma(v) \cos \frac{v\pi}{2} \left(\frac{1}{ \xi - a ^v} - \frac{1}{ \xi + a ^v} \right) \quad (v \notin \mathbf{Z})$
$\frac{\cos ax}{ x ^{1-v}}$	$\begin{cases} \sqrt{\pi/2} & (\xi < a) \\ 0 & (\xi > a) \end{cases}$
$\frac{\sin ax}{x}$	$\begin{cases} \frac{1}{\sqrt{2}} \Gamma((1/2) - v) \left(\frac{ \xi }{2a} \right)^v J_{-v}(a \xi) & (\operatorname{Re} v < 1/2, a > 0) \\ -\frac{1}{\sqrt{2}} \Gamma((1/2) - v) \left(\frac{ \xi }{2a} \right)^v N_v(a \xi) & (-1/2 < \operatorname{Re} v < 1/2, a > 0) \end{cases}$
$\frac{1}{(x^2 + a^2)^{v+(1/2)}}$	$\frac{\sqrt{2}}{\Gamma(v + (1/2))} \left(\frac{ \xi }{2a} \right)^v K_v(a \xi) \quad (\operatorname{Re} v > -1/2, a > 0)$

Case 2. $n > 1$. Let $r = \sqrt{x_1^2 + x_2^2 + \dots + x_n^2}$ and $\rho = \sqrt{\xi_1^2 + \xi_2^2 + \dots + \xi_n^2}$, where $x = (x_i) \in \mathbf{R}^n$ and $\xi = (\xi_i) \in \mathbf{R}^n$. If $f \in L_1(\mathbf{R}^n)$ depends only on r , then $\mathcal{F}[f]$ depends only on ρ and is expressed as

$$\mathcal{F}[f](\rho) = \rho^{-(n-2)/2} \int_0^\infty f(r) r^{n/2} J_{(n-2)/2}(r\rho) dr.$$

The constant C in the table stands for Euler's number.

$T (\in \mathcal{S}')$	$\mathcal{F}[T] (\in \mathcal{S}')$
$\delta(x)$	$(2\pi)^{n/2}$
$P(x)$ (polynomial)	$(2\pi)^{n/2} P(i\partial/\partial \xi) \delta(\xi)$
$\operatorname{Pf} r^\lambda$	$\frac{2^{(n/2)+\lambda} \Gamma((n+\lambda)/2)}{\Gamma(-\lambda/2)} \operatorname{Pf} \rho^{-n-\lambda} \quad (\lambda \notin 2\mathbf{Z}_+, \lambda \neq -n-2\mathbf{Z}_+)$
r^{2m}	$(2\pi)^{n/2} (-\Delta)^m \delta(\xi) \quad (m \in \mathbf{Z}_+)$
$\operatorname{Pf} r^{-n-2m}$	$\frac{(-1)^m \rho^{2m}}{2^{(n/2)+2m} \Gamma((n/2)+m)m!} \left[2 \log \frac{2}{\rho} - C + \sum_{j=1}^m \frac{1}{j} + \frac{\Gamma'((n/2)+m)}{\Gamma((n/2)+m)} \right] \quad (m \in \mathbf{Z}_+)$
$(1+r^2)^\lambda$	$\frac{\rho^{-(n/2)+\lambda} K_{(n/2)+\lambda}(\rho)}{2^{-\lambda-1} \Gamma(-\lambda)} \quad (\lambda \notin \mathbf{Z}_+)$
$(1+r^2)^m$	$(2\pi)^{n/2} (1-\Delta)^m \delta(\xi) \quad (m \in \mathbf{Z}_+)$
$\operatorname{Pf} r^\lambda \log \lambda$	$\frac{2^{(n/2)+\lambda} \Gamma((n+\lambda)/2)}{\Gamma(-\lambda/2)} \operatorname{Pf} \rho^{-n-\lambda} \left[\log \frac{2}{\rho} + \frac{1}{2} \frac{\Gamma'((n+\lambda)/2)}{\Gamma((n+\lambda)/2)} + \frac{1}{2} \frac{\Gamma'(-\lambda/2)}{\Gamma(-\lambda/2)} \right] \quad (\lambda \notin 2\mathbf{Z}_+, \lambda \neq -n-2\mathbf{Z}_+)$
$r^{2m} \log r$	$(-1)^{m-1} 2^{(n/2)+2m-1} m! \Gamma((n/2)+m) \operatorname{Pf} \rho^{-n-2m} + (2\pi)^{n/2} \left[\log 2 - \frac{1}{2} C + \frac{1}{2} \sum_{j=1}^m \frac{1}{j} + \frac{\Gamma'((n/2)+m)}{\Gamma((n/2)+m)} \right] (-\Delta)^m \delta(\xi) \quad (m \in \mathbf{Z}_+)$
$\operatorname{Pf} r^{-n-2m} \log r$	$\frac{(-1)^m}{2^{(n/2)+2m} \Gamma((n/2)+m)m!} \rho^{2m} \left[\left\{ \log \frac{2}{\rho} - \frac{1}{2} C + \frac{1}{2} \sum_{j=1}^m \frac{1}{j} + \frac{1}{2} \frac{\Gamma'((n/2)+m)}{\Gamma((n/2)+m)} \right\}^2 + \frac{\pi^2}{24} + \frac{1}{4} \sum_{j=1}^m \frac{1}{j^2} - \frac{1}{4} \frac{\Gamma''((n/2)+m)}{\Gamma((n/2)+m)} + \frac{\Gamma'((n/2)+m)^2}{\Gamma((n/2)+m)^2} \right] \quad (m \in \mathbf{Z}_+)$
e^{-ar}	$\frac{\sqrt{2^n}}{\sqrt{\pi}} \Gamma\left(\frac{n+1}{2}\right) \frac{a}{(a^2 + \rho^2)^{(n+1)/2}} \quad (a > 0)$

The Fourier transform mentioned above is a transformation in the family of complex-valued functions or distributions. Similar transformations in the family of real-valued functions are frequently used in applications:

Fourier cosine transform $f_c(u) = \int_0^\infty F(t) \cos ut \, dt.$

Inverse transform $\frac{2}{\pi} \int_0^\infty f_c(u) \cos ut \, du = \begin{cases} F(t) & (t > 0), \\ F(-t) & (t < 0). \end{cases}$

Fourier sine transform $f_s(u) = \int_0^\infty F(t) \sin ut \, dt.$

Inverse transform $\frac{2}{\pi} \int_0^\infty f_s(u) \sin ut \, du = \begin{cases} F(t) & (t > 0), \\ -F(-t) & (t < 0). \end{cases}$

The Fourier transform can be expressed in terms of these transforms. For example (in \mathbb{R}^1),

$$\mathcal{F}[f](u) = \frac{1}{\sqrt{2\pi}} \int_0^\infty [f(t) + f(-t)] \cos ut \, dt - \frac{i}{\sqrt{2\pi}} \int_0^\infty [f(t) - f(-t)] \sin ut \, dt.$$

$F(t)$	$f_c(u)$	$f_s(u)$
$\begin{cases} 1 & (0 < t < a) \\ 0 & (a < t) \end{cases}$	$\frac{\sin au}{u}$	$\frac{1 - \cos au}{u}$
t^{-1}	(diverges)	$(\pi/2) \operatorname{sgn} u$
$t^{\alpha-1} \quad (0 < \alpha < 1)$	$\Gamma(\alpha) \cos(\pi\alpha/2) u^{-\alpha}$	$\Gamma(\alpha) \sin(\pi\alpha/2) u^{-\alpha}$
$1/(a^2 + t^2)$	$\pi e^{-a u }/2a$	$[e^{-au} \operatorname{Ei}(au) - e^{au} \operatorname{Ei}(-au)]/a^{(2)}$
e^{-at}	$a/(a^2 + u^2)$	$u/(a^2 + u^2)$
$e^{-\lambda t^2} \quad (\operatorname{Re} \lambda > 0)$	$\sqrt{\pi/4\lambda} e^{-u^2/4\lambda}$	$e^{-u^2/4\lambda} \varphi(u/2\sqrt{\lambda})/\sqrt{\lambda} \quad (3)$
$e^{-\lambda t^2}$		$\sqrt{\pi/4\lambda} (u/4\lambda) e^{-u^2/4\lambda}$
$\frac{\sin at}{t} \quad (a > 0)$	$\begin{cases} \pi/2 & (0 < u < a) \\ 0 & (a < u) \end{cases}$	$\frac{1}{2} \log \left \frac{a+u}{a-u} \right $
$\tanh(\pi t/2)$		$\operatorname{cosech} u$
$\operatorname{sech}(\pi t/2)$	$\operatorname{sech} u$	
$J_\nu(t) \quad (\operatorname{Re} \nu > -1)$	$\begin{cases} \frac{\cos(\nu \arcsin u)}{\sqrt{1-u^2}} \\ -\frac{(u - \sqrt{u^2-1})^\nu}{\sqrt{u^2-1}} \sin \frac{\nu\pi}{2} \end{cases}$	$\begin{cases} \frac{\sin(\nu \arcsin u)}{\sqrt{1-u^2}} & (0 < u < 1) \\ \frac{(u - \sqrt{u^2-1})^\nu}{\sqrt{u^2-1}} \cos \frac{\nu\pi}{2} & (1 < u) \end{cases}$
$J_0(at)$	$\begin{cases} 1/\sqrt{a^2 - u^2} \\ 0 \end{cases}$	$\begin{cases} 0 & (0 \leq u < a) \\ 1/\sqrt{u^2 - a^2} & (a < u) \end{cases}$
$N_0(t)$	$\begin{cases} 0 \\ -\frac{1}{\sqrt{u^2-1}} \end{cases}$	$\begin{cases} \frac{2}{\pi} \frac{\arcsin u}{\sqrt{1-u^2}} & (0 < u < 1) \\ \frac{2}{\pi} \frac{\log(u - \sqrt{u^2-1})}{\sqrt{u^2-1}} & (1 < u) \end{cases}$
$K_0(t)$	$\pi/2 \sqrt{1+u^2}$	$(\operatorname{arc sinh} u)/\sqrt{1+u^2}$

Notes

(2) Ei is the exponential integral function (\rightarrow Table 19.II.3, this Appendix).

(3) We put $\varphi(x) = \int_0^x e^{t^2} dt$.

12. Laplace Transforms and Operational Calculus

(I) Laplace Transforms (\rightarrow 240 Laplace Transform)

Laplace transform $V(p) = \int_0^\infty e^{-pt} F(t) dt \quad (\operatorname{Re} p > 0).$

Inverse transform (Bromwich integral) $\frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} e^{pt} V(p) dp = \begin{cases} F(t) & (t > 0), \\ 0 & (t < 0). \end{cases}$

$F(t)$	$V(p)$	$F(t)$	$V(p)$
1	$1/p$	$J_\nu(t) \quad (\operatorname{Re} \nu > -1)$	$\frac{(\sqrt{1+p^2}-p)^\nu}{\sqrt{1+p^2}}$
$1(t-a)=\begin{cases} 0 & (0 \leq t < a) \\ 1 & (a \leq t) \end{cases}$	e^{-ap}/p	$\frac{1}{t}J_\nu(at) \quad (\operatorname{Re} \nu > 0)$	$\frac{(\sqrt{a^2+p^2}-p)^\nu}{\nu a^\nu}$
$[x/a] \quad (\text{integral part})$	$1/p(e^{ap}-1)$		
$t^{\alpha-1} \quad (\operatorname{Re} \alpha > 0)$	$\Gamma(\alpha)/p^\alpha$	$t^\nu J_\nu(at) \left(\operatorname{Re} \nu > -\frac{1}{2} \right)$	$\frac{(2a)^\nu \Gamma[\nu+(1/2)]}{\sqrt{\pi} (p^2+a^2)^{\nu+(1/2)}}$
e^{-at}	$1/(p+a)$		
$e^{-at}t^{\alpha-1} \quad (\operatorname{Re} \alpha > 0, \ a > 0)$	$(p+a)^{-\alpha}\Gamma(\alpha)$	$t^{\nu/2}J_\nu(x\sqrt{t}) \quad (\operatorname{Re} \nu > -1)$	$\frac{x^\nu}{2^\nu p^{\nu+1}}e^{-x^2/4p}$
$e^{-at}F(t) \quad (a > 0)$	$V(p+a)$		
$(1-e^{-t})/t$	$\log(1+p^{-1})$	$J_0(t)$	$(1+p^2)^{-1/2}$
$(\pi t)^{-1/2}e^{-x^2/4t}$	$p^{-1/2}e^{-x\sqrt{p}} \quad (x > 0)$	$J_0(x\sqrt{t})$	$e^{-x^2/4p}/p$
$\log t$	$-(\log p + C)/p^{(1)}$	$N_0(t)$	$\frac{2}{\pi} \frac{\log(\sqrt{1+p^2}-p)}{\sqrt{1+p^2}}$
$\sin at$	$a/(p^2+a^2)$	$L_n(t)^{(2)}$	$\frac{1}{p} \left(\frac{p-1}{p} \right)^n$
$\cos at$	$p/(p^2+a^2)$		
$\sin(x\sqrt{t})$	$\frac{\sqrt{\pi}}{2} \frac{x}{p^{3/2}} e^{-x^2/4p}$	$t^\alpha L_n^{(\alpha)}(t)$	$\frac{\Gamma(\alpha+n+1)}{n!} \frac{1}{p^{\alpha+1}} \left(\frac{p-1}{p} \right)^n$
$t^{-1/2} \cos(x\sqrt{t})$	$\sqrt{\pi/p} e^{-x^2/4p}$	$H_{2n+1}(\sqrt{t})^{(3)}$	$\sqrt{\frac{\pi}{2}} \frac{(2n+1)!!}{p^{n+(3/2)}} \frac{(1-p)^n}{p^{n+(3/2)}}^{(4)}$
$t^{-1} \sin xt$	$\arctan(x/p)$		
$x^{-1}(1-\cos ax)$	$\frac{1}{2} \log[1+(a^2/p^2)]$	$\frac{H_{2n}(\sqrt{t})}{\sqrt{t}}$	$\sqrt{\pi} (2n-1)!! \frac{(1-p)^n}{p^{n+(1/2)}}$
$\sinh at$	$a/(p^2-a^2)$		
$\cosh at$	$p/(p^2-a^2)$		

- Notes
- (1) C is Euler's constant.
 - (2) $L_n(t)$ is a Laguerre polynomial.
 - (3) $H_n(t)$ is a Hermite polynomial.
 - (4) $(2n+1)!! = (2n+1)(2n-1)(2n-3)\dots 5 \cdot 3 \cdot 1$.

(II) Operational Calculus (→ 306 Operational Calculus)

Heaviside function (unit function) $1(t) = \begin{cases} 0 & (t < 0) \\ 1 & (t \geq 0). \end{cases}$

Dirac delta function (unit impulse function) $\delta(t) = \lim_{\epsilon \rightarrow 0} \frac{1}{2\epsilon} [1(t+\epsilon) - 1(t-\epsilon)].$

When an operator $\Omega(p)$ operates on $1(t)$ and the result is $A(t)$ we write $\Omega(p)1(t) = A(t)$.
In the following table (i) of general formulas, we assume the relations $\Omega_i(p)1(t) = A_i(t)$ ($i = 1, 2$).

Carson's integral $\Omega(p) = p \int_0^\infty e^{-pt} A(t) dt \quad (\operatorname{Re} p > 0).$

Laplace transform $V(p) = \frac{\Omega(p)}{p} = \int_0^\infty e^{-pt} A(t) dt.$

(i) General Formulas		(ii) Examples	
$\Omega(p)$	$A(t)$	$\Omega(p)=pV(p)$	$A(t)$
$\Omega_1(p)+\Omega_2(p)$	$A_1(t)+A_2(t)$	p	$\delta(t)$
$a\Omega_1(p)$	$aA_1(t)$	$1/p^n \ (n=0,1,2,\dots)$	$(t^n/n!)\mathbf{1}(t)$
$p\Omega_1(p)$	$A_1(0)\delta(t)+A_1'(t)$	$p/(p+a)$	$(e^{-at})\mathbf{1}(t)$
		$p^2/(p^2+a^2)$	$(\cos at)\mathbf{1}(t)$
$\frac{1}{p}\Omega_1(p)$	$\int_0^t A_1(\tau) d\tau$	$ap/(p^2+a^2)$	$(\sin at)\mathbf{1}(t)$
$\Omega_1(ap)$	$A_1(t/a)$		
$[p/(p+a)]\Omega_1(p+a)$	$e^{-at}A_1(t) \ (\operatorname{Re} a \geq 0)$	$a_0+\frac{a_1}{p}+\frac{a_2}{p^2}+\dots$	$\left(a_0+a_1\frac{t}{1!}+a_2\frac{t^2}{2!}+\dots\right)\mathbf{1}(t)$
$\frac{1}{p}\Omega_1(p)\Omega_2(p)$	$\int_0^t A_1(\tau)A_2(t-\tau) d\tau$ $=\int_0^t A_1(t-\tau)A_2(\tau) d\tau$	$\sum_{k=1}^n \frac{B_k}{p-p_k}$	$\sum_{k=1}^n \frac{B_k}{p_k} (e^{p_k t}-1)\mathbf{1}(t)$ $=\Omega(0)\mathbf{1}(t)+\sum_{k=1}^n \frac{B_k}{p_k} e^{p_k t}$

13. Conformal Mappings (→ 77 Conformal Mappings)

Original Domain	Image Domain	Mapping Function
$ z <1$ (unit disk)	$ w <1$	$w=\varepsilon \frac{z-z_0}{1-\bar{z}_0 z}, \quad z_0 <1, \quad \varepsilon =1 \quad (\text{general form})$
$\operatorname{Im} z>0$ (upper half-plane)	$ w <1$	$w=\varepsilon \frac{z-z_0}{z-\bar{z}_0}, \quad \operatorname{Im} z_0>0, \quad \varepsilon =1 \quad (\text{general form})$
$\operatorname{Im} z>0$ (upper half-plane)	$\operatorname{Im} w>0$	$w=\frac{az+b}{cz+d}, \quad a,b,c,d \text{ are real; } ad-bc>0$ (general form)
$0<\arg z<\alpha$ (angular domain)	$\operatorname{Im} w>0$	$w=z^{\pi/\alpha}$
$ z <1, \operatorname{Im} z>0$ (upper semidisk)	$\operatorname{Im} w>0$	$w=\left(\frac{1+z}{1-z}\right)^2$
$0<\arg z<\alpha,$ $ z <1$ (fan shape)	$\operatorname{Im} w>0$	$w=\left(\frac{1+z^{\pi/\alpha}}{1-z^{\pi/\alpha}}\right)^2$
$\alpha<\arg \frac{z-p}{z-q}<\beta$ (circular triangle)	$0<\arg w<\gamma$	$w=\left(e^{-i\alpha}\frac{z-p}{z-q}\right)^{\frac{\gamma}{\beta-\alpha}}$
$0<\operatorname{Im} z<\eta$ (parallel strip)	$\operatorname{Im} w>0$	$w=e^{\pi z/\eta}$
$\operatorname{Re} z<0,$ $0<\operatorname{Im} z<\eta$ (semiparallel strip)	$\operatorname{Im} w>0, \quad w <1$	$w=e^{\pi z/\eta}$
$y^2>4c^2(x+c^2),$ $z=x+iy, \quad c>0$ (exterior of a parabola)	$\operatorname{Im} w>0$	$w=\sqrt{z}-ic$
$y^2<4c^2(x+c^2),$ $z=x+iy, \quad c>0$ (interior of a parabola)	$\operatorname{Im} w>0$	$w=i \sec \frac{\pi \sqrt{z}}{2ic}$

Original Domain	Image Domain	Mapping Function
$\frac{x^2}{[c + (1/c)]^2} + \frac{y^2}{[c - (1/c)]^2} > 1,$ $z = x + iy, \quad c > 1$ (exterior of an ellipse)	$ w > c$	$w = \frac{z + \sqrt{z^2 - 4}}{2}, \quad z = w + \frac{1}{w}$
$\frac{x^2}{\cos^2 \alpha} - \frac{y^2}{\sin^2 \alpha} < 4,$ $z = x + iy,$ $0 < \alpha < \pi/2$ (exterior of a hyperbola)	$\operatorname{Im} w > 0$	$w = \left(e^{-i\alpha} \frac{z + \sqrt{z^2 - 4}}{2} \right)^{\pi(\pi - 2\alpha)}$
$\frac{x^2}{\cos^2 \alpha} - \frac{y^2}{\sin^2 \alpha} > 4,$ $x > 0,$ $z = x + iy,$ $0 < \alpha < \pi/2$ (right-hand side interior of a hyperbola)	$\operatorname{Im} w > 0$	$w = \frac{1}{2i} \left[\left(\frac{z + \sqrt{z^2 - 4}}{2} \right)^{\frac{\pi}{2\alpha}} + \left(\frac{z - \sqrt{z^2 - 4}}{2} \right)^{\frac{\pi}{2\alpha}} \right]$
$ z < 1$	Slit domain with boundary $ \operatorname{Re} w \leq 2, \operatorname{Im} w = 0$	$w = z + \frac{1}{z}$
$ z < 1$	Slit domain with boundary $ w \geq 1/4, \arg w = \lambda$	$w = \frac{z}{(1 + e^{-i\lambda} z)^2}$
$ z < 1$	Slit domain with boundary $ w \geq 1/4^{1/p}$ $\arg w = \lambda + (2j\pi/p),$ $j = 0, \dots, p-1$	$w = \frac{z}{(1 + e^{-ip\lambda} z^p)^{2/p}}$
$-\pi/2 < \operatorname{Re} z < \pi/2$ (parallel strip)	Slit domain with boundary $ \operatorname{Re} w \geq 1, \operatorname{Im} w = 0$	$w = \sin z$
$-\pi < \operatorname{Im} z < \pi$ (parallel strip)	Slit domain with boundary $\operatorname{Re} w \leq -1, \operatorname{Im} w = \pm \pi$	$w = z + e^z$
Arbitrary circle or half plane	Interior of an n -gon	$w = c \int^z \prod_{j=1}^n (t - z_j)^{\alpha_j - 1} dt + c' \quad (c \neq 0,$ <p>c' are constants), where the inverse image of the vertex with the inner angle $\alpha_j \pi$ ($j = 1, \dots, n$) is $z = z_j$. When $z_n = \infty$, we omit the factor $(t - z_n)^{\alpha_n - 1}$ (Schwarz-Christoffel transformation)</p>
Arbitrary circle or half-plane	Exterior of an n -gon	$w = c \int^z (t - p)^{-2} \prod_{j=1}^n (t - z_j)^{1 - \alpha_j} dt + c'$ <p>($c \neq 0$, c' are constants), where the inverse image of the vertex with the inner angle $\alpha_j \pi$ ($j = 1, \dots, n$) is $z = z_j$, and the inverse image of ∞ is $z = p$</p>
$\operatorname{Im} z > 0$	Interior of an equilateral triangle	$w = \int_0^z \frac{1}{\sqrt[3]{t^2(1-t)^2}} dt$
$\operatorname{Im} z > 0$	Interior of an isosceles right triangle	$w = \int_0^z \frac{1}{\sqrt[5]{t^2(1-t)^3}} dt$

Original Domain	Image Domain	Mapping Function
$\text{Im } z > 0$	Interior of a right triangle with one angle $\pi/6$	$w = \int_0^z \frac{1}{\sqrt[6]{t^3(1-t)^4}} dt$
$ z < 1$	Interior of a regular n -gon	$w = \int_0^z (1-t^n)^{-2/n} dt$
$0 < \text{Re } z < \omega_1,$ $0 < \text{Im } z < \omega_3/i$ (rectangle)	$\text{Im } w > 0$	$w = \wp(z 2\omega_1, 2\omega_3)$ (\wp is the Weierstrass \wp -function)
$-K < \text{Re } z < K,$ $0 < \text{Im } z < K'$ (rectangle) ⁽¹⁾	$\text{Im } w > 0$	$w = \text{sn}(z, k),$ $z = \int_0^w \frac{1}{\sqrt{(1-t^2)(1-k^2t^2)}} dt$ (sn is Jacobi's sn function)
$v < z < 1$ $\text{Im } z < 0$ (upper half-ring domain)	$\log q < \text{Re } w < 0,$ $0 < \text{Im } w < \pi$ (rectangle)	$w = \log z$
$ z < 1$	$\frac{u^2}{A^2} + \frac{v^2}{B^2} < 1,$ $w = u + v, \quad A > B > 0;$ (interior of an ellipse)	$w = \sqrt{A^2 - B^2} \sin \left(\frac{\pi}{2K} \int_0^{2z/k(1+z^2)} \frac{dt}{\sqrt{(1-t^2)(1-k^2t^2)}} \right),$ $\frac{1}{\pi} \log \frac{A+B}{A-B} = \frac{K'}{K}$
$\text{Im } z > 0$	interior of a circular polygon	$\{w; z\} \equiv \frac{w'''}{w'} - \frac{3}{2} \left(\frac{w''}{w'} \right)^2 = R(z)$ ($R(z)$ is a rational function)
$ z < 1$	Interior of an equilateral circular triangle with inner angle $\pi/k, 1 < k < \infty$	$z = \int_0^1 t^{-\frac{1}{2} - \frac{1}{2k}} (1-t)^{-\frac{1}{6} + \frac{1}{2k}} (1-z^3t)^{-\frac{5}{6} + \frac{1}{2k}} dt$ $w = \frac{\int_0^1 t^{-\frac{1}{2} - \frac{1}{2k}} (1-t)^{-\frac{5}{6} + \frac{1}{2k}} (1-z^3t)^{-\frac{1}{6} + \frac{1}{2k}} dt}{\int_0^1 t^{-\frac{1}{2} - \frac{1}{2k}} (1-t)^{-\frac{5}{6} + \frac{1}{2k}} (1-z^3t)^{-\frac{1}{6} + \frac{1}{2k}} dt}$ The vertices are the images of $z = 1, e^{2\pi i/3},$ and $e^{4\pi i/3},$ and $\left[\frac{dw}{dz} \right]_{z=0} = \frac{\Gamma[(5/6) + (1/2k)]\Gamma(2/3)}{\Gamma[(1/6) + (1/2k)]\Gamma(4/3)}$
$\text{Im } z > 0$	Interior of a circular triangle with inner angles $\pi\alpha, \pi\beta, \pi\gamma,$ $\alpha + \beta + \gamma < 1$ ⁽²⁾	$w = \frac{\int_0^1 t^{-\frac{1+\alpha+\beta+\gamma}{2}} (1-t)^{-\frac{1+\alpha-\beta-\gamma}{2}} (1-zt)^{-\frac{1-\alpha+\beta-\gamma}{2}} dt}{\int_0^1 t^{-\frac{1+\alpha+\beta+\gamma}{2}} (1-t)^{-\frac{1-\alpha-\beta+\gamma}{2}} (1-t+zt)^{-\frac{1-\alpha+\beta-\gamma}{2}} dt}$
$ \tau > 1,$ $-1/2 < \text{Re } \tau < 0$	$\text{Im } J > 0$	$J = J(\tau), \tau = \omega_3/\omega_1, J = g_2^2/(g_2^3 - 27g_3^2)$ (the absolute invariant of the elliptic modular function); $J(e^{2\pi i/3}) = 0, J(i) = 1, J(\infty) = \infty$ $\lambda = \lambda(\tau), \tau = \omega_3/\omega_1, \lambda = (e_2 - e_3)/(e_1 - e_3);$
$ \tau + 1/2 < 1/2,$ $-1 < \text{Re } \tau < 0$	$\text{Im } \lambda < 0$	$J(\tau) \equiv \frac{4}{27} \frac{[\lambda(\tau)^2 - \lambda(\tau) + 1]^2}{\lambda(\tau)^2 [\lambda(\tau) - 1]^2},$ $\lambda(-1) = \infty, \lambda(0) = 1, \lambda(\infty) = 0$

Notes

(1) K, K', k' are the usual notations in the in the theory of elliptic integrals:

$$K \equiv \int_0^1 \frac{1}{\sqrt{(1-t^2)(1-k^2t^2)}} dt, \quad K' = \int_0^1 \frac{1}{\sqrt{(1-t^2)(1-k'^2t^2)}} dt, \quad k^2 + k'^2 = 1.$$

(2) When $\alpha + \beta + \gamma = 1$, the circular triangle is mapped into the ordinary linear triangle by a suitable linear transformation, and we can apply the Schwarz-Christoffel transformation. When $\alpha + \beta + \gamma > 1$, we have a similar mapping function replacing the integral representations of hypergeometric functions in the formula by the corresponding integral representations of the hypergeometric functions converging at α, β , and γ .

14. Ordinary Differential Equations

(I) Solution by Quadrature

a, b, c, \dots are integral constants.

(1) Solution of the First-Order Differential Equations (\rightarrow 313 Ordinary Differential Equations).

(i) Separated type $dy/dx = X(x)Y(y)$. The general solution is

$$\int^y \frac{dy}{Y(y)} = \int^x X(x) dx + c.$$

(ii) Homogeneous ordinary differential equation $dy/dx = f(y/x)$. Putting $y = ux$, we have $du/dx = [f(u) - u]/x$, and the equation reduces to type (i). The general solution is

$$x = c \exp \left[\int^u \frac{du}{f(u) - u} \right] \quad \left(u = \frac{y}{x} \right).$$

(iii) Linear ordinary differential equation of the first order. $dy/dx + p(x)y + q(x) = 0$. The general solution is

$$y = \left[c - \int q(x)P(x) dx \right] / P(x),$$

where

$$P(x) \equiv \exp \left[\int p(x) dx \right].$$

(iv) Bernoulli's differential equation $dy/dx + p(x)y + q(x)y^\alpha = 0$ ($\alpha \neq 0, 1$). Putting $z = y^{1-\alpha}$, the equation is transformed into

$$dz/dx + (1-\alpha)p(x)z + (1-\alpha)q(x) = 0,$$

which reduces to (iii).

(v) Riccati's differential equation $dy/dx + ay^2 = bx^m$. If $m = -2, 4k/(1-2k)$ (k an integer), this is solved by quadrature. In general, it is reduced to Bessel's differential equation by $ay = u'/u$.

(vi) Generalized Riccati differential equation $dy/dx + p(x)y^2 + q(x)y + r(x) = 0$. If we know one, two, or three special solutions $y = y_i(x)$, the general solution is represented as follows. When $y_1(x)$ is one known special solution,

$$y = y_1(x) + P(x) / \left[\int p(x)P(x) dx + c \right],$$

where

$$P(x) \equiv \exp \left[- \int \{ q(x) + 2p(x)y_1(x) \} dx \right].$$

When $y_1(x), y_2(x)$ are the known solutions,

$$\frac{y - y_1(x)}{y - y_2(x)} = c \exp \left[\int p(x) \{ y_2(x) - y_1(x) \} dx \right].$$

When $y_1(x), y_2(x), y_3(x)$ are known solutions,

$$\frac{y - y_1(x)}{y - y_2(x)} = c \frac{y_3(x) - y_1(x)}{y_3(x) - y_2(x)}.$$

(vii) Exact differential equation $P(x,y)dx + Q(x,y)dy = 0$. If the left-hand side is an exact differential form, the condition is $\partial P/\partial y = \partial Q/\partial x$. The general solution is

$$\int P dx + \int \left(Q - \frac{\partial}{\partial y} \int P dx \right) dy = c.$$

(viii) Integrating factors. A function $M(x,y)$ is called an integrating factor of a differential equation $P(x,y)dx + Q(x,y)dy = 0$, if $M(x,y)[P(x,y)dx + Q(x,y)dy]$ is an exact differential form $d\varphi(x,y)$. If we know an integrating factor, the general solution is given by $\varphi(x,y) = c$. If we know two independent integrating factors M and N , the general solution is given by $M/N = c$.

(ix) Clairaut's differential equation $y = xp + f(p)$ ($p = dy/dx$). The general solution is the family of straight lines $y = cx + f(c)$, and the singular solution is the envelope of this family, which is given by eliminating p from the original equation and $x + f'(p) = 0$.

(x) Lagrange's differential equation $y = x\varphi(p) + \psi(p)$ ($p \equiv dy/dx$). Differentiation with respect to x reduces the equation to a linear differential equation $[\varphi(p) - p](dx/dp) + \varphi'(p)x + \psi'(p) = 0$

with respect to x, p (see (iii)). The general solution of the original equation is given by eliminating p from the original equation and the solution of the latter linear equation. The parameter p may represent the solution. If the equation $p = \varphi(p)$ has a solution $p = p_0$, we have a solution $y = p_0x + \psi(p_0)$ (straight line). This solution is sometimes the singular solution.

(xi) Singular solutions. The singular solution of $f(x, y, p) = 0$ is included in the equation resulting from eliminating p from $f = 0$ and $\partial f / \partial p = 0$, though the eliminant may contain various curves that are not the singular solutions.

(xii) System of differential equations.

$$\text{eq. (1)} \quad dx : dy : dz = P : Q : R.$$

A function $M(x, y, z)$ is called a Jacobi's last multiplier for eq. (1) if M is a solution of a partial differential equation $(\partial MP / \partial x) + (\partial MQ / \partial y) + (\partial MR / \partial z) = 0$. If we know two independent last multipliers M and N , then $M/N = c$ is a solution of eq. (1). If we know a last multiplier M and a solution $f = a$ of eq. (1), we may find another solution of (1) as follows: solving $f = a$ with respect to z and inserting the solution into eq. (1), we see that $M(Qdx - Pdy)/f_z$ is an exact differential form $dG(x, y, a)$ in three variables x, y , and a . Then $G(x, y, f(x, y, z)) = b$ is another solution of eq. (1).

(2) Solutions of Higher-Order Ordinary Differential Equations. The following (i)–(iv) are several examples of depression.

(i) $f(x, y^{(k)}, y^{(k+1)}, \dots, y^{(n)}) = 0$ ($0 < k \leq n$). Set $y^{(k)} = z$; the equation reduces to one of the $(n - k)$ th order in z .

(ii) $f(y, y', y'', \dots, y^{(n)}) = 0$. This is reduced to $(n - 1)$ st order if we consider $y' = p$ as a variable dependent on y .

(iii) $y'' = f(y)$. The general solution is given by

$$x = a \pm \int \left[2 \int f(y) dy + b \right]^{-1/2} dy.$$

We have a similar formula for $y^{(n)} = f(y^{(n-2)})$.

(iv) Homogeneous ordinary differential equation of higher order. If the left-hand side of $F(x, y, y', \dots, y^{(n)}) = 0$ satisfies the homogeneity relation $F(x, \rho y, \rho y', \dots, \rho y^{(n)}) = \rho^\alpha F(x, y, y', \dots, y^{(n)})$, the equation is reduced to one of the $(n - 1)$ st order in u by $u = y'/y$.

If F satisfies $F(\rho x, \rho^t y, \rho^{t-1} y', \dots, \rho^{t-n} y^{(n)}) = \rho^\alpha F(x, y, y', \dots, y^{(n)})$, then $u = y/x^t$, $t = \log x$ reduces the equation to one of type (ii) not containing t .

(v) Euler's linear ordinary differential equation.

$$p_n(x)x^n y^{(n)} + p_{n-1}(x)x^{n-1} y^{(n-1)} + \dots + p_1(x)xy' + p_0(x)y = q(x)$$

is reduced to a linear equation by $t = \log x$.

(vi) Linear ordinary differential equations of higher order (exact equations). A necessary and sufficient condition that $L[y] \equiv \sum_{j=0}^n p_j(x)y^{(j)} = X(x)$ is an exact differential form is $\sum_{j=0}^n (-1)^j p_j^{(j)} = 0$, and then the first integral of the equation is given by

$$\sum_{j=0}^{n-1} \sum_{k=0}^{n-j-1} (-1)^k p_{k+j+1}^{(k)} y^{(j)} = \int X(x) dx + c.$$

(vii) Linear ordinary differential equation of higher order (depression).

$$L[y] \equiv \sum_{j=0}^n p_j(x)y^{(j)} = X(x).$$

If we know mutually independent special solutions $y_1(x), \dots, y_m(x)$ for the homogeneous linear ordinary differential equation $L[y] = 0$, the equation is reduced to the $(n - m)$ th linear ordinary differential equation with respect to z by a transformation $z = A(y)$, where $A(y) = 0$ is the m th linear ordinary differential equation with solutions $y_1(x), \dots, y_m(x)$. For example, if $m = 1$, the equation is reduced to the $(n - 1)$ st linear ordinary differential equation with respect to z by the transformation

$$y(x) = y_1(x) \int z(x) dx.$$

Also, if $n = m = 2$, the general solution is

$$y = c_1 y_1 + c_2 y_2 - y_1 \int T y_2 dx + y_2 \int T y_1 dx,$$

where $T(x) \equiv X(x)/[y_1(x)y_2'(x) - y_2(x)y_1'(x)]$. The denominator of the last expression is the Wronskian of y_1 and y_2 .

(viii) Regular singularity. For a linear ordinary differential equation of higher order,

$$\text{eq. (1)} \quad x^n y^{(n)} + x^{n-1} p_1(x) y^{(n-1)} + \dots + p_n(x) y = 0,$$

the point $x=0$ is its regular singularity if $p_1(x), \dots, p_n(x)$ are analytic at $x=0$.

We put $p_0=1$ and

$$\sum_{\nu=0}^{\infty} f_{\nu}(\rho) x^{\nu} \equiv \sum_{j=0}^n p_{n-j}(x) \rho(\rho-1) \dots (\rho-j+1).$$

If ρ is a root of the characteristic equation $f_0(\rho)=0$ and $\rho+1, \rho+2, \dots$ are not roots, we can determine the coefficients c_{ν} uniquely from

$$\text{eq. (2)} \quad \sum_{\nu=0}^m c_{\nu} f_{m-\nu}(\rho+\nu) = 0 \quad (m=1, 2, \dots),$$

starting from a fixed value $c_0 (\neq 0)$, and the series $y = x^{\rho} \sum_{\nu=0}^{\infty} c_{\nu} x^{\nu}$ converges and represents a solution of eq. (1). If the differences of all pairs of roots of the determining equation are not integers, we have n linearly independent solutions of eq. (1) applying the process for each characteristic root.

If there are roots whose differences are integers (including multiple roots), we denote such a system of roots by ρ_1, \dots, ρ_l . We arrange them in increasing order, and denote the multiplicities of the roots by e_1, \dots, e_l , respectively. Put $q_k = \rho_k - \rho_1$ ($k=1, 2, \dots, l$; $0 = q_1 < q_2 < \dots < q_l$). Take $N \geq q_l$ and a constant $c (\neq 0)$. Let λ be a parameter, and starting from $c_0 = c_0(\lambda) \equiv c \prod_{k=1}^N f_0(\lambda + k)$, we determine $c_{\nu} = c_{\nu}(\lambda)$ uniquely by the relation (2). Putting

$$m_k \equiv e_k + e_{k+1} + \dots + e_l \quad (k=1, \dots, l) \quad \text{for } h \text{ in } m_{k+1} \leq h \leq m_k - 1,$$

the series

$$\text{eq. (3)} \quad y = \left[\frac{\partial}{\partial \lambda} x^{\lambda} \sum_{\nu=0}^{\infty} c_{\nu}(\lambda) x^{\nu} \right]_{\lambda=\rho_k} = x^{\rho_k} \sum_{\nu=0}^{\infty} x^{\nu} \left[\sum_{j=0}^h \binom{h}{j} c_{\nu}^{(j)}(\rho_k) (\log x)^{h-j} \right]$$

converges and gives e_k independent solutions of eq. (1). Hence for $k=1, \dots, l$, we may have $\sum_{k=1}^l e_k = m_1$ mutually independent solutions of (1). Applying this process to every characteristic root, we have finally n independent solutions of (1) (Frobenius method).

In the practical computation of the solution, since it is known to have the expression (3), we often determine its coefficients successively by the method of undetermined coefficients.

(3) Solution of Linear Ordinary Differential Equations with Constant Coefficient (\rightarrow 252 Linear Ordinary Differential Equations). Let α_i, α_{jk} be constants. We consider the following linear ordinary differential equation of higher order (eq. (1)) and system of linear ordinary differential equations (eq. (2)).

$$\text{eq. (1)} \quad \sum_{i=0}^n \alpha_i y^{(i)} = X(x).$$

$$\text{eq. (2)} \quad y_j' = \sum_{k=1}^n \alpha_{jk} y_k + X_j(x) \quad (j=1, \dots, n).$$

(i) The general solution of the homogeneous equation (cofactor) is given by the following formulas:

$$\text{for eq. (1)} \quad y = x^j \exp \lambda_k x \quad (j=0, 1, \dots, e_k - 1; k=1, \dots, m),$$

$$\text{for eq. (2)} \quad y_j(x) = \sum_{k=1}^m p_{jk}(x) \exp \lambda_k x \quad (j=1, \dots, n),$$

where $\lambda_1, \dots, \lambda_m$ are the roots of the characteristic equation of eq. (1) or eq. (2) given by

$$\text{eq. (1')} \quad \sum_{i=0}^n \alpha_i \lambda^i = 0,$$

$$\text{eq. (2')} \quad \det(\alpha_{jk} - \lambda \delta_{jk}) = 0,$$

respectively. We denote the multiplicities of the roots by e_1, \dots, e_m ($e_1 + \dots + e_m = n$); $p_{jk}(x)$ is a polynomial of degree at most $e_k - 1$ containing e_k arbitrary constants.

If all the coefficients in the original equation are real, and the root $\lambda_k = \mu_k + i\nu_k$ is imaginary, then $\bar{\lambda}_k = \mu_k - i\nu_k$ is also a root with the same multiplicity. Then we may replace $\exp \lambda_k x$ and $\exp \bar{\lambda}_k x$ by $\exp \mu_k x \cos \nu_k x$ and $\exp \mu_k x \sin \nu_k x$, and in this way we can represent the solution using real functions.

(ii) Inhomogeneous equation. The solution of an inhomogeneous linear ordinary differential equation is given by the method of variation of parameters or by the method described in Section (2)(vii).

We explain the method of variation of parameters for eq. (2). First we use (i) to find a fundamental system of n independent solutions $y_j = \varphi_{jk}(x)$ ($k = 1, \dots, n$) by (i). Inserting $y_j = \sum_{k=1}^n c_k(x) \varphi_{jk}(x)$ into eq. (2), we have a system of linear equations in the $c'_k(x)$. Solving for the $c'_k(x)$ and integrating, we have $c_k(x)$.

Special forms of $X(x)$ or $X_j(x)$ determine the form of the solutions, and the parameters may be found by the method of undetermined coefficients. The following table shows some examples of special solutions for eq. (1). In the table, α, k, a, b, c , are constants, p_r, q_r are polynomials of degree r , and I_a is the operator defined by

$$I_a \cdot F = \frac{1}{a} \left[\sin ax \int \cos ax \cdot F(x) dx - \cos ax \int \sin ax \cdot F(x) dx \right] \quad (a \neq 0).$$

$X(x)$	Condition	Special Solution
$p_r(x)$	$\lambda = 0$ is an m -tuple root of $(1')$	$x^m q_r(x)$
$ke^{\alpha x}$	$\lambda = \alpha$ is an m -tuple root of $(1')$	$cx^m e^{\alpha x}$
$e^{\alpha x} p_r(x)$	$\lambda = \alpha$ is an m -tuple root of $(1')$	$x^m q_r(x) e^{\alpha x}$
$\left. \begin{matrix} \cos(ax+b) \\ \sin(ax+b) \end{matrix} \right\}$	$\left\{ \begin{matrix} (1') \equiv \varphi(\lambda^2) + \lambda \psi(\lambda^2), \text{ and} \\ \varphi(-a^2) + a^2 \psi(-a^2) \neq 0 \end{matrix} \right\}$	$c_1 \cos(ax+b) + c_2 \sin(ax+b)$
$\left. \begin{matrix} \cos(ax+b) \\ \sin(ax+b) \end{matrix} \right\}$	$\left\{ \begin{matrix} (1') = g(\lambda)/f(\lambda^2) \text{ and } f(\lambda^2) \\ \text{is divisible by } (\lambda^2 + a^2)^m \\ \text{(but not by } (\lambda^2 + a^2)^{m+1}) \end{matrix} \right\}$	$c(I_a)^m \left\{ \begin{matrix} \cos(ax+b) \\ \sin(ax+b) \end{matrix} \right\}$

(II) Riemann's P -Function and Special Functions (\rightarrow 253 Linear Ordinary Differential Equations (Global Theory))

(1) Some Examples Expressed by Elementary Functions. A, B are integral constants.

$$P \left\{ \begin{matrix} a & b & c \\ 0 & \mu & -\mu \\ 1 & \mu' & -\mu' \end{matrix} \middle| x \right\} = \begin{cases} A \left(\frac{x-b}{x-c} \right)^\mu + B \left(\frac{x-b}{x-c} \right)^{\mu'} & (\mu \neq \mu'), \\ \left(\frac{x-b}{x-c} \right)^\mu \left[A + B \log \left(\frac{x-b}{x-c} \right) \right] & (\mu = \mu'). \end{cases}$$

$$P \left\{ \begin{matrix} a & b & c \\ \lambda & \mu & \nu \\ 0 & 0 & 0 \end{matrix} \middle| x \right\} = A + B \int (x-a)^{\lambda-1} (x-b)^{\mu-1} (x-c)^{\nu-1} dx \quad (\lambda + \mu + \nu = 1).$$

These are for finite a, b, c . If $c = \infty$, $x - c$ should be replaced by 1.

$$P \left\{ \begin{matrix} \infty & 0 \\ \alpha & 0 & 0 \\ \alpha' & 0 & 1 \end{matrix} \middle| x \right\} = \begin{cases} Ae^{\alpha x} + Be^{\alpha' x} & (\alpha \neq \alpha'), \\ e^{\alpha x} (Ax + B) & (\alpha = \alpha'). \end{cases}$$

$$P \left\{ \begin{matrix} \infty & 0 \\ \alpha & 1-\sigma & \sigma \\ 0 & 0 & 0 \end{matrix} \middle| x \right\} = A + B \int e^{\alpha x} x^{\sigma-1} dx \quad (\alpha \neq 0).$$

$$P \left\{ \begin{matrix} \infty & 0 \\ \alpha & -\sigma & \sigma \\ \alpha' & 1-\sigma' & \sigma' \end{matrix} \middle| x \right\} = x^\sigma e^{\alpha x} \left[A + B \int e^{(\alpha'-\alpha)x} x^{\sigma'-\sigma-1} dx \right] \quad (\alpha \neq \alpha').$$

Riemann's P -function is reduced to Gauss's hypergeometric function with parameters $\alpha = \lambda + \mu + \nu$, $\beta = \lambda + \mu + \nu'$, $\gamma = 1 + \lambda + \lambda'$ by transforming a, b, c to 0, 1, ∞ by a suitable linear transformation and by putting $z = x^{-\lambda}(x-1)^{-\mu}y$.

(2) Representation of Special Functions by Riemann's P -function.

(i) Gauss's hypergeometric differential equation $x(1-x)y'' + [\gamma - (\alpha + \beta + 1)x]y' - \alpha\beta y = 0$.

$$y = P \left\{ \begin{matrix} 0 & 1 & \infty \\ 0 & 0 & \alpha \\ 1-\gamma & \gamma-\alpha-\beta & \beta \end{matrix} \middle| x \right\}.$$

A special solution is $F(\alpha, \beta, \gamma; x)$ (\rightarrow 206 Hypergeometric Functions).

(ii) Confluent hypergeometric differential equation $xy'' - (x - \mu)y' - \lambda y = 0$.

$$y = P \left\{ \begin{array}{ccc} \infty & 0 \\ 0 & \lambda & 0 \\ 1 & \mu - \lambda & 1 - \mu \end{array} \begin{array}{c} x \\ x \\ x \end{array} \right\}.$$

A special solution is

$${}_1F_1(\lambda, \mu; x) \equiv \sum_{k=0}^{\infty} \frac{\Gamma(\lambda + k)}{\Gamma(\lambda)} \frac{\Gamma(\mu)}{\Gamma(\mu + k)} \frac{x^k}{k!}.$$

(iii) Whittaker's differential equation $y'' + \left[-\frac{1}{4} + \frac{k}{x} + \frac{(1/4) - n^2}{x^2} \right] y = 0$.

$$y = P \left\{ \begin{array}{ccc} \infty & 0 \\ 1/2 & k & 1/2 + n \\ -1/2 & -k & 1/2 - n \end{array} \begin{array}{c} x \\ x \\ x \end{array} \right\}.$$

Special solutions are $M_{k,n}(x), W_{k,n}(x)$.

(iv) Bessel's differential equation $x^2y'' + xy' + (x^2 - \nu^2)y = 0$.

$$y = P \left\{ \begin{array}{ccc} \infty & 0 \\ i & 1/2 & \nu \\ -i & 1/2 & -\nu \end{array} \begin{array}{c} x \\ x \\ x \end{array} \right\}.$$

Special solutions are $J_\nu(x), N_\nu(x)$ (\rightarrow 39 Bessel Functions). When $m = 0, 1, 2, \dots, J_{m-1/2}(x) = (-1)^m 2^{m+1/2} \pi^{-1/2} x^{m-1/2} d^m(\cos x)/dx^m$.

(v) Hermite's differential equation (parabolic cylindrical equation) $y'' - 2xy' + 2ny = 0$.

$$y = P \left\{ \begin{array}{ccc} \infty & 0 \\ 0 & -n/2 & 0 \\ 1 & (n+1)/2 & 1/2 \end{array} \begin{array}{c} x^2 \\ x^2 \\ x^2 \end{array} \right\}.$$

When $n = 0, 1, 2, \dots$, the Hermite polynomial $H_n(x) = (-1)^n 2^{-n/2} e^{x^2} d^n(e^{-x^2})/dx^n$ is the solution.

(vi) Laguerre's differential equation $xy'' + (l - x + 1)y' + ny = 0$.

$$y = P \left\{ \begin{array}{ccc} \infty & 0 \\ 0 & -n & 0 \\ 1 & l + n + 1 & -l \end{array} \begin{array}{c} x \\ x \\ x \end{array} \right\}.$$

When $n = 0, 1, 2, \dots$, the Laguerre polynomial $L_n^l(x) = (1/n!) x^{-l} e^x d^n(x^{n+l} e^{-x})/dx^n$ is the solution.

(vii) Jacobi's differential equation $x(1-x)y'' + [q - (p+1)x]y' + n(n+p)y = 0$.

$$y = P \left\{ \begin{array}{ccc} 0 & 1 & \infty \\ 1-q & q-p & p+n \\ 0 & 0 & -n \end{array} \begin{array}{c} x \\ x \\ x \end{array} \right\}.$$

When $n = 0, 1, 2, \dots$, the Jacobi polynomial

$$G_n(p, q; x) = \frac{\Gamma(q)x^{1-q}(1-x)^{q-p}}{\Gamma(n+q)} \frac{d^n[x^{q+n-1}(1-x)^{p+n-q}]}{dx^n}$$

is the solution.

(viii) Legendre's differential equation $(1-x^2)y'' - 2xy' + n(n+1)y = 0$.

$$y = P \left\{ \begin{array}{ccc} 1 & -1 & \infty \\ 0 & 0 & n+1 \\ 0 & 0 & -n \end{array} \begin{array}{c} x \\ x \\ x \end{array} \right\}.$$

When $n = 0, 1, 2, \dots$, the general solution is

$$\frac{d^n}{dx^n} \left[A(x^2-1)^n + B(x^2-1)^n \int \frac{dx}{(x^2-1)^{n+1}} \right].$$

The Legendre polynomial $P_n(x) = [d^n\{(x^2-1)^n\}/dx^n]/2^n n!$ is a special solution.

(3) Solution by Cylindrical Functions of Ordinary Linear Differential Equations of the Second Order. We denote cylindrical functions by $C_\nu(x)$ (\rightarrow 39 Bessel Functions).

Equation	Solution
$y'' + \frac{1-2\alpha}{x}y' + \left[(\beta\gamma x^{\gamma-1})^2 + \frac{\alpha^2 - \nu^2\gamma^2}{x^2} \right]y = 0$	$y = x^\alpha C_\nu(\beta x^\gamma)$
$y'' + \left[\frac{1-2\alpha}{x} - 2\beta\gamma ix^{\gamma-1} \right]y' + \left[\frac{\alpha^2 - \nu^2\gamma^2}{x^2} - \beta\gamma(\gamma - 2\alpha)ix^{\gamma-2} \right]y = 0$	$y = x^\alpha \exp(i\beta x^\gamma) C_\nu(\beta x^\gamma)$
$y'' + \left[\frac{1}{x} - 2u(x) \right]y' + \left[1 - \frac{\nu^2}{x^2} + u(x)^2 - u'(x) - \frac{u(x)}{x} \right]y = 0$	$y = \exp \left[\int u(x) dx \right] C_\nu(x)$
$y'' + \alpha^2 \nu^2 x^{2\nu-2}y = 0$	$y = \sqrt{x} C_{1/2\nu}(\alpha x^\nu)$
$y'' + (e^{2x} - \nu^2)y = 0$	$y = C_\nu(e^x)$
$x^2 y'' + xy' + (\beta^2 x^2 - \nu^2)y = 0$	$y = C_\nu(\beta x)$
$x^2 y'' + xy' - (x^2 + \nu^2)y = 0$	$y = C_\nu(ix)$ (modified Bessel function)

(III) Transformation Groups and Invariants

Let $U \equiv \xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}$ be the infinitesimal transformation of a given continuous transformation group of two variables, and $U' \equiv \xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y} + \zeta \frac{\partial}{\partial p}$ be that of its extended group.

We have

$$\zeta = \frac{\partial \eta}{\partial x} + p \left(\frac{\partial \eta}{\partial y} - \frac{\partial \xi}{\partial x} \right) - p^2 \frac{d\xi}{dx}.$$

We put

$$p \equiv \frac{dy}{dx}, \quad r \equiv \frac{d^2y}{dx^2}.$$

Let α, β and γ be invariants of the 0th, first, and second order, respectively. The general form of the differential equation of the first or of the second order invariant under U is given by $\Phi(\alpha, \beta) = 0$ (or $\beta = F(\alpha)$), and $\Psi(\alpha, \beta, \gamma) = 0$ (or $\gamma = G(\alpha, \beta)$), respectively, where F, Φ, Ψ, G denote arbitrary functions of the corresponding variables.

Group With Infinitesimal Transformation U			Invariants			Note
ξ	η	ζ	0th	1st	2nd	
0	1	0	x	p	r	(1)
1	0	0	y	p	r	(1)
$-y$	x	$1 + p^2$	$x^2 + y^2$	$(y - xp)/(x + yp)$	$r/(1 + p^2)^{3/2}$	(2)
0	y	p	x	p/y	r/y	(3)
x	0	$-p$	y	xp	x^2r	(3)
x	y	0	y/x	p	xr	(4)
x	$-y$	$-2p$	xy	x^2p	x^3r	
μx	νy	$(\nu - \mu)p$	y^μ/x^ν	$x^{1-\nu/\mu}p$ or px/y	$r\mu/x^{\nu-\mu-1}$	
μ	ν	0	$\nu x - \mu y$	p	r	
0	$h(x)$	$h'(x)$	x	$h(x)p - h'(x)y$	$h(x)r - h''(x)y$	(5)
$k(y)$	0	$-k'(y)p^2$	y	$\frac{1}{p} - \frac{k'(y)}{k(y)}x$	$\frac{r}{p^3} + \frac{k''(y)}{k'(y)p}$	
0	$k(y)$	$k'(y)p$	x	$\frac{p}{k(y)}$	$\frac{r}{k(y)} - \frac{k'(y)p^2}{[k(y)]^2}$	(6)
$h(x)$	0	$-h'(x)p$	y	$h(x)p$	$(h(x))^2r + h(x)h'(x)p$	

Group With Infinitesimal Transformation U			Invariants			Note
ξ	η	ζ	0th	1st	2nd	
0	$h(x)k(y)$	$h'(x)k(y) + h(x)k'(y)p$	x	$\frac{p}{k(y)} - \frac{h'(x)}{h(x)} \int^y \frac{dy}{k(y)}$	—	(7)
$xh(x)$	$yh(x)$	$h'(x)(y - xp)$	$\frac{y}{x}$	$\left(p - \frac{y}{x}\right)h(x)$	$\left(\frac{x^2r}{xp - y} - 1\right)h(x) + h'(x)$	
y	x	$1 - p^2$	$x^2 - y^2$	$\frac{1 - p}{1 + p} \frac{x + y}{x - y}$ or $(x - yp)/(1 + p)(x - y)$	$\frac{r}{(1 - p^2)^{3/2}}$	

Notes

- (1) Parallel translation.
- (2) Rotation.
- (3) Affine transformation.
- (4) Similar transformation; the equation is a homogeneous differential equation.
- (5) Linear differential equation.
- (6) Separated variable type.
- (7) When $k(y) = y^n$, the equation is Bernoulli's differential equation.

Reference

[1] A. R. Forsyth, A treatise on differential equations, Macmillan, fourth edition, 1914.

15. Total and Partial Differential Equations

(I) Total Differential Equations (→ 428 Total Differential Equations)

Suppose we are given a system of total differential equations

$$dz_j = \sum_{k=1}^n P_{jk}(x; z) dx_k \quad (j = 1, 2, \dots, m).$$

A condition for complete integrability is given by

$$\frac{\partial P_{jk}(x; z)}{\partial x_i} + \sum_i \frac{\partial P_{jk}(x; z)}{\partial z_i} P_{il}(x; z) = \frac{\partial P_{jl}(x; z)}{\partial x_k} + \sum_i \frac{\partial P_{jl}(x; z)}{\partial z_i} P_{ik}(x; z).$$

Under this condition, the solution with the initial condition $(x_1^0, \dots, x_n^0; z_1^0, \dots, z_m^0)$ is obtained as follows: First, solve the system of differential equations $dz_j/dx_1 = P_{j1}(x_1, x_2^0, \dots, x_n^0; z)$ in x_1 with the initial condition $z_j(x_1^0) = z_j^0$, and denote the solution by $z_j = \varphi_j(x_1)$. Next, considering x_1 as a parameter, solve the system of differential equations $dz_j/dx_2 = P_{j2}(x_1, x_2, x_3^0, \dots, x_n^0; z)$ in x_2 with the initial condition $z_j(x_2^0) = \varphi_j(x_1)$, and denote the solution by $z_j = \varphi_j(x_1, x_2)$. Repeat the process, until we finally have $z_j = \varphi_j(x_1, \dots, x_n)$, which is the solution of the original equation. Or, if we have m independent first integrals $f_j(x; z) = c_j$ of the equation $dz_j/dx_1 = P_{j1}(x; z)$, we may transform the equation into $du_j = \sum_{k=1}^n Q_{jk}(x; u) dx_k$ by the transformation $u_j = f_j(x; z)$. Since the $Q_{jk}(x; u)$ do not involve x_1 and the equation is a completely integrable total differential equation, we have reduced the number of variables. We obtain the general solution by repeating this process n times.

For

$$P(x, y, z) dx + Q(x, y, z) dy + R(x, y, z) dz = 0$$

($n=3, m=1$), the complete integrability condition is

$$P\left(\frac{\partial Q}{\partial z} - \frac{\partial R}{\partial y}\right) + Q\left(\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z}\right) + R\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right) = 0.$$

(II) Solution of Partial Differential Equations of First Order (→ 322 Partial Differential Equations (Methods of Integration), 324 Partial Differential Equations of First Order)

Let z be a function of x and y , and

$$p \equiv \partial z / \partial x, \quad q \equiv \partial z / \partial y, \quad r \equiv \partial^2 z / \partial x^2, \quad s \equiv \partial^2 z / \partial x \partial y, \quad t \equiv \partial^2 z / \partial y^2.$$

We consider a partial differential equation of the first order $F(x, y, z, p, q) = 0$.

(1) The Lagrange-Charpit Method. We consider the auxiliary equation

$$\frac{dx}{F_p} = \frac{dy}{F_q} = \frac{dz}{pF_p + qF_q} = \frac{-dp}{F_x + pF_z} = \frac{-dq}{F_y + qF_z},$$

which is a system of ordinary differential equations. Let $G(x, y, z, p, q) = a$ be the solution of the auxiliary equation. Using this together with the original equation $F = 0$, we obtain $p = P(x, y, z, a)$, $q = Q(x, y, z, a)$, and the complete solution by integrating $dz = P dx + Q dy$. If we know another solution of the auxiliary equation $H(x, y, z, p, q) = b$ independent of $G = a$, we have the complete solution $z = \Phi(x, y, a, b)$ by eliminating p and q from $F = 0$, $G = a$, and $H = b$.

(2) Solution of Various Standard Forms of Partial Differential Equations of the First Order. The integration constants are a, b .

(i) $f(p, q) = 0$. The complete solution is $z = ax + \varphi(a)y + b$, where the function $t = \varphi(a)$ is defined by $f(t, a) = 0$.

(ii) $f(px, q) = 0$, $f(x, qy) = 0$, $f(p/z, q/z) = 0$. These equations reduce to (i) if $x = e^x$, $y = e^y$, $z = e^z$, respectively.

(iii) $f(x, p, q) = 0$. If we can solve for $p = F(x, q)$, the complete solution is $z = \int F(x, a) dx + ay + b$. A similar procedure applies to $f(y, p, q) = 0$.

(iv) $f(z, p, q) = 0$. Solve $f(z, t, at) = 0$ for $t = F(z, a)$. The complete solution is then given by $x + ay + b = \int dz / F(z, a)$. If we eliminate a and b from the complete solution $\Phi(x, y, z, a, b) = 0$ and $\partial \Phi / \partial a = \partial \Phi / \partial b = 0$, we have the singular solution of the original equation.

(v) Separated variable type $f(x, p) = g(y, q)$. Solve the two ordinary differential equations $f(x, p) = a$ and $g(y, q) = a$ for the solutions $p = P(x, a)$ and $q = Q(y, a)$, respectively. Then the complete solution is $z = \int P(x, a) dx + \int Q(y, a) dy + b$.

(vi) Lagrange's partial differential equation $Pp + Qq = R$. Here P, Q, R are functions of x, y , and z . Denote the solutions of the system of differential equations $dx : dy : dz = P : Q : R$ by $u(x, y, z) = a$, $v(x, y, z) = b$. Then the general solution is $\Phi(u, v) = 0$, where Φ is an arbitrary function. A similar method is applicable to

$$\sum_{j=1}^n P_j(x_1, \dots, x_n) \frac{\partial z}{\partial x_j} = R(x_1, \dots, x_n).$$

If we have n independent solutions $u_j(x) = a_j$ of a system of n differential equations $dx_j / P_j = dz / R$ ($j = 1, \dots, n$), the general solution is given by $\Phi(u_1, \dots, u_n) = 0$.

(vii) Clairaut's partial differential equation $z = px + qy + f(p, q)$. The complete solution is given by the family of planes $z = ax + by + f(a, b)$. The singular solution as the envelope of the family of planes is given by eliminating p and q from the original equation and $x = -\partial f / \partial p$ and $y = -\partial f / \partial q$.

(III) Solutions of Partial Differential Equations of Second Order (→ 322 Partial Differential Equations (Methods of Integration))

(1) Quadrature. Here φ and ψ are arbitrary functions.

(i) $r = f(x)$. The general solution is $z = \iint f(x) dx dx + \varphi(y)x + \psi(y)$. A similar rule applies to $t = f(y)$.

(ii) $s = f(x, y)$. The general solution is $z = \iiint f(x, y) dx dy + \varphi(x) + \psi(y)$.

(iii) Wave equation. $r - t = 0$. The general solution is $z = \varphi(x + y) + \psi(x - y)$.

(iv) Laplace's differential equation. $r + t = 0$. Let $x + iy = \zeta$ and φ, ψ be complex analytic functions of ζ . The general solution is $z = \varphi(\zeta) + \psi(\bar{\zeta})$, and a real solution is $z = \varphi(\zeta) + \bar{\varphi}(\bar{\zeta})$.

(v) $r + Mp = N$, where M and N are functions of x and y . The general solution is given by $z = \int [\int L(x, y) N(x, y) dx + \varphi(y)] / L(x, y) dx + \psi(y)$, $L(x, y) = \exp \int [M(x, y) dx]$. In the integration, y is considered a constant.

A similar method is applicable to $s + Mp = N$, $s + Mq = N$, and $t + Mq = N$.

(vi) Monge-Ampère partial differential equation. $Rr + Ss + Tt + U(rt - s^2) = V$, where R, S, T, U, V are functions of x, y, z, p, q .

First, in the case $U=0$, we take auxiliary equations

$$\text{eq. (1)} \quad R dy^2 + T dx^2 - S dx dy = 0,$$

$$\text{eq. (2)} \quad R dp dy + T dq dx = V dx dy.$$

Equation (1) is decomposed into two linear differential forms $X_i dx + Y_i dy = 0$ ($i=1, 2$). The combination with (2) gives a solution $u_i(x, y, z, p, q) = a_i$, $v_i(x, y, z, p, q) = b_i$ ($i=1, 2$), and we have intermediate integrals $F_i(u_i, v_i) = 0$ ($i=1, 2$) for an arbitrary function F_i . We have the solution of the original equation by solving the intermediate integrals. If $S^2 \neq 4RT$, two intermediate integrals are distinct, and hence we can solve them in the form $p = P(x, y, z)$, $q = Q(x, y, z)$, and then we may integrate $dz = P dx + Q dy$.

Next, in the case $U \neq 0$, let λ_1 and λ_2 be the solutions of $U^2 \lambda^2 + US \lambda + TR + UV = 0$. We have two auxiliary equations

$$\begin{cases} \lambda_1 U dy + T dx + U dp = 0, \\ \lambda_2 U dx + R dy + U dq = 0, \end{cases} \quad \text{or} \quad \begin{cases} \lambda_2 U dy + T dx + U dp = 0, \\ \lambda_1 U dx + R dy + U dq = 0, \end{cases}$$

and from the solutions $u_i = a_i$, $v_i = b_i$ ($i=1, 2$), we have intermediate integrals $F_i(u_i, v_i) = 0$ ($i=1, 2$). If $4(TR + UV) \neq S^2$, $\lambda_1 \neq \lambda_2$, we have two different intermediate integrals $F_i = 0$. Solving the simultaneous equations $F_i = 0$ in $p = P(x, y, z)$, $q = Q(x, y, z)$, we may also find the solution by integrating $dz = P dx + Q dy$.

(vii) Poisson's differential equation. $P = (rt - s^2)^n Q$, where $P = P(p, q, r, s, t)$ is homogeneous with respect to r, s, t and we assume that $Q = Q(x, y, z)$ satisfies $\partial Q / \partial z \neq \infty$ for x, y, z when $rt = s^2$. The equation $P(p, \varphi(p), r, r\varphi'(p), r\{\varphi'(p)\}^2) = 0$ is then an ordinary differential equation in φ as a function of p . We first solve this for φ , and then solve a partial differential equation of the first order $q = \varphi(p)$ by the method (II)(2)(i).

(2) Intermediate Integrals. Let $f(x, y, z, p, q, r, s, t)$ be polynomials with respect to r, s, t . Suppose that $f(x, y, z, p, q, r, s, t) = 0$ has the first integral $u(x, y, z, p, q) = 0$. We insert

$$r = -\left(\frac{\partial u}{\partial x} + p \frac{\partial u}{\partial z} + s \frac{\partial u}{\partial q}\right) / \frac{\partial u}{\partial p}, \quad t = -\left(\frac{\partial u}{\partial y} + q \frac{\partial u}{\partial z} + s \frac{\partial u}{\partial p}\right) / \frac{\partial u}{\partial q}$$

into the original equation, and replace all the coefficients that are polynomials of s by 0. We thus obtain a system of differential equations in u . If u and v are two independent solutions of this system, an intermediate integral of the original equation is given in the form $\Phi(u, v) = 0$.

(3) Initial Value Problem for a Hyperbolic Partial Differential Equation $L[u] \equiv u_{xy} + au_x + bu_y + cu = h$.

$$\begin{aligned} u(\xi, \eta) = & [(uR)_A + (uR)_B] / 2 + \iint_{\Delta} R(x, y; \xi, \eta) h(x, y) dx dy \\ & + \int_A^B \left[\frac{1}{2} \left(u \frac{\partial R}{\partial n'} - R \frac{\partial u}{\partial n'} \right) - \{a \cos(n, x) + b \cos(n, y)\} uR \right] ds, \end{aligned}$$

where Δ is the hatched region in Fig. 19, and the conormal n' is the mirror image of the normal n with respect to $x = y$.

$$u(\xi, \eta) = (uR)_C + \int_C^A R(u_y + au) dy + \int_C^B R(u_x + bu) dx + \iint_{\square} R(x, y; \xi, \eta) h(x, y) dx dy$$

(characteristic initial value problem).

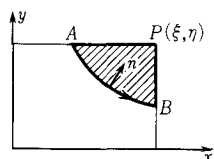


Fig. 19

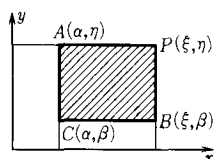


Fig. 20

Here \square is the hatched rectangular region in Fig. 20. $R(x, y; \xi, \eta)$ is the Riemann function; it satisfies

$$M[R(x, y; \xi, \eta)] = 0,$$

$$R_x - bR = 0 \quad (\text{on } x = \xi),$$

$$R_y - aR = 0 \quad (\text{on } y = \eta),$$

$$R(\xi, \eta; \xi, \eta) = 1.$$

Example (i). $u_{xy} = h(x, y)$. $R(x, y; \xi, \eta) = 1$.

$$u(\xi, \eta) = \frac{1}{2} [u_A + u_B] + \frac{1}{2} \int_A^B [u_y \cos(n, x) + u_x \cos(n, y)] ds + \iint_{\Delta} h(x, y) dx dy.$$

Example (ii). Telegraph equation $u_{xy} + cu = 0$ ($c > 0$). $R(x, y; \xi, \eta) = J_0(2\sqrt{c(x-\xi)(y-\eta)})$.

Example (iii). $u_{xy} + \frac{n}{x+y}(u_x + u_y) = 0$ ($n = \text{a constant} > 0$).

$$R(x, y; \xi, \eta) = \left(\frac{x+y}{\xi+\eta}\right)^n F\left(1-n, n; 1; -\frac{(x-\xi)(y-\eta)}{(x+y)(\xi+\eta)}\right).$$

(IV) Contact Transformations (→ 82 Contact Transformations)

We consider a transformation $(x_1, \dots, x_n; z) \rightarrow (X_1, \dots, X_n; Z)$. We put $p_j \equiv \partial z / \partial x_j$, $P_j \equiv \partial Z / \partial X_j$ ($j = 1, \dots, n$). The transformation is called a contact transformation if there exists a function $\rho(x, z, p) \neq 0$ satisfying $dZ - \sum P_j dX_j = \rho(x, z, p)(dz - \sum p_j dx_j)$.

A transformation given by $(2n+1)$ equations $\Omega = 0$, $\partial \Omega / \partial X_j + P_j \partial \Omega / \partial Z = 0$, $\partial \Omega / \partial x_j + p_j \partial \Omega / \partial z = 0$ generated by a generating function $\Omega(x, z, X, Z)$ is a contact transformation.

Generating Function	ρ	Transformation	Name
$\sum x_j X_j + z + Z$	-1	$X_j = -p_j, \quad P_j = -x_j,$ $Z = \sum p_j x_j - z$	Legendre's transformation
$\sum X_j^2 + Z^2 - \sum x_j X_j - zZ$	$Z/(2Z-z)$	$X_j = -p_j Z,$ $p_j = -(2X_j - x_j)/(2Z - z)$	Pedal transformation
$\sum (X_j - x_j)^2 + (Z - z)^2 - a^2$	1	$X_j = x_j - ap_j(1 + \sum p_j^2)^{-1/2},$ $P_j = p_j,$ $Z = z_j + a(1 + \sum p_j^2)^{-1/2}$	Similarity
$\sum (X_j - x_j)^2 - Z^2 - z^2$	$-\frac{1}{\sqrt{\sum p_j^2 - 1}}$	$X_j = x_j - p_j z,$ $P_j = -p_j(\sum p_j^2 - 1)^{-1/2},$ $Z = z(\sum p_j^2 - 1)^{1/2}$	

(V) Fundamental Solutions (→ 320 Partial Differential Equations H)

A function (or a generalized function such as a distribution) T satisfying $LT = \delta$ (δ is the Dirac delta function) for a linear differential operator L is called the fundamental (or elementary) solution of L . In the following table, we put

$$\Delta \equiv \sum_{i=1}^n \frac{\partial^2}{\partial x_i^2}, \quad \square \equiv \frac{\partial^2}{\partial x_n^2} - \sum_{i=1}^{n-1} \frac{\partial^2}{\partial x_i^2}, \quad r^2 \equiv \sum_{i=1}^n x_i^2, \quad \mathbf{1}(x) = \begin{cases} 1 & (x > 0) \\ 0 & (x \leq 0) \end{cases} \quad (\text{Heaviside function}).$$

J_ν is the Bessel function of the first kind; K_ν and I_ν are the modified Bessel functions. (→Table 19.IV, this Appendix.)

$$s = \begin{cases} \sqrt{x_n^2 - x_1^2 - \dots - x_{n-1}^2} & (\text{if } x_n > 0 \text{ and the quantity under the radical sign is positive}), \\ 0 & (\text{otherwise}). \end{cases}$$

(For Pf (finite part) → 125 Distributions and Hyperfunctions.)

Operator	Fundamental Solution
d/dx	$\mathbf{1}(x)$
$\frac{d^m}{dx^m}$	$\begin{cases} x^{m-1}/(m-1)! & (x > 0) \\ 0 & (x \leq 0) \end{cases}$
$\partial^n / \partial x_1 \partial x_2 \dots \partial x_n$	$\mathbf{1}(x_1)\mathbf{1}(x_2)\dots\mathbf{1}(x_n)$
$\frac{\partial}{\partial x} + i \frac{\partial}{\partial y}$	$\frac{1}{2\pi} \frac{1}{x+iy} \quad (i \equiv \sqrt{-1})$

Operator	Fundamental Solution
Δ	$\begin{cases} -\left[\Gamma\left(\frac{n}{2}\right)/2(n-2)\pi^{n/2}\right]\frac{1}{r^{n-2}} & (n \geq 3) \\ (1/2\pi)\log r & (n=2) \end{cases}$
Δ^m	$\begin{cases} \left[\Gamma\left(\frac{n}{2}\right)/2^m(m-1)!\pi^{n/2}\prod_{k=1}^m(2k-n)\right]\frac{1}{r^{n-2m}} \begin{pmatrix} n-2m \text{ is a positive integer} \\ \text{or a negative odd integer} \end{pmatrix} \\ \left[\Gamma\left(\frac{n}{2}\right)/2^m(m-1)!\pi^{n/2}\prod_{\substack{k=1 \\ k \neq m-h}}^m(2k-n)\right]\frac{\log r}{r^{n-2m}} \begin{pmatrix} n-2m=-2h, \\ h=0,1,2,\dots \end{pmatrix} \end{cases}$
$\left(1-\frac{\Delta}{4\pi^2}\right)^m$	$\frac{2\pi^m}{(m-1)!}r^{m-(n/2)}K_{(n/2)-m}(2\pi r)$
\square^m	$(\text{Pf } s^{2m-n})/\pi^{(n/2)-1}2^{2m-1}(m-1)!\Gamma[m+1-(n/2)]$
$(\square-\lambda)^m$ (λ is real and $\neq 0$)	$\frac{ \lambda ^{(n/4)-(m/2)}}{\pi^{(n/2)-1}(m-1)!2^{m-1+(n/2)}}\text{Pf } s^{m-(n/2)}\begin{cases} I_{m-(n/2)}(\sqrt{\lambda}s) & (\lambda > 0) \\ J_{m-(n/2)}(\sqrt{ \lambda }s) & (\lambda < 0) \end{cases}$
$\left(\frac{\partial}{\partial x_n}-\sum_{i=1}^{n-1}\frac{\partial^2}{\partial x_i^2}\right)^m$	$\begin{cases} \frac{x_n^{m-1}}{(m-1)!}\left(\frac{1}{2\sqrt{\pi x_n}}\right)^{n-1}\exp\left(-\sum_{i=1}^{n-1}x_i^2/4x_n\right) & (x_n > 0) \\ 0 & (x_n \leq 0) \end{cases}$

(VI) Solution of Boundary Value Problems (\rightarrow 188 Green's Functions, 323 Partial Differential Equations of Elliptic Type, 327 Partial Differential Equations of Parabolic Type)

$$L[u] = Au_{xx} + 2Bu_{xy} + Cu_{yy} + Du_x + Eu_y + Fu,$$

$$M[v] = (Av)_{xx} + 2(Bv)_{xy} + (Cv)_{yy} - (Dv)_x - (Ev)_y + Fv.$$

$$\text{Green's formula} \quad \int \int_D \{vL[u] - uM[v]\} dx dy = \int_C \left\{ P \left(u \frac{\partial v}{\partial n'} - v \frac{\partial u}{\partial n'} \right) + Quv \right\} ds.$$

$$\text{eq. (1)} \quad \begin{cases} A \cos(n, x) + B \cos(n, y) = P \cos(n', x), \\ B \cos(n, x) + C \cos(n, y) = P \cos(n', y). \end{cases}$$

$$Q = (A_x + B_y - D)\cos(n, x) + (B_x + C_y - E)\cos(n, y).$$

The integration contour C is the boundary of the domain D (Fig. 21), n is the inner normal of C , and n' , called the conormal, is given by (1).

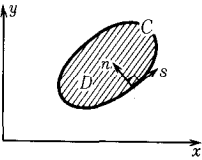


Fig. 21

(1) Elliptic Partial Differential Equation $L[u] \equiv u_{xx} + u_{yy} + au_x + bu_y + cu = h$.

$$u(\xi, \eta) = - \int_C u(s) \frac{\partial G}{\partial n} ds + \int \int_D G(x, y; \xi, \eta) h(x, y) dx dy.$$

Here $G(x, y; \xi, \eta)$ is Green's function, which satisfies $M(G(x, y; \xi, \eta)) = 0$ in the interior of D except at $(x, y) \neq (\xi, \eta)$, and

$$G(x, y; \xi, \eta) = -(1/2\pi)\log\sqrt{(x-\xi)^2 + (y-\eta)^2} + \text{a regular function},$$

$$G(x, y; \xi, \eta) = 0 \quad ((x, y) \in C).$$

(2) Laplace's Differential Equation in the 2-Dimensional Case $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$.

$$u(x, y) \equiv \tilde{u}(r, \varphi) = \operatorname{Re} f(z) \quad (z \equiv x + iy = re^{i\theta}).$$

(i) Interior of a disk ($r \leq 1$).

$$f(z) = \frac{1}{2\pi} \int_0^{2\pi} \tilde{u}(1, \varphi) \frac{e^{i\varphi} + z}{e^{i\varphi} - z} d\varphi, \quad \tilde{u}(r, \theta) = \frac{1}{2\pi} \int_0^{2\pi} \tilde{u}(1, \varphi) \frac{1 - r^2}{1 - 2r \cos(\theta - \varphi) + r^2} d\varphi$$

(Poisson's integration formula).

(ii) Annulus ($0 < q \leq r \leq 1$).

$$f(z) = \frac{\omega_1}{\pi^2 i} \left\{ \int_0^{2\pi} \tilde{u}(1, \varphi) \zeta_1(w) d\varphi - \int_0^{2\pi} \tilde{u}(q, \varphi) \zeta_3(w) d\varphi - a \log z \right\}$$

(Villat's integration formula).

$$w = \frac{\omega_1}{\pi} (i \log z + \varphi), \quad a = \left(\frac{1}{2\omega_3} - \frac{\eta_1}{\pi i} \right) \int_0^{2\pi} \{ \tilde{u}(1, \varphi) - \tilde{u}(q, \varphi) \} d\varphi, \quad \frac{\omega_3}{\omega_1} = -\frac{i}{\pi} \log q.$$

Here ζ_1 and ζ_3 are the Weierstrass ζ -functions (\rightarrow 134 Elliptic Functions) with the fundamental periods $2\omega_1$ and $2\omega_3$.

(iii) Half-plane ($y \geq 0$). $f(z) = \frac{i}{\pi} \int_{-\infty}^{\infty} \frac{u(t, 0)}{z - t} dt, \quad u(x, y) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{u(t, 0)y}{(x - t)^2 + y^2} dt.$

(3) Laplace's Differential Equation in the 3-Dimensional Case.

(i) Interior of a sphere ($r \leq 1$).

$$\tilde{u}(r, \varphi, \theta) = \frac{1}{4\pi} \int_0^\pi \int_0^{2\pi} \tilde{u}(1, \Phi, \Theta) \frac{1 - r^2}{(1 - 2r \cos \gamma + r^2)^{3/2}} \sin \Theta d\Theta d\Phi,$$

where

$$\cos \gamma = \cos \Theta \cos \theta + \sin \Theta \sin \theta \cos(\Phi - \varphi).$$

(ii) Half-space ($z \geq 0$).

$$u(x, y, z) = \frac{z}{2\pi} \int \int_{-\infty}^{\infty} \frac{u(\xi, \eta, 0)}{\{(x - \xi)^2 + (y - \eta)^2 + z^2\}^{3/2}} d\xi d\eta.$$

(4) Equation of Oscillation (Helmholtz Differential Equation) $\Delta u + k^2 u = 0$. Let u_n be the normalized eigenfunction with the same boundary condition for the eigenvalue k_n . Green's function is

$$G(P, Q) = \sum \frac{u_n(P) u_n^*(Q)}{k^2 - k_n^2}$$

Domain	Boundary Condition	Eigenvalue	Eigenfunction
rectangle $0 < x < a, \quad 0 < y < b$	$u = 0$	$k_{nm} = \pi \sqrt{\frac{n^2}{a^2} + \frac{m^2}{b^2}}$ ($n, m = 1, 2, \dots$)	$\sin n\pi \frac{x}{a} \sin m\pi \frac{y}{b}$
circle $0 < r < a$	$u = 0$	k_{nm} is the root of $J_m(kx) = 0$	$J_m(k_{nm}r) e^{\pm im\varphi}$
annulus $b < r < a$	$u = 0$	k_{nm} is the root of $J_m(ka)N_m(kb) - J_m(kb)N_m(ka) = 0$	$\left\{ \frac{J_m(k_{nm}r)}{J_m(k_{nm}a)} - \frac{N_m(k_{nm}r)}{N_m(k_{nm}a)} \right\} e^{\pm im\varphi}$
fan shape $0 < r < a, \quad 0 < \varphi < \alpha$	$u = 0$	k_{nm} is the root of $J_\mu(ka) = 0$ ($\mu = m\pi/\alpha$)	$J_\mu(k_{nm}r) \sin \mu\varphi$
rectangular parallelepiped $0 < x < a, \quad 0 < y < b, \quad 0 < z < c$	$\frac{\partial u}{\partial n} = 0$	$k_{nml} = \pi \sqrt{\frac{n^2}{a^2} + \frac{m^2}{b^2} + \frac{l^2}{c^2}}$	$\cos n\pi \frac{x}{a} \cos m\pi \frac{y}{b} \cos l\pi \frac{z}{c}$
sphere $0 < r < a$	$\frac{\partial u}{\partial n} = 0$	k_{nl} is the root of $\psi'_n(ka) = 0$, where $\psi_n(\rho) \equiv \sqrt{\pi/2} J_{n+(1/2)}(\rho)$	$\psi_n(k_{nl}r) P_n^m(\cos \theta) e^{\pm im\varphi}$

(5) Heat Equation. $\frac{\partial u}{\partial t} = \kappa \Delta u \left(\Delta = \frac{\partial^2}{\partial x_1^2} + \dots + \frac{\partial^2}{\partial x_m^2}; \kappa \text{ is a positive constant} \right)$. Boundary condi-

tion: $hu - k \partial u / \partial n = \varphi$, where h and k are nonnegative constants with $h + k = 1$, and φ is a given function.

$$u(P, t) = \int_V G(P, Q, t) u(Q, 0) dV_Q + \kappa \int_0^t d\tau \int_S \left\{ \frac{\partial G(P, Q, t - \tau)}{\partial n_Q} + G(P, Q, t - \tau) \right\} \varphi(Q, \tau) dS_Q.$$

Here V is the domain, and S is its boundary. $G(P, Q, t)$ is the elementary solution that satisfies $\partial G / \partial t = \kappa \Delta G$ in V and $k \partial G / \partial n = hG$ on S , and further in the neighborhood of $P = Q$, $t = 0$, it has the form $G(P, Q, t) = (4\pi\kappa t)^{-m/2} e^{-R^2/4\kappa t} + \text{terms of lower degree}$ ($R = \overline{PQ}$).

(i) $-\infty < x < \infty$, $G = U(x - \xi, t)$, where $U(x, t) = e^{-x^2/4\kappa t} / \sqrt{4\pi\kappa t}$ (similar in the following case (ii)).

(ii) $0 \leq x < \infty$. $u(0, t) = 0$: $G = U(x - \xi, t) - U(x + \xi, t)$.

$$\frac{\partial u}{\partial x} = hu: \quad G = U(x - \xi, t) + U(x + \xi, t) - 2he^{h\xi} \int_{-\infty}^{-\xi} e^{h\eta} U(x - \eta, t) d\eta.$$

(iii) $0 \leq x \leq l$. $u(0, t) = u(l, t) = 0$: $G = \vartheta\left(\frac{x - \xi}{2l} \middle| \tau\right) - \vartheta\left(\frac{x + \xi}{2l} \middle| \tau\right)$.

$$\frac{\partial u}{\partial x}(0, t) = \frac{\partial u}{\partial x}(l, t) = 0: \quad G = \vartheta\left(\frac{x - \xi}{2l} \middle| \tau\right) + \vartheta\left(\frac{x + \xi}{2l} \middle| \tau\right).$$

$$u(0, t) = \frac{\partial u}{\partial x}(l, t) = 0: \quad G = \vartheta\left(\frac{x - \xi}{4l} \middle| \tau\right) - \vartheta\left(\frac{x + \xi}{4l} \middle| \tau\right) + \vartheta\left(\frac{x + \xi - 2l}{4l} \middle| \tau\right) - \vartheta\left(\frac{x - \xi - 2l}{4l} \middle| \tau\right).$$

Here ϑ is the elliptic theta function: $\vartheta(x|\tau) = \vartheta_3(x, \tau) \equiv 1 + 2 \sum e^{i\pi n^2 \tau} \cos 2n\pi x$.

(iv) $0 \leq x < \infty$, $0 \leq y < \infty$. $u(x, 0, t) = u(0, y, t) = 0$:

$$G = (e^{-(x-\xi)^2/4\kappa t} - e^{-(x+\xi)^2/4\kappa t})(e^{-(y-\eta)^2/4\kappa t} - e^{-(y+\eta)^2/4\kappa t}) / 4\pi\kappa t.$$

(v) $0 \leq x \leq a$, $0 \leq y \leq b$. $u = 0$ on the boundary:

$$G = \frac{4}{ab} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \exp\left\{-\kappa\pi^2 t \left(\frac{m^2}{a^2} + \frac{n^2}{b^2}\right)\right\} \sin \frac{m\pi x}{a} \sin \frac{m\pi \xi}{a} \sin \frac{n\pi y}{b} \sin \frac{n\pi \eta}{b}.$$

(vi) $0 \leq x \leq a$, $0 \leq y \leq b$, $0 \leq z \leq c$. $u = 0$ on the boundary:

$$G = \frac{8}{abc} \sum_{l=1}^{\infty} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \exp\left\{-\kappa\pi^2 t \left(\frac{l^2}{a^2} + \frac{m^2}{b^2} + \frac{n^2}{c^2}\right)\right\} \times \sin \frac{l\pi x}{a} \sin \frac{l\pi \xi}{a} \sin \frac{m\pi y}{b} \sin \frac{m\pi \eta}{b} \sin \frac{n\pi z}{c} \sin \frac{n\pi \zeta}{c}.$$

(vii) $0 \leq r < \infty$. Spherically symmetric. $|x| = r$, $|\xi| = r'$:

$$G = (e^{-(r-r')^2/4\kappa t} - e^{-(r+r')^2/4\kappa t}) / 8\pi r r' (\pi\kappa t)^{1/2}.$$

(viii) $0 \leq r \leq a$. Spherically symmetric. $u = 0$ on the boundary:

$$G = \frac{1}{2\pi a r r'} \sum_{n=1}^{\infty} e^{-\kappa n^2 \pi^2 t / a^2} \sin \frac{n\pi r}{a} \sin \frac{n\pi r'}{a}.$$

(ix) $a \leq r < \infty$. Spherically symmetric. $k \partial u / \partial r - hu = 0$ on the boundary:

$$G = \frac{1}{8\pi r r' (\pi\kappa t)^{1/2}} \left[e^{-(r-r')^2/4\kappa t} + e^{-(r+r'-2a)^2/4\kappa t} - \frac{ah+k}{ak} (4\pi\kappa t)^{1/2} \times \exp\left\{\kappa t \left(\frac{ah+k}{ak}\right)^2 + (r+r'-2a) \frac{ah+k}{ak}\right\} \times \operatorname{erfc}\left\{\frac{r+r'-2a}{2\sqrt{\kappa t}} + \frac{ah+k}{ak} \sqrt{\kappa t}\right\} \right] \left(\operatorname{erfc} x \equiv \int_x^{\infty} e^{-t^2} dt\right).$$

(x) $0 \leq r < \infty$. Axially symmetric: $G = e^{-(r^2+r'^2)/4\kappa t} I_0(rr'/2\kappa t) / 4\pi\kappa t$.

(xi) $0 \leq r \leq a$. Axially symmetric. $k \frac{\partial u}{\partial r} - hu = 0$ on the boundary:

$$G = \frac{1}{\pi a^2} \sum_{n=1}^{\infty} \frac{J_0(r\alpha_n) J_0(r'\alpha_n)}{\{J_0(\alpha_n)\}^2 + \{J_1(\alpha_n)\}^2} e^{-\kappa\alpha_n^2 t / a^2},$$

where α_n is given by $k\alpha_n J_1(\alpha_n) - hJ_0(\alpha_n) = 0$.

16. Elliptic Integrals and Elliptic Functions

(I) Elliptic Integrals (\rightarrow 134 Elliptic Functions)

(1) Legendre-Jacobi Standard Form.

Elliptic integral of the first kind

$$F(k, \varphi) \equiv \int_0^\varphi \frac{d\psi}{\sqrt{1-k^2 \sin^2 \psi}} = \int_0^{\sin \varphi} \frac{dt}{\sqrt{(1-t^2)(1-k^2 t^2)}} \quad (k \text{ is the modulus}).$$

Elliptic integral of the second kind

$$E(k, \varphi) \equiv \int_0^\varphi \sqrt{1-k^2 \sin^2 \psi} \, d\psi = \int_0^{\sin \varphi} \sqrt{\frac{1-k^2 t^2}{1-t^2}} \, dt.$$

Elliptic integral of the third kind

$$\Pi(\varphi, n, k) = \int_0^\varphi \frac{d\psi}{(1+n \sin^2 \psi) \sqrt{1-k^2 \sin^2 \psi}} = \int_0^{\sin \varphi} \frac{dt}{(1+nt^2) \sqrt{(1-t^2)(1-k^2 t^2)}}.$$

When $\varphi = \pi/2$, elliptic integrals of the first and the second kinds are called complete elliptic integrals:

$$K(k) \equiv F\left(k, \frac{\pi}{2}\right) = \int_0^{\pi/2} \frac{d\psi}{\sqrt{1-k^2 \sin^2 \psi}} = \int_0^1 \frac{dt}{\sqrt{(1-t^2)(1-k^2 t^2)}} = \frac{\pi}{2} F\left(\frac{1}{2}, \frac{1}{2}; 1; k^2\right),$$

$$E(k) \equiv E\left(k, \frac{\pi}{2}\right) = \int_0^{\pi/2} \sqrt{1-k^2 \sin^2 \psi} \, d\psi = \int_0^1 \sqrt{\frac{1-k^2 t^2}{1-t^2}} \, dt = \frac{\pi}{2} F\left(-\frac{1}{2}, \frac{1}{2}; 1; k^2\right),$$

where F is the hypergeometric function.

$$K(k') = K(\sqrt{1-k^2}) \equiv K'(k), \quad E(k') = E(\sqrt{1-k^2}) \equiv E'(k) \quad (k'^2 = 1-k^2; k' \text{ is the complementary modulus}).$$

$$EK' + E'K - KK' = \frac{\pi}{2} \quad (\text{Legendre's relation}). \quad K\left(\frac{1}{\sqrt{2}}\right) = \frac{\Gamma(1/4)^2}{4\sqrt{\pi}}.$$

$$\frac{\partial F}{\partial k} = \frac{1}{k^2} \left(\frac{E - k'^2 F}{k} - \frac{\sin \varphi \cos \varphi}{\sqrt{1-k^2 \sin^2 \varphi}} \right), \quad \frac{\partial E}{\partial k} = \frac{E - F}{k}.$$

(2) Change of Variables.

$$\tan(\psi - \varphi) = k' \tan \varphi: \quad F\left(\frac{1-k'}{1+k'}, \psi\right) = (1+k')F(k, \varphi),$$

$$E\left(\frac{1-k'}{1+k'}, \psi\right) = \frac{2}{1+k'} [E(k, \varphi) + k' F(k, \varphi)] - \frac{1-k'}{1+k'} \sin \psi.$$

$$\sin \chi = \frac{(1+k) \sin \varphi}{1+k \sin^2 \varphi}: \quad F\left(\frac{2\sqrt{k}}{1+k}, \chi\right) = (1+k)F(k, \varphi),$$

$$E\left(\frac{2\sqrt{k}}{1+k}, \chi\right) = \frac{1}{1+k} \left[2E(k, \varphi) - k'^2 F(k, \varphi) + 2k \frac{\sin \varphi \cos \varphi}{1+k \sin^2 \varphi} \sqrt{1-k^2 \sin^2 \varphi} \right].$$

k_1	$\sin \varphi_1$	$\cos \varphi_1$	$F(k_1, \varphi_1)$	$E(k_1, \varphi_1)$
$i \frac{k}{k'}$	$k' \frac{\sin \varphi}{\sqrt{1-k^2 \sin^2 \varphi}}$	$\frac{\cos \varphi}{\sqrt{1-k^2 \sin^2 \varphi}}$	$k' F(k, \varphi)$	$\frac{1}{k'} \left[E(k, \varphi) - k^2 \frac{\sin \varphi \cos \varphi}{\sqrt{1-k^2 \sin^2 \varphi}} \right]$
k'	$-i \tan \varphi$	$\frac{1}{\cos \varphi}$	$-i F(k, \varphi)$	$i [E(k, \varphi) - F(k, \varphi) - \sqrt{1-k^2 \sin^2 \varphi} \tan \varphi]$
$\frac{1}{k}$	$k \sin \varphi$	$\sqrt{1-k^2 \sin^2 \varphi}$	$k F(k, \varphi)$	$\frac{1}{k} [E(k, \varphi) - k'^2 F(k, \varphi)]$

(3) Transformation into Standard Form.
(i) The following are reducible to elliptic integrals of the first kind (we assume $a > b > 0$ for parameters).

$AF(k, \varphi)$	A	k	φ
$\int_1^x \frac{dt}{\sqrt{t^3-1}}$	$\frac{1}{\sqrt[4]{3}}$	$\frac{\sqrt{3}-1}{2\sqrt{2}}$	$\arccos \frac{\sqrt{3}+1-x}{\sqrt{3}-1+x}$
$\int_1^x \frac{dt}{\sqrt{1-t^3}}$	$\frac{1}{\sqrt[4]{3}}$	$\frac{\sqrt{3}+1}{2\sqrt{2}}$	$\arccos \frac{\sqrt{3}-1+x}{\sqrt{3}+1-x}$
$\int_0^x \frac{dt}{\sqrt{1+t^4}}$	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\arccos \frac{1-x^2}{1+x^2}$
$\int_0^x \frac{dt}{\sqrt{(a^2-t^2)(b^2-t^2)}}$	$\frac{1}{a}$	$\frac{b}{a}$	$\arcsin \frac{x}{b}$
$\int_b^x \frac{dt}{\sqrt{(a^2-t^2)(t^2-b^2)}}$	$\frac{1}{a}$	$\sqrt{1-(b/a)^2}$	$\arcsin \sqrt{\frac{1-(b/x)^2}{1-(b/a)^2}}$
$\int_a^x \frac{dt}{\sqrt{(t^2-a^2)(t^2-b^2)}}$	$\frac{1}{a}$	$\frac{b}{a}$	$\arcsin \sqrt{\frac{1-(a/x)^2}{1-(b/x)^2}}$
$\int_0^x \frac{dt}{\sqrt{(a^2+t^2)(b^2+t^2)}}$	$\frac{1}{a}$	$\sqrt{1-(b/a)^2}$	$\arctan \frac{x}{b}$
$\int_0^x \frac{dt}{\sqrt{(a^2-t^2)(b^2+t^2)}}$	$\frac{1}{\sqrt{a^2+b^2}}$	$\frac{a}{\sqrt{a^2+b^2}}$	$\arcsin \sqrt{\frac{1+(b/a)^2}{1+(b/x)^2}}$
$\int_b^x \frac{dt}{\sqrt{(a^2+t^2)(t^2-b^2)}}$	$\frac{1}{\sqrt{a^2+b^2}}$	$\frac{a}{\sqrt{a^2+b^2}}$	$\arccos \frac{b}{x}$

(ii) The following are reducible to elliptic integrals of the second kind (we assume $a > b > 0$ for parameters).

$AE(k, \varphi)$	A	k	φ
$\int_0^x \sqrt{\frac{a^2-t^2}{b^2-t^2}} dt$	a	$\frac{b}{a}$	$\arcsin \frac{x}{b}$
$\int_x^a \sqrt{\frac{b^2+t^2}{a^2-t^2}} dt$	$\sqrt{a^2+b^2}$	$\frac{a}{\sqrt{a^2+b^2}}$	$\arccos \frac{x}{a}$
$\int_b^x \frac{1}{t^2} \sqrt{\frac{t^2+a^2}{t^2-b^2}} dt$	$\frac{\sqrt{a^2+b^2}}{b^2}$	$\frac{a}{\sqrt{a^2+b^2}}$	$\arccos \frac{b}{x}$
$\int_\infty^x t^2 \sqrt{\frac{t^2+a^2}{t^2-b^2}} dt$	$\sqrt{a^2+b^2}$	$\frac{a}{\sqrt{a^2+b^2}}$	$\arcsin \sqrt{\frac{1+(b/a)^2}{1+(x/a)^2}}$
$\int_0^x \sqrt{\frac{a^2+t^2}{(b^2+t^2)^3}} dt$	$\frac{a}{b^2}$	$\frac{\sqrt{a^2-b^2}}{a}$	$\arctan \frac{x}{b}$
$\int_b^x \frac{dt}{t^2 \sqrt{(t^2-b^2)(a^2-t^2)}}$	$\frac{1}{ab^2}$	$\frac{\sqrt{a^2-b^2}}{a}$	$\arcsin \sqrt{\frac{1-(b/x)^2}{1-(b/a)^2}}$

(II) Elliptic Theta Functions

(1) For $\text{Im } \tau > 0$, we put $q \equiv e^{i\pi\tau}$ and define

$$\vartheta_0(u, \tau) \equiv \vartheta_4(u, \tau) \equiv 1 + 2 \sum_{n=1}^\infty (-1)^n q^{n^2} \cos 2n\pi u,$$

$$\vartheta_1(u, \tau) \equiv 2 \sum_{n=0}^\infty (-1)^n q^{[n+(1/2)]^2} \sin(2n+1)\pi u,$$

$$\vartheta_2(u, \tau) \equiv 2 \sum_{n=0}^\infty q^{[n+(1/2)]^2} \cos(2n+1)\pi u,$$

$$\vartheta_3(u, \tau) \equiv 1 + 2 \sum_{n=1}^\infty q^{n^2} \cos 2n\pi u.$$

Each of the four functions ϑ_j ($j=0, 1, 2, 3$) as a function of two variables u and τ satisfies the following partial differential equation

$$\frac{\partial^2 \vartheta(u, \tau)}{\partial u^2} = 4\pi i \frac{\partial \vartheta(u, \tau)}{\partial \tau}.$$

(2) Mutual Relations.

$$\vartheta_0^4(u) + \vartheta_2^4(u) = \vartheta_1^4(u) + \vartheta_3^4(u), \quad \vartheta_0^2(u) = k\vartheta_1^2(u) + k'\vartheta_3^2(u),$$

$$\vartheta_2^2(u) = -k'\vartheta_1^2(u) + k\vartheta_3^2(u), \quad \vartheta_1^2(u) = k\vartheta_0^2(u) - k'\vartheta_2^2(u),$$

where k is the modulus such that $iK'(k)/K(k) = \tau$, and k' is the corresponding complementary modulus.

$$k = \vartheta_2^2(0)/\vartheta_3^2(0), \quad k' = \vartheta_0^2(0)/\vartheta_3^2(0).$$

$$\vartheta_1'(0) = \pi\vartheta_2(0)\vartheta_3(0)\vartheta_0(0), \quad \frac{\vartheta_1'''(0)}{\vartheta_1'(0)} = \frac{\vartheta_2''(0)}{\vartheta_0(0)} + \frac{\vartheta_3''(0)}{\vartheta_3(0)} + \frac{\vartheta_0''(0)}{\vartheta_0(0)}.$$

(3) Pseudoperiodicity. In the following table, the only variables in ϑ are u and τ . m and n are integers.

Increment of u	ϑ_0	ϑ_1	ϑ_2	ϑ_3	Exponential Factor
$m + n\tau$	$(-1)^n \vartheta_0$	$(-1)^{m+n} \vartheta_1$	$(-1)^m \vartheta_2$	ϑ_3	$\left. \begin{array}{l} \exp[-n\pi i \\ \times (2u + n\tau)] \end{array} \right\}$
$m - \frac{1}{2} + n\tau$	ϑ_3	$(-1)^{m+1} \vartheta_2$	$(-1)^{m+n} \vartheta_1$	$(-1)^n \vartheta_0$	
$m + \left(n + \frac{1}{2}\right)\tau$	$(-1)^n i \vartheta_1$	$(-1)^{m+n} i \vartheta_0$	$(-1)^m \vartheta_3$	ϑ_2	$\left. \begin{array}{l} \exp\left[-\left(n + \frac{1}{2}\right)\pi i\right] \\ \times \left\{2u + \left(n + \frac{1}{2}\right)\tau\right\} \right\}$
$m - \frac{1}{2} + \left(n + \frac{1}{2}\right)\tau$	ϑ_2	ϑ_3	$(-1)^{m+n} i \vartheta_0$	$(-1)^n i \vartheta_1$	
Zeros $u =$	m $+ \left(n + \frac{1}{2}\right)\tau$	$m + n\tau$	$m + \frac{1}{2}$ $+ n\tau$	$m + \frac{1}{2}$ $+ \left(n + \frac{1}{2}\right)\tau$	

(4) Expansion into Infinite Products. We put $Q_0 \equiv \prod_{n=1}^{\infty} (1 - q^{2n})$. Then we have

$$\begin{aligned}\vartheta_0(u) &= Q_0 \prod_{n=1}^{\infty} (1 - 2q^{2n-1} \cos 2\pi u + q^{4n-2}), \\ \vartheta_1(u) &= 2Q_0 q^{1/4} \sin \pi u \prod_{n=1}^{\infty} (1 - 2q^{2n} \cos 2\pi u + q^{4n}), \\ \vartheta_2(u) &= 2Q_0 q^{1/4} \cos \pi u \prod_{n=1}^{\infty} (1 + 2q^{2n} \cos 2\pi u + q^{4n}), \\ \vartheta_3(u) &= Q_0 \prod_{n=1}^{\infty} (1 + 2q^{2n-1} \cos 2\pi u + q^{4n-2}).\end{aligned}$$

(III) Jacobi's Elliptic Functions

(1) We express the modulus k and the complementary modulus as follows.

$$k = \frac{\vartheta_2^2(0)}{\vartheta_3^2(0)}, \quad k' = \frac{\vartheta_0^2(0)}{\vartheta_3^2(0)}, \quad k^2 + k'^2 = 1.$$

Then we have

$$K(k) = K = \frac{\pi}{2} \vartheta_3^2(0), \quad K'(k) = K' = -i\tau K.$$

The relation between q and k is

$$\begin{aligned}q &= e^{i\pi\tau} = e^{-\pi(K'/K)}, \\ q^{1/4} &= \left(\frac{k}{4}\right)^{1/2} \left[1 + 2\left(\frac{k}{4}\right)^2 + 15\left(\frac{k}{4}\right)^4 + 150\left(\frac{k}{4}\right)^6 + 1707\left(\frac{k}{4}\right)^8 + \dots \right], \\ q &= \frac{1}{2}L + \frac{2}{2^5}L^5 + \frac{15}{2^9}L^9 + \frac{150}{2^{13}}L^{13} + \frac{1707}{2^{17}}L^{17} + \dots, \quad \text{where } L = \frac{1 - \sqrt[4]{1-k^2}}{1 + \sqrt[4]{1-k^2}}.\end{aligned}$$

(2) Functions sn, cn, dn; Addition Theorem.

$$\begin{aligned}\operatorname{sn}(u, k) &\equiv \frac{1}{\sqrt{k}} \frac{\vartheta_1(u/2K)}{\vartheta_0(u/2K)}, \quad \operatorname{cn}(u, k) \equiv \sqrt{\frac{k'}{k}} \frac{\vartheta_2(u/2K)}{\vartheta_0(u/2K)}, \quad \operatorname{dn}(u, k) \equiv \sqrt{k'} \frac{\vartheta_3(u/2K)}{\vartheta_0(u/2K)}, \\ \operatorname{sn}^2 u + \operatorname{cn}^2 u &= 1, \quad \operatorname{dn}^2 u + k^2 \operatorname{sn}^2 u = 1, \\ \operatorname{sn}(u+v) &= \frac{\operatorname{sn} u \operatorname{cn} v \operatorname{dn} v + \operatorname{sn} v \operatorname{cn} u \operatorname{dn} u}{1 - k^2 \operatorname{sn}^2 u \operatorname{sn}^2 v}, \quad \operatorname{cn}(u+v) = \frac{\operatorname{cn} u \operatorname{cn} v - \operatorname{sn} u \operatorname{dn} u \operatorname{sn} v \operatorname{dn} v}{1 - k^2 \operatorname{sn}^2 u \operatorname{sn}^2 v}, \\ \operatorname{dn}(u+v) &= \frac{\operatorname{dn} u \operatorname{dn} v - k^2 \operatorname{sn} u \operatorname{cn} u \operatorname{sn} v \operatorname{cn} v}{1 - k^2 \operatorname{sn}^2 u \operatorname{sn}^2 v}, \\ \frac{d \operatorname{sn} u}{du} &= \operatorname{cn} u \operatorname{dn} u, \quad \frac{d \operatorname{cn} u}{du} = -\operatorname{sn} u \operatorname{dn} u, \quad \frac{d \operatorname{dn} u}{du} = -k^2 \operatorname{sn} u \operatorname{cn} u.\end{aligned}$$

(3) Periodicity. In the next table, m and n are integers.

Increment of u	$\operatorname{sn} u$	$\operatorname{cn} u$	$\operatorname{dn} u$
$2mK + 2niK'$	$(-1)^m \operatorname{sn} u$	$(-1)^{m+n} \operatorname{cn} u$	$(-1)^n \operatorname{dn} u$
$(2m-1)K + 2niK'$	$(-1)^{m+1} \frac{\operatorname{cn} u}{\operatorname{dn} u}$	$(-1)^{m+n} k' \frac{\operatorname{sn} u}{\operatorname{dn} u}$	$(-1)^n k' \frac{1}{\operatorname{dn} u}$
$2mK + (2n+1)iK'$	$(-1)^m k^{-1} \frac{1}{\operatorname{sn} u}$	$(-1)^{m+n+1} k^{-1} \frac{\operatorname{dn} u}{\operatorname{sn} u}$	$i(-1)^{n+1} \frac{\operatorname{cn} u}{\operatorname{sn} u}$
$(2m-1)K + (2n+1)iK'$	$(-1)^{m+1} k^{-1} \frac{\operatorname{dn} u}{\operatorname{cn} u}$	$(-1)^{m+n} i k' k^{-1} \frac{1}{\operatorname{cn} u}$	$(-1)^n i k' \frac{\operatorname{sn} u}{\operatorname{cn} u}$
Zeros $u =$	$2nK + 2miK'$	$(2n+1)K + 2miK'$	$(2n+1)K + (2m+1)iK'$
Poles $u =$	$2nK + (2m+1)iK'$	$2nK + (2m+1)iK'$	$2nK + (2m+1)iK'$
Fundamental periods	$4K, 2iK'$	$4K, 2K + 2iK'$	$2K, 4iK'$

(4) Change of Variables. In the next table, the second column, for example, means the relation $\operatorname{sn}(ku, 1/k) = k \operatorname{sn}(u, k)$.

u	k	sn	cn	dn
ku	$1/k$	$k \operatorname{sn}$	dn	cn
iu	k'	$i \frac{\operatorname{sn}}{\operatorname{cn}}$	$\frac{1}{\operatorname{cn}}$	$\frac{\operatorname{dn}}{\operatorname{cn}}$
$k'u$	$i \frac{k}{k'}$	$k' \frac{\operatorname{sn}}{\operatorname{dn}}$	$\frac{\operatorname{cn}}{\operatorname{dn}}$	$\frac{1}{\operatorname{dn}}$
iku	$i \frac{k'}{k}$	$ik \frac{\operatorname{sn}}{\operatorname{dn}}$	$\frac{1}{\operatorname{dn}}$	$\frac{\operatorname{cn}}{\operatorname{dn}}$
$ik'u$	$\frac{1}{k'}$	$ik' \frac{\operatorname{sn}}{\operatorname{cn}}$	$\frac{\operatorname{dn}}{\operatorname{cn}}$	$\frac{1}{\operatorname{cn}}$
$(1+k)u$	$\frac{2\sqrt{k}}{1+k}$	$\frac{(1+k)\operatorname{sn}}{1+k\operatorname{sn}^2}$	$\frac{\operatorname{cn} \operatorname{dn}}{1+k\operatorname{sn}^2}$	$\frac{1-k\operatorname{sn}^2}{1+k\operatorname{sn}^2}$ (Gauss's transformation)
$(1+k')u$	$\frac{1-k'}{1+k'}$	$(1+k') \frac{\operatorname{sn} \operatorname{cn}}{\operatorname{dn}}$	$\frac{1-(1+k')\operatorname{sn}^2}{\operatorname{dn}}$	$\frac{1-(1-k')\operatorname{sn}^2}{\operatorname{dn}}$ (Landen's transformation)
$\frac{(1+k')^2 u}{2}$	$\left(\frac{1-\sqrt{k'}}{1+\sqrt{k'}} \right)^2$	$\frac{1+\sqrt{k'}}{1-\sqrt{k'}} \frac{k^2 \operatorname{sn} \operatorname{cn}}{(1+\operatorname{dn})(k'+\operatorname{dn})}$	$\frac{\operatorname{dn}-\sqrt{k'}}{1-\sqrt{k'}} \times$	$\frac{\sqrt{2(1+k')}}{1+\sqrt{k'}} \frac{\operatorname{dn}+\sqrt{k'}}{\sqrt{(1+\operatorname{dn})(k'+\operatorname{dn})}}$
$\sqrt{\frac{2(1+k')}{(1+\operatorname{dn})(k'+\operatorname{dn})}}$				

Jacobi's transformation. $\operatorname{sn}(iu, k) = i \frac{\operatorname{sn}(u, k')}{\operatorname{cn}(u, k')}, \quad \operatorname{cn}(iu, k) = \frac{1}{\operatorname{cn}(u, k')},$

$\operatorname{dn}(iu, k) = \frac{\operatorname{dn}(u, k')}{\operatorname{cn}(u, k')}.$

(5) Amplitude.
The inverse function $\varphi = \operatorname{am}(u, k)$ of $u(k, \varphi) \equiv F(k, \varphi) = \int_0^\varphi \frac{d\psi}{\sqrt{1-k^2 \sin^2 \psi}}$
is called the amplitude.

$$\operatorname{sn}(u, k) = \sin \varphi = \sin \operatorname{am}(u, k), \quad \operatorname{cn}(u, k) = \cos \varphi = \cos \operatorname{am}(u, k),$$

$$\operatorname{dn}(u, k) = \sqrt{1 - k^2 \sin^2 \varphi} = \sqrt{1 - k^2 \operatorname{sn}^2(u, k)}.$$

$$u(k, x) = \int_0^x \frac{dt}{\sqrt{(1-t^2)(1-k^2 t^2)}}, \quad x = \operatorname{sn}(u, k).$$

$$\operatorname{am}(u, k) = \frac{\pi u}{2K} + \sum_{n=1}^{\infty} \frac{2q^n}{n(1+q^{2n})} \sin\left(n\pi \frac{u}{2K}\right) \quad (q = e^{i\pi\tau} = e^{-\pi(K'/K)}).$$

$$\operatorname{am}(\theta, 1) = \operatorname{gd} \theta \quad (\text{Gudermann function}).$$

(IV) Weierstrass's Elliptic Functions

(1) Weierstrass's \wp -function. For the fundamental periods $2\omega_1, 2\omega_3$, we have

$$\begin{aligned} \wp(u) &\equiv \frac{1}{u^2} + \sum'_{n,m} \left[\frac{1}{(u - 2n\omega_1 - 2m\omega_3)^2} - \frac{1}{(2n\omega_1 + 2m\omega_3)^2} \right] \\ &= \frac{1}{u^2} + \frac{g_2}{20}u^2 + \frac{g_3}{28}u^4 + \frac{g_2^2}{1200}u^6 + \frac{3g_2g_3}{6160}u^8 + \dots \\ g_2 &\equiv 60 \sum'_{n,m} \frac{1}{(2n\omega_1 + 2m\omega_3)^4}, \quad g_3 \equiv 140 \sum'_{n,m} \frac{1}{(2n\omega_1 + 2m\omega_3)^6}, \end{aligned}$$

where \sum' means the sum over all integers except $m = n = 0$.

$\wp(-u) = \wp(u)$. Putting $\omega_2 \equiv -(\omega_1 + \omega_3)$, $e_j \equiv \wp(\omega_j)$ ($j = 1, 2, 3$) we have

$$e_1 + e_2 + e_3 = 0, \quad e_1e_2 + e_2e_3 + e_3e_1 = -g_2/4, \quad e_1e_2e_3 = g_3/4.$$

$$\wp'(u) \equiv d\wp/du = -2 \sum'_{m,n} \frac{1}{(u - 2n\omega_1 - 2m\omega_3)^3}.$$

$$\wp'^2(u) = 4[\wp(u) - e_1][\wp(u) - e_2][\wp(u) - e_3] = 4\wp^3(u) - g_2\wp(u) - g_3.$$

Addition theorem

$$\begin{aligned} \wp(u+v) &= -\wp(u) - \wp(v) + \frac{1}{4} \left[\frac{\wp'(u) - \wp'(v)}{\wp(u) - \wp(v)} \right]^2, \\ \wp(u + \omega_j) &= e_j + \frac{(e_j - e_k)(e_j - e_l)}{\wp(u) - e_j} \quad (j, k, l) = (1, 2, 3). \end{aligned}$$

Using theta functions corresponding to $\tau = \omega_3/\omega_1$,

$$\begin{aligned} \wp(u) &= -\frac{\eta_1}{\omega_1} - \frac{d^2 \log \vartheta_1(u/2\omega_1)}{du^2} \quad \left(\eta_1 = \zeta(\omega_1) = -\frac{1}{12\omega_1} \frac{\vartheta_1'''(0)}{\vartheta_1'(0)} \right), \\ \wp'(u) &= -\frac{1}{4\omega_1^3} \frac{\vartheta_1'^3(0) \vartheta_2(u/2\omega_1) \vartheta_3(u/2\omega_1) \vartheta_0(u/2\omega_1)}{\vartheta_2(0) \vartheta_0(0) \vartheta_1^3(u/2\omega_1)}. \end{aligned}$$

The relations to Jacobi's elliptic functions are

$$q \equiv \exp(i\pi\omega_3/\omega_1).$$

$$\wp\left(\frac{u}{\sqrt{e_1 - e_3}}\right) = e_1 + (e_1 - e_3) \frac{\operatorname{cn}^2 u}{\operatorname{sn}^2 u} = e_2 + (e_1 - e_3) \frac{\operatorname{dn}^2 u}{\operatorname{sn}^2 u} = e_3 + (e_1 - e_3) \frac{1}{\operatorname{sn}^2 u},$$

$$\text{where the modulus is } k = \sqrt{\frac{e_2 - e_3}{e_1 - e_3}}, \quad K(k) = \omega_1 \sqrt{e_1 - e_3}.$$

(2) ζ -function.

$$\begin{aligned} \zeta(u) &\equiv \frac{1}{u} + \sum'_{n,m} \left[\frac{1}{u - 2n\omega_1 - 2m\omega_3} + \frac{u}{(2n\omega_1 + 2m\omega_3)^2} + \frac{1}{2n\omega_1 + 2m\omega_3} \right] \\ &= \frac{1}{u} - \frac{g_2}{60}u^3 - \frac{g_3}{140}u^5 - \frac{g_2^2}{8400}u^7 - \frac{g_2g_3}{18480}u^9 - \dots \\ &= (\zeta_1/\omega_1)u + d \log \vartheta_1(u/2\omega_1)/du. \\ \zeta'(u) &= -\wp(u). \end{aligned}$$

Gamma Functions and Related Functions

Pseudoperiodicity. Putting $\eta_j \equiv \zeta(\omega_j)$ ($j=1, 2, 3$) we have

$$\zeta(u + 2n\omega_1 + 2m\omega_3) = \zeta(u) + 2n\eta_1 + 2m\eta_3 \quad (n, m = 0, \pm 1, \pm 2, \dots),$$

$$\eta_1 = -\frac{1}{12\omega_1} \frac{\vartheta_1'''(0)}{\vartheta_1'(0)}, \quad \eta_1 + \eta_2 + \eta_3 = 0,$$

$$\eta_1\omega_3 - \eta_3\omega_1 = \eta_2\omega_1 - \eta_1\omega_2 = \eta_3\omega_2 - \eta_2\omega_3 = \pi i/2 \text{ (Legendre's relation).}$$

Addition theorem $\zeta(u+v) = \zeta(u) + \zeta(v) + \frac{1}{2} \frac{\zeta''(u) - \zeta''(v)}{\zeta'(u) - \zeta'(v)}$.

(3) σ -function.

$$\sigma(u) \equiv \prod'_{n,m} \left(1 - \frac{u}{2n\omega_1 + 2m\omega_3}\right) \exp \left[\frac{u}{2n\omega_1 + 2m\omega_3} + \frac{1}{2} \left(\frac{u}{2n\omega_1 + 2m\omega_3} \right)^2 \right] \quad \begin{matrix} n, m = 0, \pm 1, \\ \pm 2, \dots, \\ (n, m) \neq (0, 0) \end{matrix}$$

$$= u - \frac{g_2}{2^4 \cdot 3 \cdot 5} u^5 - \frac{g_3}{2^3 \cdot 3 \cdot 5 \cdot 7} u^7 - \frac{g_2^2}{2^9 \cdot 3^2 \cdot 5 \cdot 7} u^9 - \dots$$

$$= 2\omega_1 \left(\exp \frac{\eta_1 u^2}{2\omega_1} \right) \frac{\vartheta_1(u/2\omega_1)}{\vartheta_1'(0)}.$$

$$\zeta(u) = \sigma'(u)/\sigma(u), \quad \sigma(-u) = -\sigma(u).$$

Pseudoperiodicity. $\sigma(u + 2n\omega_1 + 2m\omega_3) = (-1)^{n+m} [\exp(2n\eta_1 + 2m\eta_3)(u + n\omega_1 + m\omega_3)] \sigma(u)$.

(4) Cosigma functions $\sigma_1, \sigma_2, \sigma_3$.

$$\sigma_j(u) \equiv -e^{\eta_j u} \frac{\sigma(u + \omega_j)}{\sigma(u)} = \left(\exp \frac{\eta_1 u^2}{2\omega_1} \right) \frac{\vartheta_{j+1}(u/2\omega_1)}{\vartheta_{j+1}(0)} \quad (j=1, 2, 3; \vartheta_4 \equiv \vartheta_0).$$

$$\wp(u) - e_j = \left[\frac{\sigma_j(u)}{\sigma(u)} \right]^2, \quad \wp'(2u) = -\frac{2\sigma_1(u)\sigma_2(u)\sigma_3(u)}{\sigma^3(u)} = -\frac{\sigma(2u)}{\sigma^4(u)}.$$

$$\operatorname{sn} u = \alpha \frac{\sigma(u/\alpha)}{\sigma_3(u/\alpha)}, \quad \operatorname{cn} u = \frac{\sigma_1(u/\alpha)}{\sigma_3(u/\alpha)}, \quad \operatorname{dn} u = \frac{\sigma_2(u/\alpha)}{\sigma_3(u/\alpha)}, \quad \text{where } \alpha = \sqrt{e_1 - e_3} = \frac{K}{\omega_1}.$$

References

- [1] W. F. Magnus, F. Oberhettinger, and R. P. Soni, Formulas and theorems for the special functions of mathematical physics, Springer, third enlarged edition, 1966.
- [2] Y. L. Luke, The special functions and their approximations I, II, Academic Press, 1969.
- [3] A. Erdelyi, Higher transcendental functions I, II, III (Bateman manuscript project) McGraw-Hill, 1953, 1955.

In particular, for hypergeometric functions of two variables see

- [4] P. Appell, Sur les fonctions hypergéométriques de plusieurs variables, Mémor. Sci. Math., Gauthier-Villars, 1925.

Also \rightarrow references to 39 Bessel Functions, 134 Elliptic Functions, 174 Gamma Function, 389 Special Functions.

17. Gamma Functions and Related Functions**(I) Gamma Functions and Beta Functions (\rightarrow 174 Gamma Function)**

In this Section (I), C means Euler's constant, B_n means a Bernoulli number, ζ means the Riemann zeta function.

(1) Gamma function. $\Gamma(z) \equiv \int_0^\infty e^{-t} t^{z-1} dt \quad (\operatorname{Re} z > 0)$

$$= \frac{1}{e^{2\pi iz} - 1} \int_\infty^{(0+)} e^{-t} t^{z-1} dt.$$

In the last integral, the integration contour goes once around the positive real axis in the positive direction.

$$\Gamma(n+1) = n! \quad (n=0, 1, 2, \dots), \quad \Gamma(1/2) = \sqrt{\pi}.$$

$$\Gamma(z+1) = z\Gamma(z), \quad \Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin \pi z}, \quad \prod_{j=0}^{n-1} \Gamma\left(z + \frac{j}{n}\right) = (2\pi)^{(n-1)/2} n^{(1/2)-nz} \Gamma(nz),$$

$$\frac{1}{\Gamma(z)} = ze^{Cz} \prod_{n=1}^{\infty} \left(1 + \frac{z}{n}\right) e^{-z/n}, \quad \log \Gamma(1+z) = -\frac{1}{2} \log \frac{\sin \pi z}{\pi z} - Cz - \sum_{n=1}^{\infty} \frac{\zeta(2n+1)}{2n+1} z^{2n+1} \quad (|z| < 1).$$

$$\left| \frac{\Gamma(x+iy)}{\Gamma(x)} \right|^2 = \prod_{n=0}^{\infty} 1 / \left(1 + \frac{y^2}{(x+n)^2} \right) \quad (x, y \text{ are real and } x > 0).$$

Asymptotic expansion (Stirling formula).

$$\begin{aligned} \Gamma(z) &\approx e^{-z} z^{z-1} \sqrt{2\pi z} \exp \left[\sum_{n=1}^{\infty} \frac{(-1)^{n-1} B_{2n} z^{1-2n}}{2n(2n-1)} \right] \quad (|\arg z| < \pi) \\ &= e^{-z} z^{z-(1/2)} \sqrt{2\pi} \left[1 + \frac{1}{12z} + \frac{1}{288z^2} - \frac{139}{51840z^3} - \frac{571}{2488320z^4} + O(z^{-5}) \right]. \end{aligned}$$

(2) Beta Function. $B(x, y) \equiv \int_0^1 t^{x-1} (1-t)^{y-1} dt \quad (\operatorname{Re} x, \operatorname{Re} y > 0)$
 $= \Gamma(x)\Gamma(y)/\Gamma(x+y).$

(3) Incomplete Gamma Function.

$$\gamma(v, x) \equiv \int_0^x t^{v-1} e^{-t} dt = \Gamma(v) - x^{(v-1)/2} e^{-x/2} W_{(v-1)/2, v/2}(x) \quad (\operatorname{Re} v > 0).$$

(4) Incomplete Beta Function. $B_\alpha(x, y) \equiv \int_0^\alpha t^{x-1} (1-t)^{y-1} dt \quad (0 < \alpha \leq 1).$

(5) Polygamma Functions. $\psi(z) \equiv \frac{d}{dz} \log \Gamma(z)$

$$= \frac{\Gamma'(z)}{\Gamma(z)} = \int_0^\infty \left[\frac{e^{-t}}{t} - \frac{e^{-zt}}{1-e^{-t}} \right] dt = -C + \sum_{n=0}^{\infty} \left(\frac{1}{n+1} - \frac{1}{z+n} \right).$$

$$\psi'(z) = \sum_{n=0}^{\infty} \frac{1}{(z+n)^2}, \quad \psi^{(k)}(z) = \sum_{n=0}^{\infty} \frac{(-1)^{k+1} k!}{(z+n)^{k+1}} \quad (k=1, 2, \dots).$$

(II) Combinatorial Problems (→ 330 Permutations and Combinations)

Factorial $n! = n(n-1)(n-2)\dots 3 \cdot 2 \cdot 1, \quad 0! = 1.$

Binomial coefficient $\binom{\alpha}{r} = \frac{\alpha(\alpha-1)\dots(\alpha-r+1)}{r!}.$

(1) Number of Permutations of n Elements Taken r at a Time.

$${}_n P_r = n(n-1)\dots(n-r+1) = n!/(n-r)!.$$

Number of combinations of n elements taken r at a time

$${}_n C_r = \frac{{}_n P_r}{r!} = \frac{n!}{r!(n-r)!} = \binom{n}{r}.$$

$${}_n C_r = {}_n C_{n-r}, \quad {}_n C_r = {}_{n-1} C_r + {}_{n-1} C_{r-1}.$$

Number of multiple permutations ${}_n \Pi_r = n^r.$

Number of multiple combinations ${}_n H_r = {}_{n+r-1} C_r = \frac{(n+r-1)!}{r!(n-1)!}.$

Number of circular permutations ${}_n P_r / r.$

(2) Binomial Theorem. $(a+b)^n = \sum_{r=0}^n \binom{n}{r} a^{n-r} b^r.$

Multinomial theorem $(a_1 + \dots + a_m)^n = \sum \frac{n!}{p_1! \dots p_m!} a_1^{p_1} \dots a_m^{p_m}.$

The latter summation runs over all nonnegative integers satisfying $p_1 + \dots + p_m = n$.

References

See references to Table 16, this Appendix.

18. Hypergeometric Functions and Spherical Functions

(I) Hypergeometric Function (\rightarrow 206 Hypergeometric Functions)

(1) Hypergeometric Function. $F(a, b; c; z) \equiv \sum_{n=0}^{\infty} \frac{\Gamma(a+n)}{\Gamma(a)} \frac{\Gamma(b+n)}{\Gamma(b)} \frac{\Gamma(c)}{\Gamma(c+n)} \frac{z^n}{n!}.$

The fundamental system of solutions of the hypergeometric differential equation

$$z(1-z) \frac{d^2 u}{dz^2} + [c - (a+b+1)z] \frac{du}{dz} - abu = 0 \text{ at } z=0 \text{ is given by}$$

$$u_1 = F(a, b; c; z), \quad u_2 = z^{1-c} F(a-c+1, b-c+1; 2-c; z) \quad (c \neq 0, -1, -2, \dots).$$

$$F(a, b; c; z) = F(b, a; c; z). \quad dF/dz = (ab/c) F(a+1, b+1; c+1; z).$$

$$F(a, b; c; 1) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} \quad (\operatorname{Re}(a+b-c) < 0).$$

$$F(a, b; c; z) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 t^{b-1} (1-t)^{c-b-1} (1-tz)^{-a} dt \quad (\operatorname{Re} c > \operatorname{Re} b > 0, \quad |z| < 1),$$

$$F(a, b; c; z) = \frac{1}{2\pi i} \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)} \int_{-\infty}^{\infty} \frac{\Gamma(a+s)\Gamma(b+s)\Gamma(-s)}{\Gamma(c+s)} (-z)^s ds.$$

(2) Transformations of the Hypergeometric Function.

$$\begin{aligned} F(a, b; c; z) &= (1-z)^{-a} F\left(a, c-b; c; \frac{z}{z-1}\right) \\ &= (1-z)^{c-a-b} F(c-a, c-b; c; z) \\ &= (1-z)^{-a} \frac{\Gamma(c)\Gamma(b-a)}{\Gamma(b)\Gamma(c-a)} F\left(a, c-b; a-b+1; \frac{1}{1-z}\right) \\ &\quad + (1-z)^{-b} \frac{\Gamma(c)\Gamma(a-b)}{\Gamma(a)\Gamma(c-b)} F\left(b, c-a; b-a+1; \frac{1}{1-z}\right) \\ &= \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} F(a, b; a+b-c+1; 1-z) \\ &\quad + (1-z)^{c-a-b} \frac{\Gamma(c)\Gamma(a+b-c)}{\Gamma(a)\Gamma(b)} F(c-a, c-b; c-a-b+1; 1-z) \\ &= \frac{\Gamma(c)\Gamma(b-a)}{\Gamma(b)\Gamma(c-a)} (-z)^{-a} F\left(a, 1-c+a; 1-b+a; \frac{1}{z}\right) \\ &\quad + \frac{\Gamma(c)\Gamma(a-b)}{\Gamma(a)\Gamma(c-b)} (-z)^{-b} F\left(b, 1-c+b; 1-a+b; \frac{1}{z}\right). \end{aligned}$$

(3) Riemann's Differential Equation (\rightarrow Table 14.II, this Appendix).

$$\frac{d^2u}{dz^2} + \left[\frac{1-\alpha-\alpha'}{z-a} + \frac{1-\beta-\beta'}{z-b} + \frac{1-\gamma-\gamma'}{z-c} \right] \frac{du}{dz} + \left[\frac{\alpha\alpha'(a-b)(a-c)}{z-a} + \frac{\beta\beta'(b-c)(b-a)}{z-b} + \frac{\gamma\gamma'(c-a)(c-b)}{z-c} \right] \frac{u}{(z-a)(z-b)(z-c)} = 0.$$

Here we have $\alpha + \alpha' + \beta + \beta' + \gamma + \gamma' = 1$ (Fuchsian relation). The solution of this equation is given by Riemann's P -function

$$u = P \left\{ \begin{matrix} a & b & c \\ \alpha & \beta & \gamma \\ \alpha' & \beta' & \gamma' \end{matrix} \right\} z = \left(\frac{z-a}{z-b} \right)^\alpha \left(\frac{z-c}{z-b} \right)^\gamma F \left(\alpha + \beta + \gamma, \alpha + \beta' + \gamma; 1 + \alpha - \alpha'; \frac{(c-b)(z-a)}{(c-a)(z-b)} \right)$$

$(\alpha - \alpha', \beta - \beta', \gamma - \gamma' \neq \text{integer}).$

We have 24 representations of the above function by interchanging the parameters $a, b, c; \alpha, \alpha'; \beta, \beta'; \gamma, \gamma'$ in the right-hand side.

(4) Barnes's Extended Hypergeometric Function.

$${}_pF_q(\alpha_1, \dots, \alpha_p; \beta_1, \dots, \beta_q; z) \equiv \sum_{n=0}^{\infty} \frac{(\alpha_1)_n \dots (\alpha_p)_n}{(\beta_1)_n \dots (\beta_q)_n} \frac{z^n}{n!}, \quad \text{where } (\alpha)_n = \alpha(\alpha+1) \dots (\alpha+n-1)$$

$$= \Gamma(\alpha+n)/\Gamma(\alpha). \quad F(a, b; c; z) = {}_2F_1(a, b; c; z). \quad {}_0F_0(x) = e^x, \quad {}_1F_0(\alpha; x) = (1-x)^{-\alpha}.$$

(5) Appell's Hypergeometric Functions of Two Variables.

$$F_1(\alpha; \beta, \beta'; \gamma; x, y) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{(\alpha)_{m+n} (\beta)_m (\beta')_n}{m! n! (\gamma)_{m+n}} x^m y^n,$$

$$F_2(\alpha; \beta, \beta'; \gamma, \gamma'; x, y) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{(\alpha)_{m+n} (\beta)_m (\beta')_n}{m! n! (\gamma)_m (\gamma')_n} x^m y^n,$$

$$F_3(\alpha, \alpha'; \beta, \beta'; \gamma; x, y) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{(\alpha)_m (\alpha')_n (\beta)_m (\beta')_n}{m! n! (\gamma)_{m+n}} x^m y^n,$$

$$F_4(\alpha; \beta; \gamma, \gamma'; x, y) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{(\alpha)_{m+n} (\beta)_{m+n}}{m! n! (\gamma)_m (\gamma')_n} x^m y^n.$$

(6) Representation of Various Special Functions by Hypergeometric Functions.

$$(1-x)^{\nu} = F(-\nu, b; b; x), \quad e^{-nx} = \left(\frac{\operatorname{sech} x}{2} \right)^n (\tanh x) F\left(1 + \frac{n}{2}, \frac{1+n}{2}; 1+n; \operatorname{sech}^2 x\right).$$

$$\log(1+x) = xF(1, 1; 2; -x), \quad \frac{1}{2} \log \frac{1+x}{1-x} = xF\left(\frac{1}{2}, 1; \frac{3}{2}; x^2\right),$$

$$\sin nx = n(\sin x) F\left(\frac{1+n}{2}, \frac{1-n}{2}; \frac{3}{2}; \sin^2 x\right),$$

$$\cos nx = F\left(\frac{n}{2}, -\frac{n}{2}; \frac{1}{2}; \sin^2 x\right) = (\cos x) F\left(\frac{1+n}{2}, \frac{1-n}{2}; \frac{1}{2}; \sin^2 x\right),$$

$$\arcsin x = xF\left(\frac{1}{2}, \frac{1}{2}; \frac{3}{2}; x^2\right), \quad \arctan x = xF\left(\frac{1}{2}, 1; \frac{3}{2}; -x^2\right).$$

$$P_{2n}(x) = (-1)^n \frac{(2n-1)!!}{(2n)!!} F\left(-n, n + \frac{1}{2}; \frac{1}{2}; x^2\right),$$

$$P_{2n+1}(x) = (-1)^n \frac{(2n+1)!!}{(2n)!!} x F\left(-n, n + \frac{3}{2}; \frac{3}{2}; x^2\right) \quad (\text{spherical function}),$$

where

$$n = 0, 1, 2, \dots; \quad m!! = \begin{cases} m(m-2)\dots 4\cdot 2 & (m \text{ even}), \\ m(m-2)\dots 3\cdot 1 & (m \text{ odd}), \end{cases} \quad 0!! = (-1)!! = 1.$$

$$K(x) = \frac{\pi}{2} F\left(\frac{1}{2}, \frac{1}{2}; 1; x^2\right), \quad E(x) = \frac{\pi}{2} F\left(-\frac{1}{2}, \frac{1}{2}; 1; x^2\right) \quad (\text{complete elliptic integral}).$$

$$J_\nu(x) = \frac{x}{2\Gamma(\nu+1)} {}_0F_1\left(\nu+1; -\frac{x^2}{4}\right) = \frac{x^\nu e^{-ix}}{2^\nu \Gamma(\nu+1)} {}_1F_1\left(\nu+\frac{1}{2}; 2\nu+1; 2ix\right).$$

$$e^x = \lim_{b \rightarrow \infty} F(a, b; a; x/b) = {}_1F_1(a; a; x) = {}_0F_0(x).$$

(II) Legendre Function (→ 393 Spherical Functions)

(1) Legendre Functions. The generalized spherical function corresponding to the rotation group of 3-dimensional space is the solution of the following differential equation.

$$(1-z^2) \frac{d^2 u}{dz^2} - 2z \frac{du}{dz} + \left[\nu(\nu+1) - \frac{\mu^2}{1-z^2} \right] u = 0.$$

When $\mu = 0$, the equation is Legendre's differential equation, and the fundamental system of solutions is given by the following two kind of functions.

$$\text{Legendre function of the first kind} \quad \mathfrak{P}_\nu(z) \equiv P_\nu(z) = {}_2F_1\left(-\nu, \nu+1; 1; \frac{1-z}{2}\right).$$

Legendre function of the second kind

$$\mathfrak{Q}_\nu(z) \equiv \frac{\Gamma(\nu+1)\sqrt{\pi}}{2^{\nu+1}\Gamma[\nu+(3/2)]} z^{-\nu-1} {}_2F_1\left(\frac{\nu+2}{2}, \frac{\nu+1}{2}; \nu+\frac{3}{2}; \frac{1}{z^2}\right).$$

$$\begin{aligned} Q_\nu(x) &\equiv \frac{1}{2} [\mathfrak{Q}_\nu(x+i0) + \mathfrak{Q}_\nu(x-i0)] \\ &= \pi \frac{(\cos \nu\pi) P_\nu(x) - P_\nu(-x)}{2 \sin \nu\pi} \quad (\nu \neq \text{integer}; -1 < x < 1). \end{aligned}$$

Recurrence formulas:

$$\mathfrak{P}_\nu(z) = \mathfrak{P}_{-\nu-1}(z), \quad \mathfrak{Q}_\nu(z) - \mathfrak{Q}_{-\nu-1}(z) = \pi(\cot \nu\pi) \mathfrak{P}_\nu(z) \quad (\nu \neq \text{integer}).$$

$$\mathfrak{P}_\nu(-z) = e^{\pm \nu \pi i} \mathfrak{P}_\nu(z) - (2/\pi)(\sin \nu\pi) \mathfrak{Q}_\nu(z), \quad \mathfrak{Q}_\nu(-z) = -e^{\pm \nu \pi i} \mathfrak{Q}_\nu(z) \quad (\pm = \text{sgn}(\text{Im } z)).$$

$$(z^2-1) d\mathfrak{P}_\nu(z)/dz = (\nu+1)[\mathfrak{P}_{\nu+1}(z) - z\mathfrak{P}_\nu(z)],$$

$$(2\nu+1)z\mathfrak{P}_\nu(z) = (\nu+1)\mathfrak{P}_{\nu+1}(z) + \nu\mathfrak{P}_{\nu-1}(z),$$

$$(z^2-1) d\mathfrak{Q}_\nu(z)/dz = (\nu+1)[\mathfrak{Q}_{\nu+1}(z) - z\mathfrak{Q}_\nu(z)],$$

$$(2\nu+1)z\mathfrak{Q}_\nu(z) = (\nu+1)\mathfrak{Q}_{\nu+1}(z) + \nu\mathfrak{Q}_{\nu-1}(z).$$

$$\begin{aligned} \mathfrak{P}_\nu(z) &= \pi^{-1/2} 2^{-\nu-1} \tan \nu\pi \frac{\Gamma(\nu+1)}{\Gamma[\nu+(3/2)]} z^{-\nu-1} {}_2F_1\left(\frac{\nu}{2}+1, \frac{\nu+1}{2}; \nu+\frac{3}{2}; \frac{1}{z^2}\right) \\ &+ \pi^{-1/2} 2^\nu \frac{\Gamma[\nu+(1/2)]}{\Gamma(\nu+1)} z^\nu {}_2F_1\left(\frac{1-\nu}{2}, -\frac{\nu}{2}; \frac{1}{2}-\nu; \frac{1}{z^2}\right). \end{aligned}$$

$$P_\nu(\cos \theta) = \frac{\sin \nu\pi}{\pi} \sum_{n=0}^{\infty} (-1)^n \left(\frac{1}{\nu-n} - \frac{1}{\nu+n+1} \right) P_n(\cos \theta) \quad (\nu \neq \text{integer}; 0 \leq \theta < \pi).$$

$$\text{Estimation:} \quad |P_\nu(\cos \theta)| \leq \frac{2}{\sqrt{\nu\pi} \sin \theta}, \quad |Q_\nu(\cos \theta)| \leq \frac{\sqrt{\pi}}{\sqrt{\nu} \sin \theta} \quad (0 < \theta < \pi; \nu > 1).$$

$$P_\nu(1) = 1, \quad P_\nu(0) = -\frac{\sin \nu\pi}{2\pi^{3/2}} \Gamma\left(\frac{\nu+1}{2}\right) \Gamma\left(\frac{-\nu}{2}\right),$$

$$Q_\nu(0) = \frac{1}{4\sqrt{\pi}} (1 - \cos \nu\pi) \Gamma\left(\frac{\nu+1}{2}\right) \Gamma\left(\frac{-\nu}{2}\right).$$

(2) The Case $\nu = n$ ($= 0, 1, 2, \dots$). In the following, the symbol !! means

$$m!! \equiv \begin{cases} m(m-2)\dots 4\cdot 2 & (m \text{ even}), \\ m(m-2)\dots 5\cdot 3\cdot 1 & (m \text{ odd}). \end{cases}$$

The function P_n is a polynomial of degree n (Legendre polynomial) and is represented as follows:

$$\begin{aligned}
 P_n(z) &\equiv \frac{1}{2^n n!} \frac{d^n}{dz^n} (z^2 - 1)^n \\
 &= \frac{(2n)!}{2^n (n!)^2} z^n {}_2F_1\left(-\frac{n}{2}, \frac{1-n}{2}; \frac{1}{2} - n; \frac{1}{z^2}\right) \\
 &= \frac{(2n-1)!!}{n!} \left[z^n - \frac{n(n-1)}{(2n-1)} z^{n-2} + \frac{n(n-1)(n-2)(n-3)}{2 \cdot 4 \cdot (2n-1)(2n-3)} z^{n-4} + \dots \right]. \\
 P_{2m}(z) &= \sum_{j=0}^m (-1)^{m-j} \frac{(2m+2j-1)!!}{(2j)!(2m-2j)!!} z^{2j}, \\
 P_{2m+1}(z) &= \sum_{j=0}^m (-1)^{m-j} \frac{(2m+2j+1)!!}{(2j+1)!(2m-2j)!!} z^{2j+1}. \\
 P_n(\cos \theta) &= \frac{(2n)!}{2^{2n} (n!)^2} e^{\mp i n \theta} {}_2F_1\left(\frac{1}{2}, -n; \frac{1}{2} - n; e^{\pm 2i\theta}\right) \\
 &= \frac{2(2n-1)!!}{(2n)!!} \left[\cos n\theta + \frac{1}{1} \frac{n}{(2n-1)} \cos(n-2)\theta + \frac{1 \cdot 3}{1 \cdot 2} \frac{n(n-1)}{(2n-1)(2n-3)} \cos(n-4)\theta \right. \\
 &\quad \left. + \frac{1 \cdot 3 \cdot 5}{1 \cdot 2 \cdot 3} \frac{n(n-1)(n-2)}{(2n-1)(2n-3)(2n-5)} \cos(n-6)\theta + \dots \right] \\
 &\quad + \begin{cases} \left[\frac{(n-1)!!}{n!!} \right]^2 & (n \text{ even}), \\ 0 & (n \text{ odd}). \end{cases} \\
 &= \frac{4}{\pi} \frac{(2n)!!}{(2n+1)!!} \left[\sin(n+1)\theta + \frac{1 \cdot (n+1)}{1 \cdot (2n+3)} \sin(n+3)\theta \right. \\
 &\quad \left. + \frac{1 \cdot 3 \cdot (n+1)(n+2)}{1 \cdot 2 \cdot (2n+3)(2n+5)} \sin(n+5)\theta + \dots (\text{ad infinitum}) \right] \quad (0 < \theta < \pi).
 \end{aligned}$$

Laplace-Mehler integral representation

$$\begin{aligned}
 P_n(\cos \theta) &= \frac{1}{\pi} \int_0^\pi (\cos \theta + i \sin \theta \cos \varphi)^n d\varphi \\
 &= \frac{\sqrt{2}}{\pi} \int_0^\theta \frac{\cos[n + (1/2)]\varphi}{\sqrt{\cos \varphi - \cos \theta}} d\varphi = \frac{\sqrt{2}}{\pi} \int_\theta^\pi \frac{\sin[n + (1/2)]\varphi}{\sqrt{\cos \theta - \cos \varphi}} d\varphi.
 \end{aligned}$$

$$P_n(x) = \frac{(-1)^n}{n!} r^{n+1} \frac{\partial^n}{\partial z^n} \left(\frac{1}{r} \right) \quad \left(x = \frac{z}{r}, r = \sqrt{z^2 + \rho^2} \right).$$

$$P_n(1) = 1, \quad P_n(-1) = (-1)^n, \quad P_{2n+1}(0) = 0,$$

$$P_{2n}(0) = (-1)^n \frac{(2n)!}{2^{2n} (n!)^2} = \frac{(-1)^n (2n-1)!!}{(2n)!!}.$$

Recurrence formulas: $nP_n(z) - (2n-1)zP_{n-1}(z) + (n-1)P_{n-2}(z) = 0$,

$$(z^2 - 1) \frac{dP_n}{dz} = n(zP_n - P_{n-1}) = \frac{n(n+1)}{2n+1} (P_{n+1} - P_{n-1}) = (n+1)(P_{n+1} - zP_n).$$

$$\begin{aligned}
 \mathfrak{D}_n(z) &= \frac{1}{2^n n!} \frac{d^n}{dz^n} \left[(z^2 - 1)^n \log \frac{z+1}{z-1} \right] - \frac{1}{2} P_n(z) \log \frac{z+1}{z-1} \\
 &= 2^n n! \int_z^\infty \dots \int_z^\infty \frac{(dz)^{n+1}}{(z^2 - 1)^{n+1}} \\
 &= 2^n \int_z^\infty \frac{(t-z)^n}{(t^2 - 1)^{n+1}} dt \\
 &= (-1)^n \frac{1}{(2n-1)!!} \frac{d^n}{dz^n} \left[(z^2 - 1)^n \int_z^\infty \frac{dt}{(t^2 - 1)^{n+1}} \right] \quad (\operatorname{Re} z > 1).
 \end{aligned}$$

$$Q_n(\cos \theta) = \frac{2 \cdot (2n)!!}{(2n+1)!!} \left[\cos(n+1)\theta + \frac{1 \cdot (n+1)}{1 \cdot (2n+3)} \cos(n+3)\theta \right. \\ \left. + \frac{1 \cdot 3 \cdot (n+1)(n+2)}{1 \cdot 2 \cdot (2n+3)(2n+5)} \cos(n+5)\theta + \dots \right] \quad (0 < \theta < \pi).$$

$$Q_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n} \left[(x^2 - 1)^n \log \frac{1+x}{1-x} \right] - \frac{1}{2} P_n(x) \log \frac{1+x}{1-x} \\ = \frac{1}{2} P_n(x) \log \frac{1+x}{1-x} - \sum_{j=1}^n \frac{1}{j} P_{j-1}(x) P_{n-j}(x).$$

$$Q_0(x) = \frac{1}{2} \log \frac{1+x}{1-x}, \quad Q_1(x) = \frac{x}{2} \log \frac{1+x}{1-x} - 1, \quad Q_2(x) = \frac{1}{4} (3x^2 - 1) \log \frac{1+x}{1-x} - \frac{3}{2} x.$$

(3) Generating Functions.

$$\frac{1}{\sqrt{1-2hz+h^2}} = \begin{cases} \sum_{n=0}^{\infty} h^n P_n(z) & (|h| < \min|z \pm \sqrt{z^2-1}|), \\ \sum_{n=0}^{\infty} \frac{1}{h^{n+1}} P_n(z) & (|h| > \max|z \pm \sqrt{z^2-1}|). \end{cases}$$

(If $-1 \leq z \leq 1$, the right-hand side is equal to 1.)

$$\frac{1}{z-t} = \sum_{n=0}^{\infty} (2n+1) \mathfrak{P}_n(t) \mathfrak{Q}_n(z) \quad (|t + \sqrt{t^2-1}| < |z + \sqrt{z^2-1}|).$$

$$\frac{1}{\sqrt{1-2tz+z^2}} \log \frac{z-t+\sqrt{1-2tz+z^2}}{\sqrt{z^2-1}} = \sum_{n=0}^{\infty} t^n \mathfrak{Q}_n(z) \quad (\operatorname{Re} z > 1, |t| < 1).$$

$$r = \sqrt{x^2 + y^2 + z^2}, \quad \cos \theta = z/r, \quad x, y \text{ real},$$

$$\frac{1}{r} + \frac{1}{2} + \sum_{m=1}^{\infty} \left[\frac{1}{\sqrt{(2m\pi+z)^2 + x^2 + y^2}} + \frac{1}{\sqrt{(2m\pi-z)^2 + x^2 + y^2}} \right]$$

(Here the square root of a complex number is taken so that its real part is positive.)

$$= \begin{cases} 1 + \sum_{n=1}^{\infty} e^{-nz} J_0(n\sqrt{x^2+y^2}) & (\operatorname{Re} z > 0), \\ \frac{1}{r} + \frac{1}{2} + \sum_{n=1}^{\infty} \frac{(-1)^n B_{2n}}{(2n)!} r^{2n-1} P_{2n-1}(\cos \theta) & (0 < \theta < 2\pi; z \text{ real}). \end{cases}$$

(4) Integrals of Legendre Polynomials.

$$\text{Orthogonal relations: } \int_{-1}^{+1} P_n(z) P_m(z) dz = \delta_{nm} \frac{2}{2n+1}.$$

$$\int_{-1}^{+1} z^k P_n(z) dz = 0 \quad (k=0, 1, \dots, n-1).$$

$$\int_0^1 z^\lambda P_n(z) dz = \begin{cases} \frac{\lambda(\lambda-2)\dots(\lambda-n+2)}{(\lambda+n+1)(\lambda+n-1)\dots(\lambda+1)} & (n \text{ even}), \\ \frac{(\lambda-1)(\lambda-3)\dots(\lambda-n+2)}{(\lambda+n+1)(\lambda+n-1)\dots(\lambda+2)} & (n \text{ odd}) \end{cases} \quad (\operatorname{Re} \lambda > -1).$$

$$\int_0^\pi P_n(\cos \theta) \sin m\theta d\theta = \begin{cases} \frac{2(m-n+1)(m-n+3)\dots(m+n-1)}{(m-n)(m-n+2)\dots(m+n)} & (m > n; m+n \text{ is odd}), \\ 0 & (\text{otherwise}). \end{cases}$$

(5) Conical Function (Kegelfunktion). This is the Legendre function corresponding to the case $\nu = -(1/2) + i\lambda$ (λ is a real parameter),

$$P_{-(1/2)+i\lambda}(\cos \theta) = 1 + \frac{4\lambda^2 + 1^2}{2^2} \sin^2 \frac{\theta}{2} + \frac{(4\lambda^2 + 1^2)(4\lambda^2 + 3^2)}{2^2 \cdot 4^2} \sin^4 \frac{\theta}{2} + \dots$$

$$P_{-(1/2)+i\lambda}(x) \equiv P_{-(1/2)-i\lambda}(x).$$

(III) Associated Legendre Functions (\rightarrow 393 Spherical Functions)

(1) Associated Legendre Functions. The fundamental system of solutions of the differential equation in (II) (1) is given by the following two kind of functions when $\mu \neq 0$.

Associated Legendre function of the first kind:

$$\mathfrak{P}_\nu^\mu(z) \equiv \frac{1}{\Gamma(1-\mu)} \left(\frac{z+1}{z-1} \right)^{\mu/2} {}_2F_1 \left(-\nu, \nu+1; 1-\mu; \frac{1-z}{2} \right),$$

where we take the branch satisfying $\arg[(z+1)/(z-1)]^{\mu/2} = 0$ for $z > 1$ in the expression raised to the $(\mu/2)$ th power.

Associated Legendre function of the second kind:

$$\mathfrak{Q}_\nu^\mu(z) \equiv \frac{e^{i\mu\pi}}{2^{\nu+1}} \frac{\Gamma(\nu+\mu+1)\sqrt{\pi}}{\Gamma[\nu+(3/2)]} (z^2-1)^{\mu/2} z^{-\nu-\mu-1} {}_2F_1 \left(\frac{\nu+\mu+2}{2}, \frac{\nu+\mu+1}{2}; \nu+\frac{3}{2}; \frac{1}{z^2} \right),$$

where we take the branch satisfying $\arg(z^2-1)^{\mu/2} = 0$ for $z > 1$ in $(z^2-1)^{\mu/2}$, and $\arg z^{-\nu-\mu-1} = 0$ for $z > 0$ in $z^{-\nu-\mu-1}$, respectively.

$$\begin{aligned} P_\nu^\mu(x) &\equiv e^{i\mu\pi/2} \mathfrak{P}_\nu^\mu(x+i0) = e^{i\mu\pi/2} \mathfrak{P}_\nu^\mu(x-i0) \\ &= \frac{1}{\Gamma(1-\mu)} \left(\frac{1+x}{1-x} \right)^{\mu/2} {}_2F_1 \left(-\nu, \nu+1; 1-\mu; \frac{1-x}{2} \right) \quad (-1 \leq x \leq 1). \end{aligned}$$

$$\begin{aligned} Q_\nu^\mu(x) &\equiv e^{-i\mu\pi} \left[e^{-i\mu\pi/2} \mathfrak{Q}_\nu^\mu(x+i0) + e^{i\mu\pi/2} \mathfrak{Q}_\nu^\mu(x-i0) \right] / 2 \\ &= \frac{\pi}{2 \sin \mu\pi} \left[P_\nu^\mu(x) \cos \mu\pi - \frac{\Gamma(\nu+\mu+1)}{\Gamma(\nu-\mu+1)} P_\nu^{-\mu}(x) \right] \quad (-1 \leq x \leq 1). \end{aligned}$$

Integral representations:

$$\mathfrak{P}_\nu^{-\mu}(z) = \frac{(z^2-1)^{\mu/2}}{2\mu\sqrt{\pi} \Gamma[\mu+(1/2)]} \int_{-1}^{+1} \frac{(1-t^2)^{\mu-1/2}}{(z+t\sqrt{z^2-1})^{\mu-\nu}} dt \quad \left(\operatorname{Re} \mu > -\frac{1}{2}, |\arg(z \pm 1)| < \pi \right).$$

$$\begin{aligned} \mathfrak{P}_\nu^{-\mu}(z) &= \frac{(z^2-1)^{\mu-2}}{2^\nu \Gamma(\mu-\nu) \Gamma(\nu+1)} \int_0^\infty \frac{(\sinh t)^{2\nu+1}}{(z + \cosh t)^{\mu+\nu+1}} dt \\ &\quad (\operatorname{Re} z > -1, |\arg(z \pm 1)| < \pi, \operatorname{Re} \nu > -1, \operatorname{Re}(\mu-\nu) > 0). \end{aligned}$$

$$\begin{aligned} \mathfrak{P}_\nu^{-\mu}(z) &= \sqrt{\frac{2}{\pi}} \frac{\Gamma[\mu+(1/2)](z^2-1)^{\mu-2}}{\Gamma(\mu+\nu+1) \Gamma(\mu-\nu)} \int_0^\infty \frac{\cosh[\nu+(1/2)]t}{(z + \cosh t)^{\mu+(1/2)}} dt \\ &\quad (\operatorname{Re} z > -1, |\arg(z \pm 1)| < \pi, \operatorname{Re}(\mu+\nu) > -1, \operatorname{Re}(\mu-\nu) > 0). \end{aligned}$$

$$\mathfrak{P}_\nu^\mu(\cosh \alpha) = \sqrt{\frac{2}{\pi}} \frac{(\sinh \alpha)^\mu}{\Gamma[-\mu+(1/2)]} \int_0^\alpha \frac{\cosh[\{\nu+(1/2)\}t] dt}{(\cosh \alpha - \cosh t)^{\mu+(1/2)}} \quad \left(\alpha > 0, \operatorname{Re} \mu < \frac{1}{2} \right).$$

$$P_\nu^\mu(\cos \theta) = \sqrt{\frac{2}{\pi}} \frac{(\sin \theta)^\mu}{\Gamma[-\mu+(1/2)]} \int_0^\theta \frac{\cos[\{\nu+(1/2)\}\varphi] d\varphi}{(\cos \theta - \cos \varphi)^{\mu+(1/2)}} \quad \left(0 < \theta < \pi, \operatorname{Re} \mu < \frac{1}{2} \right).$$

$$\begin{aligned} P_\nu^{-\mu}(\cos \theta) &= \frac{\Gamma(2\mu+1) 2^{-\mu} (\sin \theta)^\mu}{\Gamma(\mu+1) \Gamma(\mu+\nu+1) \Gamma(\mu-\nu)} \int_0^\infty \frac{t^{\mu+\nu} dt}{(1+2t \cos \theta + t^2)^{\mu+(1/2)}} \\ &\quad (\operatorname{Re}(\mu+\nu) > -1, \operatorname{Re}(\mu-\nu) > 0). \end{aligned}$$

$$P_\nu^{-\mu}(\cos \theta) = \frac{1}{\Gamma(\nu+\mu+1)} \int_0^\infty e^{-t \cos \theta} J_\mu(t \sin \theta) t^\nu dt \quad \left(0 < \theta < \frac{\pi}{2}, \operatorname{Re}(\mu+\nu) > -1 \right).$$

$$\begin{aligned} \mathfrak{Q}_\nu^\mu(z) &= \frac{e^{i\mu\pi}}{2^{\nu+1}} \frac{\Gamma(\nu+\mu+1)}{\Gamma(\nu+1)} (z^2-1)^{\mu/2} \int_{-1}^{+1} (1-t)^\nu (z-1)^{-\nu-\mu-1} dt \\ &\quad (\operatorname{Re}(\nu+\mu) > -1, \operatorname{Re} \nu > -1, |\arg(z \pm 1)| < \pi). \end{aligned}$$

$$\begin{aligned} \mathfrak{Q}_\nu^\mu(\cosh \alpha) &= \sqrt{\frac{\pi}{2}} \frac{e^{i\mu\pi} (\sinh \alpha)^\mu}{\Gamma[-\mu+(1/2)]} \int_\alpha^\infty \frac{e^{-[\nu+(1/2)]t} dt}{(\cosh t - \cosh \alpha)^{\mu+(1/2)}} \\ &\quad (\alpha > 0, \operatorname{Re} \mu < 1/2, \operatorname{Re}(\nu+\mu) > -1). \end{aligned}$$

Recurrence formulas:

$$(z^2 - 1)d\mathfrak{P}_\nu^\mu(z)/dz = (\nu - \mu + 1)\mathfrak{P}_{\nu+1}^\mu(z) - (\nu + 1)z\mathfrak{P}_\nu^\mu(z),$$

$$(2\nu + 1)z\mathfrak{P}_\nu^\mu(z) = (\nu - \mu + 1)\mathfrak{P}_{\nu+1}^\mu(z) + (\nu + \mu)\mathfrak{P}_{\nu-1}^\mu(z),$$

$$\mathfrak{P}_{\nu-1}^\mu(z) = \mathfrak{P}_\nu^\mu(z),$$

$$\mathfrak{Q}_\nu^{-\mu}(z) = e^{-2i\mu\pi} \frac{\Gamma(\nu - \mu + 1)}{\Gamma(\nu + \mu + 1)} \mathfrak{Q}_\nu^\mu(z),$$

$$(1 - x^2)dP_\nu^\mu(x)/dx = (\nu + 1)xP_{\nu+1}^\mu(x) - (\nu - \mu + 1)P_{\nu+1}^\mu(x).$$

The case when μ is an integer $m(m=0, 1, 2, \dots)$ and ν is also an integer n :

$$P_n^{m+2}(x) + 2(m+1)x(1-x^2)^{-1/2}P_n^{m+1}(x) + (n-m)(n+m+1)P_n^m(x) = 0.$$

$$(2n+1)xP_n^m(x) - (n-m+1)P_{n+1}^m(x) - (n+m)P_{n-1}^m(x) = 0 \quad (0 \leq m \leq n-2),$$

$$(x^2 - 1)dP_n^m(x)/dx - (n-m+1)P_{n+1}^m(x) + (n+1)xP_n^m(x) = 0,$$

$$P_{n-1}^m(x) - P_{n+1}^m(x) = (2n+1)\sqrt{1-x^2}P_n^{m-1}(x).$$

$$\mathfrak{P}_\nu^{-\mu}(z) = \frac{\Gamma(\nu - \mu + 1)}{\Gamma(\nu + \mu + 1)} \left[\mathfrak{P}_\nu^\mu(z) - \frac{2}{\pi} e^{-i\mu\pi} (\sin \mu\pi) \mathfrak{Q}_\nu^\mu(z) \right],$$

$$\mathfrak{Q}_\nu^\mu(z) \sin[(\nu + \mu)\pi] - \mathfrak{Q}_{\nu-1}^\mu(z) \sin[(\nu - \mu)\pi] = \pi e^{i\mu\pi} (\cos \nu\pi) \mathfrak{P}_\nu^\mu(z),$$

$$\mathfrak{P}_\nu^\mu(-z) = e^{\mp i\mu\pi} \mathfrak{P}_\nu^\mu(z) - (2/\pi) [\sin(\nu + \mu)\pi] e^{-i\mu\pi} \mathfrak{Q}_\nu^\mu(z) \quad (\mp = -\operatorname{sgn}(\operatorname{Im} z)),$$

$$\mathfrak{Q}_\nu^\mu(-z) = -e^{\pm i\mu\pi} \mathfrak{Q}_\nu^\mu(z) \quad (\pm = \operatorname{sgn}(\operatorname{Im} z)).$$

$$e^{-i\mu\pi} \mathfrak{Q}_\nu^\mu(\cosh \alpha) = \frac{\pi \Gamma(1 + \mu + \nu)}{\sqrt{2\pi} \sinh \alpha} \mathfrak{P}_{-\mu-1}^{-\nu-1/2}(\coth \alpha) \quad (\operatorname{Re} \cosh \alpha > 0).$$

$$e^{-i\mu\pi} \mathfrak{Q}_\nu^\mu(x \pm i0) = e^{\pm i\mu\pi/2} [Q_\nu^\mu(x) \mp (i\pi/2)P_\nu^\mu(x)].$$

$$Q_{\nu-1}^\mu(x) = \frac{\sin(\nu + \mu)\pi}{\sin(\nu - \mu)\pi} Q_\nu^\mu(x) - \frac{\pi \cos \nu\pi \cos \mu\pi}{\sin(\nu - \mu)\pi} P_\nu^\mu(x),$$

$$P_\nu^{-\mu}(x) = \frac{\Gamma(\nu - \mu + 1)}{\Gamma(\nu + \mu + 1)} \left[\cos \mu\pi P_\nu^\mu(x) - \frac{2}{\pi} \sin \mu\pi Q_\nu^\mu(x) \right],$$

$$P_\nu^\mu(-x) = [\cos(\nu + \mu)\pi] P_\nu^\mu(x) - (2/\pi) [\sin(\nu + \mu)\pi] Q_\nu^\mu(x),$$

$$Q_\nu^\mu(-x) = -[\cos(\nu + \mu)\pi] Q_\nu^\mu(x) + (\pi/2) [\sin(\nu + \mu)\pi] P_\nu^\mu(x).$$

$$\mathfrak{P}_\nu^m(z) = \frac{\Gamma(1 + \nu + m)(z^2 - 1)^{m/2}}{\Gamma(1 + \nu - m)m!2^m} {}_2F_1\left(m - \nu, m + \nu + 1; m + 1; \frac{1-z}{2}\right) = (z^2 - 1)^{m/2} \frac{d^m \mathfrak{P}_\nu(z)}{dz^m},$$

$$\mathfrak{P}_\nu^{-m}(z) = (z^2 - 1)^{-m/2} \int_1^z \dots \int_1^z P_\nu(z)(dz)^m.$$

$$\mathfrak{Q}_\nu^m(z) = (z^2 - 1)^{m/2} \frac{d^m \mathfrak{Q}_\nu(z)}{dx^m}, \quad \mathfrak{Q}_\nu^{-m}(z) = (-1)^m (z^2 - 1)^{-m/2} \int_z^\infty \dots \int_z^\infty \mathfrak{Q}_\nu(z)(dz)^m.$$

$$\begin{aligned} P_\nu^m(x) &= (-1)^m \frac{\Gamma(1 + \nu + m)(1 - x^2)^{m/2}}{\Gamma(1 + \nu - m)m!2^m} {}_2F_1\left(m - \nu, m + \nu + 1; m + 1; \frac{1-x}{2}\right) \\ &= (-1)^m (1 - x^2)^{m/2} \frac{d^m P_\nu(x)}{dx^m}, \end{aligned}$$

$$P_\nu^{-m}(x) = (1 - x^2)^{-m/2} \int_x^1 \dots \int_x^1 P_\nu(x)(dx)^m = (-1)^m \frac{\Gamma(\nu - m + 1)}{\Gamma(\nu + m + 1)} P_\nu^m(x).$$

$$Q_\nu^m(x) = (-1)^m (1 - x^2)^{m/2} \frac{d^m Q_\nu(x)}{dx^m}, \quad Q_\nu^{-m}(x) = (-1)^m \frac{\Gamma(\nu - m + 1)}{\Gamma(\nu + m + 1)} Q_\nu^m(x).$$

The values at the origin are

$$P_\nu^\mu(0) = \frac{\sqrt{\pi} 2^\mu}{\Gamma[(\nu - \mu)/2 + 1] \Gamma[(\nu - \mu + 1)/2]},$$

$$\frac{dP_\nu^\mu(0)}{dx} = \frac{2^{\mu+1} \sin[\pi(\nu+\mu)/2] \Gamma[(\nu+\mu+2)/2]}{\Gamma[(\nu-\mu+1)/2] \sqrt{\pi}} = \frac{\sqrt{\pi} 2^{\mu+1}}{\Gamma[(\nu-\mu+1)/2] \Gamma[(\nu-\mu)/2]},$$

$$Q_\nu^\mu(0) = -2^{\mu-1} \sqrt{\pi} \sin\left(\frac{\nu+\mu}{2}\pi\right) \frac{\Gamma[(\nu+\mu+1)/2]}{\Gamma[(\nu-\mu+2)/2]},$$

$$\frac{dQ_\nu^\mu(0)}{dx} = 2^\mu \sqrt{\pi} \cos\left(\frac{\nu+\mu}{2}\pi\right) \frac{\Gamma[(\nu+\mu+2)/2]}{\Gamma[(\nu-\mu+1)/2]}.$$

(2) Generating Functions.

$$(\cos\theta + i \sin\theta \sin\varphi)^n = P_n(\cos\theta) + 2 \sum_{m=1}^n (-i)^m \frac{n!}{(n+m)!} (\cos m\varphi) P_n^m(\cos\theta).$$

$$P_\nu^{-\mu}(\cos\theta) = \frac{\sin\nu\pi}{\pi} \sum_{n=0}^{\infty} (-1)^n \left(\frac{1}{\nu-n} - \frac{1}{\nu+n+1} \right) P_n^{-\mu}(\cos\theta) \quad (0 < \theta < \pi, \mu \geq 0).$$

(3) Orthogonal Relations.

$$\int_{-1}^{+1} P_n^m(x) P_n^m(x) dx = \frac{2}{2n+1} \frac{(n+m)!}{(n-m)!} \delta_{nn'}.$$

$$\int_0^{2\pi} d\varphi \int_0^\pi \sin\theta e^{\pm i(m-m')\varphi} P_n^m(\cos\theta) P_n^{m'}(\cos\theta) d\theta = \frac{4\pi}{2n+1} \frac{(n+m)!}{(n-m)!} \delta_{nn'} \delta_{mm'}.$$

(4) Addition Theorems.

$$\mathfrak{P}_\nu(z\xi - \sqrt{z^2-1} \sqrt{\xi^2-1} \cos\varphi) = \mathfrak{P}_\nu(z) \mathfrak{P}_\nu(\xi) + 2 \sum_{m=1}^{\infty} (-1)^m \mathfrak{P}_\nu^m(z) \mathfrak{P}_\nu^{-m}(\xi) \cos m\varphi$$

$$(\operatorname{Re} z > 0, \operatorname{Re} \xi > 0, |\arg(z-1)| < \pi, |\arg(\xi-1)| < \pi).$$

$$\mathfrak{Q}_\nu(t't' - \sqrt{t^2-1} \sqrt{t'^2-1} \cos\varphi) = \mathfrak{Q}_\nu(t) \mathfrak{P}_\nu(t') + 2 \sum_{m=1}^{\infty} (-1)^m \mathfrak{Q}_\nu^m(t) \mathfrak{P}_\nu^{-m}(t') \cos m\varphi$$

$$(t, t' \text{ real}, 1 < t' < t, \nu \neq \text{negative integer}, \varphi \text{ real}).$$

$$P_\nu(\cos\theta \cos\theta' + \sin\theta \sin\theta' \cos\varphi) = P_\nu(\cos\theta) P_\nu(\cos\theta') + 2 \sum_{m=1}^{\infty} (-1)^m P_\nu^{-m}(\cos\theta') P_\nu^m(\cos\theta) \cos m\varphi$$

$$= P_\nu(\cos\theta) P_\nu(\cos\theta') + 2 \sum_{m=1}^{\infty} \frac{\Gamma(\nu-m+1)}{\Gamma(\nu+m+1)} P_\nu^m(\cos\theta) P_\nu^m(\cos\theta') \cos m\varphi$$

$$(0 \leq \theta < \pi, 0 \leq \theta' < \pi, \theta + \theta' < \pi, \varphi \text{ real}).$$

$$Q_\nu(\cos\theta \cos\theta' + \sin\theta \sin\theta' \cos\varphi) = P_\nu(\cos\theta') Q_\nu(\cos\theta) + 2 \sum_{m=1}^{\infty} (-1)^m P_\nu^{-m}(\cos\theta') Q_\nu^m(\cos\theta) \cos m\varphi$$

$$(0 < \theta' < \pi/2, 0 < \theta < \pi, \theta + \theta' < \pi, \varphi \text{ real}).$$

$$\mathfrak{Q}_\nu(\tau\tau' + \sqrt{\tau^2+1} \sqrt{\tau'^2+1} \cosh\alpha) = \sum_{m=n+1}^{\infty} \frac{1}{(m-n-1)!(m+n)!} \mathfrak{Q}_n^m(i\tau) \mathfrak{Q}_n^m(i\tau') e^{-m\alpha}$$

$$(\tau, \tau', \alpha > 0).$$

(5) Asymptotic Expansions.

$$\mathfrak{P}_\nu^\mu(z) = \left[\frac{2^\nu \Gamma[\nu+(1/2)]}{\sqrt{\pi} \Gamma(\nu-\mu+1)} z^\nu + \frac{2^{-\nu-1} \Gamma[-\nu-(1/2)]}{\sqrt{\pi} \Gamma(-\mu-\nu)} z^{-\nu-1} \right] [1 + O(z^{-2})]$$

$$(\nu+(1/2) \neq \text{integer}, |\arg z| < \pi, |z| \gg 1).$$

$$\mathfrak{Q}_\nu^\mu(z) = \frac{\sqrt{\pi} e^{i\mu\pi}}{2^{\nu+1}} \frac{\Gamma(\nu+\mu+1)}{\Gamma[\nu+(3/2)]} z^{-\nu-1} [1 + O(z^{-2})]$$

$$(\nu+(1/2) \neq \text{negative integer}, |\arg z| < \pi, |z| \gg 1).$$

$$P_\nu^\mu(\cos\theta) = \frac{2}{\sqrt{\pi}} \frac{\Gamma(\nu+\mu+1)}{\Gamma[\nu+(3/2)]} \frac{\cos[\{\nu+(1/2)\}\theta - (\pi/4) + (\mu\pi/2)]}{\sqrt{2} \sin\theta} [1 + O(\nu^{-1})]$$

$$(\varepsilon \leq \theta \leq \pi - \varepsilon, \varepsilon > 0, |\nu| \gg 1/\varepsilon).$$

$$P_{\nu}^{\mu}(\cos \theta) = \frac{2\Gamma(\nu + \mu + 1)}{\sqrt{\pi} \Gamma[\nu + (3/2)]} \times \sum_{l=0}^{\infty} \frac{\Gamma\left(\frac{1}{2} + \mu + l\right) \Gamma\left(\frac{1}{2} - \mu + l\right) \Gamma\left(\nu + \frac{3}{2}\right)}{\Gamma\left(\frac{1}{2} + \mu\right) \Gamma\left(\frac{1}{2} - \mu\right) \Gamma\left(\nu + l + \frac{3}{2}\right) l!} \frac{\cos\left[\left(\nu + \frac{2l+1}{2}\right)\theta - \frac{(2l+1)\pi}{4} + \frac{\mu\pi}{2}\right]}{(2\sin\theta)^{l+(1/2)}},$$

$$Q_{\nu}^{\mu}(\cos \theta) = \sqrt{\pi} \frac{\Gamma(\nu + \mu + 1)}{\Gamma[\nu + (3/2)]} \times \sum_{l=0}^{\infty} (-1)^l \frac{\Gamma\left(\frac{1}{2} + \mu + l\right) \Gamma\left(\frac{1}{2} - \mu + l\right) \Gamma\left(\nu + \frac{3}{2}\right)}{\Gamma\left(\frac{1}{2} + \mu\right) \Gamma\left(\frac{1}{2} - \mu\right) \Gamma\left(\nu + l + \frac{3}{2}\right) l!} \frac{\cos\left[\left(\nu + \frac{2l+1}{2}\right)\theta + \frac{(2l+1)\pi}{4} + \frac{\mu\pi}{2}\right]}{(2\sin\theta)^{l+(1/2)}}.$$

(In the final two formulas the series converges when $\nu + \mu \neq$ negative integer, $\nu + (1/2) \neq$ negative integer, $\pi/6 < \theta < 5\pi/6$.)

$$\left[\left(\nu + \frac{1}{2}\right)\cos\frac{\theta}{2}\right]^{\mu} P_{\nu}^{-\mu}(\cos\theta) = J_{\mu}(\eta) + \sin^2\frac{\theta}{2} \left[\frac{J_{\mu+1}(\eta)}{2\eta} - J_{\mu+2}(\eta) + \frac{\eta}{6} J_{\mu+3}(\eta) \right] + O\left(\sin^4\frac{\theta}{2}\right) \\ (\eta = (2\nu + 1)\sin(\theta/2)).$$

(6) Estimation. When $\nu \geq 1$, $\nu - \mu + 1 > 0$, $\mu \geq 0$,

$$|P_{\nu}^{\pm\mu}(\cos\theta)| < \frac{\Gamma(\nu \pm \mu + 1)}{\Gamma(\nu + 1)} \left(\frac{8}{\nu\pi\sin\theta}\right)^{1/2} \frac{1}{(\sin\theta)^{\mu}},$$

$$|Q_{\nu}^{\pm\mu}(\cos\theta)| < \frac{\Gamma(\nu \pm \mu + 1)}{\Gamma(\nu + 1)} \left(\frac{2\pi}{\nu\sin\theta}\right)^{1/2} \frac{1}{(\sin\theta)^{\mu}},$$

$$|P_{\nu}^{\pm m}(\cos\theta)| < \frac{\Gamma(\nu \pm m + 1)}{\Gamma(\nu + 1)} \left(\frac{4}{\nu\pi\sin\theta}\right)^{1/2} \frac{1}{(\sin\theta)^m},$$

$$|Q_{\nu}^{\pm m}(\cos\theta)| < \frac{\Gamma(\nu \pm m + 1)}{\Gamma(\nu + 1)} \left(\frac{4}{\nu\sin\theta}\right)^{1/2} \frac{1}{(\sin\theta)^m}.$$

(7) Torus Functions. These are solutions of the differential equation

$$\frac{d^2u}{d\eta^2} + \coth\eta \frac{du}{d\eta} - \left(n^2 - \frac{1}{4} + \frac{m^2}{\sinh^2\eta}\right)u = 0.$$

The fundamental system of solutions is given by

$$\mathfrak{P}_{n-(1/2)}^m(\cosh\eta), \quad \mathfrak{Q}_{n-(1/2)}^m(\cosh\eta).$$

The asymptotic expansion when $m=0$ is

$$\mathfrak{P}_{n-(1/2)}(\cosh\eta) = \frac{(n-1)!e^{n-(1/2)\eta}}{\Gamma[n+(1/2)]\sqrt{\pi}} \left[\frac{2\Gamma^2[n+(1/2)]}{\pi n!(n-1)!} (\log 4 + \eta) e^{-2\eta} {}_2F_1\left(\frac{1}{2}, n + \frac{1}{2}; n+1; e^{-2\eta}\right) + A + B \right].$$

Here

$$A = 1 + \frac{(1/2)[n-(1/2)]}{1 \cdot (n-1)} e^{-2\eta} + \frac{(1/2)(3/2)[n-(1/2)][n-(3/2)]}{1 \cdot 2 \cdot (n-1)(n-2)} e^{-4\eta} + \dots \\ + \frac{(2n-3)!!(2n-1)!!}{[(2n-2)!!]^2} e^{-2(n-1)\eta},$$

$$B = \frac{\Gamma[n+(1/2)]}{\pi^{3/2}(n-1)!} \sum_{l=1}^{\infty} \frac{\Gamma[l+(1/2)]\Gamma[n+l+(1/2)]}{(n+l)!l!} (u_{n+l} + u_l - v_{l-(1/2)} - v_{n+l-(1/2)}) e^{-2(l+n)\eta},$$

where

$$u_r \equiv 1 + \frac{1}{2} + \dots + \frac{1}{r}, \quad v_{r-(1/2)} \equiv \frac{2}{1} + \frac{2}{3} + \frac{2}{5} + \dots + \frac{2}{2r-1} = 2u_{2r} - u_r.$$

References

See references to Table 16, this Appendix.

19. Functions of Confluent Type and Bessel Functions**(I) Hypergeometric Function of Confluent Type** (\rightarrow 167 Functions of Confluent Type)**(1) Kummer Functions.**

$$\begin{aligned}
 v(z) = {}_1F_1(a; c; z) &\equiv \sum_{n=0}^{\infty} \frac{\Gamma(a+n)}{\Gamma(a)} \frac{\Gamma(c)}{\Gamma(c+n)} \frac{z^n}{n!} \\
 &= \frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)} z^{1-c} \int_0^z e^t t^{a-1} (z-t)^{c-a-1} dt \quad (0 < \operatorname{Re} a < \operatorname{Re} c) \\
 &= \frac{\Gamma(c)2^{1-c}}{\Gamma(a)\Gamma(c-a)} e^{z/2} \int_{-1}^{+1} e^{zt/2} (1-t)^{c-a-1} (1+t)^{a-1} dt \quad (0 < \operatorname{Re} a < \operatorname{Re} c).
 \end{aligned}$$

The fundamental system of solutions of the confluent hypergeometric differential equation (Kummer's differential equation)

$$z \frac{d^2 v}{dz^2} + (c-z) \frac{dv}{dz} - av = 0,$$

when $c \neq 0, -1, -2, \dots$, is given by

$$v_1(z) \equiv {}_1F_1(a; c; z), \quad v_2(z) \equiv z^{1-c} {}_1F_1(a-c+1; 2-c; z).$$

$$d {}_1F_1(a; c; z)/dz = (a/c) {}_1F_1(a+1; c+1; z),$$

$${}_1F_1(a; c; z) = e^z {}_1F_1(c-a; c; -z),$$

$$a {}_1F_1(a+1; c+1; z) = (a-c) {}_1F_1(a; c+1; z) + c {}_1F_1(a; c; z),$$

$$a {}_1F_1(a+1; c; z) = (z+2a-c) {}_1F_1(a; c; z) + (c-a) {}_1F_1(a-1; c; z).$$

Putting $(a)_n = a(a+1)\dots(a+n-1) = \Gamma(a+n)/\Gamma(a)$ we have

$$\lim_{c \rightarrow -n} \frac{1}{\Gamma(c)} {}_1F_1(a; c; z) = \frac{z^{n+1} (a)_{n+1}}{(n+1)!} {}_1F_1(a+n+1; n+2; z) \quad (n=0, 1, 2, \dots).$$

Asymptotic expansion:

$$v_1 \approx A_1 z^{-a} \sum_{n=0}^{\infty} \frac{(a)_n (a-c+1)_n}{n!} (-z)^{-n} + B_1 e^z z^{a-c} \sum_{n=0}^{\infty} \frac{(c-a)_n (1-a)_n}{n!} z^n,$$

$$v_2 \approx A_2 z^{-a} \sum_{n=0}^{\infty} \frac{(a)_n (a-c+1)_n}{n!} (-z)^{-n} + B_2 e^z z^{a-c} \sum_{n=0}^{\infty} \frac{(c-a)_n (1-a)_n}{n!} z^n$$

$$(|z| \gg |a|, |z| \gg |c|, -3\pi/2 < \arg z < \pi/2, c \neq \text{integer}),$$

where

$$A_1 = e^{-i\pi a} \Gamma(c)/\Gamma(c-a),$$

$$B_1 = \Gamma(c)/\Gamma(a),$$

$$A_2 = e^{-i\pi(a-c+1)} \Gamma(2-c)/\Gamma(1-a),$$

$$B_2 = \Gamma(2-c)/\Gamma(a-c+1).$$

(2) The fundamental system of solutions at $z=0$ of the hypergeometric differential equation of confluent type

$$\frac{d^2 u}{dz^2} + \frac{du}{dz} + \left[\frac{\kappa}{z} + \frac{(1/4) - \mu^2}{z^2} \right] u = 0$$

is given by

$$z^{(1/2) \pm \mu} e^{-z} {}_1F_1[(1/2) \pm \mu - \kappa; \pm 2\mu + 1; z].$$

(II) Whittaker Functions (\rightarrow 167 Functions of Confluent Type)**(1) A pair of linearly independent solutions of Whittaker's differential equation**

$$\frac{d^2 W}{dz^2} + \left[-\frac{1}{4} + \frac{\kappa}{z} + \frac{(1/4) - \mu^2}{z^2} \right] W = 0$$

is given by $M_{\kappa, \pm\mu}(z) = z^{\pm\mu+(1/2)} e^{-z/2} {}_1F_1[\pm\mu - \kappa + (1/2); \pm 2\mu + 1; z]$.

Whittaker functions:

$$W_{\kappa, \mu}(z) \equiv \frac{\Gamma(-2\mu)}{\Gamma[(1/2) - \mu - \kappa]} M_{\kappa, \mu}(z) + \frac{\Gamma(2\mu)}{\Gamma[(1/2) + \mu - \kappa]} M_{\kappa, -\mu}(z) = W_{\kappa, -\mu}(z).$$

When 2μ is an integer, the above definition of $W_{\kappa, \mu}(z)$ loses meaning, but by taking the limit with respect to μ we can define it in terms of the following integrals.

$$\begin{aligned} W_{\kappa, \mu}(z) &= \frac{z^{\mu+(1/2)} e^{-z/2}}{\Gamma[\mu + (1/2) - \kappa]} \int_0^\infty e^{-z\tau} \tau^{\mu - \kappa - (1/2)} (1 + \tau)^{\mu + \kappa - (1/2)} d\tau \\ &= \frac{z^\kappa e^{-z/2}}{\Gamma[\mu + (1/2) - \kappa]} \int_0^\infty t^{\mu - \kappa - (1/2)} e^{-t} \left(1 + \frac{t}{z}\right)^{\mu + \kappa - (1/2)} dt \\ &(\operatorname{Re}[\mu + (1/2) - \kappa] > 0, \quad |\arg z| < \pi). \end{aligned}$$

$$W_{\kappa, \mu}(z) = \frac{e^{-z/2}}{2\pi i} \int_{-i\infty}^{+i\infty} \frac{\Gamma(s - \kappa) \Gamma[-s - \mu + (1/2)] \Gamma[-s + \mu + (1/2)]}{\Gamma[-\kappa + \mu + (1/2)] \Gamma[-\kappa - \mu + (1/2)]} z^s ds.$$

$$M_{l+\mu+(1/2), \mu}(z) = (-1)^l z^{\mu+(1/2)} e^{-z/2} (2\mu + 1) {}_1F_1(-l; 2\mu + 1; z) \quad (l = 0, 1, 2, \dots).$$

$$M_{\kappa, \mu}(z) = e^{-i\pi[\mu+(1/2)]} M_{-\kappa, \mu}(e^{i\pi} z).$$

$$\begin{aligned} M_{\kappa, \mu}(z) &= \frac{\Gamma(2\mu + 1)}{\Gamma[\mu + (1/2) - \kappa]} e^{i\pi\kappa} W_{-\kappa, \mu}(e^{i\pi} z) + \frac{\Gamma(2\mu + 1)}{\Gamma[\mu + (1/2) + \kappa]} e^{i\pi[\kappa - \mu - (1/2)]} W_{\kappa, \mu}(z) \\ &(-3\pi/2 < \arg z < \pi/2, \quad 2\mu \neq -1, -2, \dots). \end{aligned}$$

$$\begin{aligned} M_{\kappa, \mu}(z) &= \frac{\Gamma(2\mu + 1)}{\Gamma[\mu + (1/2) - \kappa]} e^{-i\pi\kappa} W_{-\kappa, \mu}(e^{-i\pi} z) + \frac{\Gamma(2\mu + 1)}{\Gamma[\mu + (1/2) + \kappa]} e^{-i\pi[\kappa - \mu - (1/2)]} W_{\kappa, \mu}(z) \\ &(-\pi/2 < \arg z < 3\pi/2, \quad 2\mu \neq -1, -2, \dots). \end{aligned}$$

$$\begin{aligned} W_{\kappa, \mu}(z) &= z^{1/2} W_{\kappa - (1/2), \mu - (1/2)}(z) + [(1/2) - \kappa + \mu] W_{\kappa - 1, \mu}(z) \\ &= z^{1/2} W_{\kappa - (1/2), \mu + (1/2)}(z) + [(1/2) - \kappa - \mu] W_{\kappa - 1, \mu}(z). \end{aligned}$$

$$z dW_{\kappa, \mu}(z)/dz = [\kappa - (z/2)] W_{\kappa, \mu}(z) - [\mu^2 - \{\kappa - (1/2)\}^2] W_{\kappa - 1, \mu}(z).$$

When κ is sufficiently large we have

$$\begin{aligned} M_{\kappa, \mu}(z) &\sim \pi^{-1/2} \Gamma(2\mu + 1) \kappa^{-\mu - (1/4)} z^{1/4} \cos[2(z\kappa)^{1/2} - \mu\pi - (\pi/4)], \\ W_{\kappa, \mu}(z) &\sim -(4z/\kappa)^{1/4} \exp(-\kappa + \kappa \log \kappa) \sin[2(z\kappa)^{1/2} - \pi\kappa - (\pi/4)], \\ W_{-\kappa, \mu}(z) &\sim (z/4\kappa)^{1/4} \exp(\kappa - \kappa \log \kappa - 2(z\kappa)^{1/2}). \end{aligned}$$

Asymptotic expansion:

$$\begin{aligned} W_{\kappa, \mu}(z) &\approx e^{-z/2} z^\kappa \\ &\times \left(1 + \sum_{n=1}^{\infty} \frac{[\mu^2 - \{\kappa - (1/2)\}^2][\mu^2 - \{\kappa - (3/2)\}^2] \dots [\mu^2 - \{\kappa - n + (1/2)\}^2]}{n! z^n}\right). \end{aligned}$$

(2) Representation of Various Special Functions by Whittaker Functions.

$$\begin{aligned} \text{(i) Probability integral (error function)} \quad \operatorname{erf} x &\equiv \Phi(x) \equiv \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt \\ &= 1 - \pi^{-1/2} x^{-1/2} e^{-x^2/2} W_{-1/4, 1/4}(x^2) \end{aligned}$$

$$= \frac{2x}{\sqrt{\pi}} {}_1F_1\left(\frac{1}{2}; \frac{3}{2}; -x^2\right) = \frac{2}{\sqrt{\pi}} \left(x - \frac{x^3}{1!3} + \frac{x^5}{2!5} - \frac{x^7}{3!7} \pm \dots\right).$$

Asymptotic expansion:

$$\begin{aligned} \frac{\sqrt{\pi}}{2} [1 - \Phi(x)] &\approx \frac{e^{-x^2}}{2x} \left(1 - \frac{1}{2x^2} + \frac{1 \cdot 3}{(2x^2)^2} - \frac{1 \cdot 3 \cdot 5}{(2x^2)^3} \pm \dots\right). \\ \frac{1}{1+i} \Phi\left(x \frac{1+i}{2} \sqrt{\pi}\right) &= C(x) - iS(x), \end{aligned}$$

where $C(x)$, $S(x)$ are the following Fresnel integrals.

$$C(x) \equiv \int_0^x \cos \frac{\pi}{2} t^2 dt = \frac{1}{2} + \frac{1}{\pi x} \sin \frac{\pi}{2} x^2 + O\left(\frac{1}{x^2}\right),$$

$$S(x) \equiv \int_0^x \sin \frac{\pi}{2} t^2 dt = \frac{1}{2} - \frac{1}{\pi x} \cos \frac{\pi}{2} x^2 + O\left(\frac{1}{x^2}\right).$$

(ii) Logarithmic integral

$$\begin{aligned} \text{Li } z &\equiv \int_0^z \frac{dt}{\log t} \quad (\text{When } z > 1, \text{ take Cauchy's principal value at } t = 1.) \\ &= -(\log 1/z)^{-1/2} z^{1/2} W_{-1/2, 0}(-\log z). \end{aligned}$$

$\text{Li } z$ is sometimes written as $\text{li } z$.

(3) Exponential Integral

$$\text{Ei } x \equiv \int_{-\infty}^x \frac{e^t}{t} dt \quad (\text{When } x > 0, \text{ take the Cauchy's principal value at } t = 0 \text{ while integrating.})$$

$$= C + \log|x| + \sum_{n=1}^{\infty} \frac{x^n}{n \cdot n!} \quad (x \text{ real, } \neq 0)$$

$$= e^x \sum_{n=1}^N \frac{(n-1)!}{t^n} + N! \sum_{\substack{n=0 \\ n \neq N}}^{\infty} \frac{t^{n-N}}{n!(n-N)} - \left(1 + \frac{1}{2} + \cdots + \frac{1}{N}\right) + C + \log|x|.$$

$$\text{Cosine integral} \quad \text{Ci } x \equiv - \int_x^{\infty} \frac{\cos t}{t} dt = C + \log x - \int_0^x \frac{1 - \cos t}{t} dt.$$

$$\text{Sine integral} \quad \text{Si } x \equiv \int_0^x \frac{\sin t}{t} dt,$$

$$\text{si } x \equiv - \int_x^{\infty} \frac{\sin t}{t} dt = \text{Si } x - \frac{\pi}{2}.$$

$$\text{Asymptotic expansion} \quad \text{Ei } ix = \text{Ci } x + i \text{si } x \approx e^{ix} \left(\frac{1}{ix} + \frac{1!}{(ix)^2} + \frac{2!}{(ix)^3} + \frac{3!}{(ix)^4} + \cdots \right).$$

(III) Bessel Functions (→ 39 Bessel Functions)

(1) Cylindrical Functions. A cylindrical function Z_ν is a solution of Bessel's differential equation

$$\frac{d^2 Z_\nu}{dz^2} + \frac{1}{z} \frac{dZ_\nu}{dz} + \left(1 - \frac{\nu^2}{z^2}\right) Z_\nu = 0.$$

Recurrence formulas:

$$Z_{\nu-1}(z) + Z_{\nu+1}(z) = (2\nu/z) Z_\nu(z), \quad Z_{\nu-1}(z) - Z_{\nu+1}(z) = 2dZ_\nu(z)/dz.$$

$$\int z^{\nu+1} Z_\nu(z) dz = z^{\nu+1} Z_{\nu+1}(z), \quad \int z^{-\nu} Z_{\nu+1}(z) dz = -z^{-\nu} Z_\nu(z).$$

As special solutions, we have the following three kinds of functions.

(i) Bessel function (Bessel function of the first kind).

$$J_\nu(z) \equiv \left(\frac{z}{2}\right)^\nu \sum_{l=0}^{\infty} \frac{(-1)^l}{l! \Gamma(\nu + l + 1)} \left(\frac{z}{2}\right)^{2l} = \frac{M_{0,\nu}(2iz)}{(2iz)^{1/2} 2^\nu i^\nu \Gamma(\nu + 1)} \quad (|\arg z| < \pi).$$

$$J_\nu(e^{im\pi} z) = e^{im\nu\pi} J_\nu(z).$$

$$J_{-n}(z) = (-1)^n J_n(z).$$

$$J_{n+(1/2)}(z) = \sqrt{\frac{2}{\pi}} z^{n+(1/2)} \left(-\frac{1}{z} \frac{d}{dz}\right)^n \left(\frac{\sin z}{z}\right) \quad (n=0, 1, 2, \dots).$$

(ii) Neumann function (Bessel function of the second kind).

$$N_\nu(z) \equiv \frac{1}{\sin \nu\pi} [(\cos \nu\pi) J_\nu(z) - J_{-\nu}(z)] \quad (\nu \neq \text{integer}; |\arg z| < \pi),$$

$$\begin{aligned} N_n(z) &\equiv \frac{2}{\pi} J_n(z) \left(C + \log \frac{z}{2}\right) - \frac{1}{\pi} \left(\frac{z}{2}\right)^n \sum_{l=0}^{\infty} \frac{(-1)^l}{l!(n+l)!} \left(\frac{z}{2}\right)^{2l} [\varphi(l) + \varphi(l+n)] \\ &\quad - \frac{1}{\pi} \left(\frac{z}{2}\right)^{-n} \sum_{l=0}^{n-1} \frac{(n-l-1)!}{l!} \left(\frac{z}{2}\right)^{2l} \quad \left(\varphi(l) \equiv \sum_{m=1}^l \frac{1}{m}\right), \end{aligned}$$

$$N_{-n}(z) \equiv (-1)^n N_n(z) \quad (n=0, 1, 2, \dots; \quad |\arg z| < \pi).$$

$$N_\nu(e^{im\pi}z) = e^{-im\nu\pi} N_\nu(z) + 2i(\sin m\nu\pi \cot \nu\pi) J_\nu(z).$$

$$N_{n+(1/2)}(z) = (-1)^{n+1} J_{-[n+(1/2)]}(z).$$

(iii) Hankel function (Bessel function of the third kind).

$$H_\nu^{(1)}(z) \equiv J_\nu(z) + iN_\nu(z),$$

$$H_\nu^{(2)}(z) \equiv J_\nu(z) - iN_\nu(z).$$

$$H_\nu^{(1)}(iz/2) = -2ie^{-i\nu\pi/2} (\pi z)^{-1/2} W_{0,\nu}(z).$$

$$H_{-\nu}^{(1)}(z) = e^{i\nu\pi} H_\nu^{(1)}(z), \quad H_{-\nu}^{(2)}(z) = e^{-i\nu\pi} H_\nu^{(2)}(z), \quad \overline{H_\nu^{(2)}(x)} = H_\nu^{(1)}(x) \quad (x, \nu \text{ real}).$$

(2) Integral Representation.

$$\begin{aligned} \text{Hansen-Bessel formula} \quad J_n(z) &= \frac{1}{2\pi} \int_{-\pi}^{+\pi} e^{iz \cos t} e^{in[t-(\pi/2)]} dt \\ &= \frac{i^{-n}}{\pi} \int_0^\pi e^{iz \cos t} \cos nt \, dt \\ &= \frac{1}{\pi} \int_0^\pi \cos(z \sin t - nt) dt \quad (n=0, 1, 2, \dots). \end{aligned}$$

$$\begin{aligned} \text{Mehler's formula} \quad J_0(x) &= \frac{2}{\pi} \int_0^\infty \sin(x \cosh t) dt, \\ N_0(x) &= -\frac{2}{\pi} \int_0^\infty \cos(x \cosh t) dt \quad (x > 0). \end{aligned}$$

$$\begin{aligned} \text{Poisson's formula} \quad J_\nu(z) &= \frac{2(z/2)^\nu}{\sqrt{\pi} \Gamma[\nu + (1/2)]} \int_0^{\pi/2} \cos(z \cos t) \sin^{2\nu} t \, dt \quad \left(\operatorname{Re} \nu > -\frac{1}{2}\right), \\ N_\nu(z) &= \frac{2(z/2)^\nu}{\sqrt{\pi} \Gamma[\nu + (1/2)]} \left[\int_0^{\pi/2} \sin(z \sin t) \cos^{2\nu} t \, dt - \int_0^\infty e^{-z \sinh t} \cosh^{2\nu} t \, dt \right] \\ &\quad (\operatorname{Re} z > 0, \operatorname{Re} \nu > -1/2). \end{aligned}$$

$$\text{Schläfli's formula} \quad J_\nu(z) = \frac{1}{\pi} \int_0^\pi \cos(z \sin t - \nu t) dt - \frac{\sin \nu\pi}{\pi} \int_0^\infty e^{-z \sinh t} e^{-\nu t} dt \quad (\operatorname{Re} z > 0),$$

$$N_\nu(z) = \frac{1}{\pi} \int_0^\pi \sin(z \sin t - \nu t) dt - \frac{1}{\pi} \int_0^\infty e^{-z \sinh t} [e^{\nu t} + (\cos \nu\pi) e^{-\nu t}] dt \quad (\operatorname{Re} z > 0).$$

$$J_\nu(z) = \frac{z^\nu}{2\pi i} \int_{c-i\infty}^{c+i\infty} \exp\left[\frac{1}{2}\left(t - \frac{z^2}{t}\right)\right] t^{-\nu-1} dt \quad (c > 0, \quad |\arg z| < \pi, \operatorname{Re} \nu > -1).$$

$$J_\nu(x) = \frac{2(x/2)^{-\nu}}{\sqrt{\pi} \Gamma[(1/2) - \nu]} \int_1^\infty \frac{\sin xt}{(t^2 - 1)^{\nu+(1/2)}} dt,$$

$$N_\nu(x) = -\frac{2(x/2)^{-\nu}}{\sqrt{\pi} \Gamma[(1/2) - \nu]} \int_1^\infty \frac{\cos xt}{(t^2 - 1)^{\nu+(1/2)}} dt \quad \left(x > 0, \quad -\frac{1}{2} < \operatorname{Re} \nu < \frac{1}{2}\right).$$

$$J_\nu(z) = \frac{1}{2\pi i} \int_{-\infty}^{(0+)} e^{z[t-(1/t)]/2} t^{-\nu-1} dt \quad (\operatorname{Re} z > 0).$$

(The contour goes once around the negative real axis in the positive direction.)

$$\text{Sommerfeld's formula} \quad J_\nu(z) = \frac{1}{2\pi} \int_{-\eta+i\infty}^{2\pi-\eta+i\infty} e^{iz \cos t} e^{i\nu[t-(\pi/2)]} dt,$$

$$H_\nu^{(1)}(z) = \frac{1}{\pi} \int_{-\eta+i\infty}^{\eta-i\infty} e^{iz \cos t} e^{i\nu[t-(\pi/2)]} dt,$$

$$H_\nu^{(2)}(z) = \frac{1}{\pi} \int_{\eta-i\infty}^{2\pi-\eta+i\infty} e^{iz \cos t} e^{i\nu[t-(\pi/2)]} dt \quad (-\eta < \arg z < \pi - \eta, \quad 0 < \eta < \pi).$$

$$H_\nu^{(1)}(z) = -\frac{2i}{\pi} e^{-i\nu\pi/2} \int_0^\infty e^{iz \cosh t} \cosh \nu t \, dt \quad (0 < \arg z < \pi; \text{ when } \nu = 0, \text{ it holds also at } z = 0).$$

$$\begin{aligned} H_\nu^{(1)}(z) &= -\frac{2ie^{-i\nu\pi} (z/2)^\nu}{\sqrt{\pi} \Gamma[\nu + (1/2)]} \int_0^\infty e^{iz \cosh t} \sin^{2\nu} t \, dt \\ &\quad (0 < \arg z < \pi, \operatorname{Re} \nu > -1/2; \text{ when } z = 0, -1/2 < \operatorname{Re} \nu < 1/2). \end{aligned}$$

$$H_\nu^{(1)}(z) = -i \frac{e^{-i\nu\pi/2}}{\pi} \int_0^\infty e^{iz[t-(1/t)]/2} t^{-\nu-1} dt \quad (0 < \arg z < \pi; \text{ when } \arg z = 0, -1 < \operatorname{Re} \nu < 1).$$

(3) Generating Function.

$$\exp\left[\frac{z(t-t^{-1})}{2}\right] = J_0(z) + \sum_{n=1}^{\infty} [t^n + (-t)^{-n}] J_n(z),$$

$$\exp(iz \cos \theta) = \sum_{n=-\infty}^{\infty} i^n J_n(z) e^{in\theta} = J_0(z) + 2 \sum_{n=1}^{\infty} i^n J_n(z) \cos n\theta.$$

$$\int J_\nu(z) dz = 2 \sum_{n=0}^{\infty} J_{\nu+2n+1}(z).$$

$$\text{Kapteyn's series} \quad \frac{1}{1-z} = 1 + 2 \sum_{n=1}^{\infty} J_n(nz),$$

$$\frac{1}{2} \frac{z^2}{1-z^2} = \sum_{n=1}^{\infty} J_{2n}(2nz) \quad \left(\left| \frac{z \exp \sqrt{1-z^2}}{1+\sqrt{1-z^2}} \right| < 1 \right).$$

Schlömilch's series. Supposing that $f(x)$ is twice continuously differentiable with respect to the real variable x in $0 \leq x \leq \pi$, we have

$$f(x) = \frac{1}{2} a_0 + \sum_{n=1}^{\infty} a_n J_0(nx) \quad (0 < x < \pi),$$

$$\text{where} \quad a_0 \equiv 2f(0) + \frac{2}{\pi} \int_0^\pi du \int_0^{\pi/2} f'(u \sin \varphi) d\varphi,$$

$$a_n \equiv \frac{2}{\pi} \int_0^\pi du \int_0^{\pi/2} u f'(u \sin \varphi) \cos n\varphi d\varphi.$$

$$1 = J_0(z) + 2 \sum_{n=1}^{\infty} J_{2n}(z) = [J_0(z)]^2 + 2 \sum_{n=1}^{\infty} [J_n(z)]^2.$$

(4) Addition Theorem. For the cylindrical function Z_ν , we have

$$e^{i\nu\psi} Z_\nu(kR) = \sum_{n=-\infty}^{\infty} J_n(k\rho) Z_{\nu+n}(kr) e^{in\varphi}$$

$$(R = \sqrt{r^2 + \rho^2 - 2r\rho \cos \varphi}, \quad 0 < \psi < \frac{\pi}{2}, \quad e^{2i\psi} = \frac{r - \rho e^{-i\varphi}}{r - \rho e^{i\varphi}}, \quad 0 < \rho < r,$$

k is an arbitrary complex number),

$$\frac{Z_\nu(kR)}{R^\nu} = 2^\nu k^{-\nu} \Gamma(\nu) \sum_{m=0}^{\infty} (\nu+m) \frac{J_{\nu+m}(k\rho)}{\rho^\nu} \frac{Z_{\nu+m}(kr)}{r^\nu} C_m^{(\nu)}(\cos \varphi)$$

($\nu \neq$ negative integer).

$$\frac{\exp[(-1)^{i+1} ikR]}{R} = \frac{\pi}{2} \frac{(-1)^{i+1} i}{\sqrt{r\rho}} \sum_{m=0}^{\infty} (2m+1) J_{m+(1/2)}(k\rho) H_{m+(1/2)}^{(i)}(kr) P_m(\cos \varphi)$$

($i=1, 2$).

$$\begin{aligned} e^{ik\rho \cos \varphi} &= \left(\frac{\pi}{2k\rho} \right)^{1/2} \sum_{m=0}^{\infty} i^m (2m+1) J_{m+(1/2)}(k\rho) P_m(\cos \varphi) \\ &= 2^\nu \Gamma(\nu) \sum_{m=0}^{\infty} (\nu+m) i^m J_{\nu+m}(k\rho) (k\rho)^{-\nu} C_m^{(\nu)}(\cos \varphi) \quad (\nu \neq 0, -1, -2, \dots), \end{aligned}$$

where P_m is a Legendre polynomial, and $C_m^{(\nu)}$ is a Gegenbauer polynomial.

(5) Infinite Products and Partial Fractions. Let $j_{\nu,n}$ be the zeros of $z^{-\nu} J_\nu(z)$ in ascending order with respect to the real part. We have

$$J_\nu(z) = \frac{(z/2)^\nu}{\Gamma(\nu+1)} \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{j_{\nu,n}^2} \right) \quad (\nu \neq -1, -2, -3, \dots).$$

Note that if ν is real and greater than -1 , all zeros are real.

Kneser-Sommerfeld formula

$$\frac{\pi J_\nu(xz)}{4J_\nu(z)} [J_\nu(z)N_\nu(Xz) - N_\nu(z)J_\nu(Xz)] = \sum_{n=1}^{\infty} \frac{J_\nu(j_{\nu,n}x)J_\nu(j_{\nu,n}X)}{(z^2 - j_{\nu,n}^2)J_{\nu,n}'^2(j_{\nu,n})}$$

($0 < x < X < 1$, $\operatorname{Re} z > 0$).

(6) Definite Integrals.

$$\int_0^{\pi/2} J_\nu(z \cos \theta) \cos \theta d\theta = \frac{1}{2z} \int_0^{2z} J_{2\nu}(t) dt, \quad \int_0^z J_\mu(t) dt = \frac{1}{\pi} \int_0^\pi \frac{\sin(z \sin \theta)}{\sin \theta} \cos \mu \theta d\theta.$$

$$\int_0^\infty e^{-at} J_\nu(bt) t^{\mu-1} dt = \frac{(b/2a)^\nu \Gamma(\mu + \nu)}{a^\mu \Gamma(\nu + 1)} {}_2F_1\left(\frac{\mu + \nu}{2}, \frac{\mu + \nu + 1}{2}; \nu + 1; -\frac{b^2}{a^2}\right)$$

($\operatorname{Re}(a + ib) > 0$, $\operatorname{Re}(a - ib) > 0$, $\operatorname{Re}(\mu + \nu) > 0$).

$$\int_0^\infty e^{-at} J_\nu(bt) t^\nu dt = \frac{(2x)^\nu \Gamma[\nu + (1/2)]}{(a^2 + b^2)^{\nu + (1/2)} \sqrt{\pi}} \quad \left(\operatorname{Re} \nu > -\frac{1}{2}, \operatorname{Re} a > |\operatorname{Im} b|\right).$$

$$\int_0^\infty e^{-at} J_\nu(bt) \frac{dt}{t} = \frac{(\sqrt{a^2 + b^2} - a)^\nu}{\nu b^\nu} \quad (\operatorname{Re} \nu > 0, \operatorname{Re} a > |\operatorname{Im} b|).$$

Sommerfeld's formula

$$\int_0^\infty J_0(\tau r) e^{-|x|\sqrt{\tau^2 - k^2}} \frac{\tau d\tau}{\sqrt{\tau^2 - k^2}} = \frac{e^{ik\sqrt{r^2 + x^2}}}{\sqrt{r^2 + x^2}}$$

(r, x real; $-\pi/2 \leq \arg \sqrt{\tau^2 - k^2} < \pi/2$, $0 \leq \arg k < \pi$).

Weyrich's formula

$$\frac{i}{2} \int_{-\infty}^{+\infty} e^{i\tau x} H_0^{(1)}(r\sqrt{k^2 - \tau^2}) d\tau = \frac{e^{ik\sqrt{r^2 + x^2}}}{\sqrt{r^2 + x^2}}$$

(r, x real; $0 \leq \arg \sqrt{k^2 - \tau^2} < \pi$, $0 \leq \arg k < \pi$).

Weber-Sonine formula

$$\int_0^\infty J_\nu(at) e^{-p^2 t^2} t^{\mu-1} dt = \frac{(a/2p)^\nu \Gamma[(\nu + \mu)/2]}{2p^\mu \Gamma(\nu + 1)} {}_1F_1\left(\frac{\nu + \mu}{2}; \nu + 1; -\frac{a^2}{4p^2}\right)$$

($\operatorname{Re}(\mu + \nu) > 0$, $|\arg p| < \pi/4$, $a > 0$),

$$\int_0^\infty J_\nu(at) e^{-p^2 t^2} t^{\nu+1} dt = \frac{a^\nu}{(2p^2)^{\nu+1}} e^{-a^2/4p^2} \quad (\operatorname{Re} \nu > -1, |\arg p| < \pi/4).$$

Sonine-Schafheitlin formula

$$\int_0^\infty J_\mu(at) J_\nu(bt) t^{-\lambda} dt = \frac{a^\mu \Gamma[(\mu + \nu - \lambda + 1)/2]}{2^\lambda b^{\mu - \lambda + 1} \Gamma[(-\mu + \nu + \lambda + 1)/2] \Gamma(\mu + 1)}$$

$$\times {}_2F_1\left(\frac{\mu + \nu - \lambda + 1}{2}, \frac{\mu - \nu - \lambda + 1}{2}; \mu + 1; \frac{a^2}{b^2}\right)$$

($\operatorname{Re}(\mu + \nu - \lambda + 1) > 0$, $\operatorname{Re} \lambda > -1$, $0 < a < b$).

(7) Asymptotic Expansion.

(i) Hankel's asymptotic representation. We put

$$(\nu, m) \equiv \frac{[4\nu^2 - 1^2][4\nu^2 - 3^2] \dots [4\nu^2 - (2m-1)^2]}{2^{2m} m!} \quad (m = 1, 2, 3, \dots); \quad (\nu, 0) \equiv 1.$$

For $|z| \gg |\nu|$, $|z| \gg 1$,

$$J_\nu(z) = \sqrt{\frac{2}{\pi z}} \cos\left(z - \frac{\nu\pi}{2} - \frac{\pi}{4}\right) \left[\sum_{m=0}^{M-1} (-1)^m \frac{(\nu, 2m)}{(2z)^{2m}} + O(|z|^{-2M}) \right]$$

$$- \sqrt{\frac{2}{\pi z}} \sin\left(z - \frac{\nu\pi}{2} - \frac{\pi}{4}\right) \left[\sum_{m=0}^{M-1} (-1)^m \frac{(\nu, 2m+1)}{(2z)^{2m+1}} + O(|z|^{-2M-1}) \right]$$

($-\pi < \arg z < \pi$),

$$N_\nu(z) = \sqrt{\frac{2}{\pi z}} \sin\left(z - \frac{\nu\pi}{2} - \frac{\pi}{4}\right) \left[\sum_{m=0}^{M-1} (-1)^m \frac{(\nu, 2m)}{(2z)^{2m}} + O(|z|^{-2M}) \right] \\ + \sqrt{\frac{2}{\pi z}} \cos\left(z - \frac{\nu\pi}{2} - \frac{\pi}{4}\right) \left[\sum_{m=0}^{M-1} (-1)^m \frac{(\nu, 2m+1)}{(2z)^{2m+1}} + O(|z|^{-2M-1}) \right] \quad (-\pi < \arg z < \pi),$$

$$H_\nu^{(1)}(z) = \sqrt{\frac{2}{\pi z}} \exp\left[i\left(z - \frac{\nu\pi}{2} - \frac{\pi}{4}\right)\right] \left[\sum_{m=0}^{M-1} \frac{(\nu, m)}{(-2iz)^m} + O(|z|^{-M}) \right] \quad (-\pi < \arg z < 2\pi),$$

$$H_\nu^{(2)}(z) = \sqrt{\frac{2}{\pi z}} \exp\left[-i\left(z - \frac{\nu\pi}{2} - \frac{\pi}{4}\right)\right] \left[\sum_{m=0}^{M-1} \frac{(\nu, m)}{(2iz)^m} + O(|z|^{-M}) \right] \quad (-2\pi < \arg z < \pi).$$

(ii) Debye's asymptotic representation.

$$\nu \doteq x, \quad 1 - (\nu/x) > \varepsilon, \quad \nu/x = \sin \alpha, \quad \text{when } 1 - (\nu/x) > (3/x)\nu^{1/2},$$

$$H_\nu^{(1)}(x) \sim \frac{1}{\sqrt{\pi}} \exp\left[ix\left\{\cos \alpha + \left(\alpha - \frac{\pi}{2}\right)\sin \alpha\right\}\right] \\ \times \left[\frac{e^{i\pi/4}}{X} + \left(\frac{1}{8} + \frac{5}{24}\tan^2 \alpha\right) \frac{3e^{3\pi i/4}}{2X^3} \right. \\ \left. + \left(\frac{3}{128} + \frac{77}{576}\tan^2 \alpha + \frac{385}{3456}\tan^4 \alpha\right) \frac{3 \cdot 5e^{5\pi i/4}}{2^2 X^5} + \dots \right] \\ (X = [-x \cos(\alpha/2)]^{1/2}).$$

$$\nu \doteq x, \quad (\nu/x) - 1 > \varepsilon, \quad \nu/x = \cosh \sigma, \quad \text{when } |\nu^2 - x^2|^{1/2} \gg 1, \quad |\nu^2 - x^2|^{3/2} \nu^{-2} \gg 1$$

$$H_\nu^{(1)}(x) \sim \frac{1}{\sqrt{\pi}} \exp[x(\sigma \cosh \sigma - \sinh \sigma)] \\ \times \left[\frac{1}{X} + \left(\frac{1}{8} - \frac{5}{24}\coth^2 \sigma\right) \frac{3}{2X^3} + \left(\frac{3}{128} - \frac{77}{576}\coth^2 \sigma + \frac{385}{3456}\coth^4 \sigma\right) \frac{3 \cdot 5}{2^2 X^5} + \dots \right] \\ (X = [-x \sinh(\sigma/2)]^{1/2}).$$

When $\nu \doteq x$, $|x - \nu| \ll x^{1/3}$, $x \gg 1$, $x - \nu = \delta$,

$$H_\nu^{(2)}(x) \sim \frac{6^{1/3} e^{i\pi/3}}{3^{1/2} \pi} \left[\frac{\Gamma(1/3)}{x^{1/3}} - 6^{1/3} e^{i\pi/3} \delta \frac{\Gamma(2/3)}{x^{2/3}} + \left(\frac{2}{5} \delta - \delta^3\right) \frac{\Gamma(4/3)}{x^{4/3}} \right. \\ \left. + \left(\frac{3}{140} - \frac{\delta^2}{4} + \frac{\delta^4}{4}\right) 6^{1/3} e^{i\pi/3} \frac{\Gamma(5/3)}{x^{5/3}} + \dots \right]$$

(iii) Watson-Nicholson formula. When $x, \nu > 0$, $w = [(x/\nu)^2 - 1]^{1/2}$,

$$H_\nu^{(i)}(x) = 3^{-1/2} w \exp\{(-1)^i + {}^1 i\{(\pi/6) + \nu(w - (w^3/3) - \arctan w)\}\} H_{1/3}^{(i)}(\nu w^3/3) + O|\nu^{-1}| \\ (i = 1, 2).$$

(IV) Functions Related to Bessel Functions

(1) Modified Bessel Functions.

$$I_\nu(z) \equiv e^{-i\nu\pi/2} J_\nu(e^{i\pi/2} z) \\ = \sum_{n=0}^{\infty} \frac{(z/2)^{\nu+2n}}{n! \Gamma(\nu+n+1)}, \\ K_\nu(z) \equiv \frac{i\pi}{2} e^{i\nu\pi/2} H_\nu^{(1)}(e^{i\pi/2} z) = -\frac{i\pi}{2} e^{-i\nu\pi/2} H_\nu^{(2)}(e^{-i\pi/2} z) \\ = \frac{\pi}{2} \frac{I_{-\nu}(z) - I_\nu(z)}{\sin \nu\pi} = \left(\frac{\pi}{2z}\right)^{1/2} W_{0,\nu}(2z).$$

Recurrence formulas:

$$I_{\nu-1}(z) - I_{\nu+1}(z) = (2\nu/z)I_{\nu}(z),$$

$$I_{\nu-1}(z) + I_{\nu+1}(z) = 2I'_{\nu}(z).$$

$$K_{\nu-1}(z) - K_{\nu+1}(z) = -(2\nu/z)K_{\nu}(z),$$

$$K_{\nu-1}(z) + K_{\nu+1}(z) = -2K'_{\nu}(z),$$

$$K_{-\nu}(z) = K_{\nu}(z).$$

Airy's integral:
$$\int_0^{\infty} \cos(t^3 - tx) dt = \frac{\pi}{3} \sqrt{\frac{x}{3}} \left[J_{1/3} \left(\frac{2x\sqrt{x}}{3\sqrt{3}} \right) + J_{-1/3} \left(\frac{2x\sqrt{x}}{3\sqrt{3}} \right) \right],$$

$$\int_0^{\infty} \cos(t^3 + tx) dt = \frac{1}{3} \sqrt{x} K_{1/3} \left(\frac{2x\sqrt{x}}{3\sqrt{3}} \right) \quad (x > 0).$$

H. Weber's formula:
$$\frac{1}{2p^2} e^{-(a^2+b^2)/4p^2} I_{\nu} \left(\frac{ab}{2p^2} \right) = \int_0^{\infty} e^{-p^2 t^2} J_{\nu}(at) J_{\nu}(bt) t dt$$

($\operatorname{Re} \nu > -1$, $|\arg p| < \pi/4$; $a, b > 0$).

Watson's formula:
$$J_{\mu}(z) N_{\nu}(z) - J_{\nu}(z) N_{\mu}(z) = \frac{4 \sin(\mu - \nu) \pi}{\pi^2} \int_0^{\infty} K_{\nu - \mu}(2z \sinh t) e^{(\mu + \nu)t} dt$$

($\operatorname{Re} z > 0$, $\operatorname{Re}(\mu - \nu) < 1$),

$$J_{\nu}(z) \frac{\partial N_{\nu}(z)}{\partial \nu} - N_{\nu}(z) \frac{\partial J_{\nu}(z)}{\partial \nu} = -\frac{4}{\pi} \int_0^{\infty} K_0(2z \sinh t) e^{-2\nu t} dt \quad (\operatorname{Re} z > 0).$$

Nicholson's formula:
$$J_{\nu}^2(z) + N_{\nu}^2(z) = \frac{8}{\pi^2} \int_0^{\infty} K_0(2z \sinh t) \cosh 2\nu t dt \quad (\operatorname{Re} z > 0).$$

Dixon-Ferrar formula:
$$J_{\nu}^2(z) + N_{\nu}^2(z) = \frac{8 \cos \nu \pi}{\pi^2} \int_0^{\infty} K_{2\nu}(2z \sinh t) dt$$

($\operatorname{Re} z > 0$; $-\frac{1}{2} < \operatorname{Re} \nu < \frac{1}{2}$).

(2) Kelvin Functions.
$$\begin{aligned} \operatorname{ber}_{\nu}(z) \pm i \operatorname{bei}_{\nu}(z) &\equiv J_{\nu}(e^{\pm 3\pi i/4} z), \\ \operatorname{her}_{\nu}(z) \pm i \operatorname{hei}_{\nu}(z) &\equiv H_{\nu}^{(1)}(e^{\pm 3\pi i/4} z), \\ \operatorname{ker}_{\nu}(z) &\equiv -(\pi/2) \operatorname{hei}_{\nu}(z), \\ \operatorname{kei}_{\nu}(z) &\equiv (\pi/2) \operatorname{her}_{\nu}(z). \end{aligned}$$

When ν is an integer n ,
$$\begin{aligned} \operatorname{ber}_n(x) - i \operatorname{bei}_n(x) &= (-1)^n J_n(\sqrt{i} x), \\ \operatorname{her}_n(x) - i \operatorname{hei}_n(x) &= (-1)^{n+1} H_n^{(1)}(\sqrt{i} x) \quad (x \text{ real}). \end{aligned}$$

(3) Struve Function.
$$\begin{aligned} H_{\nu}(x) &\equiv \frac{2(z/2)^{\nu}}{\Gamma[\nu + (1/2)] \sqrt{\pi}} \int_0^{\pi/2} \sin(z \cos \theta) \sin^{2\nu} \theta d\theta \\ &= \sum_{m=0}^{\infty} \frac{(-1)^m (z/2)^{\nu+2m+1}}{\Gamma[m + (3/2)] \Gamma[\nu + m + (3/2)]}. \end{aligned}$$

Anger function:
$$J_{\nu}(z) \equiv \frac{1}{\pi} \int_0^{\pi} \cos(\nu \theta - z \sin \theta) d\theta.$$

H. F. Weber function:
$$E_{\nu}(z) \equiv \frac{1}{\pi} \int_0^{\pi} \sin(\nu \theta - z \sin \theta) d\theta.$$

Putting $\nabla_{\nu} \equiv z^2 \frac{d^2}{dz^2} + z \frac{d}{dz} + z^2 - \nu^2$,

$$\nabla_{\nu} H_{\nu}(z) = \frac{4(z/2)^{\nu+1}}{\Gamma[\nu + (1/2)] \sqrt{\pi}}, \quad \nabla_{\nu} J_{\nu}(z) = \frac{(z - \nu) \sin \nu \pi}{\pi},$$

$$\nabla_{\nu} E_{\nu}(z) = -\frac{z + \nu}{\pi} - \frac{(z - \nu) \cos \nu \pi}{\pi}.$$

When ν is an integer n , $J_n(z) = J_n(z)$.

$$\int_0^z J_0(t) dt = zJ_0(z) + \frac{\pi z}{2} [J_1(z)H_0(z) - J_0(z)H_1(z)],$$

$$\int_0^z N_0(t) dt = zN_0(z) + \frac{\pi z}{2} [N_1(z)H_0(z) - N_0(z)H_1(z)].$$

(4) Neumann Polynomials. $O_n(t) = \sum_{j=0}^{\lfloor n/2 \rfloor} \frac{n(n-j-1)!}{j!(t/2)^{n-2j+1}} \quad (n \text{ is a positive integer}),$

$$O_0(t) = 1/t.$$

$$\frac{1}{t-z} \equiv 1 + 2 \sum_{n=1}^{\infty} O_n(t)J_n(z) \quad (|t| > |z|).$$

Schl\"afli polynomials: $S_n(t) \equiv \frac{2}{n} \left[tO_n(t) - \cos^2 \frac{n\pi}{2} \right] \quad (n \text{ is a positive integer}),$

$$S_0(t) \equiv 0.$$

$$\nabla_n S_n(x) = 2n + 2(x-n)\sin^2(n\pi/2) \quad (\nabla_n \text{ is the same operator defined in (3)}).$$

Lommel polynomials: $R_{m,\nu}(z) \equiv \frac{\Gamma(\nu+m)}{\Gamma(\nu)(z/2)^m} {}_2F_3\left(\frac{1-m}{2}, \frac{-m}{2}; \nu, -m, 1-\nu-m; -z^2\right)$
 $= (\pi z/2 \sin \nu\pi) [J_{\nu+m}(z)J_{-\nu+1}(z) + (-1)^m J_{-\nu-m}(z)J_{\nu-1}(z)]$
(m is a nonnegative integer).

References

See references to Table 16, this Appendix.

20. Systems of Orthogonal Functions (→ 317 Orthogonal Functions)

$$\int_a^b p_n(x)p_m(x)\varphi(x)dx = \delta_{nm}A_n$$

Name	Notation $p_n(x)$	Interval (a, b)	Weight $\varphi(x)$	Norm A_n
Legendre	$P_n(x)$	$(-1, +1)$	1	$2/(2n+1)$
Gegenbauer	$C_n^\nu(x)$	$(-1, +1)$	$(1-x^2)^{\nu-(1/2)}$	$2\pi\Gamma(2\nu+n)/2^{2\nu}(n+\nu)n![\Gamma(\nu)]^2$
Chebyshev	$T_n(x)$	$(-1, +1)$	$(1-x^2)^{-1/2}$	$\pi(n=0); \pi/2 (n \geq 1)$
Hermite	$H_n(x)$	$(-\infty, +\infty)$	e^{-x^2}	$\sqrt{\pi} \cdot n!$
Jacobi	$G_n(\alpha, \gamma; x)$	$(0, 1)$	$x^{\gamma-1}(1-x)^{\alpha-\gamma}$	$\frac{n![\Gamma(\gamma)]^2\Gamma(\alpha+n-\gamma+1)}{(\alpha+2n)\Gamma(\alpha+n)\Gamma(\gamma+n)}$
Laguerre	$L_n^\alpha(x)$	$(0, \infty)$	$x^\alpha e^{-x}$	$\Gamma(\alpha+n+1)/n!$

For Legendre polynomials $P_n(x) \rightarrow$ Table 18.II, this Appendix.

(I) Gegenbauer Polynomials (Gegenbauer Functions)

$$C_n^\nu(t) \equiv \frac{\Gamma(n+2\nu)}{n!\Gamma(2\nu)} {}_2F_1\left(n+2\nu, -n; \nu+\frac{1}{2}; \frac{1-t}{2}\right) \\ = \frac{\Gamma(2\nu+n)\Gamma[\nu+(1/2)]}{\Gamma(2\nu)n!} \left[\frac{1}{4}(t^2-1)\right]^{(1/4)-(\nu/2)} \mathfrak{P}_{n+\nu-(1/2)}^{(1/2)-\nu}(t).$$

$$\text{Generating function} \quad (1-2\alpha t + \alpha^2)^{-\nu} \equiv \sum_{n=0}^{\infty} C_n^{\nu}(t) \alpha^n, \quad C_{n-l}^{l+(1/2)}(x) = \frac{1}{(2l-1)!!} \frac{d^l P_n(t)}{dt^l}.$$

$$\text{Orthogonal relation} \quad \int_0^{\pi} (\sin^{2\nu} \theta) C_m^{\nu}(\cos \theta) C_n^{\nu}(\cos \theta) d\theta = \frac{\pi \Gamma(2\nu + n)}{2^{2\nu-1} (\nu + n) n! [\Gamma(\nu)]^2} \delta_{nm}.$$

(II) Chebyshev (Tschebyscheff) Polynomials

(1) Chebyshev Polynomial (Chebyshev Function of the First Kind)

$$\begin{aligned} T_n(x) &\equiv \cos(n \arccos x) \\ &= (1/2) \left[(x + i\sqrt{1-x^2})^n + (x - i\sqrt{1-x^2})^n \right] \\ &= F(n, -n; 1/2; (1-x)/2) \\ &= \sum_{j=0}^{[n/2]} (-1)^j \binom{n}{2j} x^{n-2j} (1-x^2)^j \\ &= \frac{(-1)^n (1-x^2)^{1/2}}{(2n-1)!!} \frac{d^n (1-x^2)^{n-(1/2)}}{dx^n}, \end{aligned}$$

Chebyshev function of the second kind

$$\begin{aligned} U_n(x) &\equiv \sin(n \arccos x) \\ &= (1/2i) \left[(x + i\sqrt{1-x^2})^n - (x - i\sqrt{1-x^2})^n \right] \\ &= \frac{(-1)^{n-1} n}{(2n-1)!!} \frac{d^{n-1} (1-x^2)^{n-(1/2)}}{dx^{n-1}}. \end{aligned}$$

$T_n(x)$, $U_n(x)$ are mutually linearly independent solutions of Chebyshev's differential equation $(1-x^2)y'' - xy' + n^2y = 0$. Recurrence relations are

$$T_{n+1}(x) - 2xT_n(x) + T_{n-1}(x) = 0, \quad U_{n+1}(x) - 2xU_n(x) + U_{n-1}(x) = 0.$$

Generating function:

$$\frac{1-t^2}{1-2tx+t^2} = T_0(x) + 2 \sum_{n=0}^{\infty} T_n(x) t^n, \quad \frac{1}{1-2tx+t^2} = \frac{1}{\sqrt{1-x^2}} \sum_{n=0}^{\infty} U_{n+1}(x) t^n.$$

Orthogonal relation:

$$\int_{-1}^{+1} \frac{T_m(x) T_n(x)}{\sqrt{1-x^2}} dx = \begin{cases} 0 & (m \neq n), \\ \pi/2 & (m = n \neq 0), \\ \pi & (m = n = 0); \end{cases} \quad \int_{-1}^{+1} \frac{U_m(x) U_n(x)}{\sqrt{1-x^2}} dx = \begin{cases} \frac{\pi}{2} & (m = n \neq 0), \\ 0 & (\text{otherwise}). \end{cases}$$

Orthogonality in finite sums. Let u_0, u_1, \dots, u_k be the zeros of $T_{k+1}(x)$. All zeros are real and situated in the interval $(-1, 1)$. Then we have

$$\sum_{i=0}^k T_m(u_i) T_n(u_i) = \begin{cases} 0 & (m \neq n, \text{ or } m = n = k+1), \\ (k+1)/2 & (1 \leq m = n \leq k), \\ k+1 & (m = n = 0). \end{cases}$$

Let $p_n(x)$ be the best approximation of x^n in $-1 \leq x \leq 1$ by polynomials of degree at most $n-1$. Then we have $x^n - p_n(x) = 2^{-n+1} T_n(x)$.

(2) Expansions by $T_n(x)$.

$$e^{ax} = I_0(a) + 2 \sum_{n=1}^{\infty} I_n(a) T_n(x),$$

$$\sin ax = 2 \sum_{n=0}^{\infty} (-1)^n J_{2n+1}(a) T_{2n+1}(x),$$

$$\cos ax = J_0(a) + 2 \sum_{n=1}^{\infty} (-1)^n J_{2n}(a) T_{2n}(x),$$

$$\log(1 + x \sin 2\alpha) = 2 \log \cos \alpha - 2 \sum_{n=1}^{\infty} \frac{1}{n} (-\tan \alpha)^n T_n(x),$$

$$\arctan x = 2 \sum_{n=1}^{\infty} (-1)^n \frac{(\sqrt{2}-1)^{2n+1}}{2n+1} T_{2n+1}(x).$$

(III) Parabolic Cylinder Functions (Weber Functions) (\rightarrow 167 Functions of Confluent Type)

Parabolic cylinder functions:

$$\begin{aligned} D_\nu(z) &\equiv 2^{(1/4)+(\nu/2)} z^{-1/2} W_{(1/4)+(\nu/2), -1/4}(z^2/2) \\ &= \sqrt{\pi} 2^{(1/4)+(\nu/2)} z^{-1/2} \left[\frac{M_{(1/4)+(\nu/2), -1/4}(z^2/2)}{\Gamma[(1-\nu)/2]} + \frac{M_{(1/4)-(\nu/2), -1/4}(z^2/2)}{\Gamma(-\nu/2)} \right] \\ &= 2^{\nu/2} e^{-z^2/4} \sqrt{\pi} \left[\frac{1}{\Gamma[(1-\nu)/2]} {}_1F_1\left(\frac{-\nu}{2}; \frac{1}{2}; \frac{z^2}{2}\right) - \frac{\sqrt{2} z}{\Gamma(-\nu/2)} {}_1F_1\left(\frac{1-\nu}{2}; \frac{3}{2}; \frac{z^2}{2}\right) \right]. \end{aligned}$$

The solutions of Weber's differential equation

$$\frac{d^2 u}{dz^2} + \left(\nu + \frac{1}{2} - \frac{z^2}{4} \right) u = 0$$

are given by

$$D_\nu(z), D_\nu(-z), D_{-\nu-1}(iz), D_{-\nu-1}(-iz),$$

and the following relations hold among them.

$$\begin{aligned} D_\nu(z) &= [\Gamma(\nu+1)/\sqrt{2\pi}] [e^{i\nu\pi/2} D_{-\nu-1}(iz) + e^{-i\nu\pi/2} D_{-\nu-1}(-iz)] \\ &= e^{-i\nu\pi} D_\nu(-z) + [\sqrt{2\pi}/\Gamma(-\nu)] e^{-i(\nu+1)\pi/2} D_{-\nu-1}(iz) \\ &= e^{i\nu\pi} D_\nu(-z) + [\sqrt{2\pi}/\Gamma(-\nu)] e^{i(\nu+1)\pi/2} D_{-\nu-1}(-iz). \end{aligned}$$

Integral representation:

$$D_\nu(z) = \frac{e^{-z^2/4}}{\Gamma(-\nu)} \int_0^\infty e^{-zt - (t^2/2)} t^{-\nu-1} dt \quad (\operatorname{Re} \nu < 0).$$

$$e^{-(z^2/4) - zt - (t^2/2)} = \sum_{n=0}^{\infty} \frac{(-t)^n}{n!} D_n(z) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} t^\nu \Gamma(-\nu) D_\nu(z) d\nu \quad (c < 0, |\arg t| < \pi/4).$$

Recurrence formula:

$$D_{\nu+1}(z) - z D_\nu(z) + \nu D_{\nu-1}(z) = 0, \quad dD_\nu(z)/dz + (1/2)z D_\nu(z) - \nu D_{\nu-1}(z) = 0.$$

$$D_\nu(0) = \frac{2^{\nu/2} \sqrt{\pi}}{\Gamma[(1-\nu)/2]}, \quad D'_\nu(0) = -\frac{2^{(\nu+1)/2} \sqrt{\pi}}{\Gamma(-\nu/2)}.$$

Asymptotic expansion:

$$D_\nu(z) \approx e^{-z^2/4} z^\nu \left(1 - \frac{\nu(\nu-1)}{2z^2} + \frac{\nu(\nu-1)(\nu-2)(\nu-3)}{2 \cdot 4z^4} \mp \dots \right) \quad (|\arg z| < \frac{3}{4}\pi).$$

$$D_{-1}(z) = e^{z^2/4} \sqrt{\frac{\pi}{2}} \left[1 - \operatorname{erf}\left(\frac{z}{\sqrt{2}}\right) \right], \quad \operatorname{erf}(x) \equiv \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt \quad (\text{error function}).$$

(IV) Hermite Polynomials

For the parabolic cylinder functions, when ν is an integer n , we have

$$D_n(z) = (-1)^n e^{z^2/4} d^n(e^{-z^2/2})/dz^n = e^{-z^2/4} H_n(z/\sqrt{2}),$$

where $H_n(x)$ is the Hermite polynomial

$$H_n(x) \equiv 2^{-n/2} (-1)^n e^{x^2/2} d^n (e^{-x^2}) / dx^n = e^{x^2/2} D_n (\sqrt{2} x).$$

A Hermite polynomial is more often defined by the following function $\text{He}_n(x)$ (e.g., in W.F. Magnus, F. Oberhettinger, and R. P. Soni [1]).

$$\text{He}_n(x) \equiv (-1)^n e^{x^2/2} d^n (e^{-x^2/2}) / dx^n = e^{x^2/4} D_n (x) = H_n(x/\sqrt{2}).$$

The function $y = H_n(x)$ is a solution of Hermite's differential equation

$$y'' - 2xy' + 2ny = 0.$$

$H_n(x)$ is a polynomial in x of degree n , and is an even or odd function according to whether n is even or odd.

$$H_{2n}(x) = (-1)^n (2n-1)!! {}_1F_1(-n; 1/2; x^2),$$

$$H_{2n+1}(x) = (-1)^n (2n+1)!! \sqrt{2} x {}_1F_1(-n; 3/2; x^2).$$

Recurrence formula:

$$H_{n+1}(x) = \sqrt{2} x H_n(x) - n H_{n-1}(x) = \sqrt{2} x H_n(x) - H'_n(x) / \sqrt{2},$$

$$H'_n(x) = \sqrt{2} n H_{n-1}(x).$$

$$H_{2n}(0) = \frac{(-1)^n (2n)!}{2^n n!} = (-1)^n (2n-1)!!, \quad H_{2n+1}(0) = 0.$$

Generating function:

$$e^{\sqrt{2} ix - (t^2/2)} = \sum_{n=0}^{\infty} H_n(x) t^n / n!.$$

Orthogonal relation:

$$\int_{-\infty}^{+\infty} H_n(x) H_m(x) e^{-x^2} dx = \delta_{nm} n! \sqrt{\pi}.$$

(V) Jacobi Polynomials

$$G_n(\alpha, \gamma; x) \equiv F(-n, \alpha+n; \gamma; x)$$

$$= x^{1-\gamma} (1-x)^{\gamma-\alpha} \frac{\Gamma(\gamma+n)}{\Gamma(\gamma)} \frac{d^n}{dx^n} [x^{\gamma+n-1} (1-x)^{\alpha+n-\gamma}].$$

These satisfy Jacobi's differential equation $x(1-x)y'' + [\gamma - (\alpha+1)x]y' + n(\alpha+n)y = 0$.

Orthogonal relation:

$$\int_0^1 x^{\gamma-1} (1-x)^{\alpha-\gamma} G_m(\alpha, \gamma; x) G_n(\alpha, \gamma; x) dx = \frac{n! \Gamma(\alpha+n-\gamma+1) \Gamma(\gamma)^2}{(\alpha+2n) \Gamma(\alpha+n) \Gamma(\gamma+n)} \delta_{mn}$$

$$(\text{Re } \gamma > 0, \quad \text{Re}(\alpha - \gamma) > -1).$$

Representation of other functions:

$$P_n(x) = G\left(1, 1; \frac{1-x}{2}\right), \quad T_n(x) = G\left(0, \frac{1}{2}; \frac{1-x}{2}\right),$$

$$C_n^\nu(x) = (-1)^n \frac{\Gamma(2\nu+n)}{\Gamma(2\nu) \cdot n!} G_n\left(2\nu, \nu + \frac{1}{2}; \frac{1+x}{2}\right).$$

(VI) Laguerre Functions**(1) Laguerre Functions.**

$$L_v^{(\alpha)}(z) \equiv \frac{\Gamma(\alpha + v + 1)}{\Gamma(\alpha + 1)\Gamma(v + 1)} z^{-(\alpha+1)/2} e^{z/2} M_{[(\alpha+1)/2]+v, \alpha/2}(z) \\ = \frac{\Gamma(\alpha + v + 1)}{\Gamma(\alpha + 1)\Gamma(v + 1)} {}_1F_1(-v; \alpha + 1; z),$$

These satisfy Laguerre's differential equation

$$z d^2 [L_v^{(\alpha)}(z)] / dz^2 + (\alpha + 1 - z) d [L_v^{(\alpha)}(z)] / dz + v L_v^{(\alpha)}(z) = 0.$$

(2) Laguerre Polynomials. When v is an integer n ($n = 0, 1, 2, \dots$), the function $L_n^{(\alpha)}(x)$ reduces to a polynomial of degree n as follows.

Laguerre polynomials:

$$L_n^{(\alpha)}(x) = \frac{e^x x^{-\alpha}}{n!} \frac{d^n}{dx^n} (e^{-x} x^{n+\alpha}) = \sum_{j=0}^n \binom{n+\alpha}{n-j} \frac{(-x)^j}{j!}. \\ L_n^{(0)}(x) = 1, \quad L_0^{(m)}(x) = 1, \quad L_{n+m}^{(-m)}(x) = \frac{(-1)^m n!}{(n+m)!} x^m L_n^{(m)}(x) \quad (m = 0, 1, 2, \dots).$$

Recurrence formulas:

$$n L_n^{(\alpha)}(x) = (-x + 2n + \alpha - 1) L_{n-1}^{(\alpha)}(x) - (n + \alpha - 1) L_{n-2}^{(\alpha)}(x), \\ x d [L_n^{(\alpha)}(x)] / dx = n L_n^{(\alpha)}(x) - (n + \alpha) L_{n-1}^{(\alpha)}(x) \quad (n = 2, 3, \dots).$$

Generating function:

$$\frac{e^{-xt/(1-t)}}{(1-t)^{\alpha+1}} = \sum_{n=0}^{\infty} L_n^{(\alpha)}(x) t^n \quad (|t| < 1).$$

Orthogonal relations:

$$\int_0^{\infty} e^{-x} x^{\alpha} L_m^{(\alpha)}(x) L_n^{(\alpha)}(x) dx = \delta_{mn} \Gamma(\alpha + n + 1) / n! = \delta_{mn} \Gamma(1 + \alpha) \binom{n + \alpha}{n}. \\ H_{2n}(x) = (-2)^n n! L_n^{(-1/2)}(x^2), \quad H_{2n+1}(x) = (-2)^n n! \sqrt{2} x L_n^{(1/2)}(x^2).$$

(3) Sonine Polynomials.

$$S_n^{(\alpha)}(x) \equiv \frac{(-1)^n}{\Gamma(\alpha + n + 1)} L_n^{(\alpha)}(x).$$

(VII) Orthogonal Polynomials

$$P_{n,m}(x) = \sum_{k=0}^n (-1)^k \binom{n}{k} \binom{n+k}{k} \frac{x(x-1)\dots(x-k+1)}{m(m-1)\dots(m-k+1)}$$

(where n, m are positive integers and $n \leq m$).

We have the same polynomials if we replace x^k in $P_n(1-2x)$ by

$$x(x-1)\dots(x-k+1) / m(m-1)\dots(m-k+1) \quad (k = 0, 1, \dots, n).$$

Orthogonality in finite sums:

$$\sum_{k=0}^m P_{n,m}(k) P_{l,m}(k) = \delta_{nl} \frac{(m+n+1)!(m-n)!}{(2n+1)(m!)^2}.$$

Chebyshev's q functions:

$$q_n(m, x) = \frac{(-1)^n (m-1)!}{2^n (m-n-1)!} P_{n,m-1}(x), \quad \xi_{n,m}(x) = [2^n (n!)^2 / (2n)!] q_n(m, x-1).$$

For given data y_k at m points $x_k = x_1 + (k-1)h$ ($k = 1, \dots, m$) that are equally spaced with step h , the least square approximation among the polynomials $Q(x)$ of degree n ($n < m$), i.e., the polynomial that minimizes the square sum of the residues $S = \sum_{k=1}^m [y_k - Q(x_k)]^2$ is given by the

following formula (→ 19 Analog Computation):

$$Q(x)=\sum_{k=0}^n\frac{B_k}{S_k}\xi_{k,m}\left(\frac{x-x_1}{h}+1\right),\quad S=\sum_{k=1}^m(y_k)^2-\sum_{k=0}^m\frac{B_k^2}{S_k},$$
$$B_k=\sum_{i=1}^m y_i\xi_{k,m}(i),\quad S_k=\sum_{i=1}^m [\xi_{k,m}(i)]^2.$$

References

See references to Table 16, this Appendix.

21. Interpolation (→ 223 Interpolation)

(1) Lagrange’s Interpolation Polynomial.

$$f(x)=\sum_{s=0}^n f(x_s)\frac{(x-x_0)(x-x_1)\dots(x-x_{s-1})(x-x_{s+1})\dots(x-x_n)}{(x_s-x_0)(x_s-x_1)\dots(x_s-x_{s-1})(x_s-x_{s+1})\dots(x_s-x_n)}.$$

Aitken’s interpolation scheme. The interpolation polynomial $f(x)$ corresponding to the value $y_s=f(x_s)$ ($s=0,1,\dots,n$) is given inductively by the following procedure. The order of x_0,x_1,\dots,x_s is quite arbitrary.

$$p_{s,0}(x)=y_s\quad (s=0,1,\dots,n),$$
$$p_{s,k+1}(x)=\left[(x_s-x)p_{k,k}(x)-(x_k-x)p_{s,k}(x)\right]/(x_s-x_k)\quad (s=k+1,k+2,\dots,n),$$
$$f(x)\equiv p_{n,n}(x).$$

(2) Interpolation for Equally Spaced Points. When the points x_k lie in the order of their subscripts at a uniform distance h ($x_s=x_0+sh$), we make the following difference table ($\Delta x=h$). Forward difference:

$$\Delta_i\equiv\Delta_i^1\equiv f_{i+1}-f_i=f(x_{i+1})-f(x_i),\quad \Delta_i^s\equiv\Delta_{i+1}^{s-1}-\Delta_i^{s-1}.$$

Variable	Value of Function	Difference				
		(1st)	(2nd)	(3rd)	(4th)	...
...
$x_0-2\Delta x$	f_{-2}	...	Δ_{-2}^2
$x_0-\Delta x$	f_{-1}	Δ_{-2}	Δ_{-1}^2	Δ_{-2}^3	Δ_{-2}^4	...
x_0	f_0	Δ_{-1}	Δ_0^2	Δ_{-1}^3	Δ_{-1}^4	...
$x_0+\Delta x$	f_1	Δ_0	Δ_1^2	Δ_0^3	Δ_0^4	...
$x_0+2\Delta x$	f_2	Δ_1	Δ_2^2	Δ_1^3
$x_0+3\Delta x$	f_3	Δ_2
...

Backward difference:

$$\bar{\Delta}_i^s\equiv\bar{\Delta}_i^{s-1}-\bar{\Delta}_{i-1}^{s-1}=\Delta_{s-i}^s.$$

Central difference:

$$\delta_i^s=\delta_{i+(1/2)}^{s-1}-\delta_{i-(1/2)}^{s-1},\quad \delta_{i+(s/2)}^s=\Delta_i^s.$$

Newton interpolation formula (forward type):

$$f(x_0+u\Delta x)=f(x_0)+\frac{u}{1!}\Delta_0+\frac{u(u-1)}{2!}\Delta_0^2+\frac{u(u-1)(u-2)}{3!}\Delta_0^3$$
$$+\frac{u(u-1)(u-2)(u-3)}{4!}\Delta_0^4+\dots$$

Gauss’s interpolation formula (forward type):

$$f(x_0+u\Delta x)=f(x_0)+\frac{u}{1!}\Delta_0+\frac{u(u-1)}{2!}\Delta_{-1}^2+\frac{u(u-1)(u+1)}{3!}\Delta_{-1}^3$$
$$+\frac{u(u-1)(u+1)(u-2)}{4!}\Delta_{-2}^4+\dots$$

Stirling's interpolation formula:

$$f(x_0 + u\Delta x) = f(x_0) + \frac{u}{1!} \frac{\Delta_{-1} + \Delta_0}{2} + \frac{u^2}{2!} \Delta_{-1}^2 + \frac{u(u^2-1)}{3!} \frac{\Delta_{-2}^2 + \Delta_{-1}^2}{2} + \frac{u^2(u^2-1)}{4!} \Delta_{-2}^4 + \dots$$

Bessel's interpolation formula:

$$f\left(\frac{x_0 + x_1}{2} + v\Delta x\right) = \frac{f(x_0) + f(x_1)}{2} + \frac{v}{1!} \Delta_0 + \frac{1}{2!} \left(v^2 - \frac{1}{4}\right) \frac{\Delta_{-1}^2 + \Delta_0^2}{2} + \frac{v}{3!} \left(v^2 - \frac{1}{4}\right) \Delta_{-1}^3 + \frac{1}{4!} \left(v^2 - \frac{1}{4}\right) \left(v^2 - \frac{9}{4}\right) \frac{\Delta_{-2}^4 + \Delta_{-1}^4}{2} + \dots$$

Everett's interpolation formula:

$$f(x_0 + u\Delta x) = f(x_1 - \xi\Delta x) = \xi f(x_0) + \frac{\xi(\xi^2-1)}{3!} \Delta_{-1}^2 + \frac{\xi(\xi^2-1)(\xi^2-4)}{5!} \Delta_{-2}^4 + \dots \\ + uf(x_1) + \frac{u(u^2-1)}{3!} \Delta_0^2 + \frac{u(u^2-1)(u^2-4)}{5!} \Delta_{-1}^4 + \dots \quad (\xi = 1-u)$$

(3) Interpolation for Functions of Two Variables. Let $x_m = x_0 + m\Delta x$, $y_n = y_0 + n\Delta y$ (m and n are integers). We define the finite differences as follows:

$$\Delta_x(x_0, y_0) \equiv f(x_1, y_0) - f(x_0, y_0),$$

$$\Delta_y(x_0, y_0) \equiv f(x_0, y_1) - f(x_0, y_0),$$

$$\Delta_x^2(x_0, y_0) \equiv \Delta_x(x_1, y_0) - \Delta_x(x_0, y_0) \equiv \delta_x^2(x_1, y_0),$$

$$\Delta_{xy}(x_0, y_0) \equiv \Delta_y(x_1, y_0) - \Delta_y(x_0, y_0) = \Delta_x(x_0, y_1) - \Delta_x(x_0, y_0),$$

$$\Delta_y^2(x_0, y_0) \equiv \Delta_y(x_0, y_1) - \Delta_y(x_0, y_0) \equiv \delta_y^2(x_0, y_1), \quad \dots$$

Newton's formula:

$$f(x_0 + u\Delta x, y_0 + v\Delta y) = f(x_0, y_0) + (u\Delta_x + v\Delta_y)(x_0, y_0) \\ + (1/2!)[u(u-1)\Delta_x^2 + 2uv\Delta_{xy} + v(v-1)\Delta_y^2](x_0, y_0) + \dots$$

Everett's formula. Putting $s \equiv 1-u$, $t \equiv 1-v$ we have

$$f(x_0 + u\Delta x, y_0 + v\Delta y) = sf(x_0, y_0) + sf(x_0, y_1) + uf(x_1, y_0) + uf(x_1, y_1) \\ - (1/6)[us(1+s)\{t\delta_x^2(x_0, y_0) + v\delta_x^2(x_0, y_1)\} + us(1+u)\{t\delta_x^2(x_1, y_0) + v\delta_x^2(x_1, y_1)\} \\ + vt(1+t)\{s\delta_y^2(x_0, y_0) + u\delta_y^2(x_1, y_0)\} + vt(1+v)\{s\delta_y^2(x_0, y_1) + u\delta_y^2(x_1, y_1)\}] + \dots$$

References

- [1] F. J. Thompson, Table of the coefficients of Everett's central-difference interpolation formula, Tracts for computers, no. V, Cambridge Univ. Press, 1921.
- [2] M. Lindow, Numerische Infinitesimalrechnung, Dummler, Berlin, 1928.
- [3] H. T. Davis, Tables of the higher mathematical functions I, Principia Press, Bloomington, 1933.
- [4] K. Hayashi and S. Moriguti, Table of higher transcendental functions (in Japanese), Iwanami, second revised edition, 1967.

22. Distribution of Typical Random Variables

(→ 341 Probability Measures, 374 Sampling Distributions)

In the following table, Nos. 1–13 are 1-dimensional continuous distributions, and Nos. 20–21 are k -dimensional continuous distributions, for which the distribution density is the one with respect to Lebesgue measure. Nos. 14–19 are 1-dimensional discrete distributions, and Nos. 22–24 are k -dimensional discrete distributions, where the density function $P(x)$ means the probability at the point x .

The characteristic function, average, and variance are given only for those represented in a simple form.

App. A, Table 22
Distribution of Typical Random Variables

No.	Name	Symbol	Density Function	Domains
1	Normal	$N(\mu, \sigma^2)$	$\frac{1}{(2\pi\sigma^2)^{1/2}} \exp\left[-\frac{(x-\mu)^2}{2\sigma^2}\right]$	$-\infty < x < \infty$
2	Logarithmic normal		$\frac{1}{(2\pi\sigma^2)^{1/2}} \frac{1}{y} \exp\left[-\frac{(\log y - \mu)^2}{2\sigma^2}\right]$	$0 < y < \infty$
3	Gamma	$\Gamma(p, \sigma)$	$[\Gamma(p)]^{-1} \sigma^{-p} x^{p-1} e^{-x/\sigma}$	$0 < x < \infty$
4	Exponential	$e(\mu, \sigma)$	$(1/\sigma) \exp(-(x-\mu)/\sigma)$	$\mu < x < \infty$
5	Two-sided exponential		$(1/2\sigma) e^{- x /\sigma}$	$-\infty < x < \infty$
6	Chi square	$\chi^2(n)$	$2^{-n/2} [\Gamma(n/2)]^{-1} x^{(n/2)-1} e^{-x/2}$	$0 < x < \infty$
7	Beta	$B(p, q)$	$[B(p, q)]^{-1} x^{p-1} (1-x)^{q-1}$	$0 < x < 1$
8	F	$F(m, n)$	$2K_F x^{(m/2)-1} [1 + (mx/n)]^{-(m+n)/2}$, $K_F \equiv [B(m/2, n/2)]^{-1} (m/n)^{m/2}$	$0 < x < \infty$
9	z	$z(m, n)$	$K_F e^{mz} [1 + (me^{2z}/n)]^{-(m+n)/2}$, $K_F \equiv [B(m/2, n/2)]^{-1} (m/n)^{m/2}$	$-\infty < z < \infty$
10	t	$t(n)$	$[\sqrt{n} B(n/2, 1/2)]^{-1} [1 + (t^2/n)]^{-(n+1)/2}$	$-\infty < t < \infty$
11	Cauchy	$C(\mu, \sigma)$	$(\pi\sigma)^{-1} \left[1 + \frac{(x-\mu)^2}{\sigma^2}\right]^{-1}$	$-\infty < x < \infty$
12	One-side stable for exponent 1/2		$c(2\pi)^{-1/2} x^{-3/2} \exp(-c^2/2x)$	$0 < x < \infty$
13	Uniform rectangular	$U(\alpha, \beta)$	$1/(\beta - \alpha)$	$\alpha < x < \beta$
14	Binomial	$Bin(n, p)$	$\binom{n}{x} p^x q^{n-x}$	$x = 0, 1, 2, \dots, n$
15	Poisson	$P(\lambda)$	$e^{-\lambda} \lambda^x / x!$	$x = 0, 1, 2, \dots$
16	Hypergeometric	$H(N, n, p)$	$\binom{Np}{x} \binom{Nq}{n-x} / \binom{N}{n}$	x integer $0 < x < Np$, $0 < n - x < Nq$
17	Negative binomial	$NB(m, p)$	$\Gamma(m+x) [\Gamma(m)x!]^{-1} p^m q^x$	$x = 0, 1, 2, \dots$
18	Geometric	$G(p)$	$p q^x$	$x = 0, 1, 2, \dots$
19	Logarithmic		$K_L q^x / x$, $K_L \equiv -1/\log p$	$x = 1, 2, 3, \dots$
20	Multidimensional normal	$N(\mu, \Sigma)$	$(2\pi)^{-k/2} \Sigma ^{-1/2}$ $\times \exp[-(x-\mu)\Sigma^{-1}(x-\mu)'/2]$, $x = (x_1, \dots, x_k)$, $\mu = (\mu_1, \dots, \mu_k)$, $\Sigma = (\sigma_{ij})$	$-\infty < x_1, \dots, x_k$ $< \infty$
21	Dirichlet		$\frac{\Gamma(p_1 + \dots + p_{k+1})}{\Gamma(p_1) \dots \Gamma(p_{k+1})} x_1^{p_1-1} \dots x_{k+1}^{p_{k+1}-1}$ $x_{k+1} = 1 - (x_1 + \dots + x_k)$	$x_1, \dots, x_k > 0$, $x_1 + \dots + x_k < 1$
22	Multinomial	$M(n, \{p_i\})$	$n! (x_1! \dots x_{k+1}!)^{-1} p_1^{x_1} \dots p_{k+1}^{x_{k+1}}$, $x_{k+1} = n - (x_1 + \dots + x_k)$	x_1, \dots, x_k $= 0, 1, \dots, n$, $x_1 + \dots + x_k \leq n$
23	Multidimensional hypergeometric	$H(N, n, \{p_i\})$	$\binom{Np_1}{x_1} \dots \binom{Np_{k+1}}{x_{k+1}} / \binom{N}{n}$, $x_{k+1} = n - (x_1 + \dots + x_k)$	x_1, \dots, x_k integers $0 < x_i < Np_i$ $(i = 1, \dots, k+1)$
24	Negative polynomial		$\frac{\Gamma(m+x_1+\dots+x_k)}{\Gamma(m)x_1! \dots x_k!} p_0^m p_1^{x_1} \dots p_k^{x_k}$	x_1, \dots, x_k $= 0, 1, 2, \dots$

Conditions for Parameters	Characteristic Function	Mean	Variance	No.
$-\infty < \mu < \infty, \sigma > 0$	$\exp\left(i\mu t - \frac{\sigma^2 t^2}{2}\right)$	μ	σ^2	1
$-\infty < \mu < \infty, \sigma > 0$		$e^{\mu + (\sigma^2/2)}$	$e^{2\mu}(e^{2\sigma^2} - e^{\sigma^2})$	2
$p, \sigma > 0$	$(1 - i\sigma t)^{-p}$	σp	$\sigma^2 p$	3
$-\infty < \mu < \infty, \sigma > 0$	$e^{i\mu t}(1 - i\sigma t)^{-1}$	$\mu + \sigma$	σ^2	4
$\sigma > 0$	$(1 + \sigma^2 t^2)^{-1}$	0	$2\sigma^2$	5
n positive integer	$(1 - 2it)^{-n/2}$	n	$2n$	6
$p, q > 0$		$\frac{p}{p+q}$	$\frac{pq}{(p+q)^2(p+q+1)}$	7
m, n positive integers		$\frac{n}{n-2} (n > 2)$	$\frac{2n^2(m+n-2)}{m(n-2)^2(n-4)} (n > 4)$	8
m, n positive integers				9
n positive integer		0 ($n > 1$)	$n/(n-2) (n > 2)$	10
$-\infty < \mu < \infty, \sigma > 0$	$\exp(i\mu t - \sigma t)$	none	none	11
$0 < c < \infty$	$\exp[-c t ^{1/2}(1 - it/ t)]$	none	none	12
$-\infty < \alpha < \beta < \infty$	$(e^{i\beta t} - e^{i\alpha t})/it(\beta - \alpha)$	$(\alpha + \beta)/2$	$(\beta - \alpha)^2/12$	13
$p + q = 1, p, q > 0, n$ positive integer	$(pe^{it} + q)^n$	np	npq	14
$\lambda > 0$	$\exp[-\lambda(1 - e^{it})]$	λ	λ	15
$p + q = 1, p, q > 0, N, Np, n$ positive integers $N > n$	$(Nq)^{[n]}(N^{[n]})^{-1} \times F(-n, -Np; Nq - n + 1; e^{it}),$ $m^{[n]} \equiv m!/(m-n)!$	np	$\frac{npq(N-n)}{N-1}$	16
$p + q = 1, p, q > 0, m > 0$	$\frac{p^m}{(1 - qe^{it})^m}$	$\frac{mq}{p}$	$\frac{mq}{p^2}$	17
$p + q = 1, p, q > 0$	$\frac{p}{1 - qe^{it}}$	$\frac{q}{p}$	$\frac{q}{p^2}$	18
$p + q = 1, p, q > 0$	$-K_L \log(1 - qe^{it})$	$K_L q/p$	$K_L q(1 - K_L q)/p^2$	19
$-\infty < \mu_1, \dots, \mu_k < \infty, \Sigma$ symmetric positive definite quadratic form	$\exp\left(i\mu' t' - \frac{t' \Sigma t'}{2}\right),$ $t = (t_1, \dots, t_k)$	$E(x_i) = \mu_i$	$V(x_i) = \sigma_{ii},$ $\text{Cov}(x_i, x_j) = \sigma_{ij}$	20
$v_1, \dots, v_{k+1} > 0$		$E(x_i) = \frac{v_i}{v_1 + \dots + v_{k+1}}$	$V(x_i) = C v_i(v_1 + \dots + v_{k+1} - v_i),$ $\text{Cov}(x_i, x_j) = -C v_i v_j,$ $C \equiv (v_1 + \dots + v_{k+1})^{-2} \times (v_1 + \dots + v_{k+1} + 1)^{-1}$	21
$p_1 + \dots + p_{k+1} = 1, p_1, \dots, p_{k+1} > 0, n$ positive integer	$(p_1 e^{it_1} + \dots + p_k e^{it_k} + p_{k+1})^n$	$E(x_i) = np_i$	$V(x_i) = np_i(1 - p_i),$ $\text{Cov}(x_i, x_j) = -np_i p_j$	22
$p_1 + \dots + p_{k+1} = 1, p_1, \dots, p_{k+1} > 0, N, Np_1, \dots, Np_k, n$ positive integers		$E(x_i) = np_i$	$V(x_i) = C np_i(1 - p_i),$ $\text{Cov}(x_i, x_j) = -C np_i p_j,$ $C \equiv \frac{N-n}{N-1}$	23
$p_0 + p_1 + \dots + p_k = 1, p_0, p_1, \dots, p_k > 0, m > 0$	$p_0^m(1 - p_1 e^{it_1} - \dots - p_k e^{it_k})^m$	$E(x_i) = \frac{mp_i}{p_0}$	$V(x_i) = mp_i(p_0 + p_i)/p_0^2,$ $\text{Cov}(x_i, x_j) = mp_i p_j/p_0^2$	24

Remarks

- 1. Reproducing property with respect to μ, σ^2 .
- 2. $X = \log Y: N(\mu, \sigma^2)$.
- 3. Reproducing property with respect to p .
- 4. $e(0, \sigma) = \Gamma(1, \sigma)$.
- 6. n is the number of degrees of freedom; reproducing property with respect to n .
- 8. m and n are the numbers of degrees of freedom.
- 9. $e^{2z} = F(m, n)$.
- 10. n is the number of degrees of freedom.
- 11. $C(0, 1) = t(1)$; reproducing property with respect to μ and σ .
- 14. Reproducing property with respect to n .
- 15. Reproducing property with respect to λ .
- 17. Reproducing property with respect to m .
- 18. $G(p) = NB(1, p)$.
- 20. Generalization of normal distribution; reproducing property with respect to μ and Σ .
- 22. Generalization of binomial distribution; reproducing property with respect to n .
- 23. Generalization of hypergeometric distribution.
- 24. Generalization of negative binomial distribution; reproducing property with respect to m .

23. Statistical Estimation and Statistical Hypothesis Testing

Listed below are some frequently used and well-investigated statistical procedures. (Concerning main probability distributions — 398 Statistical Decision Functions, 399 Statistical Estimation, 400 Statistical Hypothesis Testing). The following notations and conventions are adopted, unless otherwise stated.

Immediately after the heading number, the distribution is indicated by the symbol as defined in Table 22, this Appendix. It is to be understood that a random sample (x_1, x_2, \dots, x_n) is observed from this distribution. Where two distributions are involved, samples (x_1, \dots, x_{n_1}) and (y_1, \dots, y_{n_2}) are understood to be observed from the respective distributions.

Next, a necessary and sufficient statistic based on the sample is marked with * when it is complete, and # otherwise. Then appears the sampling distribution of this statistic. For those statistics consisting of several independent components, the distribution of these are shown. Greek lowercase letters except α and χ denote unknown parameters. Italic lowercase letters denote constants, each taking arbitrary real values. Italic capital letters denote constants whose values are specified in each procedure; repeated occurrences of the same letter under the same heading number specify a certain common real value.

Problems of point estimation, interval estimation, and hypothesis testing are presented, with corresponding estimators, confidence intervals, and tests (critical regions) as their solutions. All the confidence intervals here are those constructed from UMP unbiased tests, having $1 - \alpha$ as confidence levels. Alternative hypotheses are understood to be the negations of corresponding null hypotheses. Significance levels of all the tests are α . The following symbols are attached to each procedure to describe its properties.

For estimators:

UMV: uniformly minimum variance unbiased.

ML: maximum likelihood.

AD: admissibility with respect to quadratic loss function.

IAD: inadmissibility with respect to quadratic loss function.

For tests:

UMP: uniformly most powerful.

UMPU: uniformly most powerful unbiased.

UMPI(): uniformly most powerful invariant with respect to the product of transformation groups shown in ().

LR: likelihood ratio.

O: group of orthogonal transformations.

L: group of shift transformations.

S: group of change of scales.

AD: admissibility with respect to simple loss function.

IAD: inadmissibility with respect to simple loss function. (Note that UMPU implies AD.)

The following symbols denote $100(1 - \alpha)\%$ points of respective distributions, α being sufficiently small.

$u(\alpha)$: standard normal distribution.

$t_f(\alpha)$: t -distribution with f degrees of freedom.

$\chi_f^2(\alpha)$: χ^2 distribution with f degrees of freedom.

$F_{f_1}^{f_2}(\alpha)$: F -distribution with (f_1, f_2) degrees of freedom.

(1) $N(\mu, b^2)$. $\sum x_i^*$. $N(n\mu, nb^2)$.

Point estimation of μ . $\bar{x} = \frac{1}{n} \sum x_i$: UMV, ML, AD.

Interval estimation of μ . $\left(\bar{x} \pm u(\alpha/2) \frac{b}{\sqrt{n}} \right)$.

Hypothesis $[\mu \leq k]$. $\bar{x} > k + u(\alpha) \frac{b}{\sqrt{n}}$: UMP, LR.

Hypothesis $[h \leq \mu \leq l]$. $\bar{x} < h - C$ or $\bar{x} > l + C$: UMPU, LR.

(2) $N(a, \sigma^2)$. $\sum (x_i - a)^{2*}$. $\sigma^2 \chi_n^2$. ($\sigma^2 \chi_n^2$ is the σ^2 -multiplication of a random variable obeying the $\chi^2(n)$ distribution. We use similar notations in the following.)

Point estimation of σ^2 . $\frac{\sum (x_i - a)^2}{n}$: UMV, ML, IAD.

Interval estimation of σ^2 . $(A \sum (x_i - a)^2, B \sum (x_i - a)^2)$.

Hypothesis $[\sigma^2 \leq k]$. $\sum (x_i - a)^2 > \chi_n^2(\alpha)k$: UMP, LR.

Hypothesis $[\sigma^2 = k]$. $\sum (x_i - a)^2 < Ak$ or $\sum (x_i - a)^2 > Bk$: UMPU.

(3) $N(\mu, \sigma^2)$. $\left(\frac{\sum x_i}{\sum (x_i - \bar{x})^2} \right)^*$. $\left(\frac{N(n\mu, n\sigma^2)}{\sigma^2 \chi_{n-1}^2} \right)$.

Point estimation of μ . \bar{x} : UMV, ML, AD.

Interval estimation of μ . $\left[\bar{x} \pm t_{n-1}(\alpha/2) \frac{\sqrt{\sum (x_i - \bar{x})^2}}{\sqrt{n(n-1)}} \right]$.

Hypothesis $[\mu \leq k]$. $\frac{\bar{x} - k}{\sqrt{\sum (x_i - \bar{x})^2}} > \frac{t_{n-1}(\alpha)}{\sqrt{n(n-1)}}$: UMPU, LR.

Hypothesis $[\mu = k]$. $\frac{|\bar{x} - k|}{\sqrt{\sum (x_i - \bar{x})^2}} > \frac{t_{n-1}(\alpha/2)}{\sqrt{n(n-1)}}$: UMPU, LR, UMPI(S, O) for $k=0$.

Point estimation of σ^2 . $\frac{\sum (x_i - \bar{x})^2}{n-1}$: UMV, IAD. $\frac{\sum (x_i - \bar{x})^2}{n}$: ML, IAD.

Point estimation of σ . $\frac{\Gamma[(n-1)/2]}{\sqrt{2} \Gamma(n/2)} \sqrt{\frac{\sum (x_i - \bar{x})^2}{n-1}}$: UMV, IAD.

Interval estimation of σ^2 . $(A \sum (x_i - \bar{x})^2, B \sum (x_i - \bar{x})^2)$.

Hypothesis $[\sigma^2 \leq k]$. $\sum (x_i - \bar{x})^2 > \chi_{n-1}^2(\alpha)k$: UMP, LR.

Hypothesis $[\sigma^2 = k]$. $\sum (x_i - \bar{x})^2 < Ak$ or $\sum (x_i - \bar{x})^2 > Bk$: UMPU.

Hypothesis $[\sigma^2 \geq k]$. $\sum (x_i - \bar{x})^2 < \chi_{n-1}^2(1-\alpha)k$: UMPU, UMPI(L).

Hypothesis $\left[\frac{\mu}{\sigma} \leq k \right]$. $\frac{\bar{x}}{\sqrt{\sum (x_i - \bar{x})^2}} > E$: UMPI(S), AD.

(4) $\text{Bin}(N, \theta)$. $\sum x_i^*$. $\text{Bin}(Nn, \theta)$.

Point estimation of θ . $\frac{\bar{x}}{N}$: UMV, ML, AD.

Hypothesis $[\theta \leq k]$. $\bar{x} > A$: UMP.

Hypothesis $[h \leq \theta \leq l]$. $\bar{x} < B$ or $\bar{x} > C$: UMPU.

(5) $H(N, m, \theta)$ ($n=1$). x^* .

Point estimation of θ . $\frac{Nx}{m}$: UMV, AD.

Hypothesis $[\theta \leq k]$. $x > A$: UMP.

- (6) $NB(N, \theta)$. $\sum x_i^*$. $NB(Nn, \theta)$.
Point estimation of θ . $\frac{Nn-1}{Nn+\sum x_i-1}$ (1 when the denominator is 0): UMV, AD.

$$\frac{Nn}{Nn+\sum x_i}: \text{ML}.$$

Hypothesis $[\theta \leq k]$. $\sum x_i < A$: UMP.

Hypothesis $[h \leq \theta \leq l]$. $\sum x_i < B$ or $\sum x_i > C$: UMPU.

- (7) $P(\lambda)$. $\sum x_i^*$. $P(n\lambda)$.
Point estimation of λ . \bar{x} : UMV, ML, AD.
Hypothesis $[\lambda \leq k]$. $\bar{x} > A$: UMP.
Hypothesis $[h \leq \lambda \leq l]$. $\bar{x} < B$ or $\bar{x} > C$: UMPU.

- (8) $G(\theta)$. $\sum x_i^*$. $NB(n, \theta)$.
For the point estimation of θ and hypothesis testing \rightarrow (6).

- (9) $U[0, \theta]$. $\max x_i^*$.
Point estimation of θ . $\max x_i$: ML, IAD. $\frac{n+1}{n} \max x_i$: UMV, IAD.
Hypothesis $[\theta \leq k]$. $\max x_i > (1-\alpha)^{1/n}k$: UMP.
Hypothesis $[\theta = k]$. $\max x_i < k\alpha^{1/n}$ or $\max x_i > k$: UMP.

- (10) $U[\xi, \eta]$. $(\min x_i, \max x_i)^*$.
Point estimation of ξ . $\frac{n \min x_i - \max x_i}{n-1}$: UMV, IAD. $\min x_i$: ML, IAD.
Point estimation of $\frac{\xi + \eta}{2}$. $\frac{\min x_i + \max x_i}{2}$: UMV, AD.
Hypothesis $[\eta - \xi \leq k]$. $\max x_i - \min x_i > k\alpha^{1/n}$: UMP.

- (11) $U\left[\theta - \frac{1}{2}, \theta + \frac{1}{2}\right]$. $(\min x_i, \max x_i)^*$.
Point estimation of θ . $\frac{\min x_i + \max x_i}{2}$: ML, AD.
Hypothesis $[\theta \leq k]$. $\min x_i > k + \frac{1}{2} - \alpha^{1/n}$ or $\max x_i > k + \frac{1}{2}$: UMP.

- (12) $e(\mu, \sigma)$. $\left(\frac{\sum x_i}{\min x_i}\right)^* \cdot \left(\frac{\Gamma(n, \sigma) + n\mu}{e(\mu, \sigma/n)}\right)$.

Point estimation of σ . $\frac{\sum x_i - n \min x_i}{n-1}$: UMV, IAD. $\bar{x} - \min x_i$: ML, IAD.

Point estimation of μ . $\frac{n}{n-1} \min x_i - \frac{1}{n-1} \bar{x}$: UMV, IAD. $\min x_i$: ML, IAD.

Hypothesis $[\sigma \leq k, \mu = h]$. $\sum x_i < h$ or $\sum x_i > k \log \alpha^{-1/n} + h$: UMP.

Hypothesis $[h \leq \sigma \leq l]$. $\sum x_i - n \min x_i < A$ or $\sum x_i - n \min x_i > B$: UMPU.

Hypothesis $[\mu = k]$. $\frac{n \min x_i - k}{\sum x_i - n \min x_i} < 0$ or $\frac{n \min x_i - k}{\sum x_i - n \min x_i} > C$: UMPU.

- (13) $\Gamma(p, \sigma)$. $\sum x_i^*$. $\Gamma(np, \sigma)$.
Point estimation of σ . $\frac{\bar{x}}{p}$: UMV, ML, IAD.
Interval estimation of σ . $(C \sum x_i, D \sum x_i)$.
Hypothesis $[\sigma \leq k]$. $\sum x_i > A$: UMP.
Hypothesis $[\sigma = k]$. $\sum x_i < Ck$ or $\sum x_i > Dk$: UMPU.

- (14) $N(\mu_1, a^2)$. $\left(\frac{\sum x_i}{\sum y_i}\right)^* \cdot \left(\frac{N(n_1 \mu_1, n_1 a^2)}{N(n_2 \mu_2, n_2 b^2)}\right)$.

Point estimation of $\mu_1 - \mu_2$. $\bar{x} - \bar{y}$: UMV, ML, AD.

Interval estimation of $\mu_1 - \mu_2$. $\left(\bar{x} - \bar{y} \pm u(\alpha/2) \sqrt{\frac{a^2}{n_1} + \frac{b^2}{n_2}}\right)$.

Hypothesis $[\mu_1 - \mu_2 \leq k]$. $\bar{x} - \bar{y} > k + u(\alpha) \sqrt{\frac{a^2}{n_1} + \frac{b^2}{n_2}}$: UMP, LR.

Hypothesis $[\mu_1 - \mu_2 = k]$. $|\bar{x} - \bar{y} - k| > u(\alpha/2) \sqrt{\frac{a^2}{n_1} + \frac{b^2}{n_2}}$: UMPU, UMPI(L), LR.

$$(15) \begin{matrix} N(\mu_1, \sigma^2) \\ N(\mu_2, \sigma^2) \end{matrix} \cdot \begin{bmatrix} \sum x_i \\ \sum y_i \\ s^2 = \sum (x_i - \bar{x})^2 + \sum (y_i - \bar{y})^2 \end{bmatrix}^* \cdot \begin{bmatrix} N(n_1 \mu_1, n_1 \sigma^2) \\ N(n_2 \mu_2, n_2 \sigma^2) \\ \sigma^2 \chi_{n_1+n_2-2}^2 \end{bmatrix}.$$

Point estimation of $\mu_1 - \mu_2$. $\bar{x} - \bar{y}$: UMV, ML, AD.

Interval estimation of $\mu_1 - \mu_2$. $\left(\bar{x} - \bar{y} \pm t_{n_1+n_2-2}(\alpha/2) \sqrt{\frac{1}{n_1} + \frac{1}{n_2}} \sqrt{\frac{s^2}{n_1+n_2-2}} \right)$.

Hypothesis $[\mu_1 - \mu_2 \leq k]$. $t = \frac{(\bar{x} - \bar{y} - k) \sqrt{n_1 n_2} \sqrt{n_1 + n_2 - 2}}{\sqrt{n_1 + n_2} \sqrt{s^2}} > t_{n_1+n_2-2}(\alpha)$: UMPU,

UMPI(L), LR.

Hypothesis $[\mu_1 - \mu_2 = k]$. $|t| > t_{n_1+n_2-2}(\alpha)$: UMPU, UMPI(L), LR.

Point estimation of σ^2 . $\frac{s^2}{n_1+n_2-2}$: UMV, IAD. $\frac{s^2}{n_1+n_2}$: ML, IAD.

Interval estimation of σ^2 . (As^2, Bs^2) .

Hypothesis $[\sigma^2 \leq k]$. $s^2 > \chi_{n_1+n_2-2}^2(\alpha)k$: UMP, LR.

Hypothesis $[\sigma^2 = k]$. $s^2 < Ak$ or $s^2 > Bk$: UMPU.

Hypothesis $[\sigma^2 \geq k]$. $s^2 > \chi_{n_1+n_2-2}^2(1-\alpha)k$: UMPU, UMPI(L), LR.

$$(16) \begin{matrix} N(\mu_1, \sigma_1^2) \\ N(\mu_2, \sigma_2^2) \end{matrix} \cdot \begin{bmatrix} \sum x_i, \sum (x_i - \bar{x})^2 \\ \sum y_i, \sum (y_i - \bar{y})^2 \end{bmatrix}^*.$$

Interval estimation of $\frac{\sigma_1^2}{\sigma_2^2}$. $\left[A \frac{\sum (x_i - \bar{x})^2}{\sum (y_i - \bar{y})^2}, B \frac{\sum (x_i - \bar{x})^2}{\sum (y_i - \bar{y})^2} \right]$.

Hypothesis $\left[\frac{\sigma_1^2}{\sigma_2^2} \leq k \right]$. $\frac{(n_2-1) \sum (x_i - \bar{x})^2}{(n_1-1) \sum (y_i - \bar{y})^2} > F_{n_2-1}^{n_1-1}(\alpha)k$: UMPU, UMPI(L, S), LR.

$$(17) N(\mu_1, \mu_2, \sigma_1^2, \sigma_2^2, \rho) \cdot \begin{bmatrix} \sum x_i, \sum (x_i - \bar{x})^2, \sum (x_i - \bar{x})(y_i - \bar{y}) \\ \sum y_i, \sum (y_i - \bar{y})^2 \end{bmatrix}^*.$$

Point estimation of ρ . $r = \frac{\sum (x_i - \bar{x})(y_i - \bar{y})}{\sqrt{\sum (x_i - \bar{x})^2 \sum (y_i - \bar{y})^2}}$: ML.

Hypothesis $[\rho = 0]$. $|r| > \frac{t_{n-1}(\alpha/2)}{\sqrt{t_{n-1}(\alpha/2)^2 + n - 2}}$: UMPU, LR.

Appendix B
Numerical Tables

1	Prime Numbers and Primitive Roots
2	Indices Modulo p
3	Bernoulli Numbers and Euler Numbers
4	Class Numbers of Algebraic Number Fields
5	Characters of Finite Groups; Crystallographic Groups
6	Miscellaneous Constants
7	Coefficients of Polynomial Approximations

1. Prime Numbers and Primitive Roots (→ 297 Number Theory,
Elementary

In the following table, p is a prime number and r is a corresponding primitive root.

p	r	p	r	p	r	p	r	p	r	p	r	p	r
2		79	3	191	19	311	17	439	17	577	5	709	2
3	2	83	2	193	5	313	17	443	2	587	2	719	11
5	2	89	3	197	2	317	2	449	3	593	3	727	5
7	3	97	5	199	3	331	3	457	13	599	7	733	7
11	2	101	2	211	2	337	19	461	2	601	7	739	3
13	2	103	5	223	3	347	2	463	3	607	3	743	5
17	3	107	2	227	2	349	2	467	2	613	2	751	3
19	2	109	11	229	7	353	3	479	13	617	3	757	2
23	5	113	3	233	3	359	7	487	3	619	2	761	7
29	2	127	3	239	7	367	11	491	2	631	3	769	11
31	3	131	2	241	7	373	2	499	7	641	3	773	2
37	2	137	3	251	11	379	2	503	5	643	11	787	2
41	7	139	2	257	3	383	5	509	2	647	5	797	2
43	3	149	2	263	5	389	2	521	3	653	2	809	3
47	5	151	7	269	2	397	5	523	2	659	2	811	3
53	2	157	5	271	43	401	3	541	2	661	2	821	2
59	2	163	2	277	5	409	29	547	2	673	5	823	3
61	2	167	5	281	3	419	2	557	2	677	2	827	2
67	2	173	2	283	3	421	2	563	2	683	5	829	2
71	7	179	2	293	2	431	7	569	3	691	3	839	11
73	5	181	2	307	5	433	5	571	3	701	2	853	2

†Mersenne numbers. A prime number of the form $2^p - 1$ is called a Mersenne number. There exist 27 such p 's less than 44500: $p=2, 3, 5, 7, 13, 17, 19, 31, 61, 89, 107, 127, 521, 607, 1279, 2203, 2281, 3217, 4253, 4423, 9689, 9941, 11213, 19937, 21701, 23209, 44497$. The even perfect numbers are the numbers of the form $2^{p-1}(2^p - 1)$, where $2^p - 1$ is a Mersenne number.

2. Indices Modulo p (→ 297 Number Theory, Elementary)

Let r be a primitive root corresponding to a prime number p . The index $l = \text{Ind}_r a$ of a with respect to the basis r is the integer l in $0 \leq l < p - 1$ satisfying $r^l \equiv a \pmod{p}$. $a \equiv b \pmod{p}$ is equivalent to $\text{Ind}_r a \equiv \text{Ind}_r b \pmod{p - 1}$. The index satisfies the following congruence relations with respect to $\text{mod}(p - 1)$: $\text{Ind}_r ab \equiv \text{Ind}_r a + \text{Ind}_r b$, $\text{Ind}_r a^n \equiv n \text{Ind}_r a$, $\text{Ind}_r a \equiv \text{Ind}_r r \text{Ind}_r a$.

App. B, Table 2
Indices Modulo p

We can solve congruence equations using these relations. The following is a table of indices.

p	$p-1$	r	2	3	5	7	11	13	17	19	23	29	31	37	41	43	47
2	1		—														
3	2	2	1	—													
5	4	2	1	3	—												
7	2·3	3	2	1	5	—											
11	2·5	2	1	8	4	7	—										
13	2 ² ·3	2	1	4	9	11	7	—									
17	2 ⁴	3	14	1	5	11	7	4	—								
19	2·3 ²	2	1	13	16	6	12	5	10	—							
23	2·11	5	2	16	1	19	9	14	7	15	—						
29	2 ² ·7	2	1	5	22	12	25	18	21	9	20	—					
31	2·3·5	3	24	1	20	28	23	11	7	4	27	9	—				
37	2 ² ·3 ²	2	1	26	23	32	30	11	7	35	15	21	9	—			
41	2 ³ ·5	7	14	25	18	1	37	9	7	31	4	33	12	8	—		
43	2·3·7	3	27	1	25	35	30	32	38	19	16	41	34	7	6	—	
47	2·23	5	18	20	1	32	7	11	16	45	5	35	3	42	15	13	—
53	2 ² ·13	2	1	17	47	14	6	24	10	37	39	46	33	30	45	22	44
59	2·29	2	1	50	6	18	25	45	40	38	15	28	49	55	14	33	23
61	2 ² ·3·5	2	1	6	22	49	15	40	47	26	57	35	59	39	54	43	20
67	2·3·11	2	1	39	15	23	59	19	64	10	28	44	47	22	53	9	50
71	2·5·7	7	6	26	28	1	31	39	49	16	15	68	11	20	25	48	9
73	2 ³ ·3 ²	5	8	6	1	33	55	59	21	62	46	35	11	64	4	51	31
79	2·3·13	3	4	1	62	53	68	34	21	32	26	11	56	19	75	49	59
83	2·41	2	1	72	27	8	24	77	56	47	60	12	38	20	40	71	23
89	2 ³ ·11	3	16	1	70	81	84	23	6	35	57	59	31	11	21	29	54
97	2 ⁵ ·3	5	34	70	1	31	86	25	89	81	77	13	46	91	85	4	84
101	2 ² ·5 ²	2	1	69	24	9	13	66	30	96	86	91	84	56	45	42	58
103	2·3·17	5	44	39	1	4	61	72	70	80	24	86	57	93	50	77	85
107	2·53	2	1	70	47	43	22	14	29	78	62	32	27	38	40	59	66
109	2 ² ·3 ³	11	15	80	92	20	1	101	87	105	3	98	34	43	63	42	103
113	2 ⁴ ·7	3	12	1	83	8	74	22	5	99	41	89	50	67	94	47	31
127	2·3 ² ·7	3	72	1	87	115	68	94	38	84	121	113	46	98	80	71	60
131	2·5·13	2	1	72	46	96	56	18	43	35	23	51	29	41	126	124	105
137	2 ³ ·17	3	10	1	75	42	122	25	38	46	125	91	73	102	119	97	19
139	2·3·23	2	1	41	86	50	76	64	107	61	27	94	56	80	32	115	98
149	2 ² ·37	2	1	87	104	142	109	53	124	84	95	120	132	72	41	93	138
151	2·3·5 ²	7	10	93	136	1	82	23	124	120	145	42	34	148	3	74	128
157	2 ² ·3·13	5	141	82	1	147	28	26	40	124	135	129	62	116	21	113	92
163	2·3 ⁴	2	1	101	15	73	47	51	57	125	9	107	69	33	160	38	28
167	2·83	5	40	94	1	118	28	103	53	58	99	150	90	61	97	87	132
173	2 ² ·43	2	1	27	39	95	23	130	73	33	20	144	102	162	138	84	64
179	2·89	2	1	108	138	171	15	114	166	54	135	118	62	149	155	80	36
181	2 ² ·3 ² ·5	2	1	56	156	15	62	164	175	135	53	48	99	26	83	20	13
191	2·5·19	19	44	116	50	171	85	112	98	1	134	33	175	15	165	8	123
193	2 ⁶ ·3	5	34	84	1	104	183	141	31	145	162	123	82	5	151	24	29
197	2 ² ·7 ²	2	1	181	89	146	29	25	159	154	120	36	141	192	110	78	66
199	2·3 ² ·11	3	106	1	138	142	189	172	123	55	118	70	164	11	167	88	76
211	2·3·5·7	2	1	43	132	139	162	144	199	154	21	179	115	118	17	80	124
223	2·3·37	3	180	1	89	210	107	147	144	172	163	128	82	152	204	118	50
227	2·113	2	1	46	11	154	28	61	99	178	34	8	197	77	131	150	218
229	2 ² ·3·19	7	111	68	214	1	42	195	24	52	131	191	175	164	73	12	193

(table continued on following page)

<i>p</i>	<i>p</i> −1	<i>r</i>	2	3	5	7	11	13	17	19	23	29	31	37	41	43	47
233	2 ³ ·29	3	72	1	165	222	197	158	103	136	112	132	182	8	85	25	139
239	2·7·17	7	66	74	138	1	4	43	52	155	63	160	188	31	99	15	113
241	2 ⁴ ·3·5	7	190	182	138	1	25	47	111	85	57	154	151	73	6	219	114
251	2·5 ³	11	135	6	80	218	1	162	184	233	134	203	226	187	64	77	85
257	2 ⁸	3	48	1	55	85	196	106	120	125	28	94	242	219	19	207	61
263	2·131	5	190	50	1	79	166	62	126	43	156	221	136	170	17	154	65
269	2 ² ·67	2	1	109	208	19	230	142	105	223	176	187	259	56	200	254	32
271	2·3 ³ ·5	43	266	153	220	98	92	15	16	261	75	45	222	182	156	1	213
277	2 ² ·3·23	5	147	188	1	22	7	222	103	252	208	74	47	87	126	55	218
281	2 ³ ·5·7	3	204	1	186	182	253	9	166	221	197	172	62	135	23	132	75

3. Bernoulli Numbers and Euler Numbers

(→ 177 Generating Functions)

B_n are Bernoulli numbers; *E_n* are Euler numbers.

<i>n</i>	Numerator of <i>B_n</i>	Denominator of <i>B_n</i>	<i>B_n</i>	<i>E_n</i>
2	1	6	0.16667	1
4	1	30	0.03333	5
6	1	42	0.02381	61
8	1	30	0.03333	1385
10	5	66	0.07576	50521
12	691	2730	0.25311	2702765
14	7	6	1.16667	199360981
16	3617	510	7.09216	19391512145
18	43867	798	54.97118	2404879675441
20	174611	330	529.12424	370371188237525
22	854513	138	6192.12319	6.934887×10 ¹⁶
24	236364091	2730	86580.25311	1.551453×10 ¹⁹
26	8553103	6	1425517.16667	4.087073×10 ²¹
28	23749461029	870	27298231.06782	1.252260×10 ²⁴
30	8615841276005	14322	601580873.90064	4.415439×10 ²⁶

4. Class Numbers of Algebraic Number Fields

(I) Class Numbers of Real Quadratic Field (→ 347 Quadratic Fields)

Let $k = \mathbb{Q}(\sqrt{m})$, where m is a positive integer without square factor ($1 < m \leq 501$). h is the class number (in the wider sense) of k . The $-$ sign in the row of $N(\epsilon)$ means that the norm $N(\epsilon)$ of the fundamental unit is -1 . When $N(\epsilon) = +1$, the class number in the narrow sense is $2h$, and when $N(\epsilon) = -1$, the class number in the narrow sense is also h .

<i>m</i>	<i>h</i>	<i>N</i> (ϵ)	<i>m</i>	<i>h</i>	<i>N</i> (ϵ)	<i>m</i>	<i>h</i>	<i>N</i> (ϵ)	<i>m</i>	<i>h</i>	<i>N</i> (ϵ)	<i>m</i>	<i>h</i>	<i>N</i> (ϵ)	<i>m</i>	<i>h</i>	<i>N</i> (ϵ)
2	1	−	85	2	−	170	4	−	253	1		335	2		421	1	−
3	1		86	1		173	1	−	254	3		337	1	−	422	1	
5	1	−	87	2		174	2		255	4		339	2		426	2	
6	1		89	1	−	177	1		257	3	−	341	1		427	6	
7	1		91	2		178	2		258	2		345	2		429	2	
10	2	−	93	1		179	1		259	2		346	6	−	430	2	
11	1		94	1		181	1	−	262	1		347	1		431	1	
13	1	−	95	2		182	2		263	1		349	1	−	433	1	−
14	1		97	1	−	183	2		265	2	−	353	1	−	434	4	

<i>m</i>	<i>h</i>	<i>N</i> (ϵ)	<i>m</i>	<i>h</i>	<i>N</i> (ϵ)	<i>m</i>	<i>h</i>	<i>N</i> (ϵ)	<i>m</i>	<i>h</i>	<i>N</i> (ϵ)	<i>m</i>	<i>h</i>	<i>N</i> (ϵ)	<i>m</i>	<i>h</i>	<i>N</i> (ϵ)
15	2		101	1	—	185	2	—	266	2		354	2		435	4	
17	1	—	102	2		186	2		267	2		355	2		437	1	
19	1		103	1		187	2		269	1	—	357	2		438	4	
21	1		105	2		190	2		271	1		358	1		439	5	
22	1		106	2	—	191	1		273	2		359	3		442	8	—
23	1		107	1		193	1	—	274	4	—	362	2	—	443	3	
26	2	—	109	1	—	194	2		277	1	—	365	2	—	445	4	—
29	1	—	110	2		195	4		278	1		366	2		446	1	
30	2		111	2		197	1	—	281	1	—	367	1		447	2	
31	1		113	1	—	199	1		282	2		370	4	—	449	1	—
33	1		114	2		201	1		283	1		371	2		451	2	
34	2		115	2		202	2	—	285	2		373	1	—	453	1	
35	2		118	1		203	2		286	2		374	2		454	1	
37	1	—	119	2		205	2		287	2		377	2		455	4	
38	1		122	2	—	206	1		290	4	—	379	1		457	1	—
39	2		123	2		209	1		291	4		381	1		458	2	—
41	1	—	127	1		210	4		293	1	—	382	1		461	1	—
42	2		129	1		211	1		295	2		383	1		462	4	
43	1		130	4	—	213	1		298	2	—	385	2		463	1	
46	1		131	1		214	1		299	2		386	2		465	2	
47	1		133	1		215	2		301	1		389	1	—	466	2	
51	2		134	1		217	1		302	1		390	4		467	1	
53	1	—	137	1	—	218	2	—	303	2		391	2		469	3	
55	2		138	2		219	4		305	2		393	1		470	2	
57	1		139	1		221	2		307	1		394	2	—	471	2	
58	2	—	141	1		222	2		309	1		395	2		473	3	
59	1		142	3		223	3		310	2		397	1	—	474	2	
61	1	—	143	2		226	8	—	311	1		398	1		478	1	
62	1		145	4	—	227	1		313	1	—	399	8		479	1	
65	2	—	146	2		229	3	—	314	2	—	401	5	—	481	2	—
66	2		149	1	—	230	2		317	1	—	402	2		482	2	
67	1		151	1		231	4		318	2		403	2		483	4	
69	1		154	2		233	1	—	319	2		406	2		485	2	—
70	2		155	2		235	6		321	3		407	2		487	1	
71	1		157	1	—	237	1		322	4		409	1	—	489	1	
73	1	—	158	1		238	2		323	4		410	4		491	1	
74	2	—	159	2		239	1		326	3		411	2		493	2	—
77	1		161	1		241	1	—	327	2		413	1		494	2	
78	2		163	1		246	2		329	1		415	2		497	1	
79	3		165	2		247	2		330	4		417	1		498	2	
82	4	—	166	1		249	1		331	1		418	2		499	5	
83	1		167	1		251	1		334	1		419	1		501	1	

One can find a table of fundamental units and representatives of ideal classes for $0 < m < 2025$ in E. L. Ince, *Cycles of reduced ideals in quadratic fields*, Royal Society, London, 1968.

(II) Class Numbers of Imaginary Quadratic Fields (→ 347 Quadratic Fields)

Let $k = \mathbf{Q}(\sqrt{-m})$, where m is a positive integer without square factor ($1 \leq m \leq 509$). h is the class number of k . In the present case, there is no distinction between the class numbers in the wider and narrow senses.

<i>m</i>	<i>h</i>	<i>m</i>	<i>h</i>	<i>m</i>	<i>h</i>	<i>m</i>	<i>h</i>	<i>m</i>	<i>h</i>	<i>m</i>	<i>h</i>	<i>m</i>	<i>h</i>	<i>m</i>	<i>h</i>
1	1	65	8	129	12	193	4	255	12	319	10	389	22	447	14
2	1	66	8	130	4	194	20	257	16	321	20	390	16	449	20
3	1	67	1	131	5	195	4	258	8	322	8	391	14	451	6
5	2	69	8	133	4	197	10	259	4	323	4	393	12	453	12
6	2	70	4	134	14	199	9	262	6	326	22	394	10	454	14
7	1	71	7	137	8	201	12	263	13	327	12	395	8	455	20
10	2	73	4	138	8	202	6	265	8	329	24	397	6	457	8
11	1	74	10	139	3	203	4	266	20	330	8	398	20	458	26

<i>m</i>	<i>h</i>	<i>m</i>	<i>h</i>	<i>m</i>	<i>h</i>	<i>m</i>	<i>h</i>	<i>m</i>	<i>h</i>	<i>m</i>	<i>h</i>	<i>m</i>	<i>h</i>	<i>m</i>	<i>h</i>
13	2	77	8	141	8	205	8	267	2	331	3	399	16	461	30
14	4	78	4	142	4	206	20	269	22	334	12	401	20	462	8
15	2	79	5	143	10	209	20	271	11	335	18	402	16	463	7
17	4	82	4	145	8	210	8	273	8	337	8	403	2	465	16
19	1	83	3	146	16	211	3	274	12	339	6	406	16	466	8
21	4	85	4	149	14	213	8	277	6	341	28	407	16	467	7
22	2	86	10	151	7	214	6	278	14	345	8	409	16	469	16
23	3	87	6	154	8	215	14	281	20	346	10	410	16	470	20
26	6	89	12	155	4	217	8	282	8	347	5	411	6	471	16
29	6	91	2	157	6	218	10	283	3	349	14	413	20	473	12
30	4	93	4	158	8	219	4	285	16	353	16	415	10	474	20
31	3	94	8	159	10	221	16	286	12	354	16	417	12	478	8
33	4	95	8	161	16	222	12	287	14	355	4	418	8	479	25
34	4	97	4	163	1	223	7	290	20	357	8	419	9	481	16
35	2	101	14	165	8	226	8	291	4	358	6	421	10	482	20
37	2	102	4	166	10	227	5	293	18	359	19	422	10	483	4
38	6	103	5	167	11	229	10	295	8	362	18	426	24	485	20
39	4	105	8	170	12	230	20	298	6	365	20	427	2	487	7
41	8	106	6	173	14	231	12	299	8	366	12	429	16	489	20
42	4	107	3	174	12	233	12	301	8	367	9	430	12	491	9
43	1	109	6	177	4	235	2	302	12	370	12	431	21	493	12
46	4	110	12	178	8	237	12	303	10	371	8	433	12	494	28
47	5	111	8	179	5	238	8	305	16	373	10	434	24	497	24
51	2	113	8	181	10	239	15	307	3	374	28	435	4	498	8
53	6	114	8	182	12	241	12	309	12	377	16	437	20	499	3
55	4	115	2	183	8	246	12	310	8	379	3	438	8	501	16
57	4	118	6	185	16	247	6	311	19	381	20	439	15	502	14
58	2	119	10	186	12	249	12	313	8	382	8	442	8	503	21
59	3	122	10	187	2	251	7	314	26	383	17	443	5	505	8
61	6	123	2	190	4	253	4	317	10	385	8	445	8	506	28
62	8	127	5	191	13	254	16	318	12	386	20	446	32	509	30

There are only 9 instances of m for which $h=1$, and only 18 instances of m for which $h=2$ (Baker, Stark). All these cases are in this table.

One can find a table of structures of the ideal class groups and representatives of ideal classes for $m < 24000$ in H. Wada, *A table of ideal class groups of imaginary quadratic fields*, Proc. Japan Acad., 46 (1970), 401–403.

(III) Class Numbers of Cyclotomic Fields

Cyclotomic field $k = \mathbb{Q}(e^{2\pi i/l})$ ($1 < l < 100$; l prime). h_1 is the first factor of the class number of k (\rightarrow 14 Algebraic Number Fields).

<i>l</i>	<i>h</i> ₁	<i>l</i>	<i>h</i> ₁	<i>l</i>	<i>h</i> ₁	<i>l</i>	<i>h</i> ₁	<i>l</i>	<i>h</i> ₁	<i>l</i>	<i>h</i> ₁
3	1	13	1	29	2 ³	43	211	61	41·1861	79	5·53·377911
5	1	17	1	31	3 ²	47	5·139	67	67·12739	83	3·279405653
7	1	19	1	37	37	53	4889	71	7 ² ·79241	89	113·118401449
11	1	23	3	41	11 ²	59	3·59·233	73	89·134353	97	577·3457·206209

$h_1 > 1$ for $l > 19$ (Uchida).

5. Characters of Finite Groups; Crystallographic Groups

(I) Symmetric Groups S_n , Alternating Groups A_n ($3 \leq n \leq 7$), and Mathieu Groups M_n ($n = 11, 12, 22, 23, 24$)

(1) In each table, the first column gives the representation of the conjugate class as we represent a permutation by the product of cyclic permutations. For example, $(3)(2)^2$ means the conjugate class containing $(123)(45)(67)$.

- (2) The second column gives the order of the centralizer of the elements of the conjugate class.
 (3) In the table of S_n , the first row gives the type of Young diagram corresponding to each irreducible character. For example, $[3, 2^2, 1]$ means $T(3, 2, 2, 1)$.
 (4) In the table of A_n , when we restrict the self-conjugate character of S_n (the character with *) to A_n , it is decomposed into two mutually algebraically conjugate irreducible characters, and therefore we show only one of them. The other irreducible character of A_n is given by the restriction to A_n of the character of S_n that is not self-conjugate.
 (5) In the table of M_n , each character with a bar over the degree is one of the two mutually algebraically conjugate characters.

$$\begin{aligned} \varepsilon_1^\pm &= (-1 \pm \sqrt{-3})/2, & \varepsilon_2^\pm &= (1 \pm \sqrt{5})/2, & \varepsilon_3^\pm &= (-1 \pm \sqrt{-7})/2 \\ \varepsilon_4^\pm &= (-1 \pm \sqrt{-11})/2, & \varepsilon_5^\pm &= (-1 \pm \sqrt{-15})/2, & \varepsilon_6^\pm &= (-1 \pm \sqrt{-23})/2 \end{aligned}$$

S_3	[3]	[2, 1]*	[1 ³]
(1) 6	1	2	1
(2) 2	1	0	-1
(3) 3	1	-1	1

A_3	[2, 1]*
(1) 3	1
(3) 3	ε_1^+
(3) 3	ε_1^-

S_4	[4]	[3, 1]	[2 ²]*	[2, 1 ²]	[1 ⁴]
(1) 24	1	3	2	3	1
(2) 4	1	1	0	-1	-1
(3) 3	1	0	-1	0	1
(4) 4	1	-1	0	1	-1
(2) ² 8	1	-1	2	-1	1

A_4	[2 ²]*
(1) 12	1
(3) 3	ε_1^+
(3) 3	ε_1^-
(2) ² 4	1

S_5	[5]	[4, 1]	[3, 2]	[3, 1 ²]*	[2 ² , 1]	[2, 1 ³]	[1 ⁵]
(1) 120	1	4	5	6	5	4	1
(2) 12	1	2	1	0	-1	-2	-1
(3) 6	1	1	-1	0	-1	1	1
(4) 4	1	0	-1	0	1	0	-1
(2) ² 8	1	0	1	-2	1	0	1
(3)(2) 6	1	-1	1	0	-1	1	-1
(5) 5	1	-1	0	1	0	-1	1

A_5	[3, 1 ²]*
(1) 60	3
(3) 3	0
(2) ² 4	-1
(5) 5	ε_2^+
(5) 5	ε_2^-

S_6	[6]	[5, 1]	[4, 2]	[4, 1 ²]	[3 ²]	[3, 2, 1]*	[2 ³]	[3, 1 ³]	[2 ² , 1 ²]	[2, 1 ⁴]	[1 ⁶]
(1) 720	1	5	9	10	5	16	5	10	9	5	1
(2) 48	1	3	3	2	1	0	-1	-2	-3	-3	-1
(3) 18	1	2	0	1	-1	-2	-1	1	0	2	1
(4) 8	1	1	-1	0	-1	0	1	0	1	-1	-1
(2) ² 16	1	1	1	-2	1	0	1	-2	1	1	1
(3)(2) 6	1	0	0	-1	1	0	-1	1	0	0	-1
(5) 5	1	0	-1	0	0	1	0	0	-1	0	1
(6) 6	1	-1	0	1	0	0	0	-1	0	1	-1
(4)(2) 8	1	-1	1	0	-1	0	-1	0	1	-1	1
(2) ³ 48	1	-1	3	-2	-3	0	3	2	-3	1	-1
(3) ² 18	1	-1	0	1	2	-2	2	1	0	-1	1

A_6		$[3, 2, 1]^*$
(1)	360	8
(3)	9	-1
(2) ²	8	0
(5)	5	ϵ_2^+
(5)	5	ϵ_2^-
(4)(2)	4	0
(3) ²	9	-1

S_7	[7]	[6, 1]	[5, 2]	[5, 1 ²]	[4, 3]	[4, 2, 1]	[3 ² , 1]	[4, 1 ³]*	[3, 2 ²]	[3, 2, 1 ²]	[2 ³ , 1]	[3, 1 ⁴]	[2 ² , 1 ³]	[2, 1 ⁵]	[1 ⁷]	
(1)	5040	1	6	14	15	14	35	21	20	21	35	14	15	14	6	1
(2)	240	1	4	6	5	4	5	1	0	-1	-5	-4	-5	-6	-4	-1
(3)	72	1	3	2	3	-1	-1	-3	2	-3	-1	-1	3	2	3	1
(4)	24	1	2	0	1	-2	-1	-1	0	1	1	2	-1	0	-2	-1
(2) ²	48	1	2	2	-1	2	-1	1	-4	1	-1	2	-1	2	2	1
(3)(2)	12	1	1	0	-1	1	-1	1	0	-1	1	-1	1	0	-1	-1
(5)	10	1	1	-1	0	-1	0	1	0	1	0	-1	0	-1	1	1
(6)	6	1	0	-1	0	0	1	0	0	0	-1	0	0	1	0	-1
(4)(2)	8	1	0	0	-1	0	1	-1	0	-1	1	0	-1	0	0	1
(2) ³	48	1	0	2	-3	0	1	-3	0	3	-1	0	3	-2	0	-1
(3) ²	18	1	0	-1	0	2	-1	0	2	0	-1	2	0	-1	0	1
(5)(2)	10	1	-1	1	0	-1	0	1	0	-1	0	1	0	-1	1	-1
(3)(2) ²	24	1	-1	2	-1	-1	-1	1	2	1	-1	-1	-1	2	-1	1
(4)(3)	12	1	-1	0	1	1	-1	-1	0	1	1	-1	-1	0	1	-1
(7)	7	1	-1	0	1	0	0	0	-1	0	0	0	1	0	-1	1

A_7		$[4, 13]*$
(1)	2520	10
(3)	36	1
(2) ²	24	-2
(5)	5	0
(4)(2)	4	0
(3) ²	9	1
(3)(2) ²	12	1
(7)	7	ϵ_3^+
(7)	7	ϵ_3^-

M_{11}	(1)	g	1	10	11	55	45	44	$\overline{16}$	$\overline{10}$
	(2) ⁴	48	1	2	3	-1	-3	4	0	-2
	(4) ²	8	1	2	-1	-1	1	0	0	0
	(3) ³	18	1	1	2	1	0	-1	-2	1
	(5) ²	5	1	0	1	0	0	-1	1	0
	(8)(2)	8	1	0	-1	1	-1	0	0	$\pm i\sqrt{2}$
	(8)(2)	8	1	0	-1	1	-1	0	0	$\mp i\sqrt{2}$
	(6)(3)(2)	6	1	-1	0	-1	0	1	0	1
	(11)	11	1	-1	0	0	1	0	ϵ_4^+	-1
	(11)	11	1	-1	0	0	1	0	ϵ_4^-	-1

$g = 11 \cdot 10 \cdot 9 \cdot 8 = 7920.$

M_{12}	(1)	g	1	11	11	55	55	55	45	54	66	99	120	144	176	$\overline{16}$
	(2) ⁴	192	1	3	3	-1	-1	7	-3	6	2	3	-8	0	0	0
	(4) ²	32	1	3	-1	3	-1	-1	1	2	-2	-1	0	0	0	0
	(3) ³	54	1	2	2	1	1	1	0	0	3	0	3	0	-4	-2
	(5) ²	10	1	1	1	0	0	0	0	-1	1	-1	0	-1	1	1
	(8)(2)	8	1	1	-1	-1	1	-1	-1	0	0	1	0	0	0	0
	(6)(3)(2)	6	1	0	0	-1	-1	1	0	0	-1	0	1	0	0	0
	(11)	11	1	0	0	0	0	0	1	-1	0	0	-1	1	0	ϵ_4^+
	(11)	11	1	0	0	0	0	0	1	-1	0	0	-1	1	0	ϵ_4^-
	(2) ⁶	240	1	-1	-1	-5	-5	-5	5	6	6	-1	0	4	-4	4
	(10)(2)	10	1	-1	-1	0	0	0	0	1	1	-1	0	-1	1	-1
	(4) ² (2) ²	32	1	-1	3	-1	3	-1	1	2	-2	-1	0	0	0	0
	(3) ⁴	36	1	-1	-1	1	1	1	3	0	0	3	0	-3	-1	1
	(6) ²	12	1	-1	-1	1	1	1	-1	0	0	-1	0	1	-1	1
	(8)(4)	8	1	-1	1	1	-1	-1	-1	0	0	1	0	0	0	0

$g = 12 \cdot 11 \cdot 10 \cdot 9 \cdot 8 = 95040.$

	g	23·21	23·55	23·88	23·99	23·144	23·11·21	23·7·36	77·72	11·35·27
(1) ²⁴	g									
(2) ⁸	21·2 ¹⁰	35	49	8	21	48	49	-28	-56	-21
(3) ⁶	27·40	6	5	-1	0	0	-15	-9	9	0
(5) ⁴	60	-2	0	-1	-3	-3	3	1	-1	0
(4) ⁴ (2) ²	128	3	1	0	1	0	-3	4	0	-1
(7) ³	42	0	-2	1	2	1	0	0	0	0
(7) ³	42	0	-2	1	2	1	0	0	0	0
(8) ² (4)(2)	16	-1	1	0	-1	0	-1	0	0	1
(6) ² (3) ² (2) ²	24	2	1	-1	0	0	1	-1	1	0
(11) ²	11	-1	0	0	0	1	0	-1	0	0
(15)(5)(3)	15	1	0	-1	0	0	0	1	-1	0
(15)(5)(3)	15	1	0	-1	0	0	0	1	-1	0
(14)(7)(2)	14	0	0	1	0	-1	0	0	0	0
(14)(7)(2)	14	0	0	1	0	-1	0	0	0	0
(23)	23	0	0	0	0	0	0	0	1	-1
(23)	23	0	0	0	0	0	0	0	1	-1
(12) ²	12	0	0	0	0	0	0	0	0	0
(6) ⁴	24	0	0	0	2	-2	0	0	0	0
(4) ⁶	96	3	-3	0	-3	0	-3	0	0	3
(3) ⁸	7·72	0	8	8	6	-6	0	0	0	0
(2) ¹²	15·2 ⁹	3	-15	24	-19	16	9	36	24	-45
(10) ² (2) ²	20	-2	0	-1	1	1	-1	1	-1	0
(21)(3)	21	0	1	1	-1	1	0	0	0	0
(21)(3)	21	0	1	1	-1	1	0	0	0	0
(4) ⁴ (2) ⁴	3·2 ⁷	3	-7	8	-3	0	1	-4	-8	3
(12)(6)(4)(2)	12	0	-1	-1	0	0	1	-1	1	0

$$g = 24 \cdot 23 \cdot 22 \cdot 21 \cdot 20 \cdot 48 = 244823040.$$

(II) General Linear Groups $GL(2, q)$, Unitary Groups $U(2, q)$, and Special Linear Groups $SL(2, q)$
(q is a power of a prime) (\rightarrow 151 Finite Groups I)

(1) The notations are as follows. $\varepsilon = \exp[2\pi\sqrt{-1}/(q-1)]$, $\eta = \exp[2\pi\sqrt{-1}/(q^2-1)]$, $\sigma = \exp[2\pi\sqrt{-1}/(q+1)]$, ρ is the generator of the multiplicative group of $GF(q) - \{0\}$, ω is the generator of the multiplicative group of $GF(q^2) - \{0\}$, $\omega^{q-1} = \alpha$, B is an element of $GL(2, q)$ with order q^2-1 , and $B_1 = B^{q-1}$.

(2) The first column gives a representative of the conjugate class.

General Linear Group $GL(2, q)$.

	$X_n(1)$	$X_n(q)$	$Y_{m,n}$	Z_n
$\begin{pmatrix} \rho^a & \\ & \rho^a \end{pmatrix}$	ε^{2na}	$q\varepsilon^{2na}$	$(q+1)\varepsilon^{(m+n)a}$	$(q-1)\eta^{na(q+1)}$
$\begin{pmatrix} \rho^a & \\ 1 & \rho^a \end{pmatrix}$	ε^{2na}	0	$\varepsilon^{(m+n)a}$	$-\eta^{na(q+1)}$
$\begin{pmatrix} \rho^a & \\ & \rho^b \end{pmatrix}$	$\varepsilon^{n(a+b)}$	$\varepsilon^{n(a+b)}$	$\varepsilon^{ma+nb} + \varepsilon^{mb+na}$	0
B^c	ε^{nc}	$-\varepsilon^{nc}$	0	$-(\eta^{nc} + \eta^{ncq})$

(1) $1 \leq a \leq q-1$, $1 \leq b \leq q-1$, $a \not\equiv b \pmod{q-1}$, $1 \leq c < q^2-1$, $c \not\equiv 0 \pmod{q+1}$.

(2) We assume that $1 \leq n \leq q-1$, for $X_n(1), X_n(q)$, $1 \leq m < n \leq q-1$, for $Y_{m,n}$, $1 \leq n < q^2-1$ for Z_n , $n \not\equiv 0 \pmod{q+1}$. Here, $Z_n = Z_{n'}$ when $n \equiv n'q \pmod{q^2-1}$.

Unitary Group $U(2, q)$.

	$X'_n(1)$	$X'_n(q)$	$Y'_{m,n}$	Z'_n
$\begin{pmatrix} \alpha^s & \\ & \alpha^s \end{pmatrix}$	σ^{2ns}	$q\sigma^{2ns}$	$(q-1)\sigma^{(m+n)s}$	$(q+1)\sigma^{ns}$
$\begin{pmatrix} \alpha^s & \\ 1 & \alpha^s \end{pmatrix}$	σ^{2ns}	0	$-\sigma^{(m+n)s}$	σ^{ns}
$\begin{pmatrix} \alpha^s & \\ & \alpha^t \end{pmatrix}$	$\sigma^{n(s+t)}$	$-\sigma^{n(s+t)}$	$-(\sigma^{ms+nt} + \sigma^{mt+ns})$	0
$\begin{pmatrix} \omega^u & \\ & \omega^{-uq} \end{pmatrix}$	σ^{-nu}	σ^{-nu}	0	$\eta^{nu} + \eta^{-nuq}$

- (1) $\begin{pmatrix} \alpha^s & \\ 1 & \alpha^s \end{pmatrix}, \begin{pmatrix} \omega^u & \\ & \omega^{-uq} \end{pmatrix}$ are the canonical forms of an element of $U(2, q)$ in $GL(2, q^2)$.
(2) $1 \leq s \leq q+1, 1 \leq t \leq q+1, s \not\equiv t \pmod{q+1}, 1 \leq u < q^2-1, u \not\equiv 0 \pmod{q-1}$. When $u \equiv -u'q \pmod{q^2-1}$ u, u' gives the same conjugate class.
(3) The ranges are $1 \leq n \leq q+1$ for $X'_n(1), X'_n(q), 1 \leq m < n \leq q+1$ for $Y'_{m,n}, 1 \leq n < q^2-1$ for $Z'_n, n \not\equiv 0 \pmod{q-1}$. When $n' \equiv -nq \pmod{q^2-1}$, we have $Z'_n = Z'_{n'}$.

Special Linear Group $SL(2, 2^n)$ (the case when $q = 2^n$).

	Y_n			Z_m
$\begin{pmatrix} 1 & \\ & 1 \end{pmatrix}$	1	q	$q+1$	$q-1$
$\begin{pmatrix} 1 & \\ 1 & 1 \end{pmatrix}$	1	0	1	-1
$\begin{pmatrix} \rho^a & \\ & \rho^{-a} \end{pmatrix}$	1	1	$\varepsilon^{na} + \varepsilon^{-na}$	0
B_1^c	1	-1	0	$-(\sigma^{mc} + \sigma^{-mc})$

- (1) $1 \leq a \leq (q-2)/2, 1 \leq c \leq q/2$.
(2) $1 \leq n \leq (q-2)/2, 1 \leq m \leq q/2$.

Special Linear Group $SL(2, q)$ ($q =$ power of an odd prime number, $e = (q-1)/2, e' = (q+1)/2$).

	Y_n			Z_m	
$\begin{pmatrix} 1 & \\ & 1 \end{pmatrix}$	1	q	$q+1$	$q-1$	$\frac{q+1}{2}, \frac{q-1}{2}$
$Z = \begin{pmatrix} -1 & \\ & -1 \end{pmatrix}$	1	q	$(-1)^n(q+1)$	$(-1)^m(q-1)$	$(-1)^e \frac{q+1}{2}, (-1)^{e'} \frac{q-1}{2}$
$P_1 = \begin{pmatrix} 1 & \\ 1 & 1 \end{pmatrix}$	1	0	1	-1	μ^\pm, λ^\pm
$P_2 = \begin{pmatrix} 1 & \\ \beta & 1 \end{pmatrix}$	1	0	1	-1	μ^\mp, λ^\mp
$P_1 Z$	1	0	$(-1)^n$	$-(-1)^m$	$(-1)^e \mu^\pm, (-1)^{e'} \lambda^\pm$
$P_2 Z$	1	0	$(-1)^n$	$-(-1)^m$	$(-1)^e \mu^\mp, (-1)^{e'} \lambda^\mp$
$\begin{pmatrix} \rho^a & \\ & \rho^{-a} \end{pmatrix}$	1	1	$\varepsilon^{na} + \varepsilon^{-na}$	0	$(-1)^a, 0$
B_1^c	1	-1	0	$-(\sigma^{mc} + \sigma^{-mc})$	0, $-(-1)^c$

- (1) $1 \leq a \leq (q-3)/2, 1 \leq c \leq (q-1)/2, 1 \leq n \leq (q-3)/2, 1 \leq m \leq (q-1)/2$,
 $\lambda^\pm = \{-1 \pm [(-1)^e q]^{1/2}\}/2, \mu^\pm = \{1 \pm [(-1)^{e'} q]^{1/2}\}/2$.
(2) The last two columns mean two characters (with the same signs), respectively.

(III) Ree group $Re(q)$, Suzuki Group $Sz(q)$, and Janko Group J .

Ree group $Re(q)$ ($q = 3^{2n+1} = 3m^2$).

The order of $Re(q)$ is $q^3(q^3+1)(q-1)$, $q_0 = q^2 - q + 1, m_+ = q + 3m + 1, m_- = q - 3m + 1$.

		A				B	C	X_μ	
1	1	1	q_0	q^3	qq_0	$(q-1)mm_+/2$	$(q-1)mm_-/2$	$m(q^2-1)$	q^3+1
J	2	1	-1	q	$-q$	$-(q-1)/2$	$(q-1)/2$	0	$q+1$
X	3	1	$-(q-1)$	0	q	$-(q+m)/2$	$(q-m)/2$	$-m$	1
Y	9	1	1	0	0	m	m	$-m$	1
T	3	1	1	0	0	α	α	2α	1
T^{-1}	3	1	1	0	0	$\bar{\alpha}$	$\bar{\alpha}$	$2\bar{\alpha}$	1
YT	9	1	1	0	0	β	β	$-\beta$	1
YT^{-1}	9	1	1	0	0	$\bar{\beta}$	$\bar{\beta}$	$-\bar{\beta}$	1
JT	6	1	-1	0	0	γ	$-\gamma$	0	1
JT^{-1}	6	1	-1	0	0	$\bar{\gamma}$	$-\bar{\gamma}$	0	1
R^a	1	1	1	1	1	0	0	0	$\rho^{\mu a} + \rho^{-\mu a}$
S^b	1	3	-1	-3	1	-1	-1	0	0
JR^a	1	-1	1	-1	0	0	0	0	$\rho^{\mu a} + \rho^{-\mu a}$
JS^b	1	-1	-1	1	1	-1	-1	0	0
V^s	1	0	-1	0	-1	0	0	-1	0
W^t	1	0	-1	0	0	1	1	1	0

		X'_μ	Y_ν	Y'_λ	Z_κ	Z'_τ
1	1	q^3+1	$(q-1)q_0$	$(q-1)q_0$	$(q^2-1)m_+$	$(q^2-1)m_-$
J	2	$-(q+1)$	$3(q-1)$	$-(q-1)$	0	0
X	3	1	$2q-1$	$2q-1$	$-m_+$	$-m_-$
Y	9	1	-1	-1	-1	-1
T	3	1	-1	-1	$-3m-1$	$3m-1$
T^{-1}	3	1	-1	-1	$-3m-1$	$3m-1$
YT	9	1	-1	-1	-1	-1
YT^{-1}	9	1	-1	-1	-1	-1
JT	6	-1	-3	1	0	0
JT^{-1}	6	-1	-3	1	0	0
R^a		$\rho^{\mu a} + \rho^{-\mu a}$	0	0	0	0
S^b		0	$\sigma(\nu b)$	$\sigma'(\lambda b)$	0	0
JR^a		$-(\rho^{\mu a} + \rho^{-\mu a})$	0	0	0	0
JS^b		0	$\sigma(\nu b)$	$\sigma'(\lambda b)$	0	0
V^s		0	0	0	$-\sum_{i=0}^2 (v^{\kappa s q^i} + v^{-\kappa s q^i})$	0
W^t		0	0	0	0	$-\sum_{i=0}^2 (w^{\tau t q^i} + w^{-\tau t q^i})$

(1) The first column gives a representative of conjugate class, and the second column gives its order. The orders of R, S, V, W are $(q-1)/2, (q+1)/4, m_-, m_+$, respectively. R, S, T are commutative with J .

(2) $R^a \sim R^{-a}, V^s \sim V^{sq} \sim V^{sq^2} \sim V^{-s} \sim V^{-sq} \sim V^{-sq^2}, W^t \sim W^{tq} \sim W^{tq^2} \sim W^{-t} \sim W^{-tq} \sim W^{-tq^2}$. Here we fix an integer δ satisfying $\delta^3 \equiv 1 \pmod{(q+1)/4}, (\delta-1, (q+1)/4) = 1$.

$$S^b \sim S^{b\delta} \sim S^{b\delta^2} \sim S^{-b} \sim S^{-b\delta} \sim S^{-b\delta^2}, \quad JR^a \sim JR^{-a}, \quad JS^b \sim JS^{-b},$$

where $A \sim B$ means that A and B are mutually conjugate.

$$(3) \rho = \exp[4\pi\sqrt{-1}/(q-1)], \quad v = \exp(2\pi\sqrt{-1}/m_-), \quad w = \exp(2\pi\sqrt{-1}/m_+), \\ \sigma = \exp[8\pi\sqrt{-1}/(q+1)].$$

$$(4) 1 \leq \mu \leq (q-3)/4, 1 \leq \lambda \leq (q-3)/8.$$

Here ν is considered mod $(q+1)/4$ and

$$Y_\nu = Y_{\nu\delta} = Y_{\nu\delta^2} = Y_{-\nu} = Y_{-\nu\delta} = Y_{-\nu\delta^2},$$

κ is considered mod m_- and

$$Z_\kappa = Z_{\kappa q} = Z_{\kappa q^2} = Z_{-\kappa} = Z_{-\kappa q} = Z_{-\kappa q^2},$$

τ is considered mod m_+ and

$$Z'_\tau = Z'_{\tau q} = Z'_{\tau q^2} = Z'_{-\tau} = Z'_{-\tau q} = Z'_{-\tau q^2}.$$

$$(5) \sigma(\nu b) = -\sum_{i=0}^2 (\sigma^{\nu b \delta^i} + \sigma^{-\nu b \delta^i}), \quad \sigma'(\lambda b) = \sum_{i=0}^1 (\sigma^{\lambda b \delta^i} + \sigma^{-\lambda b \delta^i}) - (\sigma^{\lambda b \delta^2} + \sigma^{-\lambda b \delta^2}).$$

$$(6) \alpha = \frac{-m + m\sqrt{-q}}{2}, \beta = \frac{-m - \sqrt{-q}}{2}, \gamma = \frac{1 - \sqrt{-q}}{2}. \quad \text{We show one of the two mutually complex conjugate characters, for the characters } A, B, C.$$

Suzuki group $Sz(q)$. The order of $Sz(q)$ is $q^2(q^2+1)(q-1)$ ($q=2^{2n+1}, 2q=r^2$).

		X_α	Y_β	Z_γ		
1	1	q^2	q^2+1	$(q-r+1)(q-1)$	$(q+r+1)(q-1)$	$r(q-1)/2$
σ	1	0	1	$r-1$	$-r-1$	$-r/2$
ρ	1	0	1	-1	-1	$r\sqrt{-1}/2$
ρ^{-1}	1	0	1	-1	-1	$-r\sqrt{-1}/2$
π_0^i	1	1	$\varepsilon_0^{ai} + \varepsilon_0^{-ai}$	0	0	0
π_1^j	1	-1	0	$-(\varepsilon_1^{bj} + \varepsilon_1^{\beta j q} + \varepsilon_1^{-\beta j} + \varepsilon_1^{-\beta j q})$	0	1
π_2^k	1	-1	0	0	$-(\varepsilon_2^{\gamma k} + \varepsilon_2^{\gamma k q} + \varepsilon_2^{-\gamma k} + \varepsilon_2^{-\gamma k q})$	-1

(1) The first column gives a representative of the conjugate class.

(2) π_0, π_1, π_2 are the elements of order $q-1, q+r+1, q-r+1$, respectively.

(3) $\varepsilon_0, \varepsilon_1, \varepsilon_2$ are the primitive $q-1, q+r+1, q-r+1$ roots of 1, respectively.

(4) π_0^i and π_0^{-i} are mutually conjugate elements, and hence X_α and $X_{-\alpha}$ give the same character. i, α run over the representatives of mod $q-1$, and $i, \alpha \not\equiv 0 \pmod{q-1}$.

- (5) $\pi_1^j, \pi_1^{-j}, \pi_1^{jq}, \pi_1^{-jq}$ are mutually conjugate, and hence $Y_\beta, Y_{-\beta}, Y_{\beta q}, Y_{-\beta q}$ give the same character. j, β run over the representatives of $\text{mod } q+r+1$, and $j, \beta \not\equiv 0 \pmod{q+r+1}$.
(6) $\pi_2^k, \pi_2^{-k}, \pi_2^{kq}, \pi_2^{-kq}$ are mutually conjugate, and hence $Z_\gamma, Z_{-\gamma}, Z_{\gamma q}, Z_{-\gamma q}$ give the same character. k, γ run over the representatives of $\text{mod } q-r+1$, and $k, \gamma \not\equiv 0 \pmod{q-r+1}$.

Janko Group J .

1	1	77	133	209	133	77	77	133	76	76	56	56	120	120	120
2	1	5	5	1	-3	-3	-3	-3	4	-4	0	0	0	0	0
3	1	-1	1	-1	-2	2	2	-2	1	1	2	2	0	0	0
5	1	2	-2	-1	ϵ^+	$-\epsilon^+$	$-\epsilon^-$	ϵ^-	1	1	$2\epsilon^-$	$2\epsilon^+$	0	0	0
5	1	2	-2	-1	ϵ^-	$-\epsilon^-$	$-\epsilon^+$	ϵ^+	1	1	$2\epsilon^+$	$2\epsilon^-$	0	0	0
6	1	-1	-1	1	0	0	0	0	1	-1	0	0	0	0	0
7	1	0	0	-1	0	0	0	0	-1	-1	0	0	1	1	1
10	1	0	0	1	$-\epsilon^+$	$-\epsilon^+$	$-\epsilon^-$	$-\epsilon^-$	-1	1	0	0	0	0	0
10	1	0	0	1	$-\epsilon^-$	$-\epsilon^-$	$-\epsilon^+$	$-\epsilon^+$	-1	1	0	0	0	0	0
11	1	0	1	0	1	0	0	1	-1	-1	1	1	-1	-1	-1
15	1	-1	1	-1	ϵ^+	$-\epsilon^+$	$-\epsilon^-$	ϵ^-	1	1	$-\epsilon^-$	$-\epsilon^+$	0	0	0
15	1	-1	1	-1	ϵ^-	$-\epsilon^-$	$-\epsilon^+$	ϵ^+	1	1	$-\epsilon^+$	$-\epsilon^-$	0	0	0
19	1	1	0	0	0	1	1	0	0	0	-1	-1	λ_1	λ_2	λ_3
19	1	1	0	0	0	1	1	0	0	0	-1	-1	λ_2	λ_3	λ_1
19	1	1	0	0	0	1	1	0	0	0	-1	-1	λ_3	λ_1	λ_2

- (1) The order of J is $8 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 19 = 175560$.
(2) The first column gives the order of the elements of each conjugate class.
(3) $\rho = \exp(2\pi\sqrt{-1}/19)$, $\lambda_1 = \rho + \rho^7 + \rho^8 + \rho^{11} + \rho^{12} + \rho^{18}$, $\lambda_2 = \rho^2 + \rho^{14} + \rho^{16} + \rho^3 + \rho^5 + \rho^{17}$, $\lambda_3 = \rho^4 + \rho^9 + \rho^{13} + \rho^6 + \rho^{10} + \rho^{15}$, $\epsilon^\pm = (1 \pm \sqrt{5})/2$.

References

For S_n ($2 \leq n \leq 10$), A_n ($3 \leq n \leq 9$):
[1] D. E. Littlewood, The theory of group characters, second edition, Oxford Univ. Press, 1950.
For S_n ($n = 11, 12, 13$):
[2] M. Zia-ud-Din, Proc. London Math. Soc., 39(1935), 200–204, 42 (1937), 340–355.
For S_{14} :
[3] K. Kondô, Table of characters of the symmetric group of degree 14, Proc. Phys. Math. Soc. Japan, 22 (1940), 585–593.
For M_n ($n = 12, 24$):
[4] G. Frobenius, Über die Charaktere der mehrfach transitive Gruppen, S. B. Preuss. Akad. Wiss., 1904, 558–571.
For M_n ($n = 11, 22, 23$):
[5] N. Burgoyne and P. Fong, The Schur multipliers of the Mathieu groups, Nagoya Math. J., 27 (1966), 733–745.
For $GL(2, q)$, $SL(2, q)$:
[6] I. Schur, Untersuchungen über die Darstellung der endlichen Gruppen durch gebrochene lineare Substitutionen, J. Reine Angew. Math., 132 (1907), 85–137.
[7] H. E. Jordan, Group characters of various types of linear groups, Amer. J. Math., 29 (1907), 387–405.
For $GL(3, q)$, $GL(4, q)$, $GL(n, q)$:
[8] R. Steinberg, The representations of $GL(3, q)$, $GL(4, q)$, $PGL(3, q)$ and $PGL(4, q)$, Canad. J. Math. 3 (1951), 225–235.
[9] J. A. Green, The characters of the finite general linear groups, Trans. Amer. Math. Soc., 80 (1955), 402–447.
For $U(2, q)$, $U(3, q)$:
[10] V. Ennola, On the characters of the finite unitary groups, Ann. Acad. Sci. Fenn., 323 (1963), 1–34.
For $Sz(q)$, $Re(q)$, J :
[11] M. Suzuki, On a class of doubly transitive groups, Ann. of Math., (2) 75 (1962), 105–145.
[12] H. N. Ward, On Ree’s series of simple groups, Trans. Amer. Math. Soc., 121 (1966), 62–89.
[13] Z. Janko, A new finite simple group with Abelian Sylow 2-subgroups and its characterization, J. Algebra, 3 (1966), 147–186.

For other Lie-type groups:
[14] B. Srinivasan, The characters of the finite symplectic group $Sp(4, q)$. Trans. Amer. Math. Soc., 131 (1968), 488–525.
[15] H. Enomoto, The characters of the finite symplectic group $Sp(4, q)$, $q = 2^f$, Osaka J. Math., 9 (1972), 75–94.
[16] G. I. Lehrer, The characters of the finite special linear groups, J. Algebra, 26 (1973), 564–583.
[17] P. Deligne and G. Lusztig, Representations of reductive groups over finite fields, Ann. of Math., (2) 103 (1976), 103–161.
For sporadic groups:
[18] M. Hall, Jr. and D. Wales, The simple groups of order 604, 800, J. Algebra, 9 (1968), 417–450.
[19] J. S. Frame, Computation of characters of the Higman-Sims group and its automorphism group, J. Algebra, 20 (1972), 320–349.
[20] D. Fendel, A characterization of Conway’s group 3, J. Algebra, 24 (1973), 159–196.
[21] D. Wright, The irreducible characters of the simple group of M. Suzuki of order 448, 345, 497, 600, J. Algebra, 29 (1974), 303–323.

(IV) Three-Dimensional Crystal Classes (→ 92 Crystallographic Groups)

Crystal System Bravais Types	Geometric Crystal Classes			Arithmetic Crystal Classes	Number of Space Groups ⁽²⁾
	Schoenflies Notation	International Notation ⁽¹⁾			
		Short	Full		
Triclinic <i>P</i>	C_1 $S_2(C_i)$	1 $\bar{1}$	1 $\bar{1}$	(<i>P</i> , 1) (<i>P</i> , $\bar{1}$)	1 2
Monoclinic <i>P</i> , <i>C</i>	C_2 C_{1h} C_{2h}	2 <i>m</i> 2/ <i>m</i>	2 <i>m</i> $\frac{2}{m}$	(<i>P</i> , 2) (<i>C</i> , 2) (<i>P</i> , <i>m</i>) (<i>C</i> , <i>m</i>) (<i>P</i> , 2/ <i>m</i>) (<i>C</i> , 2/ <i>m</i>)	3–5 6–9 10–15
Orthorhombic <i>P</i> , <i>C</i> , <i>F</i> , <i>I</i>	$D_2(V)$ C_{2v} $D_{2h}(V_h)$	222 <i>mm</i> 2 <i>mmm</i>	222 <i>mm</i> 2 $\frac{2}{m} \frac{2}{m} \frac{2}{m}$	(<i>P</i> , 222) (<i>C</i> , 222) (<i>F</i> , 222) (<i>I</i> , 222) (<i>P</i> , <i>mm</i> 2) (<i>C</i> , <i>mm</i> 2) (<i>A</i> , <i>mm</i> 2) (<i>F</i> , <i>mm</i> 2) (<i>I</i> , <i>mm</i> 2) (<i>P</i> , <i>mmm</i>) (<i>C</i> , <i>mmm</i>) (<i>F</i> , <i>mmm</i>) (<i>I</i> , <i>mmm</i>)	16–24 25–46 47–74
Tetragonal <i>P</i> , <i>I</i>	C_4 S_4 C_{4h} D_4 C_{4v} $D_{2d}(V_d)$ D_{4h}	4 $\bar{4}$ 4/ <i>m</i> 422 4 <i>mm</i> $\bar{4}2m$ 4/ <i>mmm</i>	4 $\bar{4}$ $\frac{4}{m}$ 422 4 <i>mm</i> $\bar{4}2m$ $\frac{4}{m} \frac{2}{m} \frac{2}{m}$	(<i>P</i> , 4) ⁽³⁾ (<i>I</i> , 4) (<i>P</i> , $\bar{4}$) (<i>I</i> , $\bar{4}$) (<i>P</i> , 4/ <i>m</i>) (<i>I</i> , 4/ <i>m</i>) (<i>P</i> , 422) ⁽⁴⁾ (<i>I</i> , 422) (<i>P</i> , 4 <i>mm</i>) (<i>I</i> , 4 <i>mm</i>) (<i>P</i> , $\bar{4}2m$) (<i>P</i> , $\bar{4}m2$) (<i>I</i> , $\bar{4}m2$) (<i>I</i> , $\bar{4}2m$) (<i>P</i> , 4/ <i>mmm</i>) (<i>I</i> , 4/ <i>mmm</i>)	75–80 81–82 83–88 89–98 99–110 111–122 123–142
Trigonal <i>P</i> , <i>R</i>	C_3 $S_6(C_{3i})$ D_3 C_{3v} D_{3d}	3 $\bar{3}$ 32 3 <i>m</i> $\bar{3}m$	3 $\bar{3}$ 32 3 <i>m</i> $\bar{3} \frac{2}{m}$	(<i>P</i> , 3) ⁽⁵⁾ (<i>R</i> , 3) (<i>P</i> , $\bar{3}$) (<i>R</i> , $\bar{3}$) (<i>P</i> , 312) ⁽⁶⁾ (<i>P</i> , 321) ⁽⁷⁾ (<i>R</i> , 32) (<i>P</i> , 3 <i>m</i> 1) (<i>P</i> , 31 <i>m</i>) (<i>R</i> , 3 <i>m</i>) (<i>P</i> , $\bar{3}1m$) (<i>P</i> , $\bar{3}m1$) (<i>R</i> , $\bar{3}m$)	143–146 147–148 149–155 156–161 162–167
Hexagonal <i>P</i>	C_6 C_{3h} C_{6h} D_6	6 $\bar{6}$ 6/ <i>m</i> 622	6 $\bar{6}$ $\frac{6}{m}$ 622	(<i>P</i> , 6) ⁽⁸⁾ (<i>P</i> , $\bar{6}$) (<i>P</i> , 6/ <i>m</i>) (<i>P</i> , 622) ⁽⁹⁾	168–173 –174 175–176 177–182

Crystal System Bravais Types	Geometric Crystal Classes			Arithmetic Crystal Classes	Number of Space Groups ⁽²⁾
	Schoenflies Notation	International Notation ⁽¹⁾			
		Short	Full		
Hexagonal <i>P</i> (cont.)	C_{6v}	$6mm$	$6mm$	$(P, 6mm)$	183–186
	D_{3h}	$\bar{6}m2$	$\bar{6}m2$	$(P, \bar{6}m2) (P, \bar{6}2m)$	187–190
	D_{6h}	$6/mmm$	$\frac{6}{m} \frac{2}{m} \frac{2}{m}$	$(P, 6/mmm)$	191–194
Cubic	T	23	23	$(P, 23) (F, 23) (I, 23)$	195–199
<i>P, F, I</i>	T_h	$m\bar{3}$	$\frac{2}{m} \bar{3}$	$(P, m\bar{3}) (F, m\bar{3}) (I, m\bar{3})$	200–206
	O	432	432	$(P, 432)^{(10)} (F, 432) (I, 432)$	207–214
	T_d	$\bar{4}3m$	$\bar{4}3m$	$(P, \bar{4}3m) (F, \bar{4}3m) (I, \bar{4}3m)$	215–220
	O_h	$m\bar{3}m$	$\frac{4}{m} \bar{3} \frac{2}{m}$	$(P, m\bar{3}m) (F, m\bar{3}m) (I, m\bar{3}m)$	221–230

Notes

- (1) The notation is based upon *International tables for X-ray crystallography* I, Kynoch, 1969. In each crystal system, the lowest class is a holohedry.
- (2) These correspond to the consecutive numbers of space groups in the book cited in (1).
- (3)–(10) Enantiomorphic pairs arise from these classes: two pairs for (4), (8), (9), and one pair for the others.
- For the shapes of Bravais lattices → 92 Crystallographic Groups E, Fig. 3.

6. Miscellaneous Constants

$\sqrt{2} = 1.41421\ 35623\ 73095,$ $\sqrt{10} = 3.16227\ 76601\ 68379.$
 $\sqrt[3]{2} = 1.25992\ 10498\ 94873,$ $\sqrt[3]{100} = 4.64158\ 88336\ 12779.$
 $\log_{10} 2 = 0.30102\ 99956\ 63981 = 1/3.32192\ 80948\ 87364.$

(I) Base of Natural Logarithm *e* (1000 decimals)

e = 2.71828 18284 59045 23536 02874 71352 66249 77572 47093 69995 95749 66967 62772 40766 30353 54759
45713 82178 52516 64274 27466 39193 20030 59921 81741 35966 29043 57290 03342 95260 59563 07381
32328 62794 34907 63233 82988 07531 95251 01901 15738 34187 93070 21540 89149 93488 41675 09244
76146 06680 82264 80016 84774 11853 74234 54424 37107 53907 77449 92069 55170 27618 38606 26133
13845 83000 75204 49338 26560 29760 67371 13200 70932 87091 27443 74704 72306 96977 20931 01416
92836 81902 55151 08657 46377 21112 52389 78442 50569 53696 77078 54499 69967 94686 44549 05987
93163 68892 30098 79312 77361 78215 42499 92295 76351 48220 82698 95193 66803 31825 28869 39849
64651 05820 93923 98294 88793 32036 25094 43117 30123 81970 68416 14039 70198 37679 32068 32823
76464 80429 53118 02328 78250 98194 55815 30175 67173 61332 06981 12509 96181 88159 30416 90351
59888 85193 45807 27386 67385 89422 87922 84998 92086 80582 57492 79610 48419 84443 63463 24496
84875 60233 62482 70419 78623 20900 21609 90235 30436 99418 49164 31409 34317 38143 64054 62531
52096 18369 08887 07016 76839 64243 78140 59271 45635 49061 30310 72085 10383 75051 01157 47704
17189 86106 87396 96552 12671 54688 95703 50354.

e (in octal) = 2.55760 52130 50535 5.
 $1/e = 0.36787\ 94411\ 71442,$ $e^2 = 7.38905\ 60989\ 30650 = 1/0.13533\ 52832\ 36613,$
 $\sqrt{e} = 1.64872\ 12707\ 00128 = 1/0.60653\ 06597\ 12633.$
 $\log_e 10 = 2.30258\ 50929\ 94046 = 1/0.43429\ 44819\ 03252,$
 $\log_e 2 = 0.69314\ 71805\ 59945 = 1/1.44269\ 50408\ 88964.$

(II) The Number π (1000 decimals) (→ 328 Pi(π))

$\pi=3.14159\ 26535\ 89793\ 23846\ 26433\ 83279\ 50288\ 41971\ 69399\ 37510\ 58209\ 74944\ 59230\ 78164\ 06286\ 20899$
86280 34825 34211 70679 82148 08651 32823 06647 09384 46095 50582 23172 53594 08128 48111 74502
84102 70193 85211 05559 64462 29489 54930 38196 44288 10975 66593 34461 28475 64823 37867 83165
27120 19091 45648 56692 34603 48610 45432 66482 13393 60726 02491 41273 72458 70066 06315 58817
48815 20920 96282 92540 91715 36436 78925 90360 01133 05305 48820 46652 13841 46951 94151 16094
33057 27036 57595 91953 09218 61173 81932 61179 31051 18548 07446 23799 62749 56735 18857 52724
89122 79381 83011 94912 98336 73362 44065 66430 86021 39494 63952 24737 19070 21798 60943 70277
05392 17176 29317 67523 84674 81846 76694 05132 00056 81271 45263 56082 77857 71342 75778 96091
73637 17872 14684 40901 22495 34301 46549 58537 10507 92279 68925 89235 42019 95611 21290 21960
86403 44181 59813 62977 47713 09960 51870 72113 49999 99837 29780 49951 05973 17328 16096 31859
50244 59455 34690 83026 42522 30825 33446 85035 26193 11881 71010 00313 78387 52886 58753 32083
81420 61717 76691 47303 59825 34904 28755 46873 11595 62863 88235 37875 93751 95778 18577 80532
17122 68066 13001 92787 66111 95909 21642 01989.

π (in octal) = 3.11037 55242 10264 3.
 $1/\pi=0.31830\ 98861\ 83791,$ $\pi^2=9.86960\ 44010\ 89359=1/0.10132\ 11836\ 42338,$
 $\sqrt{\pi}=1.77245\ 38509\ 05516=1/0.56418\ 95835\ 47756,$
 $\sqrt{2\pi}=2.50662\ 82746\ 31001=1/0.39894\ 22804\ 01433,$
 $\sqrt{\pi/2}=1.25331\ 41373\ 15500=1/0.79788\ 45608\ 02865,$
 $\sqrt[3]{\pi}=1.46459\ 18875\ 61523=1/0.68278\ 40632\ 55296.$
 $\log_{10}\pi=0.49714\ 98726\ 94134,$ $\log_e\pi=1.14472\ 98858\ 49400.$

(III) Radian rad

1 rad = 57°.29577 95130 82321 = 3437'.74677 07849 393 = 20626 4".80624 70964.
1° = 0.01745 32925 19943 rad, 1' = 0.00029 08882 08666 rad, 1" = 0.00000 48481 36811 rad.

(IV) Euler's Constant C (100 decimals) (→ 174 Gamma Function)

$C=0.57721\ 56649\ 01532\ 86060\ 65120\ 90082\ 40243\ 10421\ 59335\ 93992$
35988 05767 23488 48677 26777 66467 09369 47063 29174 67495.

$e^C=1.78107\ 24179\ 90197\ 98522.$

$$S_n=\sum_{i=1}^n\frac{1}{i}.$$

n	S_n	n	S_n	n	S_n	n	S_n
3	1.83333 333	6	2.45000 000	15	3.31822 899	100	5.18737 752
4	2.08333 333	8	2.71785 714	20	3.59773 966	500	6.79282 343
5	2.28333 333	10	2.92896 825	50	4.79920 534	1000	7.48547 086

7. Coefficients of Polynomial Approximations

In this table, we give some typical examples of approximation formulas for computation of functions on a digital computer (→ 19 Analog Computation, 336 Polynomial Approximation).

(I) Exponential Function

- (1) Putting $\frac{x}{\log 2}+1=q+y+\frac{1}{2}\left(q\text{ is an integer, }-\frac{1}{2} \leq y<\frac{1}{2}\right)$, we have
 $e^x=2^{qv(y)}, v(y) \doteq \sum a_i y^i$, which gives an approximation by a polynomial of the 7th degree, where the maximal error is 3×10^{-11} .
 $a_0=0.70710\ 67811\ 6,$ $a_1=0.49012\ 90717\ 2,$ $a_2=0.16986\ 57957\ 2,$ $a_3=0.03924\ 73321\ 5,$
 $a_4=0.00680\ 09712,$ $a_5=0.00094\ 28173,$ $a_6=0.00010\ 93869,$ $a_7=0.00001\ 0826.$
- (2) An approximation by a polynomial of the 11th degree: $e^x \doteq \sum a_i x^i$ ($-1 \leq x \leq 0$).
Maximal error 1×10^{-12} .
 $a_0=0.99999\ 99999\ 990,$ $a_1=0.99999\ 99999\ 995,$ $a_2=0.50000\ 00000\ 747,$
 $a_3=0.16666\ 66666\ 812,$ $a_4=0.04166\ 66657\ 960,$ $a_5=0.00833\ 33332\ 174,$
 $a_6=0.00138\ 88925\ 998,$ $a_7=0.00019\ 84130\ 955,$ $a_8=0.00002\ 47944\ 428,$
 $a_9=0.00000\ 27550\ 711,$ $a_{10}=0.00000\ 02819\ 019,$ $a_{11}=0.00000\ 00255\ 791.$

$$(3) \quad e^x \approx 1 + \frac{x}{-\frac{x}{2} + \frac{k_0 + k_1 x^2 + k_2 x^4}{1 + k_3 x^2}} \quad (-\log \sqrt{2} \leq x \leq \log \sqrt{2}).$$

Maximal error 1.4×10^{-14} .

$$k_0 = 1.00000 \ 00000 \ 00327 \ 1, \quad k_1 = 0.10713 \ 50664 \ 56464 \ 2,$$

$$k_2 = 0.00059 \ 45898 \ 69018 \ 8, \quad k_3 = 0.02380 \ 17331 \ 57418 \ 6.$$

(II) Logarithmic Function

(1) An approximation by a polynomial of the 11th degree: $\log(1+x) \approx \sum a_i x^i$ ($0 \leq x \leq 1$).

Maximal error 1.1×10^{-10} .

$$a_0 = 0.00000 \ 00001 \ 10, \quad a_1 = 0.99999 \ 99654 \ 98, \quad a_2 = -0.49999 \ 82537 \ 98,$$

$$a_3 = 0.33329 \ 85059 \ 64, \quad a_4 = -0.24963 \ 72428 \ 65, \quad a_5 = 0.19773 \ 31015 \ 60,$$

$$a_6 = -0.15744 \ 88954 \ 13, \quad a_7 = 0.11712 \ 91156 \ 18, \quad a_8 = -0.07364 \ 03719 \ 14,$$

$$a_9 = 0.03469 \ 74937 \ 56, \quad a_{10} = -0.01046 \ 82295 \ 69, \quad a_{11} = 0.00148 \ 19917 \ 22.$$

(2) For $1 \leq x \leq 2$, and putting $y = \frac{x - \sqrt{2}}{x + \sqrt{2}} (3 + 2\sqrt{2})$ ($-1 \leq y \leq 1$), then $\log x \approx \log \sqrt{2} +$

$\sum a_i y^{2i+1}$ gives an approximation by a polynomial of the 11th degree ($0 \leq i \leq 5$), where the maximal error is 9.2×10^{-15} .

$$a_0 = 0.34314 \ 57505 \ 07610 \ 6, \quad a_1 = 0.00336 \ 70892 \ 56222 \ 5, \quad a_2 = 0.00005 \ 94707 \ 04347 \ 4,$$

$$a_3 = 0.00000 \ 12504 \ 99776 \ 2, \quad a_4 = 0.00000 \ 00285 \ 68292 \ 8, \quad a_5 = 0.00000 \ 00007 \ 43713 \ 9.$$

(III) Trigonometric Functions

(1) We put $\frac{x}{2\pi} = p + \frac{q}{2} + \frac{r}{4} + \frac{z}{8}$ (p is an integer; $q=0, 1$; $r=0, 1$; $-1 \leq z < 1$), and $s = \sin \frac{\pi z}{4}$, $c = \cos \frac{\pi z}{4}$.

If $r=0$, $\sin x = (-1)^q s$, $\cos x = (-1)^q c$,

If $r=1$, $\sin x = (-1)^q c$, $\cos x = -(-1)^q s$.

Here s and c are computed by the following approximation formulas. Putting $-z^2/2 = y$, $s(y) = \sin(\pi z/4) \approx \sum a_i y^i$, $c(y) = \cos(\pi z/4) \approx \sum b_i y^i$ gives an approximation by a polynomial of the 5th degree, where the maximal errors are $s: 2 \times 10^{-15}$, $c: 2 \times 10^{-13}$.

$$a_0 = 0.78539 \ 81633 \ 97426, \quad a_1 = 0.16149 \ 10243 \ 75338, \quad a_2 = 0.00996 \ 15782 \ 61200,$$

$$a_3 = 0.00029 \ 26094 \ 99152, \quad a_4 = 0.00000 \ 50133 \ 389, \quad a_5 = 0.00000 \ 00555 \ 1357,$$

$$b_0 = 0.99999 \ 99999 \ 999, \quad b_1 = 0.61685 \ 02750 \ 601, \quad b_2 = 0.06341 \ 73767 \ 885,$$

$$b_3 = 0.00260 \ 79335 \ 007, \quad b_4 = 0.00005 \ 74476 \ 09, \quad b_5 = 0.00000 \ 07765 \ 93.$$

(2) $\frac{\sin(\pi x/2)}{x} \approx \sum (-1)^i a_i x^{2i}$ ($-1 \leq x \leq 1$). This gives an approximation by a polynomial of 10th degree ($0 \leq i \leq 5$), where the maximal error is 2.67×10^{-11} .

$$a_0 = 1.57079 \ 63267 \ 682, \quad a_1 = 0.64596 \ 40955 \ 820, \quad a_2 = 0.07969 \ 26037 \ 435,$$

$$a_3 = 0.00468 \ 16578 \ 837, \quad a_4 = 0.00016 \ 02547 \ 767, \quad a_5 = 0.00000 \ 34318 \ 696.$$

(3) $\tan \frac{\pi x}{4} \approx x \left(k_0 + \frac{x^2}{|k_1|} + \dots + \frac{x^2}{|k_4|} \right)$ (continued fraction) ($-1 \leq x \leq 1$).

Maximal error 9.8×10^{-12} .

$$k_0 = 0.78539 \ 81634 \ 9907, \quad k_1 = 6.19229 \ 46807 \ 1350, \quad k_2 = -0.65449 \ 83095 \ 2316,$$

$$k_3 = 520.24599 \ 06398 \ 9939, \quad k_4 = -0.07797 \ 95098 \ 7751.$$

(IV) Inverse Trigonometric Functions

(1) An approximation by a polynomial of the 21st degree ($0 \leq i \leq 10$):

$$\arcsin x \approx \sum a_i x^{2i+1} \quad (|x| \leq 1/\sqrt{2}).$$

Maximal error 10^{-10} .

$$a_0 = 1.00000 \ 00005 \ 3, \quad a_1 = 0.16666 \ 65754 \ 5, \quad a_2 = 0.07500 \ 46066 \ 5, \quad a_3 = 0.04453 \ 58425 \ 7,$$

$$a_4 = 0.03175 \ 26509 \ 6, \quad a_5 = 0.01176 \ 58281 \ 9, \quad a_6 = 0.06921 \ 26185 \ 7, \quad a_7 = -0.14821 \ 09628 \ 8,$$

$$a_8 = 0.32889 \ 76635 \ 2, \quad a_9 = -0.35020 \ 41201 \ 5, \quad a_{10} = 0.19740 \ 50325 \ 0.$$

- (2) Putting $x = w + u$ ($w = \frac{1}{8}, \frac{3}{8}, \frac{5}{8}, \frac{7}{8}$; $-\frac{1}{8} \leq u \leq \frac{1}{8}$), $v = \frac{x-w}{1+xw}$ ($|v| \leq \frac{1}{2}$)

$$\arctan x = \arctan w + t(v), \quad t(v) = \arctan v.$$

The values of $\arctan w$:

$$\arctan(1/8) = 0.12435\ 49945\ 46711, \quad \arctan(3/8) = 0.35877\ 06702\ 70611,$$

$$\arctan(5/8) = 0.55859\ 93153\ 43560, \quad \arctan(7/8) = 0.71882\ 99996\ 21623.$$

$t(v)$ is computed by an approximation by a polynomial of the 9th degree ($0 \leq i \leq 4$), where

$$t(v) = \arctan v \approx \sum (-1)^i a_i v^{2i+1}.$$

Maximal error 1.6×10^{-13} .

$$a_0 = 0.99999\ 99999\ 9992, \quad a_1 = 0.33333\ 33328\ 220, \quad a_2 = 0.19999\ 97377\ 6,$$

$$a_3 = 0.14280\ 9976, \quad a_4 = 0.10763\ 60.$$

- (3) $\arctan x \approx x \left(k_0 + \frac{x^2}{|k_1|} + \dots + \frac{x^2}{|k_6|} \right)$ (continued fraction) ($-1 \leq x \leq 1$).

Maximal error 3.6×10^{-10} .

$$k_0 = 0.99999\ 99936\ 2, \quad k_1 = -3.00000\ 30869\ 4, \quad k_2 = -0.55556\ 97728\ 4,$$

$$k_3 = -15.77401\ 81127\ 3,$$

$$k_4 = -0.16190\ 80978\ 0, \quad k_5 = -44.57191\ 79508\ 8, \quad k_6 = -0.10810\ 67493\ 1.$$

(V) Gamma Function

An approximation by a polynomial of the 8th degree:

$$\Gamma(2+x) \approx \sum a_i x^i \quad (-1/2 \leq x \leq 1/2).$$

Maximal error 7.6×10^{-8} .

$$a_0 = 0.99999\ 9926, \quad a_1 = 0.42278\ 4604, \quad a_2 = 0.41184\ 9671, \quad a_3 = 0.08156\ 52323,$$

$$a_4 = 0.07406\ 48982, \quad a_5 = -0.00012\ 51376\ 7, \quad a_6 = 0.01229\ 95771, \quad a_7 = -0.00349\ 61289,$$

$$a_8 = 0.00213\ 85778.$$

(VI) Normal Distribution

- (1) $\frac{2}{\sqrt{\pi}} \int_x^\infty e^{-t^2} dt \approx \frac{1}{(1 + \sum a_i x^{i+1})^{16}}$ ($0 \leq x < \infty$). This gives an approximation by a polynomial of the 6th degree.

Maximal error 2.8×10^{-7} .

$$a_0 = 0.07052\ 30784, \quad a_1 = 0.04228\ 20123, \quad a_2 = 0.00927\ 05272,$$

$$a_3 = 0.00015\ 20143, \quad a_4 = 0.00027\ 65672, \quad a_5 = 0.00004\ 30638.$$

- (2) $P(x) \equiv \frac{1}{\sqrt{2\pi}} \int_x^\infty e^{-t^2/2} dt,$

$$4P(x)(1-P(x)) \approx \left[\exp\left(-\frac{2x^2}{\pi}\right) \right] \left[1 + x^4 \left(a_0 + \frac{a_1}{x^2 + a_2} \right) \right] \quad (0 \leq x < \infty).$$

Maximal error 2×10^{-5} .

$$a_0 = 0.0055, \quad a_1 = 0.0551, \quad a_2 = 14.4.$$

- (3) The inverse function of (2)

$$x \approx \left[y \left(a_0 + \frac{a_1}{y + a_2} \right) \right]^{1/2}, \quad y = -\log[4P(x)(1-P(x))] \quad (0 \leq y < \infty).$$

Maximal error 4.9×10^{-4} .

$$a_0 = 2.06117\ 86, \quad a_1 = -5.72622\ 04, \quad a_2 = 11.64059\ 5.$$

Statistical Tables for Reference

Statistical Tables

- [1] J. A. Greenwood and H. O. Hartley, Guide to tables in mathematical statistics, Princeton Univ. Press, 1962.
- [2] Research Group for Statistical Sciences (T. Kitagawa and M. Masuyama, eds.) New statistical tables (Japanese), explanation p. 264, table p. 214, Kawade, 1952.
- [3] R. A. Fisher and F. Yates, Statistical tables for biological, agricultural and medical research, explanation p. 30, table p. 137, Oliver & Boyd, third edition, 1948.
- [4] E. S. Pearson and H. O. Hartley, Biometrika tables for statisticians, explanation p. 104, table p. 154, Cambridge Univ. Press, third edition, 1970.
- [5] K. Pearson, Tables for statisticians and biometricians I, 1930, explanation p. 83, table p. 143; II, 1931, explanation p. 250, table p. 262, Cambridge Univ. Press.
- [6] Statistical tables JSA-1972 (Japanese), table p. 454, explanation p. 260, Japanese Standards Association, 1972.

Tables of Special Statistical Values

- [8] Harvard Univ., Tables of the cumulative binomial probability distribution, Harvard, 1955,

$$\sum_{i=r}^n \binom{n}{i} p^i q^{n-i}; 5 \text{ dec.},$$

$$p = 0.01(0.01)0.50,$$

$$n = 1(1)50(2)100(10)200(20)500(50)1000.$$

- [9] National Bureau of Standards, NBS applied mathematical series, no. 6, Tables of the binomial probability distribution, 1950, $\binom{n}{i} p^i q^{n-i}$ and the partial sum: 7 dec., $p = 0.01(0.01)0.50$, $n = 2(1)49$.

- [10] T. Kitagawa, Table of Poisson distribution (Japanese), Baihûkan, 1951, $e^{-m} m^i / i!$: 7-8 dec., $m = 0.001(0.001)1.000(0.01)10.00$.

- [11] G. J. Lieberman and D. B. Owen, Tables of the hypergeometric probability distribution, Stanford, 1961, $\binom{k}{x} \binom{N-k}{n-x} / \binom{N}{n}$: 6 dec., $N = 2(1)50(10)100(100)2000$, $n = 1(1)(N/2)$, $k = 1(1)n$.

- [12] National Bureau of Standards, NBS no. 23, Tables of normal probability functions, 1942,

$$\varphi(x) = (1/\sqrt{2\pi}) \exp(-\frac{1}{2}x^2),$$

$$\Phi(x) = \int_{-\infty}^x \varphi(x) dx: 15 \text{ dec.},$$

$$x = 0(0.00001)1.0000(0.001)8.285.$$

- [13] K. Pearson, Tables of the incomplete beta-functions, Cambridge, second edition, 1968, $I_x(p, q)$: 8 dec., $p, q = 0.5(0.5)11(1)50$.

- [14] K. Pearson, Tables of the incomplete gamma-function, Cambridge, 1922, revised edition, 1951,

$$I(u, p) = \int_0^{u\sqrt{p+1}} (1/e^v)(v^p/\Gamma(p+1)) dv:$$

$$7 \text{ dec.}, p = 0.0(0.1)5.0(0.2)50.0, u = 0.1(0.1)20.0;$$

$$p = -1.0(0.05)0.0, u = 0.1(0.1)51.3.$$

- [15] N. V. Smirnov, Tables for the distribution and density function of the t -distribution, Pergamon, 1961, 6 dec., $f = 1(1)35$, $t = 0(0.01)3.00(0.02)4.50(0.05)6.50$.

- [16] G. J. Resnikoff and G. J. Lieberman, Tables of the non-central t -distribution, Stanford, 1957.

- [17] F. N. David, Tables of the ordinates and probability integral of the distribution of the correlation coefficient in small samples, Cambridge, 1938.

- [18] D. B. Owen, The bivariate normal probability distribution, Sandia Corp., 1957,

$$T(h, a): 6 \text{ dec.}, a = 0.000(0.025)1.000, \infty, h =$$

$$0.00(0.01)3.50(0.05)4.75, T(h, a)$$

$$= \int_0^h \int_0^{ax} \frac{1}{2\pi} \exp\left(-\frac{x^2+y^2}{2}\right) dy dx.$$

- [19] National Bureau of Standards, NBS no. 50, Tables of the bivariate normal distribution function and related functions, 1959, $L(h, k, r)$

$$= \int_h^\infty \int_k^\infty \frac{1}{2\pi\sqrt{1-r^2}}$$

$$\times \exp\left[-\frac{x^2+y^2-2rxy}{2(1-r^2)}\right] dy dx: 6 \text{ dec.},$$

$$r = \pm 0.00(0.05)0.95(0.01)0.99,$$

$$h, k = 0.0(0.1)4.0.$$

Tables of Allocation

- [20] T. Kitagawa and M. Midome, Table of allocation of elements for experimental design (Japanese), Baihûkan, 1953.

- [21] R. C. Bose, W. H. Clatworthy, and S. S. Shrikhande, Tables of partially balanced designs with two associate classes, North Carolina Agric. Expt. Station Tech. Bull., 1954 (table of PBIBD).

Numerical Tables for Reference

General Tables

- [1] M. Boll, *Tables numériques universelles*, Dunod, 1947.
- [2] P. Barlow, *Barlow's tables*, Robinson, 1814, third edition, 1930.
- [3] W. Shibagaki, 0.01% table of elementary functions (Japanese), Kyôritu, 1952.
- [4] K. Hayashi, *Table of higher functions* (Japanese), Iwanami, second edition, 1967.
- [5] E. Jahnke and F. Emde, *Funktionentafeln mit Formeln und Kurven*, Teubner, second edition 1933 (English translation: *Tables of functions with formulae and curves*, Dover, fourth edition, 1945).
- [6] Y. Yoshida and M. Yoshida, *Mathematical tables* (Japanese), Baihûkan, 1958.
- [7] M. Abramowitz and I. A. Stegun (eds.), *Handbook of mathematical functions with formulas, graphs and mathematical tables*, National Bureau of Standards, 1964 (Dover, 1965).
- [8] A. Fletcher et al. (eds.), *Index of mathematical tables I, II*, Scientific Computing Service, Addison-Wesley, second edition, 1962.

Multiplication Table

- [9] A. L. Crelle, *Rechentafeln welche alles Mutiplicieren und Dividiren mit Zahlen unter 1000 ersparen, bei grösseren Zahlen aber die Rechnung erleichtern und sicherer machen*, W. de Gruyter, new edition 1944.

Table of Prime Numbers

- [10] D. N. Lehmer, *List of prime numbers from 1 to 10,006,721*, Carnegie Institution of Washington, 1914.

Series of Tables of Functions

- [11] British Association for the Advancement of Science, *Mathematical tables*, vol. 2, Emden functions, 1932; vol. 6, Bessel functions, pt. 1, 1937; vol. 8, Number-divisor tables, 1940; vol. 9, Tables of powers giving integral powers of integrals, 1940; vol. 10, Bessel functions, pt. 2, 1952.
- [12] Harvard University, Computation Laboratory, *Annals*, vol. 2, Tables of the modified Hankel functions of order one-third and their derivatives, 1945; vol. 3, Tables of the Bessel functions of the first kind of orders

zero and one, 1947; vol. 14, Orders seventy-nine through one hundred thirty-five, 1951; vol. 18, Tables of generalized sine- and cosine-integral functions, pt. 1, 1949; vol. 19, pt. 2, 1949; vol. 20, Tables of inverse hyperbolic functions, 1949; vol. 21, Tables of the generalized exponential-integral functions, 1949.

- [13] National Bureau of Standards, Applied Mathematics Series (AMS), AMS 1, Tables of the Bessel functions $Y_0(x)$, $Y_1(x)$, $K_0(x)$, $K_1(x)$, $0 \leq x \leq 1$, 1948; AMS 5, Tables of sines and cosines to fifteen decimal places at hundredths of a degree, 1949; AMS 11, Table of arctangents of rational numbers, 1951; AMS 14, Tables of the exponential function e^x (including e^{-x}), 1951; AMS 16, Tables of $n!$ and $\Gamma(n + \frac{1}{2})$ for the first thousand values of n , 1951; AMS 23, Tables of normal probability functions, 1953; AMS 25, Tables of the Bessel functions $Y_0(x)$, $Y_1(x)$, $K_0(x)$, $K_1(x)$, $0 \leq x \leq 1$, 1952; AMS 26, Tables of $\text{Arctan } x$, 1953; Tables of 10^x , 1953; AMS 32, Table of sine and cosine integrals for arguments from 10 to 100, 1954; AMS 34, Table of the gamma function for complex arguments, 1954; AMS 36, Tables of circular and hyperbolic sines and cosines for radian arguments, 1953; AMS 40, Table of secants and cosecants to nine significant figures at hundredths of a degree, 1954; AMS 41, Tables of the error function and its derivative, 1954; AMS 43, Tables of sines and cosines for radian arguments, 1955; AMS 45, Table of hyperbolic sines and cosines, 1955; AMS 46, Table of the descending exponential, 1955.

Tables of Special Functions

- [14] Akademiya Nauk SSSR, Tables of the exponential integral functions, 1954.
- [15] J. Brownlee, Table of $\log \Gamma(x)$, *Tracts for computers*, no. 9, Cambridge Univ. Press, 1923.
- [16] L. Dolansky and M. P. Dolansky, Table of $\log_2(1/p)$, $p \cdot \log_2(1/p)$ and $p \cdot \log_2(1/p) + (1-p) \log_2(1/(1-p))$, M.I.T. Research Lab. of Electronics tech. report 227, 1952.
- [17] L. M. Milne-Thomson, *Die elliptischen Funktionen von Jacobi*, Springer, 1931; English translation: *Jacobian elliptic function tables*, Dover, 1950.
- [18] H. T. Davis, Tables of the higher mathematical functions, Principia Press, vol. 1, Gamma function, 1933; vol. 2, Polygamma functions, 1935; vol. 3, Arithmetical tables, 1962.
- [19] R. G. Selfridge and J. E. Maxfield, A table of the incomplete elliptic integral of the third kind, Dover, 1958.

- [20] W. Shibagaki, 0.01% table of modified Bessel functions and the method of numerical computation for them (Japanese), Baihûkan, 1955.
 - [21] W. Shibagaki, Theory and application of gamma function, Appendix. Table of gamma function of complex variable effective up to 6 decimals (Japanese), Iwanami, 1952.
 - [22] L. J. Slater, A short table of the Laguerre polynomials, Proc. IEE, monograph no. 103c, 1955, 46–50.
 - [23] G. N. Watson, A treatise on the theory of Bessel functions, Appendix, Cambridge Univ. Press, 1922, second edition, 1958.
- See also Statistical Tables for Reference.

Tables of Approximation Formulas of Functions

- [24] S. Hitotumatu, Approximation formula (Japanese) Takeuti, 1963.
- [25] T. Uno (ed.), Approximation formulas of functions for computers 1–3, Joint research work for mathematical sciences, pt. IV, section 5, 1961–1963 (Japanese, mimeographed).
- [26] Z. Yamauti, S. Moriguti, and S. Hitotumatu (eds.), Numerical methods for electronic computers (Japanese), Baihûkan, I, 1965; II, 1967.
- [27] C. Hastings (ed.), Approximations for digital computers, Princeton, 1955.
- [28] A. J. W. Duijvestijn and A. J. Dekkers, Chebyshev approximations of some transcendental functions for use in digital computing, Philips Research Reports, 16 (1961), 145–174.
- [29] L. A. Lyusternik, O. A. Chervonenkis, and A. P. Yanpol'skiĭ, Mathematical analysis: functions, limits, series, continued fractions, Pergamon, 1965. (Original in Russian, 1963.)
- [30] J. F. Hart et al. (eds.), Computer approximations, Wiley, 1968.

Journals

Abh. Akad. Wiss. Göttingen

Abhandlungen der Akademie der Wissenschaften in Göttingen (Göttingen)

Abh. Bayer. Akad. Wiss. Math.-Nat. Kl.

Abhandlungen der Bayerischen Akademie der Wissenschaften. Mathematisch-Naturwissenschaftliche Klasse (Munich)

Abh. Math. Sem. Univ. Hamburg

Abhandlungen aus dem Mathematischen Seminar der Universität Hamburg (Hamburg)

Acta Arith.

Acta Arithmetica, Polska Akademia Nauk, Instytut Matematyczny (Warsaw)

Acta Informat.

Acta Informatica (Berlin)

Acta Math.

Acta Mathematica (Uppsala)

Acta Math. Acad. Sci. Hungar.

Acta Mathematica Academiae Scientiarum Hungaricae (Budapest)

Acta Math. Sinica

Acta Mathematica Sinica (Peking). Chinese Math. Acta is its English translation

Acta Sci. Math. Szeged.

Acta Universitatis Szegediensis, Acta Scientiarum Mathematicarum (Szeged)

Actualités Sci. Ind.

Actualités Scientifiques et Industrielles (Paris)

Advances in Math.

Advances in Mathematics (New York)

Aequationes Math.

Aequationes Mathematicae (Basel-Waterloo)

Algebra and Logic

Algebra and Logic (New York). Translation of Algebra i Logika

Algebra i Logika

Akademiya Nauk SSSR Sibirskoe Otdelenie. Institut Matematiki. Algebra i Logika (Novosibirsk). Translated as Algebra and Logic

Algebra Universalis

Algebra Universalis (Basel)

Amer. J. Math.

American Journal of Mathematics (Baltimore)

Amer. Math. Monthly

The American Mathematical Monthly (Menasha)

Amer. Math. Soc. Colloq. Publ.

American Mathematical Society Colloquium Publications (Providence)

Amer. Math. Soc. Math. Surveys

American Mathematical Society Mathematical Surveys (Providence)

Amer. Math. Soc. Proc. Symp. Pure Math.

American Mathematical Society Proceedings of Symposia in Pure Mathematics (Providence)

Amer. Math. Soc. Transl.

American Mathematical Society Translations (Providence)

Amer. Math. Soc. Transl. Math. Monographs

American Mathematical Society Translations of Mathematical Monographs (Providence)

Ann. Acad. Sci. Fenn.

Suomalaisen Tiedeakatemian Toimituksia, Annales Academiae Scientiarum Fennicae. Series A. I. Mathematica (Helsinki)

Ann. der Phys.

Annalen der Physik (Leipzig)

Ann. Fac. Sci. Univ. Toulouse

Annales de la Faculté des Sciences de l'Université de Toulouse pour les Sciences Mathématiques et les Sciences Physiques (Toulouse)

Ann. Inst. Fourier

Annales de l'Institut Fourier, Université de Grenoble (Grenoble)

Ann. Inst. H. Poincaré

Annales de l'Institut Henri Poincaré (Paris)

Ann. Inst. Statist. Math.

Annals of the Institute of Statistical Mathematics (Tokyo)

Ann. Mat. Pura Appl.

Annali di Matematica Pura ed Applicata (Bologna)

Ann. Math. Statist.

The Annals of Mathematical Statistics (Baltimore)

Ann. Math.

Annals of Mathematics (Princeton)

Ann. Physique

Annales de Physique (Paris)

Ann. Probability

The Annals of Probability (San Francisco)

Ann. Polon. Math.

Annales Polonici Mathematici. Polska Akademia Nauk (Warsaw)

Ann. Roumaines Math.

Annales Roumaines de Mathématiques. Journal de l'Institut Mathématique Roumain (Bucharest)

Ann. Sci. Ecole Norm. Sup.

Annales Scientifiques de l'Ecole Normale Supérieure (Paris)

Ann. Scuola Norm. Sup. Pisa
Annali della Scuola Normale Superiore di
Pisa, Scienze Fisiche e Matematiche (Pisa)

Ann. Statist.
The Annals of Statistics (San Francisco)

Arch. History Exact Sci.
Archive for History of Exact Sciences
(Berlin-New York)

Arch. Math.
Archiv der Mathematik (Basel-Stuttgart)

Arch. Rational Mech. Anal.
Archive for Rational Mechanics and Analysis
(Berlin)

Ark. Mat.
Arkiv för Matematik (Stockholm)

Ark. Mat. Astr. Fys.
Arkiv för Matematik, Astronomi och Fysik
(Uppsala)

Astérique
Astérique. La Société Mathématique de
France (Paris)

Atti Accad. Naz. Lincei, Mem.
Atti della Accademia Nazionale dei Lincei,
Memorie (Rome)

Atti Accad. Naz. Lincei, Rend.
Atti della Accademia Nazionale dei Lincei,
Rendiconti (Rome)

Automat. Control Comput. Sci.
Automatic Control and Computer Sciences
(New York). Translation of Avtomatika i
Vychislitel'naya Tekhnika. Akademiya Nauk
Latvijskoj SSR (Riga)

Bell System Tech. J.
The Bell System Technical Journal (New
York)

Biometrika
Biometrika, A Journal for the Statistical
Study of Biological Problems (London)

Bol. Soc. Mat. São Paulo
Boletim da Sociedade de Matemática de São
Paulo (São Paulo)

Bull. Acad. Pol. Sci.
Bulletin de l'Académie Polonaise des Sciences
(Warsaw)

Bull. Amer. Math. Soc.
Bulletin of the American Mathematical
Society (Providence)

Bull. Calcutta Math. Soc.
Bulletin of the Calcutta Mathematical Society
(Calcutta)

Bull. Math. Statist.
Bulletin of Mathematical Statistics (Fukuoka,
Japan)

Bull. Nat. Res. Council
Bulletin of the National Research Council
(Washington)

Bull. Sci. Math.
Bulletin des Sciences Mathématiques (Paris)

Bull. Soc. Math. Belg.
Bulletin de la Société Mathématique de
Belgique (Brussels)

Bull. Soc. Math. France
Bulletin de la Société Mathématique de
France (Paris)

Bull. Soc. Roy. Sci. Liège
Bulletin de la Société Royale des Sciences de
Liège (Liège)

C. R. Acad. Sci. Paris
Comptes Rendus Hebdomadaires des Séances
de l'Académie des Sciences (Paris)

Canad. J. Math.
Canadian Journal of Mathematics (Toronto)

Časopis Pěst. Mat.
Časopis pro Pěstování Matematiky, Čes-
koslovenská Akademie Věd (Prague)

Colloq. Math.
Colloquium Mathematicum (Warsaw)

Comm. ACM
Communications of the Association for
Computing Machinery (New York)

Comm. Math. Phys.
Communications in Mathematical Physics
(Berlin)

Comm. Pure Appl. Math.
Communications on Pure and Applied
Mathematics (New York)

Comment. Math. Helv.
Commentarii Mathematici Helvetici (Zurich)

Compositio Math.
Compositio Mathematica (Groningen)

Comput. J.
The Computer Journal (London)

Crelles J.
= J. Reine Angew. Math.

CWI Newslett.
Centrum voor Wiskunde en Informatica.
Newsletter (Amsterdam)

Cybernetics
Cybernetics (New York). Translation of
Kibernetika (Kiev)

Czech. Math. J.
Czechoslovak Mathematical Journal (Prague)

Deutsche Math.
Deutsche Mathematik (Berlin)

- Differentsial'nye Uravneniya
Differentsial'nye Uravneniya (Minsk). Translated as Differential Equations
- Differential Equations
Differential Equations (New York). Translation of Differentsial'nye Uravneniya
- Dokl. Akad. Nauk SSSR
Doklady Akademii Nauk SSSR (Moscow). Soviet Math. Dokl. is the English translation of its mathematics section
- Duke Math. J.
Duke Mathematical Journal (Durham)
- Econometrica
Econometrica, Journal of the Econometric Society (Chicago)
- Edinburgh Math. Notes
The Edinburgh Mathematical Notes (Edinburgh)
- Enseignement Math.
L'Enseignement Mathématique (Geneva)
- Enzykl. Math.
Enzyklopädie der Mathematischen Wissenschaften mit Einschluss ihrer Anwendungen (Berlin)
- Erg. Angew. Math.
Ergebnisse der Angewandte Mathematik (Berlin-New York)
- Erg. Math.
Ergebnisse der Mathematik und ihrer Grenzgebiete (Berlin-New York)
- Fund. Math.
Fundamenta Mathematicae (Warsaw)
- Funkcial. Ekvac.
Fako de l'Funkcialaj Ekvacioj Japana Matematika Societo. Funkcialaj Ekvacioj (Serio Internacia) (Kobe, Japan)
- Functional Anal. Appl.
Functional Analysis and its Applications (New York). Translation of Funktsional. Anal. Prilozhen.
- Funktsional. Anal. Prilozhen.
Funktsional'nyi Analiz i ego Prilozheniya. Akademiya Nauk SSSR (Moscow). Translated as Functional Anal. Appl.
- General Topology and Appl.
General Topology and its Applications (Amsterdam)
- Hiroshima Math. J.
Hiroshima Mathematical Journal. Hiroshima Univ. (Hiroshima, Japan)
- Hokkaido Math. J.
Hokkaido Mathematical Journal. Hokkaido Univ. (Sapporo, Japan)
- IBM J. Res. Develop.
IBM Journal of Research and Development (Armonk, N.Y.)
- Illinois J. Math.
Illinois Journal of Mathematics (Urbana)
- Indag. Math.
Indagationes Mathematicae = Nederl. Akad. Wetensch. Proc.
- Indian J. Math.
Indian Journal of Mathematics (Allahabad)
- Indiana Univ. Math. J.
Indiana University Mathematics Journal (Bloomington)
- Information and Control
Information and Control (New York)
- Inventiones Math.
Inventiones Mathematicae (Berlin)
- Izv. Akad. Nauk SSSR
Izvestiya Akademii Nauk SSSR (Moscow). Math. USSR-Izv. is the English translation of its mathematics section
- J. Algebra
Journal of Algebra (New York)
- J. Analyse Math.
Journal d'Analyse Mathématiques (Jerusalem)
- J. Appl. Math. Mech.
Journal of Applied Mathematics and Mechanics (New York). Translation of Prikl. Mat. Mekh.
- J. Approximation Theory
Journal of Approximation Theory (New York)
- J. Assoc. Comput. Mach. (J. ACM)
Journal of the Association for Computing Machinery (New York)
- J. Austral. Math. Soc.
The Journal of the Australian Mathematical Society (Sydney)
- J. Combinatorial Theory
Journal of Combinatorial Theory. Series A and Series B (New York)
- J. Comput. System Sci.
Journal of Computer and System Sciences (New York)
- J. Differential Equations
Journal of Differential Equations (New York)
- J. Differential Geometry
Journal of Differential Geometry (Bethlehem, Pa.)
- J. Ecole Polytech.
Journal de l'Ecole Polytechnique (Paris)
- J. Fac. Sci. Hokkaido Univ.
Journal of the Faculty of Science, Hokkaido University. Series I. Mathematics (Sapporo, Japan)

J. Fac. Sci. Univ. Tokyo
Journal of the Faculty of Science, University of Tokyo. Section I. (Tokyo)

J. Franklin Inst.
Journal of the Franklin Institute (Philadelphia)

J. Functional Anal.
Journal of Functional Analysis (New York)

J. Indian Math. Soc.
The Journal of the Indian Mathematical Society (Madras)

J. Inst. Elec. Engrs.
Journal of the Institution of Electrical Engineers (London)

J. Inst. Polytech. Osaka City Univ.
Journal of the Institute of Polytechnics, Osaka City University. Series A. Mathematics (Osaka)

J. London Math. Soc.
The Journal of the London Mathematical Society (London)

J. Math. Anal. Appl.
Journal of Mathematical Analysis and Applications (New York)

J. Math. and Phys.
Journal of Mathematics and Physics (Cambridge, Massachusetts, for issues prior to 1975; for 1975 and later, New York)

J. Math. Econom.
Journal of Mathematical Economics (Amsterdam)

J. Math. Kyoto Univ.
Journal of Mathematics of Kyoto University (Kyoto)

J. Math. Mech.
Journal of Mathematics and Mechanics (Bloomington)

J. Math. Pures Appl.
Journal de Mathématiques Pures et Appliquées (Paris)

J. Math. Soc. Japan
Journal of the Mathematical Society of Japan (Tokyo)

J. Mathematical Phys.
Journal of Mathematical Physics (New York)

J. Multivariate Anal.
Journal of Multivariate Analysis (New York)

J. Number Theory
Journal of Number Theory (New York)

J. Operations Res. Soc. Japan
Journal of the Operations Research Society of Japan (Tokyo)

J. Optimization Theory Appl.
Journal of Optimization Theory and Applications (New York)

J. Phys. Soc. Japan
Journal of the Physical Society of Japan (Tokyo)

J. Pure Appl. Algebra
Journal of Pure and Applied Algebra (Amsterdam)

J. Rational Mech. Anal.
Journal of Rational Mechanics and Analysis (Bloomington)

J. Reine Angew. Math.
Journal für die Reine und Angewandte Mathematik (Berlin). = Crelles J.

J. Res. Nat. Bur. Standards
Journal of Research of the National Bureau of Standards. Section B. Mathematics and Mathematical Physics (Washington)

J. Sci. Hiroshima Univ.
Journal of Science of Hiroshima University. Series A (Mathematics, Physics, Chemistry); Series A-I. (Mathematics) (Hiroshima)

J. Soviet Math.
Journal of Soviet Mathematics (New York). Translation of (1) Itogi Nauki—Seriya Matematika (Progress in Science—Mathematical Series); (2) Problemy Matematicheskogo Analiza (Problems in Mathematical Analysis); (3) Zap. Nauchn. Sem. Leningrad. Otdel. Mat. Inst. Steklov.

J. Symbolic Logic
The Journal of Symbolic Logic (New Brunswick)

Japan. J. Math.
Japanese Journal of Mathematics (Tokyo)

Jber. Deutsch. Math. Verein. (Jber. D.M.V.)
Jahresbericht der Deutschen Mathematiker Vereinigung (Stuttgart)

Kibernetika (Kiev)
Otdelenie Matematiki, Mekhaniki i Kibernetiki Akademii Nauk Ukrainsoi SSR. Kibernetika (Kiev). Translated as Cybernetics

Kôdai Math. Sem. Rep.
Kôdai Mathematical Seminar Reports (Tokyo)

Linear Algebra and Appl.
Linear Algebra and Its Applications (New York)

Linear and Multilinear Algebra.
Linear and Multilinear Algebra (New York)

Mat. Sb.
Matematicheskii Sbornik (Recueil Mathématique). Akademiya Nauk SSSR (Moscow). Translated as Math. USSR-Sb.

- Mat. Tidsskr. A
Matematisk Tidsskrift. A (Copenhagen)
- Mat. Zametki
Matematicheskii Zametki.
Akademiya Nauk SSSR (Moscow). Translated
as Math. Notes
- Math. Ann.
Mathematische Annalen (Berlin-Göttingen-
Heidelberg)
- Math. Comp.
Mathematics of Computation (Providence).
Formerly Math. Tables Aids Comput.
- Math. J. Okayama Univ.
Mathematical Journal of Okayama University
(Okayama, Japan)
- Math. Japonicae
Mathematica Japonicae (Osaka)
- Math. Nachr.
Mathematische Nachrichten (Berlin)
- Math. Notes
Mathematical Notes of the Academy of
Sciences of the USSR (New York). Trans-
lation of Mat. Zametki
- Math. Rev.
Mathematical Reviews (Ann Arbor)
- Math. Scand.
Mathematica Scandinavica (Copenhagen)
- Math. Student
The Mathematical Student (Madras)
- Math. Tables Aids Comput. (MTAC)
Mathematical Tables and Other Aids to
Computation (Washington). Name changed
to Mathematics of Computation in 1960
(vol. 14ff.)
- Math. USSR-Izv.
Mathematics of the USSR-Izvestiya (Provi-
dence). Translation of Izv. Akad. Nauk SSSR
- Math. USSR-Sb.
Mathematics of the USSR-Sbornik (Provi-
dence). Translation of Mat. Sb.
- Math. Z.
Mathematische Zeitschrift (Berlin-Göttingen-
Heidelberg)
- Mathematika
Mathematika, A Journal of Pure and Applied
Mathematics (London)
- Meed. Lunds Univ. Mat. Sem.
Meddelanden från Lunds Universitet
Matematiska Seminarium = Communications
du Séminaire Mathématique de l'Université
de Lund (Lund)
- Mem. Amer. Math. Soc.
Memoirs of the American Mathematical
Society (Providence)
- Mem. Coll. Sci. Univ. Kyôto
Memoirs of the College of Science, University
of Kyôto. Series A (Kyoto)
- Mem. Fac. Sci. Kyushu Univ.
Memoirs of the Faculty of Science, Kyushu
University. Series A. Mathematics (Fukuoka,
Japan)
- Mémor. Sci. Math.
Mémorial des Sciences Mathématiques (Paris)
- Michigan Math. J.
The Michigan Mathematical Journal (Ann
Arbor)
- Mitt. Math. Ges. Hamburg
Mitteilungen der Mathematischen
Gesellschaft in Hamburg (Hamburg)
- Monatsh. Math. Phys.
Monatshefte für Mathematik und Physik
(Vienna)
- Monatsh. Math.
Monatshefte für Mathematik (Vienna)
- Monograf. Mat.
Monografia Matematyczne (Warsaw)
- Moscow Univ. Math. Bull.
Moscow University Mathematics Bulletin
(New York). Translation of the mathematics
section of Vestnik Moskov. Univ., Ser. I, Mat.
Mekh.
- Nachr. Akad. Wiss. Göttingen
Nachrichten der Akademie der Wissenschaften
in Göttingen. Math.-Phys. Kl. (Göttingen)
- Nagoya Math. J.
Nagoya Mathematical Journal (Nagoya)
- Nederl. Akad. Wetensch. Proc.
Koninklijke Nederlandse Akademie van
Wetenschappen, Proceedings. Series A.
Mathematical Sciences (Amsterdam) = Indag.
Math., Proc. Acad. Amsterdam
- Nieuw Arch. Wisk.
Nieuw Archief voor Wiskunde (Groningen)
- Numerische Math.
Numerische Mathematik (Berlin-Göttingen-
Heidelberg)
- Nuovo Cimento
Il Nuovo Cimento (Bologna)
- Osaka J. Math.
Osaka Journal of Mathematics (Osaka)
- Osaka Math. J.
Osaka Mathematical Journal (Osaka)
- Pacific J. Math.
Pacific Journal of Mathematics (Berkeley)
- Philos. Trans. Roy. Soc. London
Philosophical Transactions of the Royal
Society of London. Series A (London)

- Phys. Rev.
The Physical Review (New York)
- Portugal. Math.
Portugaliae Mathematica (Lisbon)
- Prikl. Mat. Mekh.
Adademiya Nauk SSSR. Otdelenie Tekhnicheskikh Nauk. Institut Mekhaniki Prikladnaya Matematika i Mekhanika (Moscow). Translated as J. Appl. Mat. Mech.
- Proc. Acad. Amsterdam
= Nederl. Akad. Wetensch. Proc.
- Proc. Amer. Math. Soc.
Proceedings of the American Mathematical Society (Providence)
- Proc. Cambridge Philos. Soc.
Proceedings of the Cambridge Philosophical Society (Cambridge)
- Proc. Imp. Acad. Tokyo
Proceedings of the Imperial Academy (Tokyo)
- Proc. Japan Acad.
Proceedings of the Japan Academy (Tokyo)
- Proc. London Math. Soc.
Proceedings of the London Mathematical Society (London)
- Proc. Nat. Acad. Sci. US
Proceedings of the National Academy of Sciences of the United States of America (Washington)
- Proc. Phys.-Math. Soc. Japan
Proceedings of the Physico-Mathematical Society of Japan (Tokyo)
- Proc. Roy. Soc. London
Proceedings of the Royal Society of London. Series A (London)
- Proc. Steklov Inst. Math.
Proceedings of the Steklov Institute of Mathematics (Providence). Translation of Trudy Mat. Inst. Steklov.
- Prog. Theoret. Phys.
Progress of Theoretical Physics (Kyoto)
- Publ. Inst. Math.
Publications de l'Institut Mathématique (Belgrade)
- Publ. Inst. Math. Univ. Strasbourg
Publications de l'Institut de Mathématiques de l'Université de Strasbourg (Strasbourg)
- Publ. Math. Inst. HES
Publications Mathématiques de l'Institut des Hautes Etudes Scientifiques (Paris)
- Publ. Res. Inst. Math. Sci.
Publications of the Research Institute for Mathematical Sciences (Kyoto)
- Quart. Appl. Math.
Quarterly of Applied Mathematics (Providence)
- Quart. J. Math.
The Quarterly Journal of Mathematics, Oxford. Second Series (Oxford)
- Quart. J. Mech. Appl. Math.
The Quarterly Journal of Mechanics and Applied Mathematics (Oxford)
- Rend. Circ. Mat. Palermo
Rendiconti del Circolo Matematico de Palermo (Palermo)
- Rend. Sem. Mat. Univ. Padova
Rendiconti del Seminario Matematico dell'Università di Padova (Padua)
- Rev. Mat. Hisp. Amer.
Revista Matemática Hispano-Americana (Madrid)
- Rev. Mod. Phys.
Reviews of Modern Physics (New York)
- Rev. Un. Mat. Argentina
Revista de la Unión Matemática Argentina (Buenos Aires)
- Rev. Univ. Tucumán
Revista Universidad Nacional de Tucumán, Facultad de Ciencias Exactas y Tecnología. Serie A. Matemáticas y Física Teórica (Tucumán)
- Roczniki Polsk. Towar. Mat.
Roczniki Polskiego Towarzystwa Matematycznego. Serja I. Prace Matematyczne (Krakow)
- Rozprawy Mat.
Rozprawy Matematyczne, Polska Akademia Nauk, Instytut Matematyczny (Warsaw)
- Russian Math. Surveys.
Russian Mathematical Surveys (London). Translation of Uspekhi Mat. Nauk
- Sammul. Göschel
Sammulung Göschel (Leipzig)
- Sankhyā
Sankhyā, The Indian Journal of Statistics. Series A and Series B (Calcutta)
- S.-B. Berlin. Math. Ges.
Sitzungsberichte der Berliner Mathematischen Gesellschaft (Berlin)
- S.-B. Deutsch. Akad. Wiss. Berlin
Sitzungsberichte der Deutschen Akademie der Wissenschaften zu Berlin, Mathematisch-Naturwissenschaftliche Klasse (Berlin)
- S.-B. Heidelberger Akad. Wiss.
Sitzungsberichte der Heidelberger Akademie der Wissenschaften, Mathematisch-Naturwissenschaftliche Klasse (Heidelberg)

- S.-B. Math.-Nat. Kl. Bayer. Akad. Wiss.
Sitzungsberichte der Mathematisch-Naturwissenschaftlichen Klasse der Bayerischen Akademie der Wissenschaften zu München (Munich)
- S.-B. Öster. Akad. Wiss.
Sitzungsberichte der Österreichische Akademie der Wissenschaften (Vienna)
- S.-B. Phys.-Med. Soz. Erlangen
Sitzungsberichte der Physikalisch-Medizinischen Sozietät zu Erlangen (Erlangen)
- S.-B. Preuss. Akad. Wiss.
Sitzungsberichte der Preussischen Akademie der Wissenschaften. Physikalisch-Mathematische Klasse (Berlin)
- Schr. Math. Inst. u. Inst. Angew. Math. Univ. Berlin
Schriften des Mathematischen Instituts und des Instituts für Angewandte Mathematik der Universität Berlin (Berlin)
- Schr. Math. Inst. Univ. Münster
Schriftenreihe des Mathematischen Instituts der Universität Münster (Münster)
- Sci. Papers Coll. Gen. Ed. Univ. Tokyo
Scientific Papers of the College of General Education, University of Tokyo (Tokyo)
- Sci. Rep. Tokyo Kyoiku Daigaku
Science Reports of the Tokyo Kyoiku Daigaku. Section A (Tokyo)
- Scripta Math.
Scripta Mathematica. A Quarterly Journal devoted to the Philosophy, History, and Expository Treatment of Mathematics (New York)
- Sém. Bourbaki
Séminaire Bourbaki (Paris)
- SIAM J. Appl. Math.
SIAM Journal of Applied Mathematics. A Publication of the Society for Industrial and Applied Mathematics (Philadelphia)
- SIAM J. Comput.
SIAM Journal on Computing (Philadelphia)
- SIAM J. Control
SIAM Journal on Control (Philadelphia)
- SIAM J. Math. Anal.
SIAM Journal on Mathematical Analysis (Philadelphia)
- SIAM J. Numer. Anal.
SIAM Journal on Numerical Analysis (Philadelphia)
- SIAM Rev.
SIAM Review (Philadelphia)
- Siberian Math. J.
Siberian Mathematical Journal (New York).
Translation of Sibirsk. Mat. Zh.
- Sibirsk. Mat. Zh.
Akademiya Nauk SSSR. Sibirskoe Otdelenie. Sibirskii Matematicheskii Zhurnal (Moscow).
Translated as Siberian Math. J.
- Skr. Norske Vid. Akad. Oslo
Skrifter Utgitt av det Norske Videnskaps-Akademii Oslo. Matematisk-Naturvidenskapelig Klasse (Oslo)
- Soviet Math. Dokl.
Soviet Mathematics, Doklady (Providence).
Translation of mathematical section of Dokl. Akad. Nauk SSSR
- SRI J.
Stanford Research Institute Journal (Menlo Park)
- Studia Math.
Studia Mathematica. (Wrocław)
- Sûbutu-kaisi
Nihon Sûgaku-buturi-gakkai Kaisi (Tokyo)
- Sûgaku
Sûgaku, Mathematical Society of Japan (Tokyo)
- Summa Brasil. Math.
Summa Brasiliensis Mathematicae (Rio de Janeiro)
- Tensor
Tensor (Chigasaki, Japan)
- Teor. Veroyatnost. i Primenen.
Teoriya Veroyatnostei i ee Primenenie. Akademiya Nauk SSSR (Moscow). Translated as Theor. Prob. Appl.
- Theor. Prob. Appl.
Theory of Probability and Its Applications. Society for Industrial and Applied Mathematics. English translation of Teor. Veroyatnost. i Primenen. (Philadelphia)
- Tôhoku Math. J.
The Tôhoku Mathematical Journal (Sendai, Japan)
- Tôhoku-rihô
Tôhoku Teikokudaigaku Rikahôkoku (Sendai, Japan)
- Topology
Topology. An International Journal of Mathematics (Oxford)
- Trans. Amer. Math. Soc.
Transactions of the American Mathematical Society (Providence)
- Trans. Moscow Math. Soc.
Transactions of the Moscow Mathematical Society (Providence). Translation of Trudy Moskov. Mat. Obshch.
- Trudy Mat. Inst. Steklov.
Trudy Matematicheskogo Instituta im. V. A.

Steklova. Akademiya Nauk SSSR (Moscow-Leningrad). Translated as Proc. Steklov Inst. Math.

Trudy Moskov. Obshch.

Trudy Moskovskogo Matematicheskogo Obshchestva (Moscow). Translated as Trans. Moscow Math. Soc.

Tsukuba J. Math.

Tsukuba Journal of Mathematics. Univ. Tsukuba (Ibaraki, Japan)

Ukrain. Mat. Zh.

Akademiya Nauk Ukrainsoi SSR. Institut Matematiki. Ukrainskii Matematicheskii Zhurnal (Kiev). Translated as Ukrainian Math. J.

Ukrainian Math. J.

Ukrainian Mathematical Journal (New York). Translation of Ukrain. Mat. Zh.

Uspekhi Mat. Nauk

Uspekhi Matematicheskikh Nauk (Moscow-Leningrad). Translated as Russian Math. Surveys

Vestnik Moskov. Univ.

Vestnik Moskovskogo Universiteta. I, Matematika i Mekhanika (Moscow). Mathematical section translated as Moscow Univ. Math. Bull.

Vierteljschr. Naturf. Ges. Zürich

Vierteljahrsschifte der Naturforschenden Gesellschaft in Zürich (Zurich)

Z. Angew. Math. Mech. (Z.A.M.M.)

Zeitschrift für Angewandte Mathematik und Mechanik, Ingenieurwissenschaftliche Forschungsarbeiten (Berlin)

Z. Angew. Math. Phys. (Z.A.M.P.)

Zeitschrift für Angewandte Mathematik und Physik (Basel)

Z. Wahrscheinlichkeitstheorie

Zeitschrift für Wahrscheinlichkeitstheorie und Verwandte Gebiete (Berlin)

Zbl. Angew. Math.

Zentralblatt für Angewandte Mathematik (Berlin)

Zbl. Math.

Zentralblatt für Mathematik und ihre Grenzgebiete (Berlin-Göttingen-Heidelberg)

Zh. Èksper. Teoret. Fiz.

Zhurnal Èksperimental'noi i Teoreticheskoi Fiziki (Moscow)

Publishers

Academic Press

Academic Press Inc., New York-London

Addison-Wesley

Addison-Wesley Publishing Company, Inc.,
Reading (Massachusetts)-Menlo Park (California)-London-Don Mills (Ontario)

Akadémiai Kiadó

A kiadásért felös: az Adadémiai Kiadó igazatőja (Publishing House of the Hungarian Academy of Sciences), Budapest

Akademie-Verlag

Berlin

Akademische Verlag.

Akademische Verlagsgesellschaft, Leipzig

Allen

W. H. Allen & Co. Ltd., London

Allen & Unwin

Allen & Unwin, Inc., Winchester (Massachusetts)

Allyn & Bacon

Allyn & Bacon, Inc., Newton (Massachusetts)

Almqvist and Wiksell

Almqvist och Wiksell Förlag, Stockholm

Asakura

Asakura-syoten, Tokyo

Aschelhoug

H. Aschelhoug and Company, Oslo

Baihûkan

Tokyo

Benjamin

W. A. Benjamin, Inc., New York-London

Birkhäuser

Birkhäuser Verlag, Basel-Stuttgart

Blackie

Blackie & Son Ltd., London-Glasgow

Cambridge Univ. Press

Cambridge University Press, London-New York

Chapman & Hall

Chapman & Hall Ltd., London

Chelsea

Chelsea Publishing Company, New York

Clarendon Press

Oxford University Press, Oxford

Cremona

Edizioni Cremonese, Rome

de Gruyter

Walter de Gruyter and Company, Berlin

Dekker

Marcel Dekker, Inc., New York

Deutscher Verlag der Wiss.

Deutscher Verlag der Wissenschaften, Berlin

Dover

Dover Publications, Inc., New York

Dunod

Dunod, Editeur, Paris

Elsevier

Elsevier Publishing Company, Amsterdam-London-New York

Fizmatgiz

Gosudarstvennoe Izdatel'stvo Fiziko-Matematicheskoi Literatury, Moscow

Gauthier-Villars

Gauthier-Villars & C^{ie}, Editeur, Paris

Ginn

Ginn and Company, Waltham (Massachusetts)-Toronto-London

Gordon & Breach

Gordon & Breach, Science Publishers Ltd., London

Goztekhizdat

Gosudarstvennoe Izdatel'stvo Tekhniko-Teoreticheskoi Literatury, Moscow

Griffin

Charles Griffin and Company Ltd., London

Hafner

Hafner Publishing Company, New York

Harper & Row

Harper & Row Publishers, New York-Evanston-London

Hermann

Hermann & C^{ie}, Paris

Hirokawa

Hirokawa-syoten, Tokyo

Hirzel

Verlag von S. Hirzel, Leipzig

Holden-Day

Holden-Day, Inc., San Francisco-London-Amsterdam

Holt, Rinehart and Winston

Holt, Rinehart and Winston, Inc., New York-Chicago-San Francisco-Toronto-London

Interscience

Interscience Publishers, Inc., New York-London

Iwanami

Iwanami Shoten, Tokyo

Kawade

Kawade-syobô, Tokyo

Kinokoniya

Kinokoniya Company, Tokyo

Kyôritu
 Kyôritu-syuppan, Tokyo

 Lippincott
 J. B. Lippincott Company, Philadelphia

 Longman
 Longman Group, Ltd., Harlow (Essex)

 Longmans, Green
 Longmans, Green and Company, Ltd.,
 London-New York-Toronto-Bombay-
 Calcutta-Madras

 Macmillan
 The Macmillan Company, New York-London

 Maki
 Maki-syoten, Tokyo

 Maruzen
 Maruzen Company Ltd., Tokyo

 Masson
 Masson et C^{ie}, Paris

 Math-Sci Press
 Math-Sci Press, Brookline (Massachusetts)

 McGraw-Hill
 McGraw-Hill Book Company, Inc., New
 York-London-Toronto

 Methuen
 Methuen and Company Ltd., London

 MIT Press
 The MIT Press, Cambridge (Massachusetts)-
 London

 Nauka
 Izdatel'stvo Nauka, Moscow

 Noordhoff
 P. Noordhoff Ltd., Groningen

 North-Holland
 North-Holland Publishing Company,
 Amsterdam

 Oldenbourg
 Verlag von R. Oldenbourg, Munich-Vienna

 Oliver & Boyd
 Oliver & Boyd Ltd., Edinburgh-London

 Oxford Univ. Press
 Oxford University Press, London-New York

 Pergamon
 Pergamon Press, Oxford-London-Edinburgh-
 New York-Paris-Frankfurt

 Polish Scientific Publishers
 Państwowe Wydawnictwo Naukowe, Warsaw

 Prentice-Hall
 Prentice-Hall, Inc., Englewood Cliffs (New
 Jersey)

 Princeton Univ. Press
 Princeton University Press, Princeton

Random House
 Random House, Inc., New York

 Sibundô
 Tokyo

 Springer
 Springer-Verlag, Berlin (-Göttingen)-
 Heidelberg-New York

 Teubner
 B. G. Teubner Verlagsgesellschaft, Leipzig-
 Stuttgart

 Tôkai
 Tôkai-syobô, Tokyo

 Tokyo-tosyo
 Tokyo

 Tokyo Univ. Press
 Tokyo University Press, Tokyo

 Ungar
 Frederick Ungar Publishing Company, New
 York

 Univ. of Tokyo Press
 University of Tokyo Press, Tokyo

 Utida-rôkakuho
 Tokyo

 Van Nostrand
 D. Van Nostrand Company, Inc., Toronto-
 New York-London

 Vandenhoeck & Ruprecht
 Göttingen

 Veit
 Verlag von Veit & Company, Leipzig

 Vieweg
 Friedr Vieweg und Sohn Verlagsgesellschaft
 mbH, Wiesbaden

 Wiley
 Wiley & Sons, Inc., New York-London

 Wiley-Interscience
 Wiley & Sons, Inc., New York-London

 Zanichelli
 Nicola Zanichelli Editore, Bologna

Special Notation

This list contains the notation commonly and frequently used throughout this work. The symbol * means that the same notation is used with more than one meaning. For more detailed definitions or further properties of the notation, see the articles cited.

Notation	Example	Definition	Article and Section
I. Logic			
\forall	$\forall xF(x)$	Universal quantifier (for all x , $F(x)$ holds)	411B, C
\exists	$\exists xF(x)$	Existential quantifier (there exists an x such that $F(x)$ holds)	411B, C
$\wedge, \&$	$A \wedge B, A \& B$	Conjunction, logical product (A and B)	411B*
\vee	$A \vee B$	Disjunction, logical sum (A or B)	411B*
\neg	$\neg A$	Negation (not A)	411B
$\rightarrow, \supset, \Rightarrow$	$A \rightarrow B, A \Rightarrow B$	Implication (A implies B)	411B*
$\leftrightarrow, \Leftrightarrow, \rightleftharpoons$	$A \leftrightarrow B$	Equivalence (A and B are logically equivalent)	411B
II. Sets			
\in	$x \in X$	Membership (element x is a member of the set X)	381A
\notin	$x \notin X$	Nonmembership (element x is not a member of the set X)	381A
\subset	$A \subset B$	Inclusion (A is a subset of B)	381A
$\not\subset$	$A \not\subset B$	Noninclusion (A is not a subset of B)	381A
\subseteq	$A \subseteq B$	Proper inclusion (A is a proper subset of B)	381A
\emptyset		Empty set	381A
\cup, \bigcup	$A \cup B, \bigcup A_\lambda$	Union, join	381B, D*
\cap, \bigcap	$A \cap B, \bigcap A_\lambda$	Intersection, meet	381B, D*
$^c, C$	$A^c, C(A)$	Complement (of a set A)	381B
$-, \setminus$	$A - B, A \setminus B$	Difference ($A - B = A \cap B^c$)	381B
\times	$A \times B$	Cartesian product (of A and B)	381B*
R, \sim	$xRy, x \sim y$	Equivalence relation (for two elements x, y)	135A*
$/$	A/R	Quotient set (set of equivalence classes of A with respect to an equivalence relation R)	135B*
\prod	$\prod_\lambda A_\lambda$	Cartesian product (of the A_λ)	381E
Σ, \amalg	$\Sigma A_\lambda, \amalg A_\lambda$	Direct sum (of the A_λ)	381E
\mathfrak{B}	$\mathfrak{B}(A)$	Power set (set of all subsets of A)	381E
	B^A	Set of all mappings from A to B	381C
$\{ \}$	$\{x P(x)\}$	Set of all elements x with the property $P(x)$	381A

Notation	Example	Definition	Article and Section
$\{ \quad \}$	$\{a_\lambda\}_{\lambda \in \Lambda}$	Family with index set Λ	165D
	$\{a_n\}$	Sequence (of numbers, points, functions, or sets)	165D
$\bar{}, , \# $	$\bar{X}, X , \#X$	Cardinal number (of the set X)	49A*
\aleph	\aleph_β	Aleph (transfinite cardinal)	49E
\rightarrow	$f: X \rightarrow Y$	Mapping (f from X to Y)	381C*
\mapsto	$f: X \mapsto Y$	Mapping (where $f(X) = Y$, but not in the present volumes)	381C
$1, \text{id}$	$1_A, \text{id}_A$	Identity mapping (identity function)	381C
c, χ	$c_X(x), \chi_X(x)$	Characteristic function (representing function)	381C
$ $	$f A$	Restriction (of a mapping f to A)	381C*
\circ	$g \circ f$	Composite (of mappings f and g)	381C
$\limsup, \overline{\lim}$	$\limsup A_n$	Superior limit (of the sequence of sets A_n)	270C*
$\liminf, \underline{\lim}$	$\liminf A_n$	Inferior limit (of the sequence of sets A_n)	270C*
\lim	$\lim A_n$	Limit (of the sequence of sets A_n)	270C*
\varinjlim	$\varinjlim A_\lambda$	Inductive limit (of A_λ)	210B
\varprojlim	$\varprojlim A_\lambda$	Projective limit (of A_λ)	210B
III. Order			
$(,)$	(a, b)	Open interval $\{x a < x < b\}$	355C*
$[,]$	$[a, b]$	Closed interval $\{x a \leq x \leq b\}$	355C*
$(,]$	$(a, b]$	Half-open-interval $\{x a < x \leq b\}$	355C
$[,)$	$[a, b)$	Half-open interval $\{x a \leq x < b\}$	355C
\max	$\max A$	Maximum (of A)	311B
\min	$\min A$	Minimum (of A)	311B
\sup	$\sup A$	Supremum, least upper bound (of A)	311B
\inf	$\inf A$	Infimum, greatest lower bound (of A)	311B
\ll	$a \ll b$	Very large (b is very large compared to a)	
\cup, \vee	$a \cup b, a \vee b$	Join of a, b in an ordered set	243A*
\cap, \wedge	$a \cap b, a \wedge b$	Meet of a, b in an ordered set	243A*
IV. Algebra			
mod	$a \equiv b \pmod{n}$	Modulo (a and b are congruent modulo n)	297G
$ $	$a b$	Divisibility (a divides b)	297A*
\nmid	$a \nmid b$	Nondivisibility (a does not divide b)	297A
$\det, $	$\det A, A $	Determinant (of a square matrix A)	103A*

Notation	Example	Definition	Article and Section
tr, Sp	$\text{tr } A, \text{Sp } A$	Trace (of a square matrix A)	269F
$'^{\text{T}}, ' ^{\text{T}}$	$'A; A^{\text{T}}, A^{\text{T}}, A'$	Transpose (of a matrix A)	269B
I	I_n	Unit matrix (of degree n)	269A
E_{ij}		Matrix unit (matrix whose (i,j) -component is 1 and all others are 0)	269B
\otimes	$A \otimes B$	Kronecker product (of two matrices A and B)	269C*
\cong	$M \cong N$	Isomorphism (of two algebraic systems M and N)	256B
$/$	M/N	Quotient space (of an algebraic system M by N)	256F*
\dim	$\dim M$	Dimension (of a linear space, etc.)	256C
Im	$\text{Im } f$	Image (of a mapping f)	277E*
Ker	$\text{Ker } f$	Kernel (of a mapping f)	277E
Coim	$\text{Coim } f$	Coimage (of a mapping f)	277E
Coker	$\text{Coker } f$	Cokernel (of a mapping f)	277E
δ_{ij}, δ_i^j		Kronecker delta ($\delta_{ii} = 1$ and $\delta_{ij} = 0$ for $i \neq j$)	269A
$(\ , \), \cdot$	$(\mathbf{a}, \mathbf{b}), \mathbf{a} \cdot \mathbf{b}$	Inner product (of two vectors \mathbf{a} and \mathbf{b})	442B*
$[\ , \], \times$	$[\mathbf{a}, \mathbf{b}], \mathbf{a} \times \mathbf{b}$	Vector product (of two 3-dimensional vectors \mathbf{a} and \mathbf{b})	442C*
\otimes	$M \otimes N$	Tensor product (of two modules M and N)	277J, 256I*
Hom	$\text{Hom}(M, N)$	Set of all homomorphisms (from M to N)	277B
Hom_A	$\text{Hom}_A(M, N)$	Set of all A -homomorphisms (of an A -module M to an A -module N)	277E
Tor	$\text{Tor}_n(M, N)$	Torsion product (of M, N)	200D
Ext	$\text{Ext}^n(M, N)$	Extension (of M, N)	200G
\wedge, \wedge^p	$\wedge M, \wedge^p M$	Exterior algebra (of a linear space M), p th exterior product (of M)	256O
V. Algebraic Systems			
\mathbf{N}		Set of all natural numbers	294A
\mathbf{Z}		Set of all rational integers	294A
\mathbf{Z}_m		$\mathbf{Z}/m\mathbf{Z}$ (set of all residue classes modulo m)	297G*
\mathbf{Q}		Set of all rational numbers	294A
\mathbf{R}		Set of all real numbers	294A
\mathbf{C}		Set of all complex numbers	294A
\mathbf{H}		Set of all quaternions	29B
$GF(q), \mathbf{F}_q$		Finite field (with q elements)	149M

Notation	Example	Definition	Article and Section
\mathbf{Q}_p		p -adic number field (p is a prime)	439F
\mathbf{Z}_p		Ring of p -adic integers	439F
$[\]$	$k[x_1, \dots, x_n]$	Polynomial ring (of variables x_1, \dots, x_n with coefficients in k)	369A
$(\)$	$k(x_1, \dots, x_n)$	Field extension (of k by x_1, \dots, x_n)	149D
$[[\]], \{ \ }$	$k[[x_1, \dots, x_n]]$	Formal power series ring (with coefficients in k). Note: The symbols \mathbf{N} , \mathbf{Z} , \mathbf{Q} , \mathbf{R} , \mathbf{C} , and \mathbf{H} stand for sets, each with its own natural mathematical structure	370A
VI. Groups			
GL	$GL(V), GL(n, K)$	General linear group (over V , or over K of degree n)	60B
SL	$SL(n, K)$	Special linear group (over K of degree n)	60B
PSL	$PSL(n, K)$	Projective special linear group (over K of degree n)	60B
U	$U(n)$	Unitary group (of degree n)	60F
SU	$SU(n)$	Special unitary group (of degree n)	60F
O	$O(n)$	Orthogonal group (of degree n)	60I
SO	$SO(n)$	Special orthogonal group, rotation group (of degree n)	60I
$Spin$	$Spin(n)$	Spinor group (of degree n)	61D
Sp	$Sp(n)$	Symplectic group (of degree n)	60L
[For $PGL(n, K), LF(n, K), PU(n), Sp(n), PSp(n, K) \rightarrow$ 60 Classical Groups]			
VII. Topology (Convergence)			
\rightarrow	$a_n \rightarrow a$	Convergence (sequence a_n converges to a)	87B, E*
\downarrow, \searrow	$a_n \downarrow a, a_n \searrow a$	Convergence monotonically decreasing	87B
\uparrow, \nearrow	$a_n \uparrow a, a_n \nearrow a$	Convergence monotonically increasing	87B
\lim	$\lim a_n$	Limit (of a sequence a_n)	87B, E*
$\limsup, \overline{\lim}$	$\limsup a_n, \overline{\lim} a_n$	Superior limit (of a sequence a_n)	87C*
$\liminf, \underline{\lim}$	$\liminf a_n, \underline{\lim} a_n$	Inferior limit (of a sequence a_n)	87C*
$^a, \bar{}, Cl$	$E^a, \bar{E}, Cl E$	Closure (of a set E)	425B
$^i, \circ, \text{Int}$	$E^i, E^\circ, \text{Int } E$	Interior (of a set E)	425B
ρ, d	$\rho(x, y), d(x, y)$	Distance (between two points x and y)	273B*
$\ \ \ $	$\ x\ $	Norm (of x)	37B
l.i.m.	$\text{l.i.m. } f_n$	Limit in the mean (of a sequence f_n)	168B

Notation	Example	Definition	Article and Section
s-lim	s-lim x_n	Strong limit (of a sequence x_n)	37B
w-lim	w-lim x_n	Weak limit (of a sequence x_n)	37E
\simeq	$f \simeq g$	Homotopy (of two mappings f and g)	202B
\approx	$X \approx Y$	Homeomorphism (of two topological spaces X and Y)	425G

VIII. Geometry and Algebraic Topology

E^n		Euclidean space (of dimension n)	140
P^n		Projective space (of dimension n)	343B
S^n		Spherical surface (of dimension n)	140
T^n		Torus (of dimension n)	422E
H^n	$H^n(X, A)$	n -dimensional cohomology group (of X with coefficients in A)	201H
H_n	$H_n(X, A)$	n -dimensional homology group (of X with coefficients in A)	201G
	$H_n(C)$	(of chain complex C)	201B
π_n	$\pi_n(X)$	n -dimensional homotopy group (of X)	202J, 170
∂	∂C	Boundary (of C)	201B
δ	δf	Coboundary (of f)	201H*
Sq	$Sq^i x$	Streenrod square (of x)	64B
\mathscr{P}	$\mathscr{P}_p^r(x)$	Steenrod p th power (of x)	64B
\smile	$z_1 \smile z_2$	Cup product (of z_1 and z_2)	201I
\frown	$z_1 \frown z_2$	Cap product (of z_1 and z_2)	201K
\wedge	$\omega \wedge \eta$	Exterior product (of two differential forms ω and η)	105Q*
d	$d\omega$	Exterior derivative (of a differential form ω)	105Q
grad	grad φ	Gradient (of a function φ)	442D
rot	rot \mathbf{u}	Rotation (of a vector \mathbf{u})	442D
div	div \mathbf{u}	Divergence (of a vector \mathbf{u})	442D
Δ	$\Delta\varphi$	Laplacian (of a function φ)	323A
\square	$\square\varphi$	d'Alembertian (of a function φ)	130A
D	$D\varphi$	Differential operator	112A*
$\frac{D(u_1, \dots, u_n)}{D(x_1, \dots, x_n)}, \left \frac{\partial u_i}{\partial x_j} \right , \det \left(\frac{\partial u_i}{\partial x_j} \right)$		Jacobian determinant (of (u_1, \dots, u_n) with respect to (x_1, \dots, x_n))	208B
$\frac{\partial(u_1, \dots, u_n)}{\partial(x_1, \dots, x_n)}, \left(\frac{\partial u_i}{\partial x_j} \right)$		Jacobian matrix (of (u_1, \dots, u_n) with respect to (x_1, \dots, x_n))	208B

IX. Function Spaces

C	$C(\Omega)$	Space of continuous functions (on Ω)	168B(1)
-----	-------------	--	---------

Notation	Example	Definition	Article and Section
L_p	$L_p(\Omega), L_p(a, b)$	Space of functions such that $ f ^p$ is integrable on Ω	168B(2)
C^l	$C^l(L)(1 \leq l \leq \infty)$	Space of functions of class C^l	168B(9)
\mathcal{D}	$\mathcal{D}(\Omega)$	Space of C^∞ functions with compact support	168B(13)
\mathcal{E}	$\mathcal{E}(\Omega)$	Space of C^∞ functions	168B(13)

[For $\mathcal{A}(\Omega), A(\Omega), A_p(\Omega), \mathcal{B}(\Omega)(=D_{L^2}(\Omega)), BMO(\mathbf{R}^n), BV(\Omega), c, C, C_0(\Omega), C_\infty(\Omega), C_0^l(\Omega), \mathcal{D}_{L^p}(\Omega), \mathcal{D}_{(M^p)}(\Omega), \mathcal{D}_{(M^p)}(\Omega), \mathcal{E}_{(M^p)}(\Omega), \mathcal{E}_{(M^p)}(\Omega), H_p(\mathbf{R}^n), H^l(\Omega), H_0^l(\Omega), \Lambda^S(\mathbf{R}^n), \bigcap \lambda(\alpha^{(k)}), \sum \lambda^\times(\alpha^{(k)}), l_p, L_{(p,q)}(\Omega), m, M(\Omega), \mathcal{O}(\Omega), \mathcal{O}_p(\Omega), \mathcal{S}, s, S(\Omega), W_p^l(\Omega) \rightarrow 168$ Function Spaces. For $\mathcal{B}(\Omega)$ (Space of Sato hyperfunctions), $\mathcal{D}'(\Omega), \mathcal{E}'(\Omega), \mathcal{O}_c, \mathcal{O}_M, \mathcal{S}'(\mathbf{R}^n) \rightarrow 125$ Distributions and Hyperfunctions]

X. Functions

$ $	$ z $	Absolute value (of a complex number z)	74B*
Re	$\operatorname{Re} z$	Real part (of a complex number z)	74A
Im	$\operatorname{Im} z$	Imaginary part (of a complex number z)	74A*
$-$	\bar{z}	Complex conjugate (of a complex number z)	74A
arg	$\arg z$	Argument (of a complex number z)	74C
$[\]$	$[\alpha]$	Gauss symbol (greatest integer not exceeding a real number α)	83A
O	$f(x)=O(g(x))$	Landau's notation ($f(x)/g(x)$ is bounded for $x \rightarrow \alpha$)	87G
o	$f(x)=o(g(x))$	Landau's notation ($f(x)/g(x)$ tends to 0 for $x \rightarrow \alpha$)	87G
\sim	$f(x) \sim g(x)$	Infinite or infinitesimal of the same order (for $x \rightarrow \alpha$)	87G*
D	$D(T)$	Domain (of an operator T)	251A
R	$R(T)$	Range (of an operator T)	251A
supp	$\operatorname{supp} f$	Support (of a function f)	168B(1)
p.v.	$\text{p.v.} \int_a^b f(x) dx$	Cauchy's principal value (of an integral)	216D
Pf	$\operatorname{Pf} \int f(x) dx$	Finite part (of an integral)	125C
δ	$\delta(x), \delta_x$	Dirac's delta function (measure or distribution)	125C*
exp	$\exp x$	Exponential function ($\exp x = e^x$)	113D, 269H
log, Log	$\log x, \operatorname{Log} x$	Natural logarithmic function and its principal value, respectively	131D, G
$\sin x, \cos x, \tan x, \sec x, \operatorname{cosec} x, \cotan x$		Trigonometric functions	131E
$\arcsin x, \arccos x, \arctan x$		Inverse trigonometric functions	131E
$\operatorname{Arcsin} x, \operatorname{Arccos} x, \operatorname{Arctan} x$		Principal value of inverse trigonometric functions	131E

Notation	Example	Definition	Article and Section
$\sinh x, \cosh x, \tanh x$		Hyperbolic functions	131F
$\binom{n}{r}, C$	$\binom{n}{r}, {}_nC_r$	Binomial coefficient, combination	330
P	${}_nP_r$	Permutation	330
$!$	$n!$	Factorial (of n)	330
φ	$\varphi(n)$	Euler function	295C*
μ	$\mu(n)$	Möbius function	295C
ζ	$\zeta(z)$	Riemann zeta function	450B*
J_v	$J_v(z)$	Bessel function of the first kind	39B
Γ	$\Gamma(x)$	Gamma function	174A
B	$B(x, y)$	Beta function	174C
F	$F(\alpha, \beta, \gamma; z)$	Gauss's hypergeometric function	206A
P	$P \left\{ \begin{matrix} a & b & c \\ \lambda & \mu & \nu \\ \lambda' & \mu' & \nu' \end{matrix} \right\} x$	Riemann's P function	253B
Li	$\text{Li}(x)$	Logarithmic integral	167D
XI. Probability			
P, Pr	$P(E), \text{Pr}(e)$	Probability (of an event)	342B*
E	$E(X)$	Mean or expectation (of a random variable X)	342C
V, σ^2	$V(X), \sigma^2(X)$	Variance (of a random variable X)	342C
ρ	$\rho(X, Y)$	Correlation coefficient (of two random variables X and Y)	342C*
$P()$	$P(E F)$	Conditional probability (of an event E under the condition F)	342E
$E()$	$E(X Y)$	Conditional mean (of a random variable X under the condition Y)	342E
N	$N(m, \sigma^2)$	One-dimensional normal distribution (with mean m and variance σ^2)	Appendix A, Table 22
	$N(\boldsymbol{\mu}, \boldsymbol{\Sigma})$	Multidimensional normal distribution (with mean vector $\boldsymbol{\mu}$ and variance matrix $\boldsymbol{\Sigma}$)	Appendix A, Table 22
P	$P(\lambda)$	Poisson distribution (with parameter λ)	Appendix A, Table 22*

Systematic List of Articles

I

Logic and Foundations

1. Foundations of Mathematics	Art. 156
2. Axiom Systems	35
3. Paradoxes	319
4. Symbolic Logic	411
5. Axiomatic Set Theory	33
6. Model Theory	276
7. Nonstandard Analysis	293
8. Gödel Numbers	185
9. Recursive Functions	356
10. Decision Problem	97
11. Constructive Ordinal Numbers	81
12. Analytic Sets	22

II

Sets, General Topology, and Categories

1. Sets	381
2. Relations	358
3. Equivalence Relations	135
4. Functions	165
5. Axiom of Choice and Equivalents	34
6. Cardinal Numbers	49
7. Structures	409
8. Permutations and Combinations	330
9. Numbers	294
10. Real Numbers	355
11. Complex Numbers	74
12. Ordering	311
13. Ordinal Numbers	312
14. Lattices	243
15. Boolean Algebras	42
16. Topological Spaces	425
17. Metric Spaces	273
18. Plane Domains	333
19. Convergence	87
20. Connectedness	79
21. Dimension Theory	117
22. Uniform Spaces	436
23. Uniform Convergence	435
24. Categories and Functors	52
25. Inductive Limits and Projective Limits	210
26. Sheaves	383

III

Algebra

1. Algebra	8
2. Matrices	269
3. Determinants	103
4. Polynomials	337
5. Algebraic Equations	10
6. Fields	149
7. Galois Theory	172
8. Linear Spaces	256
9. Rings	368
10. Associative Algebras	29
11. Commutative Rings	67
12. Noetherian Rings	284
13. Rings of Polynomials	369
14. Rings of Power Series	370

15. Quadratic Forms	Art. 348
16. Clifford Algebras	61
17. Differential Rings	113
18. Witt Vectors	449
19. Valuations	439
20. Adeles and Ideles	6
21. Cayley Algebras	54
22. Jordan Algebras	231
23. Modules	277
24. Homological Algebra	200
25. Hopf Algebras	203
Appendix A, Table 1. Algebraic Equations	

IV

Group Theory

1. Groups	190
2. Abelian Groups	2
3. Free Groups	161
4. Finite Groups	151
5. Classical Groups	60
6. Topological Groups	423
7. Topological Abelian Groups	422
8. Compact Groups	69
9. Lie Groups	249
10. Lie Algebras	248
11. Algebraic Groups	13
12. Homogeneous Spaces	199
13. Symmetric Riemannian Spaces and Real Forms	412
14. Discontinuous Groups	122
15. Crystallographic Groups	92
16. Representations	362
17. Unitary Representations	437
18. Invariants and Covariants	226
Appendix A, Table 5. Lie Algebras, Symmetric Riemannian Spaces, and Singularities	
Appendix B, Table 5. Characters of Finite Groups and Crystallographic Groups	

V

Number Theory

1. Number Theory	296
2. Number Theory, Elementary	297
3. Continued Fractions	83
4. Number-Theoretic Functions	295
5. Additive Number Theory	4
6. Partitions of Numbers	328
7. Distribution of Prime Numbers	123
8. Lattice-Point Problems	242
9. Diophantine Equations	118
10. Geometry of Numbers	182
11. Transcendental Numbers	430
12. Quadratic Fields	347
13. Algebraic Number Fields	14
14. Class Field Theory	59
15. Complex Multiplication	73
16. Fermat's Problem	145
17. Local Fields	257
18. Arithmetic of Associative Algebras	27
19. Zeta Functions	450
Appendix B, Table 1. Prime Numbers and Primitive Roots	
Appendix B, Table 2. Indices Modulo p	

Appendix B, Table 4. Class Numbers of Algebraic Number Fields

VI

Euclidean and Projective Geometry

1. Geometry Art. 181
2. Foundations of Geometry 155
3. Euclidean Geometry 139
4. Euclidean Spaces 140
5. Geometric Construction 179
6. Regular Polyhedra 357
7. π (π) 332
8. Trigonometry 432
9. Conic Sections 78
10. Quadric Surfaces 350
11. Convex Sets 89
12. Vectors 442
13. Coordinates 90
14. Projective Geometry 343
15. Affine Geometry 7
16. Non-Euclidean Geometry 285
17. Conformal Geometry 76
18. Erlangen Program 137
19. Continuous Geometry 85
20. Curves 93
21. Surfaces 410
22. Four-Color Problem 157

Appendix A, Table 2. Trigonometry

Appendix A, Table 3. Vector Analysis and Coordinate Systems

VII

Differential Geometry

1. Differential Geometry 109
 2. Differentiable Manifolds 105
 3. Riemannian Manifolds 364
 4. Connections 80
 5. Tensor Calculus 417
 6. Geodesics 178
 7. Symmetric Spaces 413
 8. G-Structures 191
 9. Complex Manifolds 72
 10. Kähler Manifolds 232
 11. Harmonic Integrals 194
 12. Differential Geometry of Curves and Surfaces 111
 13. Riemannian Submanifolds 365
 14. Minimal Submanifolds 275
 15. Harmonic Mappings 195
 16. Morse Theory 279
 17. Differential Geometry in Specific Spaces 110
 18. Finsler Spaces 152
 19. Integral Geometry 218
 20. Siegel Domains 384
 21. Spectral Geometry 391
 22. Pseudoconformal Geometry 344
 23. Global Analysis 183
- Appendix A, Table 4. Differential Geometry

VIII

Algebraic Geometry

1. Algebraic Geometry 12
2. Algebraic Curves 9

3. Algebraic Surfaces Art. 15
4. Algebraic Varieties 16
5. Abelian Varieties 3
6. Riemann-Roch Theorems 366

IX

Topology

1. Topology 426
 2. Fundamental Groups 170
 3. Covering Spaces 91
 4. Degree of Mapping 99
 5. Complexes 70
 6. Homology Theory 201
 7. Fixed-Point Theorems 153
 8. Cohomology Operations 64
 9. Homotopy Theory 202
 10. Fiber Spaces 148
 11. Obstructions 305
 12. Topology of Lie Groups and Homogeneous Spaces 427
 13. Fiber Bundles 147
 14. Characteristic Classes 56
 15. K -Theory 237
 16. Knot Theory 235
 17. Combinatorial Manifolds 65
 18. Differential Topology 114
 19. Transformation Groups 431
 20. Theory of Singularities 418
 21. Foliations 154
 22. Dynamical Systems 126
 23. Shape Theory 382
 24. Catastrophe Theory 51
- Appendix A, Table 5. Lie Algebra, Symmetric Riemannian Spaces, and Singularities
- Appendix A, Table 6. Topology
- Appendix A, Table 7. Knot Theory

X

Analysis

1. Analysis 20
2. Continuous Functions 84
3. Inequalities 211
4. Convex Analysis 88
5. Functions of Bounded Variation 166
6. Differential Calculus 106
7. Implicit Functions 208
8. Elementary Functions 131
9. C^∞ -Functions and Quasi-Analytic Functions 58
10. Integral Calculus 216
11. Curvilinear Integrals and Surface Integrals 94
12. Measure Theory 270
13. Integration Theory 221
14. Invariant Measures 225
15. Set Functions 380
16. Length and Area 246
17. Denjoy Integrals 100
18. Series 379
19. Asymptotic Series 30
20. Polynomial Approximation 336
21. Orthogonal Functions 317
22. Fourier Series 159

- 23. Fourier Transform Art. 160
- 24. Harmonic Analysis 192
- 25. Almost Periodic Functions 18
- 26. Laplace Transform 240
- 27. Integral Transforms 220
- 28. Potential Theory 338
- 29. Harmonic Functions and Subharmonic Functions 193
- 30. Dirichlet Problem 120
- 31. Capacity 48
- 32. Calculus of Variations 46
- 33. Plateau's Problem 334
- 34. Isoperimetric Problems 228
- 35. Variational Inequalities 440

Appendix A, Table 8. Inequalities

Appendix A, Table 9. Differential and Integral Calculus

Appendix A, Table 10. Series

Appendix A, Table 11. Fourier Analysis

Appendix A, Table 12. Laplace Transforms and Operational Calculus

Appendix A, Table 20. Systems of Orthogonal Functions

XI

Complex Analysis

- 1. Holomorphic Functions 198
 - 2. Power Series 339
 - 3. Dirichlet Series 121
 - 4. Bounded Functions 43
 - 5. Univalent and Multivalent Functions 438
 - 6. Transcendental Entire Functions 429
 - 7. Meromorphic Functions 272
 - 8. Distribution of Values of Functions of a Complex Variable 124
 - 9. Cluster Sets 62
 - 10. Algebraic Functions 11
 - 11. Algebroidal Functions 17
 - 12. Riemann Surfaces 367
 - 13. Ideal Boundaries 207
 - 14. Conformal Mappings 77
 - 15. Quasiconformal Mappings 352
 - 16. Teichmüller Spaces 416
 - 17. Kleinian Groups 234
 - 18. Extremal Length 143
 - 19. Function-Theoretic Null Sets 169
 - 20. Analytic Functions of Several Complex Variables 21
 - 21. Analytic Spaces 23
 - 22. Automorphic Functions 32
- Appendix A, Table 13. Conformal Mappings

XII

Functional Analysis

- 1. Functional Analysis 162
- 2. Hilbert Spaces 197
- 3. Banach Spaces 37
- 4. Ordered Linear Spaces 310
- 5. Topological Linear Spaces 424
- 6. Function Spaces 168
- 7. Distributions and Hyperfunctions 125
- 8. Vector-Valued Integrals 443
- 9. Linear Operators 251

- 10. Compact and Nuclear Operators Art. 68
- 11. Interpolation of Operators 224
- 12. Spectral Analysis of Operators 390
- 13. Perturbation of Linear Operators 331
- 14. Semigroups of Operators and Evolution Equations 378
- 15. Differential Operators 112
- 16. Microlocal Analysis 274
- 17. Banach Algebras 36
- 18. Function Algebras 164
- 19. Operator Algebras 308
- 20. Operational Calculus 306
- 21. Nonlinear Functional Analysis 286

XIII

Differential, Integral, and Functional Equations

- 1. Differential Equations 107
- 2. Ordinary Differential Equations 313
- 3. Ordinary Differential Equations (Initial Value Problems) 316
- 4. Ordinary Differential Equations (Boundary Value Problems) 315
- 5. Ordinary Differential Equations (Asymptotic Behavior of Solutions) 314
- 6. Linear Ordinary Differential Equations 252
- 7. Linear Ordinary Differential Equations (Local Theory) 254
- 8. Linear Ordinary Differential Equations (Global Theory) 253
- 9. Nonlinear Ordinary Differential Equations (Local Theory) 289
- 10. Nonlinear Ordinary Differential Equations (Global Theory) 288
- 11. Nonlinear Oscillation 290
- 12. Nonlinear Problems 291
- 13. Stability 394
- 14. Integral Invariants 219
- 15. Difference Equations 104
- 16. Functional-Differential Equations 163
- 17. Total Differential Equations 428
- 18. Contact Transformations 82
- 19. Partial Differential Equations 320
- 20. Partial Differential Equations (Methods of Integration) 322
- 21. Partial Differential Equations (Initial Value Problems) 321
- 22. Partial Differential Equations of First Order 324
- 23. Monge-Ampère Equations 278
- 24. Partial Differential Equations of Elliptic Type 323
- 25. Partial Differential Equations of Hyperbolic Type 325
- 26. Partial Differential Equations of Parabolic Type 327
- 27. Partial Differential Equations of Mixed Type 326
- 28. Green's Functions 188
- 29. Green's Operators 189
- 30. Integral Equations 217
- 31. Integrodifferential Equations 222
- 32. Special Functional Equations 388
- 33. Pseudodifferential Operators 345

Appendix A, Table 14. Ordinary Differential Equations

Appendix A, Table 15. Total and Partial Differential Equations

XIV

Special Functions

1. Special Functions Art. 389
2. Generating Functions 177
3. Elliptic Functions 134
4. Gamma Function 174
5. Hypergeometric Functions 206
6. Spherical Functions 393
7. Functions of Confluent Type 167
8. Bessel Functions 39
9. Ellipsoidal Harmonics 133
10. Mathieu Functions 268

Appendix A, Table 16. Elliptic Integrals and Elliptic Functions

Appendix A, Table 17. Gamma Function and Related Functions

Appendix A, Table 18. Hypergeometric Functions and Spherical Functions

Appendix A, Table 19. Functions of Confluent Type and Bessel Functions

Appendix B, Table 3. Bernoulli Numbers and Euler Numbers

XV

Numerical Analysis

1. Numerical Methods 300
 2. Interpolation 223
 3. Error Analysis 138
 4. Numerical Solution of Linear Equations 302
 5. Numerical Solution of Algebraic Equations 301
 6. Numerical Computation of Eigenvalues 298
 7. Numerical Integration 299
 8. Numerical Solution of Ordinary Differential Equations 303
 9. Numerical Solution of Partial Differential Equations 304
 10. Analog Computation 19
 11. Evaluation of Functions 142
- Appendix A, Table 21. Interpolation
- Appendix B, Table 6. Miscellaneous Constants

XVI

Computer Science and Combinatorics

1. Automata 31
2. Computers 75
3. Coding Theory 63
4. Cybernetics 95
5. Random Numbers 354
6. Simulation 385
7. Data Processing 96
8. Mathematical Models in Biology 263
9. Complexity of Computations 71
10. Combinatorics 66
11. Latin Squares 241
12. Graph Theory 186

XVII

Probability Theory

1. Probability Theory Art. 342
2. Probability Measures 341
3. Limit Theorems in Probability Theory 250
4. Stochastic Processes 407
5. Markov Processes 261
6. Markov Chains 260
7. Brownian Motion 45
8. Diffusion Processes 115
9. Additive Processes 5
10. Branching Processes 44
11. Martingales 262
12. Stationary Processes 395
13. Gaussian Processes 176
14. Stochastic Differential Equations 406
15. Ergodic Theory 136
16. Stochastic Control and Stochastic Filtering 405
17. Probabilistic Methods in Statistical Mechanics 340

XVIII

Statistics

1. Statistical Data Analysis 397
 2. Statistical Inference 401
 3. Statistic 396
 4. Sampling Distributions 374
 5. Statistical Models 403
 6. Statistical Decision Functions 398
 7. Statistical Estimation 399
 8. Statistical Hypothesis Testing 400
 9. Multivariate Analysis 280
 10. Robust and Nonparametric Methods 371
 11. Time Series Analysis 421
 12. Design of Experiments 102
 13. Sample Survey 373
 14. Statistical Quality Control 404
 15. Econometrics 128
 16. Biometrics 40
 17. Psychometrics 346
 18. Insurance Mathematics 214
- Appendix A, Table 22. Distribution of Typical Random Variables
- Appendix A, Table 23. Statistical Estimation and Statistical Hypothesis Testing

XIX

Mathematical Programming and Operations Research

1. Mathematical Programming 264
2. Linear Programming 255
3. Quadratic Programming 349
4. Nonlinear Programming 292
5. Network Flow Problems 281
6. Integer Programming 215
7. Stochastic Programming 408
8. Dynamic Programming 127
9. Game Theory 173
10. Differential Games 108
11. Control Theory 86

- 12. Information Theory Art. 213
- 13. Operations Research 307
- 14. Inventory Control 227
- 15. Scheduling and Production Planning
376

XX**Mechanics and Theoretical Physics**

- 1. Systems of Units 414
- 2. Dimensional Analysis 116
- 3. Variational Principles 441
- 4. Mechanics 271
- 5. Spherical Astronomy 392
- 6. Celestial Mechanics 55
- 7. Orbit Determination 309
- 8. Three-Body Problem 420
- 9. Hydrodynamics 205
- 10. Hydrodynamical Equations 204
- 11. Magnetohydrodynamics 259
- 12. Turbulence and Chaos 433
- 13. Wave Propagation 446
- 14. Oscillations 318
- 15. Geometric Optics 180
- 16. Electromagnetism 130
- 17. Networks 282
- 18. Thermodynamics 419
- 19. Statistical Mechanics 402
- 20. Boltzmann Equation 41
- 21. Relativity 359
- 22. Unified Field Theory 434
- 23. Quantum Mechanics 351
- 24. Lorentz Group 258
- 25. Racah Algebra 353
- 26. Scattering Theory 375
- 27. Second Quantization 377
- 28. Field Theory 150
- 29. S-Matrices 386
- 30. Feynman Integrals 146
- 31. Elementary Particles 132
- 32. Renormalization Group 361
- 33. Nonlinear Lattice Dynamics 287
- 34. Solitons 387
- 35. Approximation Methods in Physics
25
- 36. Inequalities in Physics 212

XXI**History of Mathematics**

- 1. Ancient Mathematics 24
- 2. Greek Mathematics 187
- 3. Roman and Medieval Mathematics
372
- 4. Arab Mathematics 26
- 5. Indian Mathematics 209
- 6. Chinese Mathematics 57
- 7. Japanese Mathematics (Wasan) 230
- 8. Renaissance Mathematics 360
- 9. Mathematics in the 17th Century 265
- 10. Mathematics in the 18th Century 266
- 11. Mathematics in the 19th Century 267
- 12. Abel, Niels Henrik 1
- 13. Artin, Emil 28

- 14. Bernoulli Family Art. 38
- 15. Cantor, Georg 47
- 16. Cartan, Elie 50
- 17. Cauchy, Augustin Louis 53
- 18. Dedekind, Julius Wilhelm Richard 98
- 19. Descartes, René 101
- 20. Dirichlet, Peter Gustav Lejeune 119
- 21. Einstein, Albert 129
- 22. Euler, Leonhard 141
- 23. Fermat, Pierre de 144
- 24. Fourier, Jean Baptiste Joseph 158
- 25. Galois, Evariste 171
- 26. Gauss, Carl Friedrich 175
- 27. Gödel, Kurt 184
- 28. Hilbert, David 196
- 29. Jacobi, Carl Gustav Jacob 229
- 30. Klein, Felix 233
- 31. Kronecker, Leopold 236
- 32. Lagrange, Joseph Louis 238
- 33. Laplace, Pierre Simon 239
- 34. Lebesgue, Henri Léon 244
- 35. Leibniz, Gottfried Wilhelm 245
- 36. Lie, Marius Sophus 247
- 37. Newton, Isaac 283
- 38. Pascal, Blaise 329
- 39. Poincaré, Henri 335
- 40. Riemann, Georg Friedrich Bernhard
363
- 41. Takagi, Teiji 415
- 42. Viète, François 444
- 43. Von Neumann, John 445
- 44. Weierstrass, Karl 447
- 45. Weyl, Hermann 448

Alphabetical List of Articles

1. Abel, Niels Henrik
2. Abelian Groups
3. Abelian Varieties
4. Additive Number Theory
5. Additive Processes
6. Adeles and Ideles
7. Affine Geometry
8. Algebra
9. Algebraic Curves
10. Algebraic Equations
11. Algebraic Functions
12. Algebraic Geometry
13. Algebraic Groups
14. Algebraic Number Fields
15. Algebraic Surfaces
16. Algebraic Varieties
17. Algebroidal Functions
18. Almost Periodic Functions
19. Analog Computation
20. Analysis
21. Analytic Functions of Several Complex Variables
22. Analytic Sets
23. Analytic Spaces
24. Ancient Mathematics
25. Approximation Methods in Physics
26. Arab Mathematics
27. Arithmetic of Associative Algebras
28. Artin, Emil
29. Associative Algebras
30. Asymptotic Series
31. Automata
32. Automorphic Functions
33. Axiomatic Set Theory
34. Axiom of Choice and Equivalents
35. Axiom Systems
36. Banach Algebras
37. Banach Spaces
38. Bernoulli Family
39. Bessel Functions
40. Biometrics
41. Boltzmann Equation
42. Boolean Algebras
43. Bounded Functions
44. Branching Processes
45. Brownian Motion
46. Calculus of Variations
47. Cantor, Georg
48. Capacity
49. Cardinal Numbers
50. Cartan, Elie
51. Catastrophe Theory
52. Categories and Functors
53. Cauchy, Augustin Louis
54. Cayley Algebras
55. Celestial Mechanics
56. Characteristic Classes
57. Chinese Mathematics
58. C^∞ -Functions and Quasi-Analytic Functions
59. Class Field Theory
60. Classical Groups
61. Clifford Algebras
62. Cluster Sets
63. Coding Theory
64. Cohomology Operations
65. Combinatorial Manifolds
66. Combinatorics
67. Commutative Rings
68. Compact and Nuclear Operators
69. Compact Groups
70. Complexes
71. Complexity of Computations
72. Complex Manifolds
73. Complex Multiplication
74. Complex Numbers
75. Computers
76. Conformal Geometry
77. Conformal Mappings
78. Conic Sections
79. Connectedness
80. Connections
81. Constructive Ordinal Numbers
82. Contact Transformations
83. Continued Fractions
84. Continuous Functions
85. Continuous Geometry
86. Control Theory
87. Convergence
88. Convex Analysis
89. Convex Sets
90. Coordinates
91. Covering Spaces
92. Crystallographic Groups
93. Curves
94. Curvilinear Integrals and Surface Integrals
95. Cybernetics
96. Data Processing
97. Decision Problem
98. Dedekind, Julius Wilhelm Richard
99. Degree of Mapping
100. Denjoy Integrals
101. Descartes, René
102. Design of Experiments
103. Determinants
104. Difference Equations
105. Differentiable Manifolds
106. Differential Calculus
107. Differential Equations
108. Differential Games
109. Differential Geometry
110. Differential Geometry in Specific Spaces
111. Differential Geometry of Curves and Surfaces
112. Differential Operators
113. Differential Rings
114. Differential Topology
115. Diffusion Processes
116. Dimensional Analysis

117. Dimensional Theory
118. Diophantine Equations
119. Dirichlet, Peter Gustav Lejeune
120. Dirichlet Problem
121. Dirichlet Series
122. Discontinuous Groups
123. Distribution of Prime Numbers
124. Distribution of Values of Functions of a Complex Variable
125. Distributions and Hyperfunctions
126. Dynamical Systems
127. Dynamic Programming
128. Econometrics
129. Einstein, Albert
130. Electromagnetism
131. Elementary Functions
132. Elementary Particles
133. Ellipsoidal Harmonics
134. Elliptic Functions
135. Equivalence Relations
136. Ergodic Theory
137. Erlangen Program
138. Error Analysis
139. Euclidean Geometry
140. Euclidean Spaces
141. Euler, Leonhard
142. Evaluation of Functions
143. Extremal Length
144. Fermat, Pierre de
145. Fermat's Problem
146. Feynman Integrals
147. Fiber Bundles
148. Fiber Spaces
149. Fields
150. Field Theory
151. Finite Groups
152. Finsler Spaces
153. Fixed-Point Theorems
154. Foliations
155. Foundations of Geometry
156. Foundations of Mathematics
157. Four-Color Problem
158. Fourier, Jean Baptiste Joseph
159. Fourier Series
160. Fourier Transform
161. Free Groups
162. Functional Analysis
163. Functional-Differential Equations
164. Function Algebras
165. Functions
166. Functions of Bounded Variation
167. Functions of Confluent Type
168. Function Spaces
169. Function-Theoretic Null Sets
170. Fundamental Groups
171. Galois, Evariste
172. Galois Theory
173. Game Theory
174. Gamma Function
175. Gauss, Carl Friedrich
176. Gaussian Processes
177. Generating Functions
178. Geodesics
179. Geometric Construction
180. Geometric Optics
181. Geometry
182. Geometry of Numbers
183. Global Analysis
184. Gödel, Kurt
185. Gödel Numbers
186. Graph Theory
187. Greek Mathematics
188. Green's Functions
189. Green's Operator
190. Groups
191. G-Structures
192. Harmonic Analysis
193. Harmonic Functions and Subharmonic Functions
194. Harmonic Integrals
195. Harmonic Mappings
196. Hilbert, David
197. Hilbert Spaces
198. Holomorphic Functions
199. Homogeneous Spaces
200. Homological Algebra
201. Homology Theory
202. Homotopy Theory
203. Hopf Algebras
204. Hydrodynamical Equations
205. Hydrodynamics
206. Hypergeometric Functions
207. Ideal Boundaries
208. Implicit Functions
209. Indian Mathematics
210. Inductive Limits and Projective Limits
211. Inequalities
212. Inequalities in Physics
213. Information Theory
214. Insurance Mathematics
215. Integer Programming
216. Integral Calculus
217. Integral Equations
218. Integral Geometry
219. Integral Invariants
220. Integral Transforms
221. Integration Theory
222. Integrodifferential Equations
223. Interpolation
224. Interpolation of Operators
225. Invariant Measures
226. Invariants and Covariants
227. Inventory Control
228. Isoperimetric Problems
229. Jacobi, Carl Gustav Jacob
230. Japanese Mathematics (Wasan)
231. Jordan Algebras
232. Kähler Manifolds
233. Klein, Felix
234. Kleinian Groups
235. Knot Theory
236. Kronecker, Leopold

237. *K*-Theory
238. Lagrange, Joseph Louis
239. Laplace, Pierre Simon
240. Laplace Transform
241. Latin Squares
242. Lattice-Point Problems
243. Lattices
244. Lebesgue, Henri Léon
245. Leibniz, Gottfried Wilhelm
246. Length and Area
247. Lie, Marius Sophus
248. Lie Algebras
249. Lie Groups
250. Limit Theorems in Probability Theory
251. Linear Operators
252. Linear Ordinary Differential Equations
253. Linear Ordinary Differential Equations (Global Theory)
254. Linear Ordinary Differential Equations (Local Theory)
255. Linear Programming
256. Linear Spaces
257. Local Fields
258. Lorentz Group
259. Magnetohydrodynamics
260. Markov Chains
261. Markov Processes
262. Martingales
263. Mathematical Models in Biology
264. Mathematical Programming
265. Mathematics in the 17th Century
266. Mathematics in the 18th Century
267. Mathematics in the 19th Century
268. Mathieu Functions
269. Matrices
270. Measure Theory
271. Mechanics
272. Meromorphic Functions
273. Metric Spaces
274. Microlocal Analysis
275. Minimal Submanifolds
276. Model Theory
277. Modules
278. Monge-Ampère Equations
279. Morse Theory
280. Multivariate Analysis
281. Network Flow Problems
282. Networks
283. Newton, Isaac
284. Noetherian Rings
285. Non-Euclidean Geometry
286. Nonlinear Functional Analysis
287. Nonlinear Lattice Dynamics
288. Nonlinear Ordinary Differential Equations (Global Theory)
289. Nonlinear Ordinary Differential Equations (Local Theory)
290. Nonlinear Oscillation
291. Nonlinear Problems
292. Nonlinear Programming
293. Nonstandard Analysis
294. Numbers
295. Number-Theoretic Functions
296. Number Theory
297. Number Theory, Elementary
298. Numerical Computation of Eigenvalues
299. Numerical Integration
300. Numerical Methods
301. Numerical Solution of Algebraic Equations
302. Numerical Solution of Linear Equations
303. Numerical Solution of Ordinary Differential Equations
304. Numerical Solution of Partial Differential Equations
305. Obstructions
306. Operational Calculus
307. Operations Research
308. Operator Algebras
309. Orbit Determination
310. Ordered Linear Spaces
311. Ordering
312. Ordinal Numbers
313. Ordinary Differential Equations
314. Ordinary Differential Equations (Asymptotic Behavior of Solutions)
315. Ordinary Differential Equations (Boundary Value Problems)
316. Ordinary Differential Equations (Initial Value Problems)
317. Orthogonal Functions
318. Oscillations
319. Paradoxes
320. Partial Differential Equations
321. Partial Differential Equations (Initial Value Problems)
322. Partial Differential Equations (Methods of Integration)
323. Partial Differential Equations of Elliptic Type
324. Partial Differential Equations of First Order
325. Partial Differential Equations of Hyperbolic Type
326. Partial Differential Equations of Mixed Type
327. Partial Differential Equations of Parabolic Type
328. Partitions of Numbers
329. Pascal, Blaise
330. Permutations and Combinations
331. Perturbation of Linear Operators
332. Pi (π)
333. Plane Domains
334. Plateau's Problem
335. Poincaré, Henri
336. Polynomial Approximation
337. Polynomials
338. Potential Theory
339. Power Series
340. Probabilistic Methods in Statistical Mechanics
341. Probability Measures
342. Probability Theory
343. Projective Geometry
344. Pseudoconformal Geometry

345. Pseudodifferential Operators
346. Psychometrics
347. Quadratic Fields
348. Quadratic Forms
349. Quadratic Programming
350. Quadric Surfaces
351. Quantum Mechanics
352. Quasiconformal Mappings
353. Racah Algebra
354. Random Numbers
355. Real Numbers
356. Recursive Functions
357. Regular Polyhedra
358. Relations
359. Relativity
360. Renaissance Mathematics
361. Renormalization Group
362. Representations
363. Riemann, Georg Friedrich Bernhard
364. Riemannian Manifolds
365. Riemannian Submanifolds
366. Riemann-Roch Theorems
367. Riemann Surfaces
368. Rings
369. Rings of Polynomials
370. Rings of Power Series
371. Robust and Nonparametric Methods
372. Roman and Medieval Mathematics
373. Sample Survey
374. Sampling Distributions
375. Scattering Theory
376. Scheduling and Production Planning
377. Second Quantization
378. Semigroups of Operators and Evolution Equations
379. Series
380. Set Functions
381. Sets
382. Shape Theory
383. Sheaves
384. Siegel Domains
385. Simulation
386. S-Matrices
387. Solitons
388. Special Functional Equations
389. Special Functions
390. Spectral Analysis of Operators
391. Spectral Geometry
392. Spherical Astronomy
393. Spherical Functions
394. Stability
395. Stationary Processes
396. Statistic
397. Statistical Data Analysis
398. Statistical Decision Functions
399. Statistical Estimation
400. Statistical Hypothesis Testing
401. Statistical Inference
402. Statistical Mechanics
403. Statistical Models
404. Statistical Quality Control
405. Stochastic Control and Stochastic Filtering
406. Stochastic Differential Equations
407. Stochastic Processes
408. Stochastic Programming
409. Structures
410. Surfaces
411. Symbolic Logic
412. Symmetric Riemannian Spaces and Real Forms
413. Symmetric Spaces
414. Systems of Units
415. Takagi, Teiji
416. Teichmüller Spaces
417. Tensor Calculus
418. Theory of Singularities
419. Thermodynamics
420. Three-Body Problem
421. Time Series Analysis
422. Topological Abelian Groups
423. Topological Groups
424. Topological Linear Spaces
425. Topological Spaces
426. Topology
427. Topology of Lie Groups and Homogeneous Spaces
428. Total Differential Equations
429. Transcendental Entire Functions
430. Transcendental Numbers
431. Transformation Groups
432. Trigonometry
433. Turbulence and Chaos
434. Unified Field Theory
435. Uniform Convergence
436. S-Matrices.
437. Unitary Representations
438. Univalent and Multivalent Functions
439. Valuations
440. Variational Inequalities
441. Variational Principles
442. Vectors
443. Vector-Valued Integrals
444. Viète, François
445. Von Neumann, John
446. Wave Propagation
447. Weierstrass, Karl
448. Weyl, Hermann
449. Witt Vectors
450. Zeta Functions

Contributors to the Second Edition

Abe Eiichi
Akahira Masafumi
Akaike Hirotugu
Akaza Touru
Amari Shun-ichi
Ando Tsuyoshi
Araki Huzihiro
Araki Shôrô
Arimoto Suguru
Asai Akira
Fujiï'e Tatuo
Fujiki Akira
Fujisaki Genjiro
Fujita Hiroshi
Fujiwara Daisuke
Fujiwara Masahiko
Fukushima Masatoshi
Furuya Shigeru
Hamada Yûsaku
Hasumi Morisuke
Hattori Akio
Hayashi Kiyoshi
Hida Takeyuki
Hijikata Hiroaki
Hirose Ken
Hitotumatu Sin
Huzii Mituaki
Ibaraki Toshihide
Ihara Shin-ichiro
Iitaka Shigeru
Ikawa Mitsuru
Ikebe Teruo
Ikebe Yasuhiko
Ikeda Nobuyuki
Inagaki Nobuo
Inoue Atsushi
Iri Masao
Itaya Nobutoshi
Itô Kiyosi
Itô Seizô
Ito Yuji
Itoh Mitsuhiro
Iwahori Nobuyoshi
Iyanaga Shôkichi
Kadoya Norihiko
Kageyama Sampei
Kamae Teturo
Kancko Akira
Kaneyuki Soji
Kasahara Kôji
Kataoka Shinji
Kato Junji
Kato Mitsuyoshi
Kawai Takahiro
Kawasaki Tetsuro
Kimura Tosihusa
Kishi Masanori

Kobayashi Shoshichi
Kodama Yukihiro
Komatsu Hikosaburo
Kômura Yukio
Kondo Takeshi
Konishi Yoshio
Konno Hiroshi
Kotani Shinichi
Kubota Yoshihisa
Kumano-go Hajime
Kunita Hiroshi
Kuramoto Yuki
Kuroda Shige-Toshi
Kuroda Tadashi
Kusama Tokitake
Kusano Takaâi
Kusunoki Yukio
Mabuchi Toshiki
Maeda Shûichiro
Maeda Yoshiaki
Maehara Shôji
Makabe Hajime
Maruyama Masaki
Matsumoto Kikuji
Matsumoto Yukio
Matsumura Mutsuhide
Mitsui Takayoshi
Mizohata Sigeru
Mizutani Tadayoshi
Mori Masatake
Morimoto Akihiko
Morimura Hidenori
Morita Shigeyuki
Morita Yasuo
Motohashi Nobuyoshi
Murasugi Kunio
Nagami Keio
Nagata Masayoshi
Nakagawa Hisao
Nakamura Tokushi
Nakaoka Minoru
Namba Kanji
Namba Makoto
Namikawa Yukihiro
Naruki Isao
Niïro Fumio
Nishikawa Seiki
Nishimura Junichi
Nishitani Kensaku
Nisio Makiko
Noguchi Hiroshi
Nozaki Akihiro
Ôaku Toshinori
Obata Morio
Ochiai Takushiro
Ogiue Koichi
Ohta Masami
Ohya Yujiro
Oikawa Kôtarô
Oka Mutsuo
Okabe Yasunori

Okuno Tadakazu
Omori Hideki
Oshima Toshio
Ozawa Mitsuru
Saito Kyoji
Saito Masahiko
Saito Tosiya
Sakai Fumio
Sakawa Yoshiyuki
Sato Fumitaka
Sato Ken-iti
Sawashima Ikuko
Shibata Katsuyuki
Shibata Keiichi
Shiga Koji
Shiga Tokuzo
Shikata Yoshihiro
Shimada Nobuo
Shimakura Norio
Shimizu Ryoichi
Shioda Tetsuji
Shiohama Katsuhiko
Shiraiwa Kenichi
Shiratani Katsumi
Sibuya Masaaki
Sugie Toru
Sugiura Mitsuo
Sugiura Shigeaki
Suito Nobuyuki
Sumihiro Hideyasu
Suzuki Masuo
Suzuki Mitsuo
Takahashi Tsuneo
Takakuwa Shoichiro
Takami Hideo
Takasu Toru
Takeuchi Kei
Takeuchi Masaru
Takeuti Gaishi
Tamura Itiro
Tanabe Kunio
Tanaka Hiroshi
Tanaka Masatsugu
Tanaka Shunichi
Tanno Shukichi
Tatsumi Tomomasa
Terada Toshiaki
Toda Hideo
Toda Hiroshi
Toda Morikazu
Toda Nobushige
Tomiya Jun
Totoki Haruo
Tugé Tosiya
Uchida Fuichi
Uchiyama Saburô
Ueno Kenji
Ukai Seiji
Ura Shoji
Ushijima Teruo
Wakabayashi Isao

Washizu Kyuichiro
Watanabe Kimio
Watanabe Shinzo
Watanabe Takesi
Yamaguchi Masaya
Yamamoto Koichi
Yamamoto Sumiyasu
Yamamoto Tetsuro
Yamazaki Masao
Yamazaki Tadashi
Yanagawa Takashi
Yanagiwara Hiroshi
Yanai Haruo
Yano Tamaki
Yokonuma Takeo
Yoshikawa Atsushi
Yoshida Junzo
Yosida Kôzaku

Contributors to the First Edition

Aizawa Sadakazu
Akao Kazuo
Akizuki Yasuo
Amemiya Ayao
Anzai Hirotada
Aoki Kiyoshi
Araki Huzihiro
Araki Shôrô
Arima Akito
Asaka Tetsuichi
Asano Keizo
Asatani Teruo
Azumaya Gorô
Fujinaka Megumu
Fujita Hiroshi
Fujiwara Daisuke
Fukuda Takeo
Fukutomi Setsuo
Furuya Shigeru
Goto Morikuni
Gotô Motinori
Gotô Yûzo
Hagihara Yusuke
Harada Manabu
Hasimoto Hidenori
Hattori Akio
Hattori Akira
Hayashi Chikio
Hayashi Keiichi
Hayashi Tsuyoshi
Hida Takeyuki
Hidaka Koji
Hirakawa Junkô
Hirano Tomoharu
Hirasawa Yoshikazu
Hirayama Akira
Hirose Ken
Hisatake Masao
Hitotumatu Sin
Hokari Shisanji
Homma Kiyomi
Homma Tatsuo
Honbu Hitoshi
Hong Imsik
Hosokawa Fujitsugu
Hukuhara Masuo
Husimi Kozi
Huzii Mituaki
Ichida Asajiro
Ihara Shin-ichiro
Ihara Yasutaka
Iitaka Shigeru
Ikawa Mitsuru
Ikeda Mineo
Ikeda Nobuyuki
Ikehara Shikao
Imai Isao

Inaba Eiji
Inagaki Takeshi
Inoue Masao
Inui Teturo
Iri Masao
Irie Shoji
Ise Mikio
Iseki Kaneshiroo
Ishida Yasushi
Ishii Goro
Isizu Takehiko
Itô Juniti
Itô Kiyosi
Itô Noboru
Itô Seizô
Itô Teiiti
Ito Tsutomu
Itoh Makoto
Iwahashi Ryôsuke
Iwahori Nagayoshi
Iwamura Tsurane
Iwano Masahiro
Iwasawa Kenkichi
Iwata Giiti
Iyanaga Shôkichi
Izumi Shin-ichi
Kaburaki Masaki
Kamae Keiko
Kametani Shunji
Kanbe Tsunekazu
Kanitani Jôyô
Kasahara Koji
Katase Kiyoshi
Kato Tosio
Katuura Sutezo
Kawada Yukiyo
Kawaguchi Akitsugu
Kawai Saburo
Kawakami Riiti
Kawata Tatuo
Kihara Taro
Kimura Motoo
Kimura Tosihusa
Kinoshita Shin'ichi
Kitagawa Toshio
Kiyasu Zen'ichi
Koba Ziro
Kobayashi Mikio
Kobori Akira
Koda Akira
Kodaira Kuniyuko
Koga Yukiyo
Komatsu Hikosaburo
Komatsu Atuo
Komatsu Yûsaku
Kondo Jiro
Kondo Kazuo
Kondô Motokiti
Kondô Ryôji
Kondo Takeshi
Kondo Yôitsu

Kôta Osamu
 Kotake Takesi
 Kotani Masao
 Kozai Yoshihide
 Kubo Izumi
 Kubo Ryogo
 Kubota Tadahiko
 Kudo Akio
 Kudô Hirokichi
 Kudo Tatsuji
 Kuga Michio
 Kunisawa Kiyonori
 Kunita Hiroshi
 Kunugi Kinjiro
 Kuranishi Masatake
 Kuroda Shige-Toshi
 Kuroda Sigekatsu
 Kuroda Sige-Nobu
 Kusama Tokitake
 Kusano Takasi
 Kusunoki Yukio
 Maehara Shôji
 Maruyama Gisiro
 Mashio Shinji
 Masuda Kyûya
 Masuyama Motosaburo
 Matsukuma Ryozaï
 Matsumoto Kikuji
 Matsumoto Makoto
 Matsumoto Toshizô
 Matsumura Hideyuki
 Matsumura Mutsuhide
 Matsushima Yozo
 Matsuyama Noboru
 Matuda Tizuko
 Matusita Kameo
 Matuzaka Kazuo
 Mikami Masao
 Mimura Yukio
 Minagawa Takizo
 Mita Hiroo
 Mitome Michio
 Mitsui Takayoshi
 Miyake Kazuo
 Miyasawa Koichi
 Miyashita Totaro
 Miyazawa Hironari
 Miyazima Tatuoki
 Mizohata Sigeru
 Mizuno Katsuhiko
 Mogi Isamu
 Mori Shigeo
 Moriguti Sigeiti
 Morimoto Haruki
 Morimoto Seigo
 Morimura Hidenori
 Morishima Taro
 Morita Kiiti
 Morita Yuzo
 Moriya Mikao
 Moriya Tomijiro

Motohashi Nobuyoshi
 Motoo Minoru
 Murakami Shingo
 Muramatsu Tosinobu
 Murata Tamotsu
 Nabeya Seiji
 Nagano Tadashi
 Nagao Hiroshi
 Nagashima Takashi
 Nagata Jun-iti
 Nagata Masayoshi
 Nagumo Mitio
 Nakai Mitsuru
 Nakai Yoshikazu
 Nakamori Kanji
 Nakamura Koshiro
 Nakamura Masahiro
 Nakamura Tokushi
 Nakanishi Shizu
 Nakano Shigeo
 Nakaoka Minoru
 Nakayama Tadasi
 Nakayama Takashi
 Namikawa Yukihiko
 Nanba Kanji
 Narita Masao
 Narumi Seimatsu
 Niino Fumio
 Nikaido Hukukane
 Nishi Mico
 Nishimura Toshio
 Noguchi Hiroshi
 Nomizu Katsumi
 Nomoto Hisao
 Nomura Yukichi
 Nonaka Toshio
 Noshiro Kiyoshi
 Nozaki Akihiro
 Nozaki Yasuo
 Ochiai Kiichiro
 Ogasawara Tôjirô
 Ogawa Junjiro
 Ogawara Masami
 Ogino Shusaku
 Ôhasi Saburo
 Ohtsuka Makoto
 Oikawa Kotaro
 Ôishi Kiyoshi
 Oka Syoten
 Okamoto Kiyosato
 Okamoto Masashi
 Okubo Kenjiro
 Okubo Tanjiro
 Okugawa Kôtarô
 Ônari Setsuo
 Ono Akimasa
 Ono Katuzi
 Ono Shigeru
 Ono Takashi
 Onoyama Takuji
 Oshida Isao

Oshima Nobunori
 Osima Masaru
 Otsuki Tominosuke
 Ozaki Shigeo
 Ozawa Mitsuru
 Ozeki Hideki
 Saito Kinichiro
 Saito Tosiya
 Saito Yoshihiro
 Sakai Eiichi
 Sakai Takuzo
 Sakamoto Heihachi
 Sakata Shôichi
 Sasaki Shigeo
 Satake Ichiro
 Sato Ken-iti
 Sato Ryoichiro
 Sato Tokui
 Sato Yumiko
 Sawashima Ikuko
 Seimiya Toshio
 Seki Setsuya
 Shibagaki Wasao
 Shiga Kôji
 Shimada Nobuo
 Shimizu Hideo
 Shimizu Tatsujiro
 Shimura Goro
 Shintani Takuro
 Shizuma Ryoji
 Shoda Kenjiro
 Shono Shigekata
 Sibuya Masaaki
 Simauti Takakazu
 Sirao Tunekiti
 Sucoka Seiichi
 Suetuna Zyoiti
 Sugawara Masahiro
 Sugawara Masao
 Sugiura Mitsuo
 Sugiura Nariaki
 Sugiyama Shohei
 Suita Nobuyuki
 Sumitomo Takeshi
 Sunouchi Genichiro
 Suzuki Michio
 Suzuki Yoshindo
 Takada Masaru
 Takagi Teiji
 Takagi Yutaka
 Takahashi Hidetoshi
 Takahashi Tsunero
 Takenouchi Osamu
 Takenouchi Tadao
 Takeuchi Kei
 Takeuti Gaisi
 Takizawa Seizi
 Tamagawa Tsuneo
 Tamura Itiro
 Tamura Jirô
 Tanaka Chuji

Tanaka Hiroshi
 Tanaka Hisao
 Tanaka Minoru
 Tanaka Sen-ichiro
 Tannaka Tadao
 Tatsumi Tomomasa
 Tatuzawa Tikao
 Terasaka Hidetaka
 Toda Hideo
 Toda Hiroshi
 Toda Morikazu
 Tôki Yukinari
 Tomotika Susumu
 Totoki Haruo
 Toyama Hiraku
 Tsuboi Chuji
 Tsuchikura Tamotsu
 Tsuji Masatsugu
 Tsujioka Kunio
 Tsukamoto Yôtarô
 Tsuzuku Toshio
 Tugué Tosiyuki
 Tumura Yosiro
 Uchiyama Saburo
 Uchiyama Tatsuo
 Udagawa Masatomô
 Ueno Kenji
 Ueno Tadashi
 Ugaeri Tadashi
 Umegaki Hisaharu
 Umezawa Hiroomi
 Umezawa Toshio
 Uno Toshio
 Ura Shoji
 Ura Taro
 Urabe Minoru
 Wada Yasaku
 Washimi Shinichi
 Washio Yasutoshi
 Washizu Kyuichiro
 Watanabe Hiroshi
 Watanabe Hisao
 Watanabe Masaru
 Watanabe Shinzo
 Yamada Isamu
 Yamaguti Masaya
 Yamamoto Koichi
 Yamanaka Takesi
 Yamanoshita Tsuneyo
 Yamanouti Takahiko
 Yamashita Hideo
 Yamauti Ziro
 Yamazaki Keijiro
 Yanagihara Kichiji
 Yano Kentaro
 Yano Shigeki
 Yoneda Nobuo
 Yoshizawa Hisaaki
 Yosida Kôsaku
 Yosida Setuzô
 Yosida Yôiti

Translators (for the First Edition)

Adachi Masahisa
Aizawa Sadakazu
Akizuki Yasuo
Akô Kiyoshi
Araki Shôrô
Arima Akito
Asano Keizo
Azumaya Goro
Chiba Katsuhiko
Fujisawa Takehisa
Fujita Hiroshi
Fukada Ichiro
Fukuyama Masaru
Furuya Shigeru
Hashimoto Yasuko
Hasimoto Hidenori
Hasumi Morisuke
Hattori Akio
Hattori Akira
Hayashi Kazumichi
Hida Takeyuki
Hinata Shigeru
Hirasawa Yoshikazu
Hirose Ken
Hitotumatu Sin
Honda Taira
Hong Imsik
Hosoi Tsutomu
Hukuhara Masuo
Ihara Shin-ichiro
Ihara Yasutaka
Ikeda Mineo
Ikeda Nobuyuki
Imai Isao
Inoue Masao
Iri Masao
Ishii Goro
Itô Seizô
Ito Yuji
Iwahashi Ryôsuke
Iwahori Nagayoshi
Iwarmura Tsurane
Iwano Masahiro
Iwata Giichi
Iyanaga Kenichi
Iyanaga Shôkichi
Kametani Shunji
Kasahara Koji
Kato Junji
Kawada Yukiyo
Kawano Sanehiko
Kimura Tosihusa
Kitagawa Toshio
Kobori Akira
Kodaira Kunihiko
Komatsu Hikosaburo
Komatu Atuo

Komatu Yûsaku
Kondo Ryôji
Kondo Takeshi
Kozai Yoshihide
Kubo Ryogo
Kubota Tomio
Kudo Akio
Kudo Hirokichi
Kumano-go Hitoshi
Kunisawa Kiyonori
Kunita Hiroshi
Kurita Minoru
Kuroda Shige-Toshi
Kusama Tokitake
Kusano Takaši
Maruyama Gisiro
Matsuda Michihiko
Matsumoto Kikuji
Matsumoto Makoto
Matsumura Hideyuki
Matsushima Yozo
Mimura Yukio
Miyazawa Hironari
Mizohata Sigeru
Mori Toshio
Morimoto Haruki
Morimura Hidenori
Morita Kiiti
Morita Yasuo
Motoo Minoru
Murakami Shingo
Muramatsu Tosinobu
Murase Ichiro
Nagao Hiroshi
Nagasawa Masao
Nagata Masayoshi
Nakai Mitsuru
Nakai Yoshikazu
Nakamura Tokushi
Nakanishi Shizu
Nakano Shigeo
Nakaoka Minoru
Namikawa Yukihiko
Namioka Isaac
Nanba Kanji
Niuro Fumio
Nikaido Hukukane
Nishi Mieo
Nishijima Kazuhiko
Nishimiya Han
Noguchi Hiroshi
Nonaka Toshio
Noshiro Kiyoshi
Nozaki Akihiro
Ôbayashi Tadao
Ogino Shusaku
Ohtsuka Makoto
Oikawa Kotaro
Okamoto Masashi
Okubo Kenjiro
Okugawa Kôtarô

Ono Katuzi
Ono Shigeru
Osima Masaru
Ozawa Mitsuru
Ozeki Hideki
Saito Tosiya
Sakai Shoichiro
Sakamoto Minoru
Sasaki Shigeo
Satake Ichiro
Sato Kenkichi
Sawashima Ikuko
Shibuya Masaaki
Shiga Kôji
Shimada Nobuo
Shintani Takuro
Shioda Tetsuji
Shiraiwa Kenichi
Shiratani Katsumi
Shizuma Ryoji
Simauti Takakazu
Sugawara Masahiro
Sugiura Mitsuo
Sugiyama Shohei
Suita Nobuyuki
Sumitomo Takeshi
Sunouchi Genichiro
Suzuki Michio
Takahashi Moto-o
Takahashi Reiji
Takahashi Tsunero
Takami Hideo
Takaoka Seiki
Takenouchi Osamu
Takeuchi Kei
Takeuti Gaisi
Tamura Itiro
Tanaka Hiroshi
Tanaka Hisao
Tanaka Minoru
Tatsumi Tomomasa
Tatuzawa Tikao
Terasaka Hidetaka
Toda Hideo
Toda Hiroshi
Tsuka-da Nobutaka
Tsukamoto Yôtarô
Tugué Tosiya
Uchiyama Saburo
Uesu Tadahiro
Umegaki Hisaharu
Umezawa Toshio
Uno Toshio
Ura Shoji
Ura Taro
Urabe Minoru
Uzawa Hirobumi
Wada Junzo
Watanabe Hiroshi
Watanabe Hisao
Watanabe Shinzo

Watanabe Takesi
Yamaguti Masaya
Yamamoto Koichi
Yamanaka Takesi
Yamazaki Keijiro
Yano Kentaro
Yano Shigeaki
Yoshizawa Taro
Yosida Kôzaku
Yosida Setuzô

Name Index

Note. Citation is to article and section, not to page.
r = references; STR = Statistical Tables for Reference
(to be found after App. B); NTR = Numerical Tables
for Reference. * means that the article is not divided
into sections.

A

Abadie, Jean M. (1919–) 292.r
Abe, Eiichi (1927–) 13.R
Abe, Kinetsu (1941–) 365.L
Abel, Niels Henrik (1802–29) 1 2.A, B, C 3.A, B, G,
J, L, M 8 10.D 11.B, C, E 20 21.B 52.B, N 60.L
73.A 121.D 136.B 172.A, B, G, H 190.A, H, Q
201.R 202.N 217.A, L 240.G 267 308.E 339.B
367.H 379.D, F, K, N 383.B 388.D 422.A, E
Abers, Ernest S. 132.r
Aberth, Oliver George (1929–) 301.F
Abhyankar, Shreeram Shankar (1930–) 15.B 16.L
23.r
Abikoff, William (1944–) 234.E
Abraham, C. T. 96.F
Abraham, Ralph H. (1936–) 126.J, r 183 271.r 316.r
420.r
Abramov, Leonid Mikhailovich (1931–) 136.E
Abramowitz, Milton J. NTR
Abrikosov, Aleksei Alekseevich (1928–) 402.r
Abūl Wafā (940–98) 432.C
Achenbach, J. D. 446.r
Ackermann, Wilhelm (1896–1962) 97.*, r 156.E, r
356.B 411.J, r
Ackoff, Russell Lincoln (1919–) 307.A
Aczél, János D. (1924–) 388.r
Adams, Douglas Payne 19.r
Adams, John Couch (1819–92) 303.E
Adams, John Frank (1930–) 64.C 200.r 202.S, T, V
237.A, E, I 249.r 426
Adamson, Iain Thomas 200.M
Addison, John West, Jr. (1930–) 22.D, F, G, H 81.r
356.H, r
Adem, José (1921–) 64.B, C 305.A App. A, Table
6.II
Adler, Mark 387.C
Adler, Roy L. (1931–) 126.K 136.E, H
Adler, Stephen L. (1939–) 132.C, r
Ado, Igor Dmitrievich 248.F, r
Adriaan, Anthonisz (c. 1543–1620) 332
Agmon, Shmuel (1922–) 112.F, H, Q 323.H, r
375.B, C
Agnesi, Maria Gaetana (1718–99) 93.H
Aguilar, Joseph 331.F
Ahern, Patrick Robert (1936–) 164.K
Ahlberg, John Harold (1927–) 223.r
Ahlfors, Lars Valerian (1907–) 17.D 21.N 43.G, K
74.r 77.E, F, r 122.I 124.B, r 143.A 169.E 198.r
234.D, E, r 272.I, J, L 352.A, B, C, F 367.B, G, I, r
416 429.D 438.r
Aho, Alfred V. 31.r 71.r 75.r 186.r
Aida, Yasuaki (1747–1817) 230
Airy, Sir George Biddell (1801–92) 325.L App. A,
Table 19.IV
Aitken, Alexander Craig (1895–1967) 223.B App.
A, Table 21
Aizawa, Sadakazu (1934–) 286.X
Aizenman, Michael (1945–) 136.G 340.r 402.G
Akahira, Masafumi (1945–) 128.r 399.K, O 400.r
Akahori, Takao (1949–) 72.r

Akaike, Hirotugu (1927–) 421.D
Akaza, Tohru (1927–83) 234.r
Akbar-Zadeh, Hassan (1927–) 152.C
Akcoglu, Mustafa A. (1934–) 136.B
Akemann, Charles A. (1941–) 36.K
Akhiezer, Naum Ilich (1901–80) 197.r 251.r 336.r
390.r
Akizuki, Yasuo (1902–84) 8 59.H 284.F, G 368.F
Alaoglu, Leonidas (1914–) 37.E 424.H
Albanese, Giacomo (1890–1947) 16.P 232.C
al-Battānī, Mohamed ibn Gabis ibn Sinan, Abu
Abdallah (858?–929) 26 432.C
Albert, Abraham Adrian (1905–72) 29.F, r 149.r
231.r
Albertus Magnus (1193?–1280) 372
Alcuin (735–804) 372
Aleksandrov, Aleksandr Danilovich (1912–) 111.r
178.A 255.D 365.H 425.r
Aleksandrov (Alexandroff), Pavel Sergeevich (1896–
1982) 22.I 65.r 93.r 99.r 117.A, E, F, r 201.A, r
207.C 273.K 425.S–V, r 426. *, r
Alekseev, Vladimir Mikhailovich (1932–80) 420.r
Alekseevskii, Dmitrii Vladimirovich (1940–) 364.r
Alexander, Herbert James (1940–) 344.F
Alexander, James Waddell (1888–1971) 65.G
201.A, J, M, O, P 235.A, C, D, E 426
Alexits, György (1899–1978) 317.r
Alfsen, Erik Magnus (1930–) 351.L
Alfvén, Hannes (1908–) 259. *, r
Alinhac, Serge (1948–) 345.A
al-Khwarizmi (Alkwarizmi), Mohammed ibn Musa
(c. 780–c. 850) 26
Allard, William Kenneth (1941–) 275.G
Allendoerfer, Carl Bennett (1911–74) 109 365.E
Almgren, Frederick Justin, Jr. (1933–) 275.A, F, G,
r 334.F
Altman, Allen B. 16.r
Amari, Shun-ichi (1936–) 399.O
Ambrose, Warren (1914–) 136.D
Amemiya, Ichiro (1923–) 72.r
Amitsur, Shimshon A. (1921–) 200.P
Ampère, André-Marie (1775–1836) 82.A 107.B
278.A
Amrein, Werner O. 375.B, r
Ananda-Rau, K. 121.D
Andersen, Erik Sparre (1919–) 260.J
Anderson, Brian D. O. 86.r
Anderson, Joel H. (1935–) 36.J
Anderson, Richard Louis (1915–) 19.r
Anderson, Robert Murdoch (1951–) 293.D, r
Anderson, Theodore Wilbur (1918–) 280.r 374.r
421.D
Andersson, Karl Gustav (1943–) 274.I
Ando, Tsuyoshi (1932–) 310.H
Ando, Y. 304.r
Andreotti, Aldo (1924–80) 32.F 72.r
Andrews, David F. 371.H
Andrews, Frank C. 419.r
Andrianov, Anatolii N. (1936–) 32.F
Andronov, Aleksandr Aleksandrovich (1901–52)
126.A, I, r 290.r 318.r
Anger, Carl Theodor (1803–58) 39.G App. A,
Table 19.IV
Anikin, S. A. 146.A
Anosov, Dmitrii Viktorovich (1936–) 126.A, J
136.G
Antiphon (fl. 430? B.C.) 187
Antoine, Louis August (1888–1971) 65.G
Anzai, Hirotada (1919–55) 136.E
Apéry, Roger (1916–) 182.G

Apollonius (of Perga)

- Apollonius (of Perga) (262–c. 200 B.C.) 179.A 181
187 App. A, Table 3.V
- Apostol, T. M. 106.r 216.r
- Appel, Kenneth I. (1932–) 157.A 186.r
- Appell, Paul-Emile (1855–1930) 11.r 206.D, r
393.E, r 428.r App. A, Table 18.I App. A,
Table 20.r
- Arakelov, S. Yu. 9.r 118.E
- Arakelyan, Norair Unanovich 164.J
- Araki, Huzihiro (1932–) 150.D, E 212.B, r 308.I, r
351.L 377.r 402.G
- Araki, Shōrō (1930–) 427.B
- Aramata, Hideo (1905–47) 450.D
- Aramovich, I. 198.r
- Arbib, Michael Anthony (1940–) 75.r 95.r
- Arbuthnot, John (1667–1735) 371.A
- Archibald, Raymond Clare (1875–1955) 187.r
- Archimedes (c. 287–212 B.C.) 20 78.F 93.H 149.N
155.B, D 187 243.G 310.C 332 355.B 439.C
- Archytas (of Taras) (c. 430–c. 365 B.C.) 187
- Arens, Richard Friedrich (1919–) 36.M 424.N
- Arf, Cahit (1910–) 114.J
- Argand, Jean Robert (1768–1822) 74.C
- Arima, Reiko (→ Sakamoto, Reiko)
- Arima, Yoriyuki (1714–83) 230
- Arimoto, Suguru (1936–) 213.E, F, r
- Aristotle (384–322 B.C.) 187
- Ariyama, Masataka (1929–) 446.r
- Arkhangel'skiĭ, Aleksandr Vladimirovich (1938–)
273.K, r 425.F, S, Y, CC, r
- Arnauld, Antoine (1612–94) 265
- Arnoff, E. Leonard (1922–) 307.A
- Arnold, Leslie K. 136.C
- Arnol'd, Vladimir Igorevich (1937–) 82.r 126.A, L,
M, r 136.r 196 219.r 271.r 402.r 418.E 420.G App.
A, Table 5, r
- Aronszajn, Nachman (1907–80) 112.H 188.r 273.K
323.J 338.E
- Arraut, José Luis (1938–) 126.H
- Arrow, Kenneth Joseph (1921–) 227.r 292.E, r
- Arsevin, Vasilii Yakovlevich 22.C, F, r
- Artin, Emil (1898–1962) 4.A 6.E, F, r 7.r 8 14.F,
K, O, R, S, U, r 28 29.r 59.A, C, F, H, r 60.r 65.G
118.F 123.F 149.N 151.I 155.G 172.F, r 174.r 196
198.B, r 200.N 235.F 257.H, r 277.I 284.A, G 295.E
343.r 368.F, r 439.L, r 450.A, G, P, R
- Artin, Michael (1934–) 15.r 16.U, W, r 126.K 210.r
418.C 450.Q, r
- Arveson, William B. (1934–) 36.r 308.r
- Āryabhata (Ārya-Bhatta) (c. 476–c. 550) 209 332
432.C
- Arzelà, Cesare (1847–1912) 168.B 216.B 435.D
- Asada, Kenji (1946–) 274.r 345.B
- Asano, Keizo (1910–) 8 29.I
- Asano, Kiyoshi (1938–) 41.D
- Aschbacher, Michael George (1944–) 151.J
- Ascher, Edgar 92.r
- Ascoli, Giulio (1843–96) 168.B 435.D
- Ash, Avner Dolnick (1949–) 16.r
- Assmus, Edward F., Jr. (1931–) 200.K
- Athreya, Krishna B. 44.C
- Atiyah, Michael Francis (1929–) 16.r 20 68.F 80.G,
r 109 114.E 147.O 153.C 183 237.A, H, r 323.K
325.J 345.A 366.A–D, r 390.I, J 391.L, N, r 426
431.D 437.X
- Atkin, Arthur O. L. (1935–) 328
- Atsugi, Masahiko (1922–) 425.Y
- Aubin, Jean-Pierre 286.X
- Aubin, Thierry Emilien (1942–) 183.r 232.C
364.H, r
- Audley, Robert John (1928–) 346.r
- Auerbach, Herman (1902–42) 270.J
- Aumann, Robert John (1930–) 173.D, E, r 443.A
- Auslander, Joseph (1930–) 126.D
- Auslander, Louis (1928–) 105.r 136.G 152.C 279.C
437.U
- Auslander, Maurice (1926–) 29.K 200.K, L 284.G
- Avery, J. 377.r
- Avez, André 126.r 136.r 402.r
- Ax, James B. 14.D 118.B, F 276.E, r 450.J
- Ayoub, Raymond George (1923–) 4.r 123.r 295.r
328.r
- Azencott, Robert Guy (1943–) 136.G
- Azima, Naonobu (1739–98) 230
- Aziz, Abdul Kadir 303.r
- Azra, Jean-Pierre (1935–) 171.r
- Azumaya Goro (1920–) 8.*, r 29.I, K. r 67.D 172.r
200.L 362.r 368.r

B

- Babbage, Charles (1792–1871) 75.A
- Bachelier, Louis (1870–1946) 45.A
- Bachet de Méziriac, Claude Gaspar (1581–1638)
296.A
- Bachmann, Paul Gustav Heinrich (1837–1920)
297.I
- Bacon, Francis (1561–1626) 401.E
- Baer, Reinhold (1902–79) 2.F 122.B 200.I, K
- Bagemihl, Frederick (1920–) 62.C–E
- Bahadur, Raghu Raj (1924–) 396.r 398.r 399.N, r
400.K, r
- Bahmann, H. 97.B
- Bailey, Norman T. J. 40.r
- Baillon, Jean-Bernard (1951–) 286.Y
- Baily, Walter Lewis, Jr. (1930–) 16.Z 32.F, H 122.r
194.r
- Baiocchi, Claudio 440.r
- Baire, René Louis (1874–1932) 20 21 C, L 84.D, r
126.H 273.B, J 425.N
- Bairstow, L. 301.E
- Baker, Alan (1939–) 118.D 182.G, r 196 347.E
430.D, r
- Baker, George Allen, Jr. (1932–) 142.r
- Baker, Henry Frederick (1866–1956) 9.r 15.r 78.r
350.r
- Baker, Kenneth R. 376.r
- Bakhshāli (c. 3rd century) 209
- Balaban, Tadeusz 325.K
- Balakrishnan, A. V. (1922–) 378.D
- Balas, Egon 215.C, r
- Baldwin, John T. (1944–) 276.F
- Balian, Roger 386.r
- Ball, W. W. Rouse 157.r
- Banach, Stefan (1892–1945) 20 23.G 36.A, F 37.A,
B, E, F, H, I, O, r 105.Z 162 168.r 246.G 286.K, Z
310.F, I 424.C, H, J, X 442.r
- Banerjee, Kali S. (1914–) 102.r
- Bang, Thøger Sophus Vilhelm (1917–) 58.F
- Banica (Bănică), Constantin (1942–) 23.r
- Baouendi, M. Salah (1937–) 323.N 345.A
- Barankin, Edward William (1920–) 396.r 399.D, r
- Barban, Mark Borisovich (1935–) 123.E
- Barbey, Klaus 164.r
- Barbosa, João Lucas Marquês 275.B
- Barbu, Viorel (1941–) 88.r 440.r
- Barden, Dennis 65.C
- Bardos, Claude Williams (1940–) 204.E
- Bargmann, Valentine (1908–) 258.r 437.EE
- Bar-Hillel, Yehoshua (1915–75) 96.r

- Bari, Nina Karlovna (1901–61) 159.J
 Barlow, Peter (1776–1862) NTR
 Barlow, William (1845–1934) 92.F
 Barnes, Ernest William (1874–1953) 206.C App. A, Table 18.I
 Barr, Michael (1937–) 200.r
 Barrow, Isaac (1630–77) 265 283
 Barth, Wolf Paul (1942–) 16.r
 Bartle, Robert Gardner (1927–) 68.M 443.A, G
 Bartlett, Maurice Stevenson (1910–) 40.r 44.r 280.J 407.r 421.C, r
 Barwise, Jon 356.r
 Bashforth, F. 303.E
 Bass, Hyman (1932–) 122.F 200.r 237, J, r
 Bass, Robert Wauchope (1930–) 289.D
 Bastin, J. 351.r
 Batchelder, Paul M. 104.r
 Batchelor, George Keith (1920–) 205.r 433.C, r
 Bateman, Paul Trevier (1919–) 4.D 348.K
 Bauer, Friedrich Ludwig (1924–) 302.r
 Bauer, Heinz (1928–) 193.U
 Baum, Paul Frank (1936–) 366.E 427.B
 Bayer, Pilar 450.r
 Bayes, Thomas (1702–61) 342.A, F 396.J 398.B 399.F 401.B, E 403.G 405.I
 Bazilevich, Ivan Evgen'evich 438.B
 Beale, E. M. L. 292.r 349.r
 Beals, Richard William (1938–) 320.r 345.A, B
 Beardon, Alan Frank (1940–) 234.r
 Beatley, Ralph 139.r
 Beauville, Arnaud (1947–) 15.r
 Bebutov, M. 126.E
 Becchi, C. M. 150.G
 Beck, James V. 200.Q, r
 Beckenbach, Edwin Ford (1906–82) 211.r
 Becker, Oskar Joachim (1889–1964) 156.r 187.r
 Beckmann, Petr (1924–) 332
 Bede Venerabilis (673–735) 372
 Beer, Stafford (1926–) 95.r
 Beeson, H. 275.C
 Beez, R. 365.E
 Behnke, Heinrich (1898–1979) 21.H, Q 23.E 198.r 367.B, G, I, r
 Behrends, Ralph Eugene (1926–) 132.r
 Behrens, W. V. 400.G
 Belardinelli, Giuseppe 206.r
 Belavin, A. A. 80.r
 Belinfante, Frederik J. 150.B
 Bell, Eric Temple (1883–1960) 177.D
 Bell, James Frederick (1914–) 33.r
 Bell, John Stewart (1928–) 351.L
 Bell, Steve 344.D
 Bellissard, Jean Vincent (1946–) 351.L
 Bellman, Richard (Ernest) (1920–84) 86.A, F 127.A, D, E, G, r 163.B 211.r 291.r 314.r 394.r 405.B, r
 Belov, Nikolaï Vasil'evich (1891–) 92.r
 Beltrami, Eugenio (1835–1900) 109 194.B 285.A 352.B App. A, Table 4.II
 Belyaev, Yurii Konstantinovich (1932–) 176.G
 Bendat, Julius S. (1923–) 212.r
 Bender, Helmut (1942–) 151.J
 Benders, J. F. 215.r
 Bendikson (Bendixson), Ivar Otto (1861–1936) 107.A 126.I
 Bengel, Günter (1939–) 112.D
 Bénilan, Philippe (1940–) 162
 Bensoussan, Alain (1940–) 405.r
 Bérard-Bergery, Lionel (1945–) 364.r
 Berens, Hubert (1936–) 224.E, r 378.r
 Berezin, Feliks Aleksandrovich (1931–80) 377.r
 Berg, Christian (1944–) 338.r
 Berg, Ira David (1931–) 331.E 390.I
 Berge, Claude (1926–) 186.r 281.r 282.r
 Berger, Charles A. (1937–) 251.K, L
 Berger, James Orvis (1950–) 398.r
 Berger, M. (1926–) 109.*, r, 178.A, C 391.B, C, r
 Berger, Melvyn (1939–) 286.r
 Berger, Toby (1940–) 213.E
 Bergh, Jöran (1941–) 224.r
 Bergman, Stefan (1895–1977) 21.Q 77.r 188.G, r 326.C
 Berkovitz, Leonard D. (1924–) 86.F 108.A, B
 Berlekamp, Elwyn R. (1940–) 63.r
 Bernays, Paul Isaak (1888–1977) 33.A, C, r 97.* 156.r 411.J, r
 Bernoulli family 20 38 107.A 266
 Bernoulli, Daniel (1700–82) 20 38 205.B 301.J 342.A 396.B
 Bernoulli, Jakob (1654–1705) 38 46.A 93.H 136.D–F 177.B 250.A 342.A 379.I App. A, Table 14.I App. B, Table 3.I
 Bernoulli, Jakob (1759–89) 38
 Bernoulli, Johann (1667–1748) 38 46.A 93.H 163.B 165.A
 Bernoulli, Johann (1710–90) 38
 Bernoulli, Johann (1744–1807) 38
 Bernoulli, Nikolaus (1687–1759) 38
 Bernoulli, Nikolaus (1695–1726) 38
 Bernshtein, I. N. 125.EE 154.G 418.H, r
 Bernshtein, Sergei Natanovich (1880–1968) 49.B 58.E 196 240.E 255.D 261.A 275.A, F 323.I 334.C 336.A, C, F
 Bernshtein, Vladimir 121.r
 Bernstein, Allen R. 276.E, r 293.D
 Bernstein, Felix (1878–1956) 228.A
 Berry, G. G. 319.B
 Berry, L. Gérard (1948–) 40.D
 Bers, Lipman (1914–) 21.r 23.r 111.r 122.I, r 204.G 234.D, r 275.A 320.r 326.r 327.r 352.B–E, r 367.r 416.*, r
 Berstel, Jean (1941–) 31.r
 Berthelot, Pierre (1943–) 16.r 366.r 450.Q
 Bertini, Eugenio (1846–1933) 15.C
 Bertrand, Joseph Louis François (1822–1900) 111.F 123.A
 Berwald, L. (1883–?) 152.C
 Bertziss, Alfs T. 96.r
 Besikovich (Besicovitch), Abram Samoilovich (1891–1970) 18.A, C 246.K
 Besov, Oleg Vladimirovich (1933–) 168.B, r
 Bessaga, Czesław (1932–) 286.D 443.D
 Besse, Arthur L. 109.r 178.r
 Besse, J. 198.N
 Bessel, Friedrich Wilhelm (1784–1846) 39.A, B, D, G 197.C 223.C App. A, Tables 14.II, 19.III, IV, 21.III, IV
 Besson, Gerard (1955–) 391.F
 Betti, Enrico (1823–92) 105.A 200.K 201.A, B 426
 Beurling, Arne (Karl-August) (1905–) 62.B, E 125.A, U 143.A 164.G, I 169.E 192.Q 251.L 338.Q, r 352.C
 Bézout, Étienne (1739–83) 9.B 12.B
 Bhāskara (1114–85?) 209 296.A
 Bhatia, Nam Parshad (1932–) 86.r 126.r
 Bhattacharya, Rabindra Nath (1937–) 374.F, r
 Bhattacharyya, A. 399.D, r
 Bianchi, Luigi (1856–1928) 80.J 365.J 417.B
 Bickel, Peter John (1940–) 371.r
 Bidal, Pierre 194.F
 Bieberbach, Ludwig (1886–1982) 43.r 77.E 89.C

Biedenbarn, Lawrence C.

- 92.F, r 107.r 179.B, r 198.r 254.r 288.r 339. 429.r
438.B, C
- Biedenbarn, Lawrence C. (1922–) 353.r
- Biezeno, Cornelis Benjamin (1888–1975) 19.r
- Biggeri, Carlos 121.C
- Biggs, Norman Linstead (1941–) 157.r
- Billera, Louis J. (1943–) 173.E
- Billingsley, Patrick P. (1925–) 45.r 136.r 250.r
341.r 374.r
- Binet, Jacques Philippe Marie (1786–1856) 174.A
295.A
- Bing, Rudolf H. (1914–86) 65.F, G 79.D 273.K
382.D 425.AA
- Birch, Bryan John (1931–) 4.E 118.C–E 450.S
- Birkeland, R. 206.D
- Birkhoff, Garrett (1911–) 8.r 87.r 103.r 183.r 243.r
248.J 310.A 311.r 343.r 443.A, E, H
- Birkhoff, George David (1884–1944) 30.r 107.A
109 111.I 126.A, F 136.A, B 139.r 153.B, D 157.A
162 253.C 254.D 279.A 286.D 420.F
- Birman, Joan S. (1922–) 235.r
- Birman, Mikhail Shlemovich (1928–) 331.E
- Birnbaum, Allan (1923–76) 399.C, r 400.r
- Birtel, Frank Thomas (1932–) 164.r
- Bisconcini, Giulio 420.C
- Bishop, Errett A. (1928–83) 164.D–F, J, K 367.r
- Bishop, Richard L. (1931–) 105.r 178.r 417.r
- Bishop, Yvonne M. M. 280.r 403.r
- Bitsadze, Andrei Vasil'evich (1916–) 326.r
- Bjerknes, Carl Anton (1825–1903) 1.r
- Björck, Åke (1934–) 302.r
- Björck, Göran (1930–) 125.r
- Björk, Jan-Erik 112.r 125.EE 274.r
- Bjorken, James Daniel (1934–) 132.r 146.A, C 150.r
- Blackman, R. B. 421.r
- Blackwell, David (Harold) (1919–) 22.H 398.r
399.C
- Blahut, Richard E. (1937–) 213.r
- Blair, David E. (1940–) 110.E 364.G
- Blakers, Albert Laurence (1917–) 202.M
- Blanchard, André (1928–) 72.r
- Blanc-Lapierre, André Joseph (1915–) 395.r
- Bland, Robert G. (1948–) 255.C
- Blaschke, Wilhelm (1885–1962) 43.F 76.r 89.C, r
109.*, r 110.C, r 111.r 178.G 218.A, C, H 228.r
- Blatt, John Markus 353.r
- Blattner, Robert James (1931–) 437.W, EE
- Bleaney, B. I. 130.r
- Bleuler, Konrad (1912–) 150.G
- Bloch, André (1893–1948) 21.N, O 77.F 272.L
429.D
- Bloch, Felix (1905–) 353.r 402.H
- Bloch, Spencer 16.R
- Block, Henry David (1920–78) 420.C
- Bloxham, M. J. D. 386.C
- Blum, Julius Rubin (1922–82) 136.E
- Blum, Manuel 71.D, r
- Blumenthal, Ludwig Otto (1876–1944) 32.G 122.E
- Blumenthal, Robert McCallum (1931–) 5.r 261.B, r
- Boas, Ralph Philip, Jr. (1912–) 58.r 220.D 240.K
429.r
- Bôcher, Maxime (1867–1918) 107.A 167.E 193.D
- Bochner, Salomon (1899–1982) 5.r 18.A, r 21.Q, r
36.L 80.r 109.*, r 125.A 160.C, r 164.G 192.B, O
194.G 261.F 327.r 341.C, J, r 367.F 378.D 443.A,
C, H
- Bodewig, Ewald 298.r
- Boerner, Hermann (1906–82) 362.r
- Boetius, Anicius Manlius Torquatus Severinus
(c. 480–524) 372
- Bogolyubov, Nikolai Nikolaevich (1909–) 125.W
136.H 146.A 150.r 212.B 290.A, D 361.r 402.J
- Böhme, Reinhold (1944–) 275.C
- Bohnenblust, (Henri) Frederic (1906–) 28 310.A, G
- Bohr, Harald (1887–1951) 18.A, B, H, r 69.B 121.B,
C 123.r 450.I
- Bohr, Niels Henrik David (1885–1962) 351.A
- Bokshtein (Bockstein), Meer Feliksovich (1913–)
64.B 117.F
- Bol, Gerrit (1906–) 110.r
- Boll, Marcel (1886–) NTR
- Bolley, Pierre (1943–) 323.N
- Boltyanskii, Vladimir Grigor'evich (1925–) 86.r
89.r 117.F 127.G
- Boltzmann, Ludwig (1844–1906) 41.A, B, r 136.A
402.B, H, r 403.B, r
- Bolyai, János (Johann) (1802–60) 35 A 181 267
285.A
- Bolza, Oskar (1857–1942) 46.r
- Bolzano, Bernard (1781–1848) 140 273.F
- Bombieri, Enrico (1940–) 15.r 72.K 118.B 123.D,
E, r 151.J 275.F 438.C 450.P, Q
- Bompiani, Enrico (1889–1975) 110.B
- Bonnesen, Tommy (1873–) 89.r 228.A
- Bonnet, Ossian Pierre (1819–92) 109 111.H 275.A,
C 364.D App. A, Table 4.I
- Bonsall, Frank Featherstone (1920–) 310.H
- Bony, Jean-Michel (1942–) 274.r
- Book, D. L. 304.r
- Boole, George (1815–64) 33.E 42.A–D, r 104.r
156.B 243.E 267 379.J 411.A, r
- Boone, William Werner (1920–83) 97.*, r 161.B
- Boothby, William M. (1918–) 110.E
- Borchardt, Carl Wilhelm (1817–80) 229.r
- Borchers, Hans-Jürgen (1926–) 150.E
- Borel, Armand (1923–) 12.B 13.A, G, r 16.Z 32.H,
r 56.r 73.r 122.F, G, r 147.K 148.E, r 199.r 203.A
248.O 249.J, V, r 366.D 383.r 384.D 427.B, r 431.r
437.Q 450.r App. A, Table 6.V
- Borel, Emile (1871–1956) 20 21.O 22.A, G 58.D
83.B 124.B 156.C 198.Q, r 261.D 270.B, C, G, J
272.E, F 273.F 339.D 342.A, B 379.O 429.B
- Borevich, Zenon Ivanovich (1922–) 14.r 297.r 347.r
- Borges, Carlos J. Rego (1939–) 273.K 425.Y
- Borisovich, Yurii Grigor'evich (1930–) 286.r
- Born, Max (1882–1970) 402.J 446.r
- Borovkov, Aleksandr Alekseevich (1931–) 260.H
- Borsuk, Karol (1905–82) 79.C, r 153.B 202.B, I
382. A, C
- Bortolotti, Ettore (1866–1947) 417.E
- Bose, Raj Chandra (1901–) 63.D 241.B STR
- Bose, Satyendra Nath (1894–1974) 132.A, C 351.H
377.B 402.E
- Bott, Raoul (1923–) 105.r 109 153.C 154.F–H
178.G 202.V, r 237.D, H, r 248.r 272.L 279.D
325.J 345.A 366.r 391.N, r 413.r 427.E, r 437.Q
App. A, Table 6.VII
- Bouligand, Georges (1889–?) 120.D
- Bouquet, Jean-Claude (1819–85) 107.A 111.F
288.B 289.B
- Bourbaki, Nicolas 8 13.r 20.r 22.r 34.r 35.r 60.r 61.r
67.r 74.r 84.r 87.r 88.r 103.r 105.r 106.r 122.r 131.r
135.r 149.r 162 168.C 172.r 187.r 216.r 221.r 225.r
248.r 249.r 256.r 265.r 266.r 267.r 270.r 277.r 284.r
310.I 311.r 312.r 337.r 348.r 355.r 360.r 362.r 368.r
379.r 381.r 409.r 423.r 424.I, r 425.S, W, Y, CC, r
435.r 436.r 443.A
- Bourgin, David G. 201.r
- Bourgne, Robert 171.r
- Bourguignon, Jean-Pierre (1947–) 80.r 364.r

Bourion, Georges 339.E
 Boussinesq, Joseph (1842–1929) 387.B, F
 Bouteroux, Pierre Léon (1880–1922) 265.r 288.B, C, r
 Bowen, Rufus (1947–78) 126.A, J, K, r 136.C, G, r 234.r
 Bowman, Frank (1891–1983) 39.r
 Box, George E. P. (1919–) 102.r 128.r 301.L 371.A 421.G, r
 Boyle, James M. 298.r
 Bradley, G. J. 92.r
 Bradley, Ralph Allan (1923–) 346.C
 Brahmagupta (598–660) 118.A 209
 Bram, Joseph (1926–) 291.r
 Brams, Steven John (1940–) 173.r
 Brandt, Heinrich (1886–1954) 190.P 241.C
 Branges, Louis de (1932–) 176.K 438.C
 Bratteli, Ola (1946–) 36.H, K, r 308.r 402.G, r
 Brauer, Richard Dagobert (1901–77) 14.E 27.D, E 29.E, F, K 118.C 151.J, r 362.G, I, r 427.B 450.D, G, L
 Braun, Hel (1914–) 32.H 122.E 231.r
 Brauner, Karl 418.r
 Bravais, Auguste (1811–63) 92.B, F App. B, Table 5.IV
 Bredikhin, Boris Maksimovich (1920–) 123.E
 Bredon, Glen E. (1932–) 383.r 431.r
 Breiman, Leo (1928–) 260.r 342.r
 Bretol, Marcel (1903–) 120.C, E, r 193.J, L, N, U 207.C 338.G, H
 Bremermann, Hans-Joachim (1926–) 21.D, I
 Bremmer, Hendricus (1904–) 240.r
 Brent, Richard Peirce (1946–) 123.B 142.A 450.I
 Breuer, Manfred 390.J
 Brewster, Sir David (1781–1868) 283.r
 Brézis, David 88.E
 Brézis, Haïm (1944–) 88.E 162.*, r 286.C, X 440.r
 Brianchon, Charles Julien (1785–1864) 78.K 343.E
 Brieskorn, Egbert (1936–) 16.r 418.C, D, r App. A, Table 5.r
 Bringham, E. Oran (1940–) 142.r
 Brill, Alexander Wilhelm von (1842–1935) 9.E, r 11.B, r 12.B
 Brillinger, David Ross (1937–) 421.r
 Brillouin, Léon Nicolas (1889–1969) 25.B 446.r
 Brin, Matthew G. 136.G
 Briot, Charles Auguste Albert (1817–82) 107.A 288.B 289.B
 Broadbent, S. R. 340.r
 Bröcker, Theodor 51.r
 Broderick, Norma 92.r
 Brodskii, Mikhail Samoilovich (1913–) 251.r 390.H
 Brody, Robert 21.O
 Bromwich, Thomas John l'Anson (1875–1929) 240.D 322.D 379.r App. A, Table 12.I
 Bronshtein, M. D. 325.I
 Bros, Jacques (1934–) 150.D 274.D, I 386.B, C
 Brosilow, C. B. 303.r
 Brouncker, Lord William (1620–84) 332
 Brouwer, Dirk (1902–66) 55.r
 Brouwer, Luitzen Egbertus Jan (1881–1966) 65.G 79.D 99.A 117.A, D, r 153.B 156.A, C 202.B 305.A 426
 Browder, Andrew (1931–) 164.r
 Browder, Felix Earl (1927–) 112.F, Q 286.C, X, r 323.H
 Browder, William (1934–) 114.J, L, r 427.B
 Brown, Edgar H., Jr. (1926–) 202.T
 Brown, Harold 92.F
 Brown, Lawrence David (1940–) 396.r

Brown, Lawrence G. (1943–) 36.J 390.J, r
 Brown, Leon 43.G
 Brown, Morton (1931–) 65.G
 Brown, Robert (1773–1858) 5.D 45.A–C, F, I 176.C, I 250.F 406.B, G
 Brown, Robert Freeman (1935–) 153.r
 Brown, Scott W. (1937–) 251.L
 Brownlee, John (1868–1927) NTR
 Bruck, Richard Hubert (1914–) 190.P, r 241.D
 Bruhat, François (1929–) 13.K, Q, R 437.O, EE
 Brumer, Armand 182.r 450.J
 Brun, Viggo (1885–1978) 4.A, C 123.D, E
 Brune, O. 282.r
 Brunel, Antoine 136.C
 Brunn 89.E
 Brunovský, Pavol 126.M
 Bruns, Heinrich (1848–1919) 126.A 420.A
 Brunschvicg, Léon (1869–1944) 329.r
 Bruter, Claude Paul (1937–) 281.r
 Buchholz, Herbert (1895–1971) 167.r
 Buchner, Michael Anthony (1947–) 126.L
 Buchsbaum, David Alvin (1929–) 284.G
 Buck, R. Creighton (1920–) 43.F, r 106.r
 Bückner, Hans 217.r
 Bucur, Ion (1930–76) 52.r
 Bucy, Richard S. (1935–) 86.E 95.r 405.G, r
 Buerger, Martin J. 92.r
 Buffon, Georges Louis Leclerc, Comte de (1707–88) 218.A 342.A 385.C
 Buhler, Joe P. 450.G
 Bühlmann, Hans (1930–) 214.r
 Bulirsch, Roland (1932–) 303.F
 Bülow, Rolf 92.F
 Bunimovich, Leonid Abramovich 136.G
 Bunyakovskii, Viktor Yakovlevich (1804–89) 211.C App. A, Table 8
 Burali-Forti, Cesare (1861–1931) 319.B
 Burchnall, Joseph Langley (1892–1975) 387.C
 Burckhardt, Johann Jakob 92.F
 Burd, Vladimir Shpeselevich (1938–) 290.r
 Burgess, David Albert (1935–) 295.E
 Burghlea, Dan (1943–) 105.r 183
 Bürgi, Joost (1552–1632) 265
 Burgoyne, N. 150.D App. B, Table 5.r
 Burkholder, Donald L. (1927–) 168.B 262.B
 Burkill, John Charles (1900–) 100.A
 Burns, Daniel Matthew, Jr. (1946–) 344.C–E
 Burnside, William Snow (1852–1927) 151.D, H, J, r 161.C 190.Q, r 267 431.F
 Busemann, Herbert (1905–) 178.F, H, r
 Bush, Robert R. 96.r 346.G
 Bush, Vannevar (1890–1974) 19.E
 Bustab, A. 4.A, C 123.E
 Butkovskii, A. G. 86.r
 Butzer, Paul L. (1928–) 224.E, r 378.r
 Byrne, George D. 303.r

C

Cabannes, Henri (1923–) 259.r
 Caflisch, Russel E. 41.D, E
 Caianiello, Eduardo R. (1921–) 291.F, r
 Cairns, Stewart Scott (1904–82) 114.A, C 426
 Calabi, Eugenio (1923–) 122.F 232.C 275.H 365.G, L
 Calderón, Alberto-Pedro (1920–) 36.M 217.J, r 224.A, F 251.O 274.B, I 321.F 323.J 345.A
 Calkin, John Williams (1909–) 36.J 390.I
 Callan, Curtis G. (1942–) 132.C 361.B, r
 Callen, Herbert Bernard (1919–) 419.r

- Campanato, Sergio (1930–) 168.B
Campbell, George Ashley (1870–?) 220.r 249.R
Camus, Jacques (1942–) 323.N
Cannon, James W. (1943–) 65.A, C, F
Cannonito, Frank Benjamin (1926–) 97.r
Cantelli, Francesco Paolo (1875–) 342.B 374.E
Cantor, Georg (1845–1918) 20.33.A 34.r 47.r
49.D, r 79.D 93.D 98.r 117.A 156.A 159.J 267
273.F 294.A, E, r 312.A, C 355.r 381.F, r 426
Cantor, Moritz Benedikt (1829–1920) 20.r 26.r 38.r
187.r 209.r 265.r 266.r 296.r 360.r 372.r
Cantwell, John C. 154.H
Cappell, Sylvain E. (1946–) 65.D 114.K, r
Carathéodory, Constantin(e) (1873–1950) 20.r
21.O, Q 43.J, K, r 46.r 74.r 77.C, r 82.r 136.A, C
180.r 198.r 246.G 255.D, E 270.E 316.F 320.r 321.r
324.r 333.B, C, r 334.E 438.B
Cardano, Giròlamo (1501–76) 8.10.D 294.A, r
360.*, r 444 App. A, Table I
Cardoso da Silva, Fernando Antonio Figueiredo
(1939–) 320.r
Carleman, Torsten (1892–1949) 20.30.A 41.D 43.H
58.F 68.L 112.N 125.A 160.r 164.J 168.B 217.J, r
240.K 321.F 323.J, M 336.I App. A, Table 8
Carleson, Fritz (1888–1952) 121.C
Carleson, Lennart A. E. (1928–) 43.F, G, r 48.r
77.E 124.C 159.H 164.I 168.B 169.r 352.F
Carlson, Bille Chandler (1924–) 389.r
Carlson, James A. 21.N
Carmeli, Moshe (1933–) 359.r
Carnahan, Brice (1933–) 304.r
Carnot, Lazare Nicholas Marguerite (1753–1823)
181.266
Carrell, James Baldwin (1940–) 226.r
Carrol, J. B. 346.E, r
Carroll, Robert Wayne (1930–) 378.r
Carson, John R. App. A, Table 12.II
Cartan, Elie (1869–1951) 13.H 21.P 50.80.A, M, N
90.r 105.r 109.*, r 110.B, r 111.r 137.*, r 147.A
152.C 178.A, B 183.191.E, H, I 218.D, E 219.B,
r 248.F, I, K, N, W, r 249.E, I, R, S, V, r 285.r
344.A–C, F 362.I 364.F 365.B, I 384.A 412.*,
r 413.F, r 417.r 427.B 428.E, G 434.B 437.X
App. A, Table 5.I
Cartan, Henri (1904–) 3.r 17.C, r 20.*, r 21.E, H,
I, L, O, Q, r 23.B, E, r 28.r 32.r 36.L 50.52.r 58.D
64.B 70.F, r 72.E 80.r 87.r 94.r 105.r 124.B, r 192.r
198.r 200.I, r 201.J 208.r 210.r 225.r 272.J 277.r
338.E, L, M, P 383.J, r 426
Carter, Roger William (1934–) 151.D, r
Cartier, Pierre Emil (1932–) 9.E 12.B 16.E 203.H
Cartwright, Mary L. 62.E
Case, James H. 108.C
Casimir, 248.J
Casorati, Felice (1835–90) 104.D 198.D
Cassandro, M. 402.G
Casselman, William Allen (1941–) 450.r
Cassels, John William Scott (1922–) 14.r 59.r 118.r
182.r 257.r
Cassini, Jean Dominique (1625–1712) 93.H
Casson, Andrew J. 114.K
Cassou-Noguès, Pierrette (1945–) 450.J
Castaing, Charles (1932–) 443.A
Castelnuovo, Guido (1865–1952) 3.E 9.H, r 12.B, r
15.B, E, G, H
Castillon, Giovanni Francesco Mauro Melchior
Salvemini de (1708–91) 179.A
Catalan, Eugène Charles (1814–94) App. A,
Table 10.III
Cauchy, Augustin Louis (1789–1857) 4.D 5.F 20
21.C 53.87.C 100.E 107.A, B 164.J 165.A, r 190.Q
198.A, B, E, F, Q 211.C 216.D, E 267.273.J 274.G
284.B 286.X, Z 294.E 296.301.G 316.A, C, G
320.B, D, I 321.A, B 339.A 341.D 344.A 379.A,
B, F, K 388.B 436.G App. A, Tables 8, 9, 10.II
Cauer, D. (1889–1918) 179.B
Cauer, Wilhelm (1900–45) 282.r
Cavalieri, (Francesco) Bonaventura (1598–1647)
20.265
Cayley, Arthur (1821–95) 12.B 54.105.A 137.151.H
157.A 190.Q 226.G 251.I 267.269.F, J 285.A
Cazenave, Thierry 286.Y
Čebyšev → Chebyshev
Čech, Edouard (1893–1960) 110.B, r 117.E 201.A,
M, P 207.C 383.F 425.T, r 426.436.I
Ceder, Jack G. (1933–) 425.Y
Cerf, Jean (1928–) 114.I
Cesari, Lamberto (1910–) 246.r 290.r 314.D, r
394.r
Cesáro, Ernesto (1859–1906) 297.D 379.K, M
Ceva, Giovanni (1647?–1734?) 7.A
Chaber, Józef 273.K
Chacon, Rafael Van Severen (1931–) 136.B, H 162
Chadan, Khosrow (1930–) 375.r
Chaikin, S. E. → Khaikin, S. E.
Chaitin, Gregory J. 71.r 354.D
Chakravarti, Indra-Mohan (1928–) 102.I
Chandler, Colston 274.D, I 386.C
Chandrasekhar, S. 433.r
Chandrasekharan, Komaravolu (1920–) 121.r 123.r
160.r 379.r 450.r
Chang Chen-Chung (1927–) 276.r 293.r
Chang, J. J. 346.E, r
Chang Sun-Yang (1948–) 164.I
Chaplygin, Sergei Alekseevich (1869–1942) 326.B
Chapman, D. G. (1920–) 399.D
Chapman, Sydney (1888–1970) 41.E 260.A 261.A
379.M 402.H, r
Chapman, Thomas A. (1940–) 65.C 382.B, D
Charnes, Abraham (1917–) 255.D, E 408.r
Charpit, Paul (?–1774) 82.C 320.D 322.B App. A,
Table 15.II
Charzyński, Zygmunt (1914–) 438.C
Chase, A. B. 24.r
Chase, Stephen Urban (1932–) 29.r 172.A, K
Chasles, Michel (1793–1880) 12.B 78.J 267.350.C
Châtelet, François (1912–) 118.D
Chaudhuri, Jyoti 63.D
Chaundy, T. 387.C
Chauvenet, William (1820–70) 392.r
Chazarain, Jacques (1942–) 274.r 325.H, M 345.B
378.F
Chazy, Jean (1882–1955) 288.D 420.D
Chebotarev, Nikolai Grigor'evich (1894–1947)
14.S 172.r
Chebyshev, Pafnutii L'vovich (1821–94) 19.G
123.A 223.A 299.A 317.D 336.B, H, J 342.C
App. A, Tables 20.II, VII
Cheeger, Jeff (1943–) 178.B, r 391.D, M, r
Chen Bang-Yen (1943–) 365.F, H, O, r 417.r
Chen Jing-Run (1933–) 4.C, r 123.E 242.A
Chen, T.-C. 142.C, r
Cheney, Elliott Ward (1929–) 142.r
Cheng, J. H. 365.L
Cheng Shiu-Yuen (1948–) 275.H 391.F, H
Ch'êng Ta-Wei (fl. 1592) 57.C
Chern Shiing-Shen (1911–) 21. N, P 50.r 56.C, F, r
80.r 90.r 109.*, r 110.E 111.r 147.A, N 152.C
218.D, E, r 237.B 272.L 275.A, E 279.C 344.B
365.B, H, L, O, r

- Chernoff, Herman (1923–) 371.A, C, H, r 374.r 400.K
- Chervonenkis, O. A. NTR
- Cherwell, Lord (Lindemann, Frederick Alexander) 291.F
- Cheung, Fan-Bill 386.r
- Chevalier, Alfred 171
- Chevalier, Jacques 329.r
- Chevalley, Claude (1909–84) 6.A 11.r 12.B 13.B, F, I, N, r 27.r 50.r 59.A, r 60.L, r 61.r 69.D 105.r 118.B, F 125.M 151.I 200.O 248.Q, r 249.U, V, r 256.r 258.r 277.r 284.G 368.r 409.r 423.r 427.B
- Chew, Geoffrey Foucar (1924–) 132.r 386.C, r
- Chien, Robert Tienwen (1931–) 282.r
- Ch'in Chiu-Shao (fl. 1250) 57.B
- Ching Wai-Mee 308.F
- Chisini, Oscar 9.r
- Chittenden, Edward Wilson (1895–1977) 273.K
- Choi Man-Duen (1945–) 36.J 308.r
- Cholesky 298.G, 302.B, D
- Chomsky, Noam (1928–) 31.D 75.E
- Choong, K. Y. 332
- Choquet, Gustave (1915–) 20.r 48.F, H, r 89.r 120.E 139.r 164.C 193.J, N, r 207.C 255.E 338.C, D, H, I, L–O 407.B 424.U, r
- Chow Wei-Liang (1911–) 3.B 12.B 13.F 16.H, R, S 72.F, H
- Chowla, Sarvandaman D. (1907–) 123.D 450.I, K, r
- Chretien, Max (1924–) 150.r
- Christian, Ulrich Hans Richard Otto (1932–) 32.F, H
- Christoffel, Elwin Bruno (1829–1900) 77.D 80.L 109 111.H 317.D 417.D App. A, Tables 4.II, 13.II
- Chu Hsin 126.N
- Chu Lan Jen (1913–) 133.r
- Chu Shih-Chieh (fl. 1300) 57.B
- Chudakov, Nikolai Grigor'evich (1904–) 4.C
- Chudnovskii, Grigoriĭ V. 430.D, r
- Chung Kai Lai (1917–) 45.r 260.J 342.B
- Church, Alonzo (1903–) 22.G 31.B 75.D 81.A, r 97.*, r 354.r 356.A, C, E, G, r
- Churchman, C. West (1913–) 307.A
- Ciarlet, Phillipe G. (1938–) 300.r 304.r
- Ciesielski, Zbigniew (1934–) 176.G
- Clagett, Marshall (1916–) 187.r 372.r
- Clairaut, Alexis Claude (1713–65) 20 107.A 165.A App. A, Tables 14.I, 15.II
- Clancey, Kevin F. 251.r
- Clark, Charles Edgar (1935–) 376.r
- Clarke, Douglas Albert 356.H, r
- Clarkson, James Andrew (1906–) 443.H
- Clatworthy, Willard H. STR
- Clausius, Rudolf Julius Emmanuel (1822–88) 419.A
- Clebsch, Rudolf Friedrich Alfred (1833–72) 11.B 226.G 353.B
- Clemence, Gerald Maurice (1908–) 55.r 392.r
- Clemens, Charles Herbert (1939–) 16.J
- Clenshaw, Charles William (1926–) 299.A
- Clifford, Alfred H. (1908–) 190.r 243.G
- Clifford, William Kingdon (1845–79) 9.C 61.A, D 275.F
- Clough, Ray William, Jr. (1920–) 304.r
- Coates, John H. (1945–) 118.D 182.r 450.J, r
- Cochran, William Gemmel (1909–80) 102.r 373.r 374.B
- Codazzi, Delfino (1824–73) 111.H 365.C 417.F App. A, Table 4.I
- Coddington, Earl Alexander (1925–) 107.r 252.r 253.r 254.r 314.r 315.r 316.r 394.r
- Coffman, Charles V. 246.J
- Cohen, Eckford (1920–) 121.A
- Cohen, Irvin Sol (1917–) 284.A, D, G
- Cohen, Jacob Willem (1923–) 227.r
- Cohen, Marshall M. (1937–) 91.r
- Cohen, Paul Joseph (1934–) 22.F 33.D, r 49.D 192.P, Q
- Cohen, S. G. 353.r
- Cohn, Paul Moritz (1924–) 249.r
- Cohn, Richard M. (1919–) 104.r
- Cohn-Vossen, Stefan (1902–36) 109 111.I 178.F, H 357.r 365.E 410.r
- Coifman, Ronald R. 168.B 251.r
- Cole, B. 164.D
- Cole, J. D. 25.r
- Coleman, Sidney Richard (1937–) 146.C
- Colin de Verdière, Yves (1945–) 391.J
- Collatz, Lothar Otto (1910–) 217.r 298.r
- Collingwood, Edward Foyle (1900–70) 62.C, D, r
- Collins, P. D. B. 386.r
- Combes, Jean-Michel Christian (1941–) 331.F
- Combescur, Édouard (c. 1819 (24?)–?) 111.F
- Commichau, Michael 72.r
- Condon, Edward U. 353.r
- Conforto, Fabio (1909–54) 3.r
- Conley, Charles Cameron (1933–84) 126.E
- Conlon, Laurence William (1933–) 154.H
- Conner, Pierre Euclide, Jr. (1932–) 237.r 431.E, r
- Connes, Alain (1947–) 136.F 308.H, I, r 351.L
- Constantine, Alan Graham 374.r
- Constantinescu, Corneliu (1929–) 193.U 207.C, D, r 367.E, G, r
- Conti, Roberto (1923–) 290.r
- Conway, John Horton 151.I 235.A
- Conway, Richard W. 376.r
- Cook, Joseph M. (1924–) 375.A
- Cook, Roger John (1947–) 118.D
- Cook, Stephen Arthur (1939–) 71.E, r
- Cooke, George Erskine (1932–) 201.r
- Cooke, Kenneth Lloyd (1925–) 163.B
- Cooke, Richard G. 379.r
- Cooley, James William (1926–) 142.D, r 304.r
- Cooper, William (1935–) 255.D, E 408.r
- Copernicus, Nicolaus (1473–1543) 360
- Coppel, William Andrew 314.r
- Corbató, Fernando J. 133.r
- Cordes, Heinz O. (1925–) 345.A
- Coriolis, Gaspard Gustav de (1792–1843) 271.D
- Cornea, Aurel (1933–) 193.U 207.C, D, r 367.E, G, r
- Cornish, Edmund Alfred (1909–73) 374.F
- Cornu, Marie Alfred (1841–1902) 93.H 167.D
- Corwin, Lawrence Jay (1943–) 132.r
- Coster, Joseph 386.C
- Cotes, Roger (1682–1716) 299.A
- Cottle, Richard Warren (1934–) 292.D
- Coulson, Charles Alfred (1910–74) 446.r
- Courant, Richard (1888–1972) 20.*, r 46.r 77.E, r 82.r 106.r 107.r 112.r 120.r 134.r 188.r 189.r 197.r 198.r 204.G 205.r 216.r 217.r 222.r 275.A, C, r 300.r 304.C, F, r 317.r 320.r 321.G, r 322.r 323.E, r 324.r 325.M, r 327.r 334.C, D 389.r 391.H 441.r 446.r
- Cousin, Pierre (1867–1933) 20 21.K, Q
- Cowen, Michael J. (1945–) 124.r
- Cowling, Thomas George (1906–) 259.r 402.r
- Cox, David Roxbee (1924–) 40.r 403.r
- Cox, Gertrude Mary (1900–78) 102.r
- Coxeter, Harold Scott Macdonald (1907–) 13.R 92.r 122.H 151.r 161.r 248.S 285.r 357.r

Cracknell, Arthur P.

Cracknell, Arthur P. 92.r
 Cramer, Gabriel (1704–52) 179.A 269.M 302.A
 Cramér, Harald (1893–1985) 123.r 214.C 242.A
 250.r 341.E, r 374.r 395.r 399.D, M, r 400.r
 Crandall, Michael G. (1940–) 162 286.T, X, r
 Crandall, Stephen Harry (1920–) 298.r
 Crapper, Gordon David (1935–) 205.F
 Crawford, Frank Stevens (1923–) 446.r
 Crelle, August Leopold (1780–1855) 1 NTR
 Cremona, Antonio Luigi Gaudenzio Giuseppe
 (1830–1903) 16.I
 Crittenden, Richard J. (1930–) 413.r
 Crofton, Morgan William (1826–1915) 218.B
 Cronin, Jane (Smiley) (1922–) 153.r
 Crout, Prescott D. 302.B
 Crow, James Franklin (1916–) 115.D 263.r
 Crowell, Richard Henry (1928–) 235.r
 Cryer, Colin W. 303.G
 Csiszár, Imre 213.r
 Čuda, Karel (1947–) 293.E, r
 Curtis, Charles Whittlesey (1926–) 29.r 92.r 151.r
 277.r 362.r
 Curtis, E. 70.r
 Curtis, John H. 299.A
 Cutkosky, Richard Edwin (1928–) 146.A, C 386.C
 Cutland, Nigel John (1944–) 293.r
 Czuber, Emanuel (1851–1925) 19.B

D

Dade, Everett C. (1937–) 92.F
 Dahlquist, Germund (1925–) 303.G
 Dakin, R. J. 215.r
 d'Alembert, Jean le Rond (1717–83) 20 107.B
 130.A 205.C 239 252.F 325.D App. A, Table 10.II
 Damerell, Robert Mark (1942–) 450.M
 Daniell, Percy John (1889–1946) 310.I
 Danilevskii, A. M. 298.D
 Danilov, V. I. 16.Z
 Dankner, Alan (1945–) 126.J
 Dantzig, George Bernard (1914–) 255.C, E, r 264.r
 292.D 408.r
 Darboux, Jean Gaston (1842–1917) 50 109.*, r
 110.B 126.L 158.r 216.A 275.A 317.D 320.C 428.A
 Darmon, Georges 374.H
 Darwin, Charles Robert (1809–82) 40.B
 Dashen, Roger Frederick (1938–) 132.r
 Date, Eturo (1950–) 287.C
 D'Atri, Joseph Eugene (1938–) 364.r 384.E
 Datta, Bibhutibhusan 209.r
 Davenport, Harold (1907–69) 4.E 118.D 192.P
 David, Florence Nightingale (1909–) STR
 David, Herbert Aron (1925–) 346.r 374.r
 Davidenko, Dmitrii Fëdorovich (1922–) 301.M
 Davie, Alexander M. 164.J
 Davies, Laurie 374.r
 Davis, Burgess J. (1944–) 262.B
 Davis, Chandler (1926–) 212.r
 Davis, Harold T. App. A, Table 21.r NTR
 Davis, Martin (David) (1928–) 22.H 31.r 97.*, r
 173.O 293.r 356.H
 Davis, Philip J. (1923–) 223.r 299.r
 Davis, William Jay (1939–) 68.M 443.H
 Davisson, Lee D. (1936–) 213.E
 Day, Mahlon Marsh (1913–) 37.r 310.r
 Daykin, David Edward 332.r
 De Alfaro, Vittorio (1933–) 132.r 375.r
 de Baggis, F. S. 126.A
 Debiard, Amédée 115.r
 Debreu, Gerard (1921–) 173.E 443.A, I

Debye, Peter Joseph William (1884–1966) 25.C, r
 30.C 39.E App. A, Table 19.III
 Dedekind, Julius Wilhelm Richard (1851–1916)
 11.B, r 12.B 14.C–E, J, U 47.49.F, r 67.K 98
 156.A, r 172.A, r 243.F 267 284.G 294.A, E, r
 328 347. H 355.A, r 363.r 379.D 450.A, D, K
 De Giorgi, Ennio 275.F 323.L
 de Haan, David Bierens (1822–95) App. A,
 Table 9.r
 Dehn, Max (1878–1952) 65.E 155.F 196
 Dejon, Bruno F. (1930–) 301.G, r
 Dekkers, A. J. NTR
 Delaunay, Charles Eugène (1816–72) 93.H
 de la Vallée-Poussin, Charles Jean (1866–1962)
 20.r 48.A, B 123.B 379.S 437.r 450.B, I
 Deleanu, Aristide (1932–) 52.r
 Delens, Paul Clément (1889–) 110.r
 de l'Hôpital, Guillaume François Antoine (1661–
 1704) 20
 Deligne, Pierre (1944–) 9.r 12.B 16.V, r 32.D 118.B
 418.r 428.H 450.A, G, H, J, M, Q, S, r App. B,
 Table 5
 Dellacherie, Claude (1943–) 22.r 261.r 262.r 407.B, r
 Deltheil, Robert 218.r
 Demazure, Michel (1937–) 13.r 16.I, Z, r
 de Miatello, I. D. 384.E
 Deming, William Edwards (1900–) 280.J 373.F, r
 Democritus (c. 460–c. 370 B.C.) 187
 de Moivre, Abraham (1667–1754) 74.C 250.A
 342.A
 De Morgan, Augustus (1806–71) 42.A 156.B 157.A
 381.B 411.A, r
 Dénes, József (1932–) 241.r
 Denjoy, Arnaud (1884–1974) 58.F 79.D 100.A, D
 126.I 154.D, H, r 159.I 168.B 429.D
 Denker, Manfred (1944–) 136.H
 Deny, Jacques (1916–) 338.M–P, r
 de Oliveira, Mário Moreria Carvalho 126.J
 De Paris, Jean-Claude 321.G
 de Possel, René (1905–) 77.E 367.F
 Deprit, André Albert (1926–) 420.G
 Deprit-Bartholomé, Andrée 420.G
 de Rham, Georges-William (1903–) 12.B, 105.R,
 V, r, 109.*, r 114.L 125.A, R 194.F, r 201.A, H, I
 237.H 249.V 274.G, 364.E, r, 417.r
 Desargues, Gérard (1593–1662) 155.E 181 265 329
 343.C
 Descartes, René (1596–1650) 7.C 10.E 20 93.H 101
 180.A 181 265 426
 Deser, Stanley (1931–) 150.r
 De-Shalit, Amos (1926–) 353.r
 Desoer, Charles A. (1926–) 86.D
 d'Espagnat, Bernard (1921–) 351.r
 DeTurck, Dennis M. 364.r
 Deuring, Max Friedrich (1907–84) 27.r 29.r 73.A,
 r 123.D 257.r 439.L 450.S, r
 Deutsch, Robert William (1924–) 55.r
 de Vries, G. 387.B
 De Wilde, Marc (1940–) 424.X, r
 DeWitt, Bryce Seligman (1923–) 359.r
 DeWitt (DeWitt-Morette), Cécile (1922–) 150.r
 359.r
 DiCastro, C. 361.r
 Dickinson, Bradley W. (1948–) 86.D
 Dickson, Leonard Eugene (1874–1954) 4.E 10.r
 54.r 60.K 118.A 151.I, r 296.r 297.r
 Didenko, Viktor Pavlovich 326.r
 Dido (Didon, Belus Elissa) 228.A
 Dienes, Paul (1882–1952) 339.r
 Diestel, Joseph (1943–) 37.r 443.A

- Dieudonné, Jean (1906–) 10.r 12.r 13.C 20.r 60.K,
r 139.r 151.r 183.r 200.r 203.r 226.r 321.r 355.r
389.r 424.r 425.S, X, Y 435.r 436.I
- Dijkstra, Edsger Wybe (1930–) 281.C
- Dikii, Leonid Aleksandrovich 387.C
- Diller, Justus (1936–) 155.r
- Dilworth, Robert Palmer (1914–) 281.E
- Dinaburg, E. I. 126.K
- Dinculeanu, Nicolae (1925–) 443.r
- Dinghas, Alexander (1908–74) 124.r 198.r
- Dini, Ulisse (1845–1918) 39.D 111.I 159.B 314.D
435.B
- Dimits, E. A. 281.r
- Dinkelbach, Werner 264.r
- Dinostratus (fl. 350? B.C.) 187
- Diocles (200 B.C.) 93.H
- Diophantus (c. 246–c. 330 or c. 1st century) 118.A
182.F 187 296.A
- Dippolito, Paul Randall (1948–) 154.D
- Dirac, Paul Adrien Maurice (1902–84) 125.C
132.A 150.A 270.D 297.r 351.G, r 359.C 377.B, C, r
App. A, Table 12.II
- Dirichlet, Peter Gustav Lejeune (1805–59) 14.D, U
84.D 98 119.r 120.A, F 121.A 123.D 159.B 160.B
164.B, 165.A, r 182.A 193.F 234.C 242.A 261.C 267
295.D 296.A, B 323.C, E 334.C 338.Q 341.D 347.E,
H, r 348.M 379.C, D 440.B 450.A, C, K App. A,
Table 9.V
- Dixmier, Jacques (1924–) 18.I 36.r 68.I 308.F, r
351.C 437.r
- Dixon, A. C. (1865–1936) App. A, Table 19.IV
- Dmitriev, N. A. 44.r
- Dobrushin, Roland L'vovich (1929–) 250.r 340.B,
r 402.G 407.B
- d'Ocagne, Maurice (1862–1938) 19.D, r
- do Carmo, Manfredo Perdigão 111.r 275.A, B
365.G, r
- Doetsch, Gustav (1892–1977) 43.E 208.r 240.r
379.M
- Doi, Koji (1934–) 450.L
- Doig, Alison G. 215.D
- Dolansky, Ladislav NTR
- Dolbeault, Pierre (1924–) 72.D
- Dolbeault-Lemoine, Simone 365.E
- Dold, Albrecht E. (1928–) 70.F, r 201.r
- Dolgachëv, Igor V. App. A, Table 5.V
- Dollard, John Day (1937–) 375.B
- Domb, Cyril 361.r 402.r
- Donaldson, Simon K. 114.K, r
- Dongarra, Jack J. (1950–) 298.r
- Donin, Iosif Failovich (1945–) 72.G
- Donnelly, Harold Gerard (1951–) 391.N
- Donoghue, William F., Jr. 212.r
- Donsker, Monroe David (1924–) 250.E 340.r
- Doob, Joseph Leo (1910–) 5.r 45.r 62.E 86.E 115.r
136.B 162 193.T 207.C 250.r 260.J 261.A, F, r
262.A, B, D 341.r 342.r 395.r 406.A 407.A, r
- Doolittle, M. H. 302.B
- Doplicher, Sergio (1940–) 150.E
- Dorfmeister, Josef F. (1946–) 384.r
- Dörge, Karl (1898–1975) 337.F
- Dorn, William Schroeder (1928–) 349.r
- Dornhoff, Larry 362.r
- Douady, Adrien (1935–) 23.G 72.G
- Douglas, Jesse (1897–1965) 77.E 109 152.C 275.A,
C 334.C, D, F, r
- Douglas, R. G. (1938–) 36.J, r 164.I 251.r 390.J, r
- Douglis, Avron (1918–) 112.H 323.H
- Dowker, Clifford Hugh (1912–82) 117.E 201.M
425.S, Y
- Drach, Jules (1871–1941) 107.A
- Drake, Frank Robert (1936–) 33.r
- Draper, Norman Richard (1931–) 102.r
- Drasin, David (1940–) 272.K, r
- Dreitlein, Joseph F. (1934–) 132.r
- Drell, Sidney David (1926–) 132.r 150.r
- Dreyfus, Stuart Ernest (1931–) 127.r
- Driver, Rodney D. (1932–) 163.B
- Dryden, H. L. 433.r
- Dubinsky, Ed (1935–) 168.B
- de Bois-Reymond, Paul David Gustave (1831–89)
159.H 379.D
- Dubreil, Paul 243.r
- Dubreil-Jacotin, Marie-Louise 243.r
- Dubrovín, B. A. 387.r
- Dubyago, Alexander Dmitrievich (1903–59) 309.r
- Dudley, Richard Mansfield (1938–) 176.G
- Duffin, Richard J. 264.r
- Duffing, G. 290.C
- Dufresnoy, Jacques 17.C 124.B
- Dugundji, James (1919–85) 425.r
- Duhem, Pierre Maurice Marie (1861–1916) 419.B
- Duijvestijn, A. J. W. (1927–) NTR
- Duistermaat, Johannes Jisse (1942–) 274.B, I
345.B, r 391.J
- Dulac, M. H. 289.C, D, r
- Dunford, Nelson (1906–) 37.r 68.M 112.I, O 136.B,
r 162.*, r 168.r 240.r 251.G, r 310.r 315.r 331.r
378.B, r 390.K, r 443.A, F–H, r
- Dupin, Pierre Charles François (1784–1873) 111.H
- Durand, E. 301.F
- Durand, William Frederick (1859–1958) 222.r
- Duren, Peter Larkin (1935–) 43.r 438.r
- Dürer, Albrecht (1471–1528) 360
- Durfee, Alan H. (1943–) 154.B 418.r
- Duschek, Adalbert (1895–1957) 111.r
- Du Val, Patrick (1903–) 418.C
- Duvaut, Georges (1934–) 440.r
- Dvoretzky, Aryeh (1916–) 45.r 443.D
- Dwork, Bernard M. (1923–) 450.G, Q
- Dwyer, Paul Summer (1901–) 298.r
- Dydak, Jerzy 382.A, C
- Dye, Henry Abel (1926–) 136.F, r
- Dyer, S. Eldon, Jr. (1929–) 237.r
- Dym, Harry (1938–) 176.K
- Dynkin, Evgenii Borisovich (1924–) 115.A, r 248.S,
r 250.r 261.A, C 270.B App. A, Table 5.I
- Dyson, Freeman John (1923–) 132.C 146.A 150.A
182.G 212.B 402.G
- Dzyaloshinskii, Igor Ekhiel'vich (1931–) 402.r
- E
- Easton, William B. 33.F, r
- Eberlein, Ernst 136.H
- Eberlein, Patrick Barry (1944–) 178.r
- Eberlein, William Frederick (1917–) 37.G 162 424.V
- Ebin, David G. (1942–) 178.B 183.r
- Eckmann, Beno (1917–) 200.I
- Eddington, Sir Arthur Stanley (1882–1944) 109
App. A, Table 4.II
- Eden, Richard John (1922–) 146.A, C, r 386.C, r
- Edgeworth, Francis Ysidro (1845–1926) 374.F
- Edmond, C. 432.r
- Edmonds, Alan Robert (1922–) 353.r
- Edrei, Albert (1914–) 17.D 272.K, r
- Edwards, Harold M. (1936–) 123.r
- Edwards, Robert Duncan (1942–) 65.A, C, F 154.H
- Eells, James (1926–) 105.r 114.B 183.*, r 194.r
195.E, r

Effros, Edward George

- Effros, Edward George (1935–) 36.H, J 308.r
 Efron, Bradley (1938–) 399.O
 Egorov, Dmitrii Fëdorovich (1869–1931) 270.J
 Egorov, Ivan Petrovich (1915–) 364.F
 Egorov, Yurii Vladimirovich (1938–) 112.D 274.C, I 345.A, B
 Ehrenfest, Paul (1880–1933) 260.A 402.r
 Ehrenfest, Tatiana Alekseevna Afanaseva 260.A 402.r
 Ehrenpreis, Leon (1930–) 112.B, C, R 125.S 320.H 437.EE
 Ehresmann, Charles (1905–79) 80.A, r 90.r 109 154.A, r
 Ehrlich, Louis W. (1927–) 301.F
 Ehrlich, Paul Ewing (1948–) 301.r 364.r
 Eichler, Martin Maximilian Emil (1912–) 11.B, r 13.P 27.D, r 32.D, H, r 60.r 61.r 348.r 450.A, L, M, S
 Eicken, W. 75.r
 Èidel'man, Samuil Davidovich (1921–) 112.B 327.H
 Eilenberg, Samuel (1913–) 31.r 52.r 70.E–G, r 75.r 91.r 200.K, M, O, r 201.A, C, E, G, J, Q 202.B, T 210.r 277.r 305.A 426.*, r
 Einstein, Albert (1879–1955) 45.A 109 129 132.A 137 150.A 256.J 285.A 351.A 359.A, B, D, r 364.D, I 434.C, r App. A, Table 4.II
 Eisenberg, Edmund (?–1965) 292.D
 Eisenhart, Luther Pfahler (1876–1965) 109.*, r 111.r 417.r
 Eisenstein, Ferdinand Gotthold Max (1823–52) 14.O 32.C, F 296.A 337.F 339.E 450.T
 Ejiri, Norio (1953–) 275.A 364.F 365.G 391.C
 Elias, Peter (1923–) 213.F
 Eliasson, Halldór Ingimar (1939–) 364.H
 Elliott, George Arthur (1945–) 36.H, K, r
 Elliott, Peter D. T. A. 295.r
 Ellis, George F. R. (1939–) 359.r
 El'sgol'ts, Lev Ernestovich (1909–67) 163.r
 Elworth, Kenneth David (1940–) 183 286.D 406.r
 Emch, Gerald Gustav (1936–) 351.r
 Emde, Fritz 389.r NTR
 Emden, Robert (1862–1940) 291.F
 Endo Shizuo (1933–) 200.K
 Eneström, Gustav (1852–1923) 10.E
 Enflo, Per (1944–) 37.G, L
 Engel, Friedrich (1861–1941) 247.r 248.F 249.r
 Engelking, Ryszard 117.r 273.r 425.r 436.r
 Enneper, A. 275.A, B
 Ennola, Veikko Olavi (1932–) App. B. Table 5.r
 Enoki Ichiro (1956–) 72.K
 Enomoto Hikoe (1945–) App. B, Table 5
 Enright, Wayne H. 303.r
 Enriques, Federigo (1871–1946) 9.r 12.B, r 15.B, E, G, H, r 72.K
 Enskog, David (1884–1947) 41.E 217.N 402.H
 Enss, Volker (1942–) 375.D
 Epstein, David Bernard Alper (1937–) 154.H
 Epstein, Henri (1932–) 125.W 150.D 212.B 386.B, r
 Epstein, Paul Sophus (1883–) 450.A, K
 Eratosthenes (275–194 B.C.) 187 297.B
 Erbacher, Joseph A. 365.H
 Erdélyi, Artur (1908–77) 25.r 30.r 220.r 254.r 389.r App. A, Table 20.r
 Erdős Paul (1913–) 4.A 45.r 123.C, r 241.E 250.r 295.E 328 336.E 342.B
 Erlang, A. K. 260.H 307.C
 Ernst, B. 359.E
 Ernst, Bruno (1947–) 424.r
 Escobal, Pedro Ramon 309.r
 Eskin, Grigorii Il'ich (1936–) 274.C, I 345.B
 Estermann, Theodor 4.C, D 123.D
 Estes, William Kaye 346.G
 Ethier, Stewart N. 263.r
 Euclid (Eukleides) (c. 303–c. 275 B.C.) 13.R 24.C 35.A 67.L 70.B 93.A 139.A, B, E 140 150.F 155.A 179.A 180.A 181 187 285.A, C 296.A 297.A, B 332 337.D 364.B 412.H 423.M 425.V
 Eudoxus (c. 408–c. 355 B.C.) 20 187
 Euler, Leonhard (1707–83) 4.C 16.E 20 38 46.A, B 56.B, F 65.A 83.A 90.C 93.C 107.A, B 126.A 131.D, G 141 145 165.A, r 174.A, C 177.C, D 181 186.A, F 201.B, F, N 204.E 205.A, B 240.A 241.B 266 271.E, F 275.A 294.A 295.C, E 296.A 297.D, H 303.D, E 320.D 332 379.I–K 419.B 420.B 432.C 441.B 450.B App. A, Tables 3.V, 14.I App. B, Tables 3.I, 6.IV
 Evans, Griffith Conrad (1887–1973) 48.E 120.D 338.H
 Evens, Leonard (1933–) 200.M
 Everett, J. D. 223.C App. A, Table 21
 Ewens, Warren J. 263.r
- F**
- Faber, Georg (1877–) 228.B 336.I 391.D 438.B
 Fabry, Eugène (1856–1944) 339.D
 Faddeev, Dmitrii Konstantinovich (1907–) 112.P 302.r
 Faddeev, Lyudvig Dmitrievich (1934–) 132.C 150.G 375.F 387.G
 Faddeeva, Vera Nikolaevna (1906–) 302.r
 Fagnano, Giulio Carlo (1682–1766) 20
 Falb, Peter L. (1936–) 86.D
 Fälthammar, Carl-Gunne (1931–) 259.r
 Faltings, Gerd (1954–) 118.E 145
 Fan Ky (1914–) 153.D
 Fannes, Marcus Marie-Paul (1950–) 402.G
 Fano, Gino (1871–1952) 12.B 137.r
 Fano, Robert Mario (1917–) 130.r 213.F
 Fano, Ugo (1912–) 353.r
 Fantappiè, Luigi 125.A
 Faraday, Michael (1791–1867) 150.A
 Farey, J. 4.B
 Farkas, Julius (1847–1930) 255.B, E
 Farquhar, Ian E. 402.r
 Farrell, O. J. 164.J
 Farrell, Roger Hamlin (1929–) 398.r
 Fary, Istvan (1922–?) 111.r 365.O
 Fathi, Albert 126.N
 Fatou, Pierre (1878–1929) 21.Q 43.D 221.C 272.D 339.D
 Fattorini, Hector O. 378.D
 Favard, Jean (1902–) 336.C
 Fazar, W. 376.r
 Federer, Herbert (1920–) 246.r 275.A, G, r 334.F
 Fedorov, Evgraf Stepanovich (1853–1919) 92.F 122.H
 Fedorov, Vyatseslav Vasil'evich 102.r
 Feferman, Solomon (1928–) 81.A
 Fefferman, Charles L. (1949–) 21.P, Q 168.B, r 224.E 262.B 320.r 344.D, F 345.A, B
 Feinberg, Stephen E. 403.r
 Feinstein, Amiel 213.F
 Feit, Walter (1930–) 151.D, J, r 362.r
 Fejér, Lipót (Leopold) (1880–1959) 43.J 77.B 159.C 255.D
 Fekete, Mihály (Michael) (1886–1957) 48.D 445
 Feldblum 179.B
 Feldman, Jacob (1928–) 136.F

Fel'dman, Naum Il'ich 118.D 430.D, r
 Fell, James Michael Gardner (1923–) 308.M
 Feller, William (1906–70) 112.J 115.A, r 250.r 260.J
 261.A, B 263.r 341.r 342.r 378.B
 Fenchel, Werner (1905–) 89.r 109 111.F 365.O
 Fendel, D. App. B, Table 5
 Fenstad, Jens Erik 356.F, r
 Fermat, Pierre de (1601–65) 4.D 20 109 144 145
 180.A 265 296.A 297.F, G 329 342.A 441.C
 Fermi, Enrico (1901–54) 132, 287.A, r 351.H 377.B
 402.E
 Fernique, Xavier 176.G
 Ferrar, William Leonard (1893–) App. A, Table
 19. IV
 Ferrari, Ludovico (1522–65) 8 10.D 360 444
 App. A, Table I
 Ferrero, Bruce 14.L 450.J
 Ferrers, N. M. 393.C
 Ferus, Dirk 365.E, J, N, O, r
 Feshbach, Herman (1917–) 25.r
 Fet, Abram Il'ich (1924–) 279.G
 Feynman, Richard Phillips (1918–) 132.C 146.A, B
 150.A, F 351.F 361.A
 Fibonacci (Leonardo da Pisa, Leonardo Pisano)
 (c. 1174–c. 1250) 295.A 372
 Fiedler, Wilhelm 350.r
 Fienberg, Stephen Elliott (1942–) 280.r
 Fierz, Markus Edoward (1912–) 150.A
 Fife, Paul C. 95.r 263.D
 Figiel, Tadeusz 68.K, M
 Figueira, Mário Sequeira Rodrigues 286.Y
 Filippov, Aleksei Fedorovich (1923–) 22.r
 Fillmore, Peter Arthur (1936–) 36.J 390.J, r
 Finn, Robert (1922–) 204.D, r 275.A, D
 Finney, David John (1917–) 40.r
 Finney, Ross L. 201.r
 Finsler, Paul (1894–1970) 109 152.A 286.L
 Fischer, Arthur Elliot 364.H
 Fischer, Bernd (1936–) 151.I, J
 Fischer, Ernst (1875–1956) 168.B 317.A
 Fischer, Gerd 23.r
 Fischer-Colbrie, Doris Helga (1949–) 275.F
 Fisher, Michael Ellis (1931–) 361.r
 Fisher, Ronald Aylmer (1890–1962) 19.r 40.B
 102.A, E, r 263.E 371.A, C 374.B, F 397.r 399.D,
 K, N, O, r 400.G 401.B, C, F, G, r 403.A, F, r STR
 Fisher, Stephen D. 43.G 77.E
 Fix, George Joseph (1939–) 300.r 304.r
 Flammer, Carson (1919–) 133.r
 Flanders, Harley (1925–) 94.r 432.r 442.r
 Flaschka, Hermann (1945–) 287.B, r
 Fleissner, William G. 273.K
 Fleming, Wendell Helms (1928–) 108.A 275.A, G
 334.F 405.r
 Flon, L. 75.r
 Floquet, G. 107.A 252.J 268.B
 Floyd, Edwin E. (1924–) 237.r 431.E, r
 Focken, C. M. 116.r
 Föder, Géza (1927–77) 33.r
 Fodor, Jerry A. 96.r
 Fogarty, John 226.r
 Fogels, E. 123.D, F
 Foguel, Shaul R. 136.C
 Foias, Ciprian (1933–) 251.N
 Fok (Fock), Vladimir Aleksandrovich (1898–1974)
 105.C 377.A, r
 Fokker, Adriaan Daniël 115.A 402.I
 Fomin, Sergei Vasil'evich (1917–75) 2.F 46.r
 136.G
 Fong, Paul 151.J App. B, Table 5.r

Name Index

Frobenius, Ferdinand Georg

Ford, Lester R. (1896–1971) 234.C, r 281.r 282.r
 Ford, Walter Burton (1874–1971) 30.r
 Forelli, Frank (1932–) 164.G, H, K
 Forrester, Jay Wright (1918–) 385.B, r
 Forst, Gunner 338.r
 Forster, Otto F. (1937–) 72.r
 Forsyth, Andrew Russell (1858–1942) 289.B 428.r
 App. A, Table 14.r
 Forsythe, George Elmer (1917–72) 302.r 304.r
 Fort, Marion Kirkland, Jr. (1921–) 65.r
 Fort, T. 104.r
 Fortet, Robert Marie (1912–) 395.r
 Fortuin, C. M. 212.A
 Foster, Ronald Martin (1896–) 220.r
 Fotiadi, Dimitri 146.A, C
 Fourier, Jean-Baptiste-Joseph (1768–1830) 10.E
 18.B 20 36.L 39.D 125.O, P, BB 142.D 158 159.A
 160.A–D 176.I 192.B, D, F, K, O 197.C 220.B
 255.E 266 267 274.C 317.A 327.B 345.B 437.Z
 App. A, Tables 11. I, II
 Fowler, Kenneth Arthur (1916–) 151.J
 Fowler, Ralph Howard (1889–1944) 402.r
 Fox, Ralph Hunter (1913–73) 65.G 235.A, C, G, r
 382.A
 Fraenkel, Abraham Adolf (1891–1965) 33.A, B, D,
 r 47.r 381.r
 Frame, J. S. App. B, Table 5
 Francis, J. G. F. 298.F
 Frank, Philipp (1884–1966) 129.r
 Frankel, Theodore T. (1929–) 364.D
 Franklin, Philip (1898–1965) 157.A, E
 Franklin, Stanley P. (1931–) 425.CC
 Franks, John M. (1943–) 126.J, K
 Franz, Wolfgang (1905–) 91.r 337.F
 Fraser, Donald Alexander Stuart (1925–) 396.r
 401.r
 Frautschi, Steven Clark (1933–) 386.r
 Fréchet, René Maurice (1878–1973) 37.O 87.K, r
 117.H 246.A, I 273.A, r 286.E, K 424.I 425.Q, S,
 CC, r 426
 Fredholm, Erik Ivar (1866–1927) 20 68.A, E, F, K,
 L 120.A 162 217.A, E, F, r 222.A 251.D 286.E
 339.D
 Freedman, David A. (1938–) 250.r
 Freedman, Michael Hartley (1951–) 114.K, r
 Frege, Friedrich Ludwig Gottlob (1848–1925)
 156.B 411.A, r
 Freitag, Eberhard (1942–) 32.F
 Frenet, Jean-Frédéric (1816–1900) 110.A 111.D
 App. A, Table 4.I
 Fresnel, Augustin Jean (1788–1827) 167.D App. A,
 Tables 9.V, 19.II
 Freudenthal, Hans (1905–) 162 178.F 202.A, U, r
 248.r 249.r 265 310.A, D
 Freyd, Peter John (1936–) 52.r 200.r
 Fricke, Robert (1861–1930) 32.r 73.r 122.r 233.r
 234.r
 Friedberg, Richard Michael (1935–) 356.D
 Friedman, Avner (1932–) 108.A, B 115.D 286.r
 320.r 322.r 327.r 406.r 440.r
 Friedman, James W. 173.E
 Friedman, Lawrence 307.r
 Friedman, Nathaniel A. (1938–) 136.E, r
 Friedrichs, Kurt Otto (1901–83) 112.D, I, S 125.A
 162 204.G 205.r 252.r 300.r 304.F 323.H, r 325.G,
 r 326.D 331.A 345.A 351.K 375.A
 Fristedt, Bert (1937–) 5.r
 Frobenius, Ferdinand Georg (1849–1917) 1.r 2.B
 3.A, D, N 14.K 29.H 107.A 145 151.H 154.B
 190.Q, r 191.B 257.D 267 269.I, N 280.F 286.H

- 297.I 310.H 362.E, G 390.B 428.A, D 437.EE
 450.P App. A, Table 14.I
- Fröhlich, Albrecht (1916–) 14.r 59.r
- Fröhlich, Jürg M. (1946–) 402.G
- Froissart, Marcel (1934–) 146.A 386.B
- Frolík, Zdeněk (1933–) 425.Y, CC 436.r
- Fronsdal, Christian (1931–) 132.r
- Frostman, Otto Albin (1907–77) 48.A, G 120.D
 338.C, r
- Froude, William (1810–79) 116.B
- Fu, James Chuan (1937–) 399.M
- Fubini, Guido (1879–1943) 109 110.B, r 221.E
 270.L
- Fubini, Sergio Piero (1928–) 132.r
- Fuchs, Immanuel Lazarus (1833–1902) 32.B 107.A
 119.r 122.C 178.F 234.B 288.B App. A, Table 18.I
- Fuchs, Ladislav 2.r
- Fuchs, Maximilian Ernst Richard (1873–) 253.A, E
 288.D
- Fuchs, Wolfgang Heinrich (1915–) 17.D 272.K, r
- Fueter, Karl Rudolf (1880–1950) 73.r
- Fuglede, Bent (1925–) 48.H 143.B 338.E
- Fujii, Hiroshi (1940–) 304.D
- Fuji'i'e, Tatuo (1930–) 143.r
- Fujiki, Akira (1947–) 23.G 72.H 232.C
- Fujikoshi, Yasunori (1942–) 280.r
- Fujimagari, Tetsuo (1943–) 44.E
- Fujimoto, Hirotaka (1937–) 21.M, N
- Fujisaki, Genjiro (1930–) 6.F 450.L
- Fujisaki, Masatoshi (1943–) 86.E 405.r
- Fujisawa, Rikitarō (1861–1933) 267
- Fujishige Satoru (1947–) 66.r 281.r
- Fujita, Hiroshi (1928–) 204.B–D 304.r 378.J
- Fujita, Takao (1949–) 9.r 15.H 72.I
- Fujiwara, Daisuke (1939–) 323.H 345.B
- Fujiwara, Masahiko (1943–) 118.C, D
- Fujiwara, Matsusaburō (1881–1946) 89.E 230.*, r
 240.B App. A, Table 9.r
- Fukamiya, Masanori (1912–) 162
- Fuks, Boris Abramovich (1907–75) 21.r 198.r
- Fuks, Dmitrii Borisovich 105.AA, r 154.G
- Fukushima Masatoshi (1935–) 115.C 261.r
- Fulkerson, Delbert Ray (1924–76) 376.r
- Fuller, Francis Brock 126.G, N
- Fulton, William (1939–) 9.r 16.I 366.E, r 418.r
- Funk, Paul 46.r 109
- Furlan, Giuseppe (1935–) 132.r
- Fürstenberg, Hillel 136.C, H
- Furtwängler, Philipp (1869–1940) 14.L, O, R 59.A,
 D, F 145
- G**
- Gaal, S. A. 425.r
- Gabriel, Pierre (1933–) 13.r 52.r
- Gabrielov, A. M. App. A, Table 5.V
- Gackstatter, Fritz (1941–) 275.E
- Gagliardo, Emilio 224.A
- Gaier, D. 77.r
- Gaifman, Haim (1934–) 33.r
- Galambos, János (1940–) 374.r
- Galanter, Eugene H. 96.r
- Gale, David (1921–) 22.H
- Galerkin, Boris Grigor'evich (1871–1945) 303.I
 304.B
- Galilei, Galileo (1564–1642) 78.F 107.A 126.A 265
 271.A 359.C 360
- Gallager, Robert G. 63.r 213.E, F
- Gallagher, Patrick Ximenes (1935–) 123.E
- Gallavotti, Giovanni 136.G
- Gallot, Sylvestre (1948–) 364.G
- Galois, Évariste (1811–32) 8 83.C 137 149.M 151.D
 171.*, r 172.A, B, G, H 190.Q 200.N 267
- Galton, Francis (1822–1911) 40.B 44.B, C, r 401.E
- Galvin, Fred (1936–) 33.F
- Gambier, Bertrand Olivier (1879–1954) 288.C
- Gamelin, Theodore Williams (1939–) 36.r 43.r
 164.G, H, J, K, r 169.r
- Gamkrelidze, Revaz Valerianovich 86.r
- Gandy, Robin Oliver 81.A 356.r
- Gangolli, Ramesh A. (1935–) 5.r 437.EE
- Gantmakher, Feliks Ruvimovich (1908–64) 103.r
 248.r
- Garabedian, Paul Roesel (1927–) 77.E 107.r 438.C
- Garbow, Burton S. 298.r
- Garcia, August 136.B
- Gårding, Lars (1919–) 112.G, N 189.B, r 321.G
 323.H 325.F, J, r 345.A
- Gardner, Clifford S. (1924–) 387.B, D
- Garfinkel, Robert S. (1939–) 215.r
- Garland, Howard (1937–) 122.G
- Garnett, John B. (1940–) 164.F, J 169.E, r
- Garnier, René (1887–) 253.E 288.C, D 334.C
- Garnir, Henri (1921–85) 189.r
- Garsia, Adriano M. (1928–) 262.r
- Garside, F. A. 235.F
- Garside, G. R. 301.N
- Gaschütz, Wolfgang (1920–) 151.D
- Gass, S. I. 255.r
- Gastwirth, Joseph L. (1938–) 371.H
- Gâteaux, R. 286.E
- Gauduchon, Paul (1945–) 109.r 391.r
- Gauss, Carl Friedrich (1777–1855) 3.A 10.E 11.B
 14.U, r 16.V 73.A 74.A, C, r 77.B 83.A 94.F 99.C
 107.A 109 111.F–H 118.A 120.A 123.A 136.C
 149.I 174.A 175 176.A–C, F, H 179.A 180.B 193.D
 206.A 223.C 242.A 253.B 267 275.C 279.C 285.A
 294.A 295.D 296.A, B 297.I 299.A 302.B, C 309.C,
 r 327.D 337.D 338.J 341.D 347.H 357.A 364.D, H
 365.C 401.E 403 407.D 417.F 426 441.B 450.C
 App. A, Tables 3.III, 4.I, 10, 14.II, 16.III, 21
- Gaveau, Bernard (1950–) 115.r
- Gear, Charles William (1935–) 303.E, G, r
- Gegenbauer, Leopold (1849–1903) 317.D 393.E
 App. A, Table 20.I
- Gel'fand, Izrail' Moiseevich (1913–) 32.r 36.E, G, L
 46.r 82.r 105.AA, r 112.O, r 125.A, Q, S 136.G
 154.G 160.r 162.*, r 168.B 183.r 192.G 218.F, G
 256.r 258.r 287.C 308.D 341.r 375.G 387.C, D
 395.r 407.C 424.T, r 437.W, EE, r 443.A, F, H
 450.A, T
- Gel'fond, Aleksandr Osipovich (1906–68) 182.G
 196 295.r 430.A, D, r
- Gell-Mann, Murray (1929–) 132.A, D, r 150.C
 361.r
- Gentzen, Gerhard (1909–45) 156.E 411.J, r
- Geöcze, Zoard de 246.D, E
- Geoffrion, Arthur M. 215.D, r
- Georgescu, V. 375.B
- Gérard, Raymond (1932–) 428.H, r
- Gerbert (Silvester II) (940?–1003) 372
- Gerhardt, Carl Immanuel (1816–99) 245.r
- Germain, Sophie (1776–1831) 111.H 145 App. A,
 Table 4.I
- Gerstner, Franz Joseph von (1789–1823) 205.F
- Getoor, Ronald Kay (1929–) 5.r 261.r 262.r
- Gevrey, Maurice-Joseph (1884–1957) 58.G, r
 125.U 168.B 325.I 327.C
- Ghosh, Javanta Kumar (1937–) 374.F 396.r 399.O
- Ghosh, Sakti P. 96.r

- Ghouila-Houri, Alain 281.r 282.r
 Giacaglia, Giorgio Eugenio Oscare 290.r
 Gibbs, Josiah Willard (1839–1903) 136.A, C 159.D
 340.B 402.B, D, G 419.B, C, r
 Gierer, Alfred (1929–) 263.D
 Giesecker, David (1943–) 16.Y 72.K
 Giga Yoshikazu (1955–) 204.C
 Gikhman, Iosif Il'ich (1918–?) 406.r
 Gilbert, Edgar N. (1923–) 63.B
 Gilford, Dorothy Morrow (1919–) 227.r
 Gilkey, Peter Belden (1946–) 391.N, r
 Gill, Stanley Jensen (1929–) 303.D
 Gillies, Donald Bruce (1928–75) 114.D 173.D
 Gillman, Leonard (1917–) 425.r
 Gindikin, Semen Grigor'evich (1937–) 384.C, r
 Gini, Corrado 397.E
 Ginibre, Jean (1938–) 212.A, r 375.F
 Giraud, Georges 323.C, F
 Giraud, Jean (1936–) 200.M
 Giri, Narayan C. (1928–) 280.r
 Girsanov, Igor Vladimirovich (1934–) 115.D 136.D
 406.B
 Girshick, M. A. 398.r 399.F, r
 Giusti, Enrico (1940–) 275.F
 Givens, James Wallace, Jr. (1910–) 298.D
 Glaeser, Georges (1918–) 58.C, r
 Glaser, V. 150.D 386.B
 Glashow, Sheldon Lee (1932–) 132.D
 Glauber, Roy J. 340.C
 Glauberman, George Isaac (1941–) 151.J, r
 Glauert, Hermann (1892–1934) 205.B, D
 Glazman, Izrail' Markovich (1916–68) 197.r 251.r
 390.r
 Gleason, Andrew Mattei (1921–) 164.F 196 351.L
 Gleser, Leon Jay (1939–) 399.M
 Glezerman, M. 201.r
 Glicksberg, Irving Leonard (1925–83) 86.r 425.T
 Glimm, James Gilbert (1934–) 36.H 150.F, r
 204.G, r 308.L
 Glivenko, Valerii Ivanovich (1896–1940) 374.E
 411.F, J, r
 Glowinski, Roland (1937–) 440.r
 Gluck, Herman R. (1937–) 183.r
 Gnanadesikan, Ramanathan 280.r
 Gnedenko, Boris Vladimirovich (1912–) 250.r
 341.G 374.G
 Godbillon, Claude 154.G, r 201.r
 Gödel, Kurt (1906–78) 22.H 31.B 33.A, C, D, F, r
 49.D 97.* 156.E, r 184 185.A, C, r 276.D 356.A–C, r
 411.J, r
 Godement, Roger (1921–) 36.L 103.r 192.r 200.r
 201.r 256.r 277.r 337.r 368.r 383.E, r 450.A, L, T
 Godwin, A. N. 51.r
 Gokhberg (Gohberg), Izrail' Tsudikovich (1928–)
 68.A, J, r 251.r 390.H, r
 Goldbach, Christian (1690–1764) 4.C
 Gol'dberg, Anatolii Asirovich (1930–) 17.C 272.K, r
 Goldberg, Richard R. 160.r
 Goldberg, Samuel I. (1923–) 105.r 110.E 194.r
 364.F 417.r
 Golden, Sidney (1917–) 212.B
 Goldman, Oscar (1925–) 29.K 200.L
 Goldschmidt, David M. 151.J
 Gol'dsheid (Goldseid), I. J. 340.r
 Goldstein, Sheldon (1947–) 136.G
 Goldstein, Sydney (1903–) 205.r 268.C
 Goldstine, Herman Heine (1913–) 75.r 138.r
 Goldstone, Jeffrey (1933–) 132.C
 Golod, Evgenii Solomonovich (1934–) 59.F 161.C
 Golub, Gene Howard (1932–) 302.r
 Goluzin, Gennadii Mikhailovich (1906–52) 48.r
 77.r 438.B, C, r
 Gomory, Ralph E. (1929–) 215.B, C, r
 González-Acuña, Francisco 235.E
 Good, Irving John (1916–) 403.G, r
 Goodier, James Norman (1905–) 271.r
 Goodman, Timothy N. T. (1947–) 126.K 136.H
 Goodner, Dwight Benjamin (1913–) 37.M
 Goodwyn, L. Wayne 126.K 136.H
 Göpel, Gustav Adolph (1812–47) 3.A
 Goppa, V. D. 63.E
 Gordan, Paul Albert (1837–1912) 11.B 226.G
 255.E 353.B
 Gordon, Marilyn K. 303.r
 Gordon, Walter (1893–1940) 351.G 377.C 387.A
 Gorenstein, Daniel (1923–) 151.J, r 200.K
 Gor'kov, Lev Petrovich (1929–) 402.r
 Gorny, A. 58.D
 Gorry, G. Anthony 215.r
 Gottschalk, W. D. 126.r
 Goulaouic, Charles (?–1983) 323.N
 Goursat, Edouard Jean-Baptiste (1858–1936) 11.r
 20.r 46.r 92.F 94.F 193.O 198.B 217.F, r 278.r
 320.r 321.r 322.r 324.r 428.r
 Govorov, Nikolai Vasil'evich (1928–) 272.K
 Grad, Harold (1923–) 41.B
 Graeffe (Gräffe), Carl Heinrich (1799–1873) 301.N
 Graeb, Werner 256.r
 Graev, Mark Iosifovich (1922–) 125.r 162.r 183.r
 218.r 437.r
 Graff, Karl F. 446.r
 Gragg, William Bryant (1936–) 303.F
 Graham, Ronald Lewis (1935–) 376.r
 Gram, Jørgen Pedersen (1850–1916) 103.G 208.E
 226.E 302.E 317.A
 Grammel, Richard (1889–) 19.r
 Granoff, Barry (1938–) 325.L
 Grant, J. A. 301.L
 Grashof, Franz (1826–93) 116.B
 Grassmann, Hermann Günther (1809–77) 90.B
 105.A 147.I 199.B 256.O 267
 Grau, Albert A. (1918–) 301.E
 Grauert, Hans (1930–) 20 21.I, L, Q, r 23.E–G, r
 32.F 72.E, G, H, J, r 118.E 147.r 178.F 194.F 232.r
 418.B
 Graunt, John (1620–74) 40.A 401.E
 Gray, Andrew 39.r
 Gray, John Walker (1931–) 110.E
 Gray, Robert M. (1943–) 213.E, F
 Graybill, Franklin Arno (1921–) 403.r
 Green, Albert E. 271.r
 Green, George (1793–1841) 45.D 94.F 105.W
 120.A 188.A 189.A, B 193.D, J, N 252.K 315.B
 327.D App. A, Tables 3.III, 4.II, 15.VI
 Green, H. E. 291.F
 Green, Herbert Sydney (1920–) 402.J, r
 Green, James Alexander (1926–) App. B, Table 5.r
 Green, Leon W. (1925–) 136.G 178.G
 Green, Mark L. (1947–) 21.N
 Green, Melville S. 361.r 402.r
 Greenberg, Bernard G. (1919–85) 374.r
 Greenberg, Leon (1931–) 234.D, r
 Greenberg, Marvin Jay (1935–) 93.r 118.r 201.r
 Greene, John M. (1928–) 387.B
 Greene, Robert Everist (1943–) 178.r 365.B
 Greenwood, J. Arthur STR
 Gregory, James (1638–75) 332
 Gregory, R. T. 301.r
 Grenander, Ulf (1923–) 395.r 421.r
 Griess, Robert Louis, Jr. (1945–) 151.I

Griffiths, Phillip A. (1938–) 9.E, r 16.J, r 21.N 72.G
124.r 218.E 272.L
Griffiths, Robert Budington (1937) 212.A
Grigelionis, Bronyus Igno (1935–) 262.r
Grillenberger, Christian (1941–) 136.r
Grisvard, Pierre (1940–) 224.E
Grobman, David Matveevich (1922–) 126.G
Gromoll, Detlef (1938–) 109.r 178.r 279.G
Gromov, Mikhael L. 154.F 178.r 364.H
Gross, Oliver Alfred (1919–) 86.r
Gross, W. (?–1918) 62.E 246.G 272.I 429.D
Grothendieck, Alexander (1928–) 3.N, r 12.B, r
13.B, r 16.E, U, Y, AA, r 29.K 52.r 68.A, K, M,
N, r 125.r 168.B, r 200.I, r 203.H 210.r 237.A,
B, J 325.J 366.A, D, r 383.E, r 424.L, S, X, r 426
443.A, D 450
Grötschel, Martin 215.C
Grötzsch, Herbert (1902–) 352.A, C 438.B
Grove, Karsten 178.r
Groves, G. W. 92.r
Grünbaum, Branko (1929–) 16.r 89.r
Grünbaum, F. Alberto 41.C
Grunsky, Helmut (1904–86) 77.E, F, r 226.r 438.B
Grünwald, Géza (1910–42) 336.E
Grushin, Viktor Vasil'evich (1938–) 323.K, N
345.A
Guckenheimer, John M. (1945–) 126.K, N
Guderley, Karl Gottfried (1910–) 205.r
Gudermann, Christoph (1798–1852) 131.F 447
App. A, Tables 16, 16.III
Guerra, Francesco (1942–) 150.F
Guest, Philip George (1920–) 19.r
Gugenheim, Victor K. A. M. (1923–) 65.D
Guggenheim, Edward Armand (1901–) 419.r
Guignard, Monique M. 292.B
Guilford, Joy Paul (1897–) 346.r
Guillemin, Victor W. (1937–) 105.r 191.r 274.I, r
325.L 391.J, N 428.F, G 431.r
Guiraud, Jean-Pierre 41.D
Gulliver, Robert D., II (1945–) 275.C 334.F
Gumbel, Emil J. (1891–) 374.r
Gundlach, Karl-Bernhard (1926–) 32.G
Gundy, Richard Floyd (1933–) 168.B 262.B
Gunning, Robert Clifford (1931–) 21.r 23.r
367.G, r
Gunson, Jack 386.C
Gupta, Suraj Narayan (1924–) 150.G
Guthrie, Francis 157.A
Guthrie, Frederick 157.A
Guttman, Louis 346.r
Györy, Kálmán (1940–) 118.D
Gysin, Werner 114.G 148.E 201.O

H

Haag, Rudolf (1922–) 150.C–E 351.K 402.G
Haar, Alfred (1885–1933) 142.B, r 225.C, r 255.E
317.C 321.C 323.E 334.C 336.B
Haberman, Shelby J. 280.r
Haboush, William J. (1942–) 16.W
Hadamard, Jacques Salomon (1865–1963) 20.r
43.E 46.A 58.F, r 109 111.I 121.C 123.B 124.B
125.A 126.J 159.J 178.A, B 208.D 272.E 320.r
321.A, G 323.B 325.B, r 339.A, D 357.r 429.B
450.B, I App. A, Table 8
Hadley, George F. 227.r
Haefliger, André (1929–) 105.r 114.D 154.A, C–H
Haff, L. R. 280.D, r
Hagihara Yusuke (1897–1979) 133.r 420.r
Hagis, Peter H., Jr. (1926–) 297.D

Hahn, Frank John (1929–) 136.G
Hahn, Hans (1879–1934) 37.F 93.D 160.F 390.G
424.C
Hájek, Jaroslav (1926–74) 371.r 399.N, r
Hajian, Arshag B. (1930–) 136.C, F
Hajós, György (Georg) (1912–72) 2.B
Haken, Hermann 95
Haken, Wolfgang R. G. (1928–) 157.A, D 186.r
235.A
Halanay, Aristide (1924–) 163.r 394.r
Halász, Gábor (1941–) 123.E
Halberstam, Heini (1926–) 123.r
Hale, Jack Kenneth (1928–) 163.B, H, r 286.r 290.r
Hall, G. 303.r
Hall, Marshall, Jr. (1910–) 29.H 66.r 151.I, r 161.C,
r 190.r 241.C App. B, Table 5
Hall, Philip (1904–82) 151.B, D–F 190.G
Hall, William Jackson (1929–) 396.r
Hallen, E. G. 130.r
Hällström, Gunnar af 124.C
Halmos, Paul Richard (1916–) 42.r 136.E, H 197.r
225.r 251.r 256.r 270.r 276.E 381.r 390.r
Halphen, Georges Henri (1844–89) 110.B 134.r
Hamachi, Toshihiro (1942–) 136.F, r
Hamada, Noboru (1940–) 96.r
Hamada, Yūsaku (1931–) 321.G
Hamburger, Hans (1889–1956) 240.K 450.M
Hamel, George (1877–1954) 270.J 388.B
Hamilton, Richard S. (1943–) 195.E 352.C
Hamilton, William Rowan (1805–65) 20 29.B
108.B 126.A, L 151.B 186.F 219.C 267 269.F
271.F 294.F, r 324.E 351.D 441.B 442.D
Hamm, Helmut A. 418.I
Hammersley, John Michael (1920–) 340.r 385.r
Hammerstein, H. 217.M
Hamming, Richard W. (1915–) 63.B, C 136.E
223.r
Hampel, Frank R. 371.A, I, r 431.r
Hanai, Sitiro (1908–) 273.K 425.CC
Hanania, Mary I. 346.r
Hancock, H. 134.r
Handel, Michael (1949–) 126.J
Handelman, David E. (1950–) 36.H
Handscorn, David Christopher (1933–) 385.r
Haneke, Wolfgang 123.C
Hankel, Hermann (1839–73) 39.B 174.A 220.B
App. A, Table 19.III
Hanks, R. 224.E
Hannan, Edward James (1921–) 421.r
Hanner, Olof (Olaf) (1922–) 79.r
Hano, Jun-ichi (1926–) 364.F 365.L
Hansen, Frank (1950–) 212.C
Hansen, Johan Peder (1951–) 16.I
Hansen, Peter Andreas (1795–1874) App. A,
Table 19.III
Hanson, David Lee (1935–) 136.E
Hanson, Richard J. (1938–) 302.r
Happel, H. 420.r
Hara, Kōkiti (1918–) 329.r
Harada, Koichiro (1941–) 151.I, J
Harada, Manabu (1931–) 200.K
Harari, Haim (1940–) 132.r
Harary, Frank (1921–) 186.r
Hardorp, Detlef 154.H
Hardt, Robert Miller (1945–) 275.C
Hardy, Godfrey Harold (1877–1947) 4.C, D, r 20.r
43.E, F 83.r 88.r 106.r 121.r 123.C, r 159.G, H, r
164.G 168.B 211.r 216.r 220.B 224.E 242.A, B, r
295.r 317.B 328.*, r 339.B, C 379.F, M. S, r 450.B,
I App. A, Table 8

- Harish-Chandra (1923–83) 13.r 32.r 122.F, G
249.V 308.M 437.V, X, AA, CC, EE 450.T
- Harle, Carlos Edgard 365.E
- Harman, Harry Horace (1913–76) 346.F, r
- Harnack, Carl Gustav Axel (1851–88) 100.E 193.I
- Harris, Joseph 9.F, r 16.r
- Harris, Theodore Edward (1919–) 44.A 260.J
340.C
- Harris, William Ashton (1930–) 289.E
- Hart, J. F. 142.r NTR
- Hartley, Herman Otto (1912–80) STR
- Hartley, Richard Ian 235.E
- Hartman, Philip (1915–) 107.r 126.G 195.E 252.r
254.r 313.r 314.r 315.r 316.r 365.J
- Hartmanis, Juris (1928–) 71.r 75.r
- Hartogs, Friedrich (1874–1943) 20 21.C, F, H, Q
- Hartshorne, Robin (1938–) 9.r 16.R, r 343.r
- Harvey, F. Reese (1941–) 112.C, D 125.Y, Z
- Harvey, William James (1941–) 234.r
- Hasegawa, Hiroshi (1782–1838) 230
- Hashimoto, Isao (1941–) 280.r
- Hashimoto, Takeshi (1952–) 213.E
- Hashimoto, Tsuyoshi (1948–) 173.E
- Hasse, Helmut (1898–1979) 9.E 12.B 14.E, L, O–R,
U, r 27.D, E 29.G 59.A, G, H, r 73.A 113 118.C, F
149.r 242.B 257.F, H, r 295.r 297.I 347.r 348.D, G
449.r 450.A, L, P, S
- Hastings, Cecil (1920–) 142.r NTR
- Hastings, D. W. 40.D
- Hasumi, Morisuke (1932–) 125.BB 164.K
- Hatakeyama, Yoji (1932–) 110.E, r
- Hattori, Akio (1929–) 237.H 431.D
- Hattori, Akira (1927–86) 200.K, M
- Haupt, Otto (1887–) 268.E
- Hausdorff, Felix (1868–1942) 79.A 117.G 169.D
224.E 234.E 240.K 246.K 249.R 273.J 317.B 381.r
388.r 423.B 425.A, N, P, Q, AA 426 436.C 443.I
- Hawking, Stephen W. (1942–) 359.r
- Hayashi, Chihiro (1911–) 290.r
- Hayashi, Chikio (1918–) 346.E
- Hayashi, Kazumichi (1925) 207.C, r
- Hayashi, Keiichi (1879–1957) NTR
- Hayashi, Mikihiro (1948–) 164.K
- Hayashi, Tsuruichi (1873–1935) 230 267
- Hayman, Walter Kurt (1926–) 17.D 124.r 193.r
272.K, r 391.D 438.C, E
- Haynal, A. 33.F, r
- Heath, David Clay (1942–) 173.E
- Heath, Sir Thomas Little (1861–1940) 181.r 187.r
- Heath-Brown, David Rodney 118.D 123.C, E
- Heaviside, Oliver (1850–1925) 125.E 306.A, B
App. A, Table 12.II
- Heawood, P. J. (1861–1955) 157.A, E
- Hecht, Henryk (1946–) 437.X
- Hecke, Erich (1887–1947) 6.D 11.B 14.r 29.C
32.C, D, H, r 73.B 123.F, r 348.L, r 450.A, D–F,
M, O
- Hector, Gilbert Joseph (1941–) 154.H
- Hedlund, Gustav Arnold (1904–) 126.A, r 136.G
- Heegaard, Poul (1871–1948) 65.C 247.r
- Heesch, Heinrich (1906–) 92.F 157.A, D
- Heiberg, Johan Ludvig (1854–1928) 181.r 187.r
- Heilbronn, Hans Arnold (1908–75) 123.D 347.E
450.K
- Heine, Heinrich Eduard (1821–81) 206.C 273.F
393.C
- Heins, Maurice Haskell (1915–) 77.F 164.K 198.r
207.C 367.E, G, r
- Heintze, Ernst 178.r
- Heinz, Erhard (1924–) 323.J
- Heisenberg, Werner Karl (1901–76) 150.A 351.C,
D 386.C
- Heitsch, James Lawrence (1946–) 154.G
- Held, A. 359.r
- Held, Dieter (1936–) 151.I
- Helgason, Sigurdur (1927–) 109.r 199.r 218.G 225.r
248.r 249.r 412.r 413.r 417.r 437.Y, AA, EE
- Hellerstein, Simon (1931–) 272.K, r
- Hellinger, Ernst D. (1883–1950) 197.r 217.r 390.G
- Helly, Eduard (1884–1943) 89.B 94.B
- Helmholtz, Hermann von (1821–94) 139.A 188.D
205.B 419.C 442.D App. A, Tables 3, 15.VI
- Helms, Lester L. 120.r 193.r
- Helson, Henry (1927–) 164.G, H, r 192.P–R, r
251.r
- Hemmingsen, Erik (1917–) 117.E
- Hempel, John Paul (1935–) 65.E
- Henderson, David William (1939–) 117.I
- Henkin → Khenkin
- Henkin, Leon (Albert) (1921–) 276.D
- Hénon, Michel 287.B, r
- Henrici, Peter K. (1923–) 138.r 300.r 301.r 303.r
- Henry, Charles 144.r
- Henry, N. F. M. 92.r
- Hensel, Kurt (1861–1941) 11.r 12.B 14.U 118.C
236.r 370.C 439.L
- Henstock, Ralph (1923–) 100.r
- Hepp, Klaus (1936–) 146.A 150.r
- Herbrand, Jacques (1908–31) 14.K 59.A, E, H, r
156.E, r 200.N 356.A, E
- Herglotz, Gustav (1881–1953) 43.I 192.B 325.J
- Hering, Christoph H. 151.J
- Herman, Michael-Robert (1942–) 126.I, N 154.G
- Hermann, Carl Heinrich (1898–1961) 92.F
- Hermes, Hans (1912–) 31.r 97.r 356.r
- Hermite, Charles (1822–1901) 14.B 60.O 107.A
131.D 167.C 176.I 182.A 199.A 217.H 223.E 232.A
251.E, O 256.Q 269.I 299.A 317.D 344.F 348.F
412.E, G 430.A App. A, Tables 14.II, 20.IV
- Herodotus (c. 484–c. 425 B.C.) 181
- Heron (between 150 B.C. and A.D. 200) 187 App. A,
Tables 2.II, III
- Hersch, Joseph (1925–) 143.A 391.E
- Hertz, Heinrich Rudolf (1857–94) 441.B
- Hertzog, David (1932–) 151.I
- Hervé, Michel André (1921–) 23.r 62.B
- Herz, Carl Samuel (1930–) 206.E, r
- Herzberger, Maximilian Jacob (1899–1981) 180.r
- Hesse, Ludwig Otto (1811–74) 9.B 139.H 226.D
279.B, E, F
- Hessel, Johann Friedrich Christian (1796–1872)
92.F
- Hessenberg, Gerhard 155.r
- Hessenberg, K. (1874–1925) 298.D
- Hestenes, Magnus R. 302.D
- Heun, Karl 303.D
- Hewitt, Edwin (1920–) 192.P, r 342.G 422.r 425.S,
BB
- Hey, Käte 27.F 450.A, L
- Heyting, Arend (1898–1980) 156.r 411.J, r
- Hicks, Noel J. (1929–79) 111.r
- Hida, Takeyuki (1927–) 176.r
- Higgs, Peter Ware (1929–) 132.D
- Higman, Donald Gordon 151.I
- Higman, Graham (1917–) 97.*, r 151.A, F, I, r
190.M
- Higuchi, Teiichi (1933–) 23.r
- Higuchi, Yasunari (1949–) 340.G, r
- Hijikata, Hiroaki (1936–) 13.O, P, R
- Hikita, Teruo (1947–) 75.r

Hilb, Emil

- Hilb, Emil (1882–1929) 253.r
 Hilbert, David (1862–1943) 4.A 9.F 11.B 14.J–L, R, U 15.H 16.E, S 20.*, r 32.G 35.A, r 41.E 46.A, C, E, r 59.A 68.A, C, I, L 73.B 77.B, E 82.r 92.F 93.J 105.Z 107.A, r 111.I 112.D, r 120.A 126.I 150.G 155.A–C, F, G, H 156.A, C, D, r 160.D 162 172.J 179.B 181 188.r 189.r 196.A, B 197.A, B, r 217.H–J, r 220.E 222.r 226.G, r 253.D 267 284.A 285.A, 286.K 304.r 317.r 320.r 321.r 322.r 323.E, I, r 324.r 325.M 327.r 337.F 347.G, H, r 356.A 357.r 364.I 365.J 369.D, F 375.F 377.D 382.B 389.r 402.H 410.r 411.J, r 423.N 424.W 430.A 441.r 443.A 446.r App. A, Table 8
 Hildebrandt, Stefan O. W. (1936–) 195.E 275.B 334.F
 Hildebrandt, Theophil Henry (1888–1980) 310.r 443.A
 Hildreth, C. 349.r
 Hilferly, M. M. 374.F
 Hilfinger, P. N. 75.r
 Hill, Edward Lee (1904–) 150.B
 Hill, George William (1838–1914) 107.A 268.B, E
 Hill, Rodney (1921–) 271.r
 Hille, Einar (1894–1980) 106.r 107.r 115.A 136.B 160.E 162 198.r 216.r 251.r 252.r 253.r 254.r 286.X, r 288.r 289.r 313.r 315.r 316.r 336.r 378.B, D, r 438.B
 Hillier, Frederick S. 215.r
 Hilton, Peter John (1923–) 70.r 201.r 202.P, U
 Hinčin → Khinchin
 Hinman, Peter G. (1937–) 22.F, r 356.r
 Hinohara, Yukitoshi (1930–) 200.K
 Hipparchus (190?–125? B.C.) 187 432.C
 Hippias (late 5th century B.C.) 187
 Hippocrates (of Chios) (470?–430? B.C.) 187
 Hirai, Takeshi (1936–) 437.W
 Hirano, Sugayasu (1930–) 301.G
 Hirashita, Yukio (1946–) 164.C
 Hirayama, Akira (1904–) 230.r
 Hironaka, Heisuke (1931–) 12.B 16.L, Z 21.L 23.D 72.H 232.C 418.B
 Hirose, Hideo (1908–81) 230.r
 Hirota, Ryogo (1932–) 387.D
 Hirsch, Guy Charles (1915–) 201.J 427.E
 Hirsch, Morris William (1933–) 114.C, D, J, r 126.J, r 279.C
 Hirschfeld, Joram 276.r
 Hirschman, Isidore I. (1922–) 220.r
 Hirzebruch, Friedrich Ernst Peter (1927–) 12.B 15.D, G, H 20 32.F 56.G, r 72.K, r 109 114.A 147.r 237.A 366.A, B, D, r 383.r 418.r 426 431.D, r
 Hitchcock, Frank Lauren (1875–1957) 301.E
 Hitchin, Nigel James (1946–) 80.r 364.r
 Hitchins, G. D. 301.L
 Hitotumatu, Sin (1926–) 186.r 301.D 389.r NTR
 Hitsuda, Masuyuki (1938–) 176.H
 Hlavatý, Václav (1894–?) 434.r
 Hlawka, Edmund (1916–) 182.D
 Ho, B. L. 86.D
 Hobson, Ernest William (1856–1933) 133.r 393.C, r
 Hochschild, Gerhard Paul (1915–) 6.E 13.r 59.H 200.K–M, O, Q, 249.r
 Hochster, Melvin (1943–) 16.Z
 Hocking, John Gilbert (1920–) 79.r 201.r
 Hocquenghem, Alexis (1908–) 63.D
 Hodge, Sir William Vallance Douglas (1903–75) 12.B 15.D 16.V, r 20 109.*, r 194.B, r 232.A, B, D 343.r
 Hodges, Joseph Lawson (1922–) 371.A, H 399.E, H, N, P, r
 Hodgkin, Alan Lloyd (1914–) 291.F
 Hodozi, Yosi (1820–68) 230
 Hocffding, Wassily (1914–) 371.A 374.I 400.r
 Hoffman, Banesh H. (1906–) 359.D 434.C
 Hoffman, David Allen (1944–) 275.r 365.H
 Hoffman, Kenneth Myron (1930–) 43.r 164.F, G, I, r
 Hoffmann-Jørgensen, Jørgen (1942–) 22.r
 Hogg, Robert Vincent, Jr. (1924–) 371.r
 Hölder, Otto (1859–1937) 84.A 104.F 168.B 190.G 211.C 277.I 288.D 379.M App. A, Table 8
 Holland, Paul W. (1940–) 280.r 403.r
 Holley, Richard Andrews (1943–) 44.E 340.r
 Holm, Per (1934–) 418.r
 Holmgren, Erik Albert (1872–) 125.DD 321.F 327.C
 Holmstedt, Tord 224.C
 Homma, Tatsuo (1926–) 65.E 235.A
 Honda, Taira (1932–75) 3.C 450.Q, S, r
 Hong Imsik (1916–) 228.B, r
 Hong Sing Leng 365.N
 Hood, William Clarence (1921–) 128.r
 Hooke, Robert (1635–1703) 271.G
 Hooker, Percy Francis 214.r
 Hooley, Christopher (1928–) 123.E, r 295.E
 Hopcroft, John E. (1939–) 31.r 71.r 75.r 186.r
 Hopf, Eberhard (1902–83) 111.I 126.A, M 136.B 162.B, C, G, r 204.B, C, r 222.C 234.r 270.E 286.U, X 433.B, C
 Hopf, Heinz (1894–1971) 65.r 72.K 93.r 99.r 109 111.I, r 126.G 147.E 153.B 178.A 201.r 202.A, B, I, Q, S, U, V, r 203.A, C, D, H 249.V 305.A 365.H 425.r 426.*, r
 Hopkins, Charles 368.F
 Horikawa, Eiji (1947–) 72.K, r
 Hörmander, Lars Valter (1931–) 20 21.I, r 107.r 112.B–D, H, K, L, R, r 115.D 125.A 164.K 189.C 274.D, I 286.J 320.I 321.r 323.M 325.H 345.A, B
 Horn, Jacob (1867–1946) 107.A 206.D 314.A
 Horner, William George (1786–1837) 301.C
 Horowitz, Ellis (1944–) 71.r
 Horrocks, Geoffrey (1932–) 16.r
 Hosokawa Fujitsugu (1930–) 235.D
 Hotelling, Harold (1895–1973) 280.B 374.C
 Hotta, Ryoshi (1941–) 437.X
 Householder, Alston Scott (1904–) 298.D 301.r 302.E
 Houseman, E. E. 19.r
 Howard, Ronald Arthur (1934–) 127.E
 Howarth, Leslie (1911–) 205.r
 Hrbáček, Karel 33.r 293.E, r
 Hsiang Wu-Chung (1935–) 114.J, K 431.D, r
 Hsiang Wu-Yi (1937–) 275.F 365.K 431.D, r
 Hsiung Chuan-Chih (1916–) 364.F 365.H
 Hsü Kwang-Ch'i (1562–1633) 57.C
 Hu Sze-Tsen (1914–) 79.r 91.r 148.r 201.r 277.r
 Hu Te Chiang 281.r
 Hua Loo-Keng (Hua Luo K'ang) (1910–85) 4.A, E, r 122.E 242.A, r 295.E
 Huang, Kerson (1928–) 402.r
 Huber, Peter J. 371.A, H, J, r 399.H, P, r
 Huber-Dyson, Verena 362.r
 Hudson, John F. P. 65.C, D
 Huff, Robert E. (1942–) 443.H
 Hugenholtz, Nicholaas Marinus (1924–) 308.H
 Hugoniot, Pierre Henri (1851–87) 51.E 204.G 205.B
 Hukuhara, Masuo (1905–) 30.C, r 88.A 254.D 288.B, r 289.B–D 314.A, C, D 315.C 316.E 388.B 443.A

Hull, Thomas Edward (1922–) 206.r 303.r
 Humbert, Georges (1859–1921) 83.D
 Humphreys, James E. (1939–) 13.r 248.r
 Hunt, Gilbert Agnew (1916–) 5.H 162 176.G 260.J,
 r 261.A, B 338.N, O 400.F 407.B
 Hunt, Richard A. 159.H 168.B, r 224.E
 Huntley, H. E. 116.r
 Huppert, Bertram 151.D, r
 Hurewicz, Witold (1904–56) 117.A, C, r 136.B
 148.D 202.A, B, N, r 426
 Hurley, Andrew Crowther (1926–) 92.F
 Hurst, Charles A. (1923–) 212.A 402.r
 Hurwicz, Leonid 255.D, E, r 292.E, F, r
 Hurwitz, Adolf (1859–1919) 3.K 9.I 10.E 11.D
 83.B 134.r 198.r 339.D 367.B 450.B, r
 Husemoller, Dale H. (1933–) 15.r 56.r 147.r
 Huxley, Andrew Fielding (1917–) 291.F
 Huxley, Martin Neil (1944–) 123.E, r
 Huygens, Christiaan (1629–95) 93.H 245 265
 325.B, D 446
 Huzita, Sadasuke (1734–1807) 230
 Hwa, Rudolph C. (1931–) 146.r
 Hypatia (370?–418) 187

I

Iagolnitzer, Daniel (1940–) 146.C 274.D, I 386.C, r
 Ibragimov, Il'dar Abdulovich (1932–) 176.r 250.r
 Ibuki, Kimio (1918–) 75.D
 Ibukiyama, Tomoyoshi (1948–) 450.S
 Ichihara, Kanji (1948–) 115.D
 Igari, Satoru (1936–) 224.E
 Igusa, Jun-ichi (1924–) 3.r 9.J 12.B 15.E, F 16.Z, r
 32.F 118.C, D
 Ihara, Yasutaka (1938–) 59.A, r 450.A, M, Q, S, U
 Itaka, Shigeru (1942–) 72.I, r
 Iizuka, Kenzo (1923–) 362.I
 Ikawa, Mitsuru (1942–) 325.K 345.A
 Ikebe, Teruo (1930–) 112.E, P 375.B, C
 Ikebe, Yasuhiko (1934–) 298.r
 Ikeda, Akira (1947–) 391.C
 Ikeda, Hideto (1948–) 96.r
 Ikeda, Masatoshi (1926–) 200.K, L
 Ikeda, Nobuyuki (1929–) 44.r 45.r 115.r 250.r 262.r
 399.r 406.F, r
 Ikeda, Tsutomu (1950–) 304.D
 Ikehara, Shikao (1904–84) 123.B 160.G
 Il'in, Arlen Mikhailovich (1932–) 327.r
 Illusie, Luc (1940–) 366.r
 Im Hof, Hans-Christoph 178.r
 Imai, Kazuo (1952–) 41.D
 Imanishi, Hideki (1942–) 154.H
 Inaba, Eizi (1911–) 337.F
 Inaba, Takashi (1951–) 154.D
 Inagaki, Nobuo (1942–) 399.N
 Ince, Edward Lindsay (1891–1941) 107.r 252.r
 254.r 268.C, D 288.r 289.r
 Infeld, Leopold (1898–1968) 206.r 359.D
 Ingham, Albert Edward (1900–) 123.C, r 242.A
 Inose, Hiroshi (1951–79) 16.r 450.S
 Inoue, Masahisa (1946–) 72.K
 Inoue, Masao (1915–) 338.r
 Iochem, Bruno 351.L
 Ionescu-Tulcea, Alexandra 136.B, C
 Iooss, Gérard (1944–) 126.M
 Ipsen, D. C. 116.r
 Iri, Masao (1933–) 66.r 186.r 281.r 299.B 301.F
 303.E, r
 Irie Seiiti (1911–) 62.E

Irwin, Michael C. 65.D 126.G, r
 Isaacs, Rufus Philip (1914–81) 108.A
 Isbell, John Rolfe (1930–) 436.r
 Iseki, Kanesiroo (1920–) 328
 Iseki, Shō (1926–) 328
 Iseki, Tomotoki (fl. 1690) 230
 Ishida, Masanori (1952–) 16.Z
 Ishihara, Shigeru (1922–) 110.r 364.F 365.H
 Ishihara, Tôru (1942) 195.r
 Isii, Keiiti (1932–) 255.D 399.r
 Ising, Ernest (1900–) 340.B, C
 Iskovskikh, Vitalii Alekseevich 16.J
 Ismagilov, R. S. 183.r
 Isozaki, Hiroshi (1950–) 375.B
 Israel, Robert B. (1951–) 402.r
 Israel, Werner (1931–) 359.r
 Iss'sa, Hej 367.G
 Itaya, Nobutoshi (1933–) 204.F
 Itô, Kiyosi (1915–) 5.E, r 45.G, r 115.A, C 176.I, r
 261.A, r 395.C, r 406.A–D, G, r 407.A, C, r
 Itô, Masayuki (1940–) 338.O
 Ito, Noboru (1925–) 151.H, J
 Itô, Seizō (1927–) 204.B 270.r 327.r
 Ito, Shunji (1943–) 126.K 136.C, r
 Ito, Takashi (1926–) 192.r
 Itô, Teiiti (1898–) 92.F
 Ito, Yoshifumi (1940–) 125.BB
 Ito, Yuji (1935–) 136.C, F
 Itoh, Mitsuhiro (1947–) 80.r
 Itoh, Takehiro (1943–) 275.A 365.G
 Iversen, F. 62.E 272.I
 Ivory, Sir James (1765–1842) 350.E
 Ivrii, V. Ya. 325.H
 Iwahori, Nagayoshi (1926–) 13.R, r 249.r 442.r
 Iwamura, Tsurane (1919–) 85.A, r
 Iwaniec, Henryk 123.C, E
 Iwano, Masahiro (1931–) 254.D 289.D, E
 Iwasawa, Kenkichi (1917–) 6.D, F 14.L 32.r 243.G
 248.F, V 249.S, T, V, r 257.H 384.C 450.A, F, J, L,
 N, r
 Iyanaga, Shôkichi (1906–) 6.r 7.r 14.Q, r 59.D, E, r
 60.r 149.r 161.r 200.r 277.r 294.r 343.r 362.r 368.r
 409.r
 Izumi, Shin-ichi (1904–) 121.r 160.B, F 310.r

J

Jackiw, Roman Wildmir (1939–) 80.r
 Jackson, Dunham (1888–1946) 336.C, E, r
 Jackson, John David (1925–) 130.r
 Jackson, Kenneth R. 303.r
 Jacob, Maurice R. 132.r 386.r
 Jacobi, Carl Gustav Jacob (1804–51) 3.A, G, L 4.D
 9.E, F 11.B, C 20.46.C 105. 107.B 108.B 126.A 134.A,
 C, I, J, r 178.A 182.H 202.P 208.B 229 248.A 267
 271.F 296.A 297.I 298.B 302.C 317.D 324.D, E
 348.K 390.G 420.A, F 428.C App. A, Tables 14.I,
 II, 16.I, III, 20.V
 Jacobowitz, Howard (1944–) 286.J 344.B
 Jacobs, Konrad (1928–) 136.H, r
 Jacobson, Florence D. 231.r
 Jacobson, Nathan (1910–) 27.r 29.r 54.r 67.D 149.r
 172.A, K, r 231.r 248.r 256.r 368.H, r 499.r
 Jacod, Jean M. (1944–) 262.r
 Jacquet, Hervé Michel (1933–) 32.r 437.r 450.A,
 N, O
 Jaekel, Louis A. 371.H, J, r
 Jaffe, Arthur Michael (1937–) 150.C, F, r
 Jaglom → Yaglom

Jahnke, Paul Rudolf Eugene

Jahnke, Paul Rudolf Eugene (1863–1921) 389.r
 NTR
 James, Alan Treleven (1924–) 102.r 374.r
 James, Ioan M. (1928–) 202.Q, U
 James, Ralph Duncan (1909–79) 100.A, r
 James, Robert Clarke (1918–) 37.G
 James, W. 280.D, r 398.r 399.G
 Jancel, Raymond (1926–) 402.r
 Janes, F. T. 403.C
 Janet, Maurice (1888–) 365.B
 Janiszewski, Zygmund (1888–1920) 426
 Janko, Zvonimir (1932–) 151.I, J App. B,
 Table 5.III
 Janner, Aloysio 92.r
 Jans, James Patrick (1927–) 368.r
 Janzen, O. 246.G
 Jarratt, Peter (1935–) 301.N
 Jauch, Josef-Maria (1914–74) 375.A, r
 Jayne, J. E. 22.r
 Jech, Thomas (1944–) 22.r 33.F, r
 Jeffreys, B. S. 25.r
 Jeffreys, Harold (1891–) 25.B, r
 Jelinek, Fredrick (1932–) 213.E, r
 Jenkins, Gwilym M. 128.r 421.D, G, r
 Jenkins, Howard B. 275.A, D
 Jenkins, James Allister (1923–) 77.F 143.r 438.B, C
 Jensen, Johann Ludwig Wilhelm Waldemar (1859–
 1925) 88.A 121.A 164.K 198.F
 Jensen, K. L. 145
 Jensen, Ronald B. 33.F, r 356.r
 Jentsch, Robert 339.E
 Jerison, Meyer (1922–) 425.r
 Jessen, Raymond J. (1910–) 373.r
 Jeulin, Thierry 406.r
 Jewett, Robert Israel (1937–) 136.H
 Jimbo, Michio (1951–) 253.E 387.C
 Jimbo, Toshiya (1941–) 164.r
 Jiřina, Miloslav 44.E
 Joffe, Anatole 44.C
 John, Fritz (1910–) 112.B 168.B 218.F 262.B 274.F
 292.B 300.r 304.r 320.r 321.r 323.r 324.r 325.r 327.r
 John, Peter W. M. (1923–) 102.r
 Johns, M. Vernon, Jr. (1925–) 371.H, r
 Johnson, Norman Lloyd (1917–) 374.r
 Johnson, Wells 14.L
 Johnson, William B. (1944–) 68.K, M
 Johnston, John (1923–) 128.r
 Jolley, Leonard Benjamin William (1886–) 379.r
 Joly, Jean-René Benoît (1938–) 118.r
 Jona-Lasinio, Giovanni 361.r
 Jonckheere, A. R. 346.r
 Jones, B. W. 347.r 348.r
 Jones, Floyd Burton (1910–) 273.K
 Jones, John D. S. 80.r
 Jones, William (1675–1749) 332
 Jordan, C. W. 214.r
 Jordan, Camille (1838–1922) 20.r 79.A 92.A, F
 93.A, B, F, K 104.r 151.H 159.B 166.B 190.G, Q, r
 267 269.G 270.D, G 277.I 302.B 310.B 333.A
 362.K 380.C App. A, Table 8
 Jordan, Ernst Pascual (1902–) 150.A 231.B 351.L
 377.B
 Jordan, Herbert E. App. B, Table 5.r
 Jöreskog, Karl G. (1935–) 403.r
 Joseph, Peter D. 405.r
 Jost, Res Wilhelm (1918–) 150.r 386.B
 Joule, James Prescott (1818–89) 130.B
 Julia, Gaston Maurice (1893–1978) 21.Q, r 43.K
 124.B 198.r 272.F 429.C 435.E
 Jung, Heinrich Wilhelm Ewald (1876–1953) 15.r

Jurečková, Jana (1940–) 371.J, r
 Jurkat, Wolfgang (Bernhard) (1929–) 123.D
 Jutila, Matti Ilmari (1943–) 123.E
 Juzvinskiĭ → Yuzvinskiĭ

K

Kac, I. S. → Kats
 Kac, Mark (1914–84) 41.C 115.C 150.F 250.r 261.r
 287.C 295.E 340.r 341.r 351.F 391.C, r
 Kaczmarz, Stefan (1895–1939) 317.r
 Kadanoff, Leo Philip (1937–) 361.r
 Kadison, Richard Vincent (1925–) 36.G, K 308.r
 Kadomtsev, Boris Borisovich (1928–) 387.F
 Kagan, Abram Meerovich (1936–) 374.H
 Kahan, William M. 302.r
 Kahane, Jean-Pierre (1926–) 159.H, r 192.Q, r
 Kähler, Erich (1906–) 109 191.I 199.A 232.A 365.L
 428.E, r
 Kailath, Thomas (1935–) 86.D, r
 Kainen, Paul C. 157.r
 Kaiser, Henry F. 346.F, r
 Kakeshita Shin-ichi (1934–) 371.A
 Takeya, Sôichi (1886–1947) 10.E 89.E
 Kakutani, Shizuo (1911–) 37.N 136.B–D, F 153.D
 162 286.D 310.A, G 352.A 367.D 398.G
 Kalashnikov, Anatolii Sergeevich (1934–) 327.r
 Kall, Peter 408.r
 Källén, Anders Olof Gunnar (1926–68) 150.D
 Kallianpur, Gopinath (1925–) 86.r 250.r 405.r
 Kalman, Rudolf Emil (1930–) 86.A, C–F 95
 405.G, r
 Kalmár, László (1905–76) 97.*
 Kaluza, Theodor, Jr. (1910–) 434.C
 Kamae, Teturo (1941–) 136.H 354.r
 Kambayashi, Tatsuji (1933–) 15.r
 Kamber, Franz W. (1936–) 154.G, H, r
 Kamenskii, Georgii Aleksandrovich (1925–) 163.r
 Kametani, Shunji (1910–) 62.E 124.C
 Kamke, Erich (1890–1961) 316.r
 Kampé de Fériet, Joseph (1893–1982) 206.D
 393.E, r 428.r
 Kan, Daniel M. 70.E
 Kanamori, Akihiro (1948–) 33.r
 Kaneda, Eiji (1948–) 365.E
 Kaneko, Akira (1945–) 162
 Kanel', Yakob Isaakovich (1932) 204.F, r
 Kaneyuki, Soji (1936–) 384.r
 Kanitani, Jōyō (1893–) 110.B, r
 Kannan, Rangachary (1946–) 290.r
 Kano, Tadayoshi (1941–) 286.Z
 Kantor, William M. 151.J
 Kantorovich, Leonid Vital'evich (1912–86) 46.r
 162 217.r 255.E 304.r 310.A
 Kaplan, Wilfred (1915–) 106.r 216.r 313.r
 Kaplansky, Irving (1917–) 2.E, r 107.r 113.r 200.K
 241.E 248.r 308.C, r
 Kapteyn, Willem (1849–?) 39.D App. A, Table 19.III
 Karacuba, Anatolii Alekseevich (1937–) 4.E
 Karhunen, Kari (1915–) 395.r
 Kariya, Takeaki (1944–) 280.r
 Karlin, Samuel (1923–) 222.r 227.r 260.J 263.E
 310.H 336.r 374.r 399.G, r 400.r
 Karp, Carol (1926–72) 356.r
 Karrass, Abe 161.r
 Karzanov, A. V. 281.r
 Kas, Arnold S. 15.H 16.R
 Kasahara, Kenkiti (1935–) 21.M
 Kasahara, Kōji (1932–) 325.H
 Kasai, Takumi (1946–) 71.r

Kasch, Friedrich (1921–) 29.H
 Kashiwara, Masaki (1947–) 68.F 125.DD, EE,
 146.A, C 162 274.I 386.C 418.H 428.H 437.r
 Kasteleyn, P. W. (1924–) 212.A
 Kastler, Daniel (1926–) 150.E 351.K 402.G
 Kataoka, Kiyōmi (1951–) 125.DD, r
 Kataoka, Shinji (1925–) 408.r
 Katětov, Miroslav 117.A, D, E
 Kato, Junji (1935–) 163.r
 Kato, Kazuhisa (1946–) 126.J
 Kato, Masahide (1947–) 72.K
 Kato, Mitsuyoshi (1942–) 65.D 147.Q 418.r
 Kato, Tosio (1917–) 68.r 162 204.B, C, E 289.E
 304.r 331.A, B, E, r 345.A 351.D 375.A–C
 378.D, E, H–J 390.r
 Katok, Anatolii Borisovich (1944–) 126.N 136.E,
 F, H, r
 Kats, Izrail' Samoïlovich (1926–) 115.r
 Katsurada, Yoshie (1911–80) 365.H
 Katz, Jerrold J. 96.r
 Katz, Nicholas M. (1943–) 16.r 450.J, Q, r
 Katsnelson, Yitzhak (1934–) 136.E 159.H, I 192.r
 Kaufman, Sol (1928–) 399.N, r
 Kaul, Helmut (1936–) 195.E
 Kaup, Wilhelm 384.r
 Kawada, Yukiyo (1916–) 59.H
 Kawaguchi, Akitsugu (1902–84) 152.C
 Kawai, Sōichi (1937–) 72.r
 Kawai, Takahiro (1945–) 125.BB, DD, r 146.A, C
 162 274.I 386.C 428.H
 Kawai, Toru (1945–) 293.E, r
 Kawakami, Hiroshi (1941–) 126.N
 Kawakubo, Katsuo (1942–) 431.D
 Kawamata, Yujiro (1952–) 15.r 21.N 72.I, r 232.D
 Kawashima, Shuichi (1953–) 41.r
 Kawata, Tatsuo (1911–) 374.H 395.r
 Kazama, Hideaki (1945–) 21.L
 Kazarinoff, Nicholas D. (1929–) 211.r
 Kazdan, Jerry L. (1937–) 364.H, r
 Kazhdan, David A. 122.G 391.N
 Kazhikov, A. V. 204.F
 Kechris, Alexander S. (1946–) 22.C, H, r
 Keedwell, Anthony Donald (1928–) 241.r
 Keesling, James Edgar (1942–) 382.C
 Keim, Dieter 424.r
 Keisler, H. Jerome (1936–) 33.r 276.E, r 293.D, r
 Keldysh, Mstislav Vsevolodovich (1911–78) 336.F
 Keller, Herbert Bishop (1925–) 303.r
 Keller, Joseph Bishop (1923–) 274.C, r
 Kelley, D. G. 212.A
 Kelley, John Ernst, Jr. (1937–) 376.r
 Kelley, John Leroy (1916–) 37.M 87.r 381.r 424.r
 425.r 435.r 436.r
 Kellogg, Oliver Dimon (1878–1932) 120.B, D, r
 153.D 193.r 286.D
 Kelly, Anthony 92.r
 Kelvin, Lord (Thomson, William) (1824–1907)
 19.B 39.G 120.A 193.B App. A, Table 19.IV
 Kemeny, John George (1926–) 260.J
 Kempe, Alfred Bray 157.A, D
 Kempf, George R. (1944–) 9.r 16.r
 Kempisty, Stefan (1892–1940) 100.A, r
 Kempthorne, Oscar (1919–) 102.r
 Kendall, David G. (1918–) 44.A 218.r 260.H, J
 Kendall, Maurice George (1907–83) 102.r 280.r
 346.r 371.K 397.r 400.r
 Kenmotsu Katsuei (1942–) 275.F
 Kennedy, P. B. 193.r
 Kepler, Johannes (1571–1630) 20 78.D 126.A 265
 271.B 309.B 432.C

Kerékjártó, Szerkeszti Béla (1898–1946) 207.C
 410.r
 Kerner, Immo O. 301.F
 Kerr, Roy Patrick (1934–) 359.E
 Kervaire, Michel André (1927–) 65.C 114.A, B,
 I–K 235.G
 Kerzman, Norberto Luis Maria (1943–) 164.K
 Kesten, Harry (1931–) 5.G 44.r 340.r
 Khachiyan, L. G. 71.D 255.C
 Khaikin, Semen Emanuilovich 290.r 318.r
 Khas'minskii, Rafail Zalmanovich (1931–) 115.D
 Khatri, Chinubhai Ghelabhai (1931–) 280.r
 Khavinson, Semëon Yakovlevich (1927–) 77.E
 Khayyām, Omar (c. 1040–c. 1123(24?)) 26
 Khenkin (Henkin), Gennadii Markovich (1942–)
 164.K 344.F
 Khinchin (Hinčin), Aleksandr Yakovlevich (1894–
 1959) 4.A 45.r 83.r 100.A 115.D 213.F 250.C
 307.C 332.r 341.G 342.D 395.B 402.r
 Khovanskii, Aleksei Nikolaevich (1916–) 83.r
 Kiefer, Jack Carl (1924–81) 399.D
 Kikuchi, Fumio (1945–) 304.r
 Kikuti, Dairoku (1855–1917) 230 267
 Killing, Wilhelm Karl Joseph (1847–1923) 50
 248.B 279.C 364.F
 Kim Wan Hee (1926–) 282.r
 Kimura, Motoo (1924–) 115.D 263.E
 Kimura, Tatsuo (1947–) 450.V
 Kimura, Toshihisa (1929–) 30.r 288.B–D 289.r
 Kinderlehrer, David S. (1941–) 105.r
 Kingman, John Frank Charles (1939–) 136.B, r
 Kinnersley, William 205.F
 Kinney, John R. 115.A
 Kino, Akiko (1934–83) 81.D 356.G
 Kinoshita, Shin'ichi (1925–) 235.A, C, H
 Kinoshita, Toichiro (1925–) 146.B
 Kirby, Robin Cromwell (1938–) 65.A, C 70.C
 114.J–L
 Kirchhoff, Gustav Robert (1824–87) 255.D 282.B
 Kirillov, Aleksandr Aleksandrovich (1936–) 437.T
 Kirkwood, John Gamble (1907–59) 402.J
 Kiselev, Andrei Alekseevich 204.C
 Kishi, Masanori (1932–) 48.H 338.I, J, M
 Kishimoto, Akitaka (1947–) 36.K 402.G
 Kister, James Milton (1930–) 147.r
 Kitada, Hitoshi (1948–) 375.B
 Kitagawa, Tosio (1909–) STR
 Kizner, William 301.D
 Klainerman, Sergiu 286.J
 Klee, Victor La Rue, Jr. (1925–) 89.r 286.D
 Kleene, Stephen Cole (1909–) 22.G 31.B, C, r
 81.A, r 97.r 156.r 185.r 276.r 319.r 356.A, C–H, r
 411.r
 Kleiman, Steven L. (1942–) 9.E, r 16.E, r 450.r
 Klein, Felix (1849–1925) 1.r 7.E 11.B 32.r 53.r 83.D
 90.B, r 109 119.r 122.C, r 137.*, r 139.A, r 151.G
 167.E 171.r 175.r 181 190.Q 196.r 206.r 229.r 233
 234.A, D, r 267.*, r 285.A, C, r 343.F 363.r 410.B
 447.r
 Klein, Jacob 444.r
 Klein, Oskar Benjamin (1895–1977) 212.B 351.G
 377.C
 Klema, Virginia C. 298.r
 Klingen, Helmut P. (1927–) 32.H 450.E
 Klingenberg, Wilhelm P. (1924–) 109.*, r 111.r
 178.C, r 279.G
 Kloosterman, Hendrik D. (1900–) 4.D 32.C, G
 Klotz, Tilla 365.H
 Kluvánek, Igor (1931–) 443.A, G
 Knapowski, Stanisław (1931–67) 123.D

Knapp, Anthony William

- Knapp, Anthony William (1941–) 260.r 437.EE
 Knaster, Bronisław (1893–1980) 79.D
 Kneser, Hellmuth (1898–1973) 198.r
 Kneser, Julius Carl Christian Adolf (1862–1930)
 107.A 314.A 316.E App. A, Table 19.III
 Kneser, Martin L. (1928–) 13.O–Q 60.K 391.C
 Knopp, Konrad (1882–1957) 208.C 339.C 379.I,
 M, r
 Knopp, Marvin Isadore (1933–) 328.r
 Knorr, Knut R. K. (1940–) 72.r
 Knowles, Greg 443.G
 Knudsen, Finn Faye (1942–) 16.r
 Knus, Max-Albert (1942–) 29.r
 Knuth, Donald Ervin (1938–) 71.r 96.r 354.r
 Knutson, Donald 16.r
 Kobayashi, Osamu (1955–) 364.H
 Kobayashi, Shoshichi (1932–) 21.N, O, Q, r 80.r
 105.r 109.r 275.A 364.r 365.K, O, r 412.r 413.r
 417.r 431.r
 Kobayashi, Zen-ichi (1906–) 367.D
 Koch, John A. 157.A 186.r
 Kochen, Simon Bernard (1934–) 118.F 276.E, r
 Köcher, Max (1924–) 32.F 231.r
 Kodaira, Kunihiko (1915–) 12.B, r 15.B, E, F
 16.V, r 20.21.N 72.F, G, I–K 109 112.I, O 147.O
 194.r 232.D, r 366.A, r
 Kodama, Akio (1949–) 384.F
 Kodama, Laura Ketchum 164.K
 Kodama, Yukihiko (1929–) 117.F 273.r 382.D 425.r
 Koebe, Paul (1882–1945) 77.B, E 193.D, E 196
 438.B, C
 Kogut, John Benjamin (1945–) 361.r
 Kohn, Joseph John (1932–) 112.H 274.I 345.A
 Koiso, Norihito (1951–) 364.r
 Koizumi, Shoji (1923–) 3.N
 Kojima, Tetsuzo (1886–1921) 121.B 240.B 379.L
 Koksma, Jurjen Ferdinand (1904–) 182.r 430.C
 Kolbin, Vyacheslav Viktorovich (1941–) 408.r
 Kolchin, Ellis Robert (1916–) 13.F 113 172.A
 Kolesov, Yurii Serafimovich (1939–) 290.r
 Kolmogorov, Andrei Nikolaevich (1903–) 20 44.r
 45.F 58.D 71.r 100.A 115.A, D 126.A, L 136.E
 159.H 196 201.A, M, P 205.E 213.E 214.C 250.F, r
 260.A, F 261.A 341.G, I, r 342.A, D, G 354.D
 371.F 374.E 407.A, r 425.Q 426 433.C
 Komatsu, Hikosaburo (1935–) 68.F 107.r 112.D, R
 162 168.B 224.C, E 254.D 378.D, I
 Komatu, Atuo (1909–) 305.A 425.U
 Komatu, Yūsaku (1914–) 77.E App. A, Table 14.r
 Kōmura, Takako (1930–) 168.B 378.F 424.S
 Kōmura, Yukio (1931–) 162 168.B 286.X 378.F
 424.S, W
 Kondo, Kazuo (1911–) 282.r
 Kondō, Motokiti (1906–80) 22.C, F
 Konheim, Alan G. 126.K 136.H
 König, Dénes (1884–1944) 33.F
 König, Heinz J. (1929–) 164.G, r
 Königs, G. 44.B
 Konishi, Yoshio (1947–) 286.X, Y
 Konno, Hiroshi (1940–) 264.r
 Kōno, Norio (1940–) 176.G 437.CC
 Koopman, Bernard Osgood (1900–81) 420.F
 Koopmans, Tjalling Charles (1910–85) 128.r
 255.E 376
 Koosis, Paul J. 164.r
 Korányi, Adam (1932–) 413.r
 Korn, Granino Arthur (1922–) 19.r
 Korn, T(h)eresa Mikhailovich 19.r
 Körner, Otto Herman (1934–) 4.F
 Korobov, Nikolaï Mikhailovich (1917–) 4.E
 Korolyuk, Vladimir Semenovich (1925–) 250.r
 Kortanek, Kenneth O. (1936–) 255.D, E
 Korteweg, Diederik Johannes (1848–1941) 387.B
 Kortum, Ludwig Hermann (1836–1904) 179.B
 Koshiba, Zen'ichiro (1926–) 304.F
 Kostant, Bertram (1928–) 248.Z 287.r 387.C
 437.U, EE
 Kostrikin, Aleksei Ivanovich (1929–) 161.C
 Koszul, Jean Louis (1921–) 200.J 412.r 413.r
 Kotake, Takeshi (1932–) 112.D 321.G
 Kotani, Shinichi (1946–) 176.F
 Köthe, Gottfried (1905–) 29.I 125.Y 168.B, r 424.r
 Kotz, Samuel (1930–) 374.r
 Kovalevskaya (Kowalewskaja), Sof'ya Vasil'evna
 (1850–91) 107.B 267 286.Z 320.I 321.A, B
 Kowalewski, Gerhard (1876–1950) 103.r
 Kowata, Atsutaka (1947–) 437.r
 Kra, Irwin (1937–) 122.r 234.r
 Krahn, E. 228.B 391.D
 Kraichnan, Robert H. 433.C
 Krakus, Bronislav 178.r
 Kramers, Hendrik Anthony (1894–1952) 25.B
 Krasinkiewicz, Józef (1944–) 382.C
 Krasner, Marc (1912–85) 145
 Krasnosel'skii, Mark Aleksandrovich (1920–) 251.r
 286.r 290.r
 Krasovskii, Nikolaï Nikolaevich (1924–) 163.B
 Kraus, Fritz (1903–1980) 212.r
 Krazzer, Karl Adolf Joseph (1858–1926) 3.r
 Krée, Paul (1933–) 224.C
 Kreider, Donald Lester (1931–) 81.*, r
 Krein, Mark Grigor'evich (1907–) 37.E 68.A, J, r
 89.r 115.r 162 176.K 251.I, r 310.H 390.H, r 424.O,
 U, V 443.H
 Krein, Selim Grigor'evich (1917–) 162 224.A 378.r
 Kreisel, Georg (1923–) 356.H
 Kreiss, Heinz-Otto (1930–) 304.F 325.K
 Krelle, Wilhelm 292.r
 Krenkel, Ulrich (1937–) 136.B, F, r
 Krichever, I. M. 387.F
 Krieger, Wolfgang 136.E, F, H, r 308.I
 Kripke, A. 411.F
 Kripke, Saul 356.G, r
 Krogh, Fred T. (1937–) 303.r
 Krohn, Kenneth 31.r
 Kronecker, Leopold (1823–91) 2.B 10.B 14.L, U
 15.C 47 73.A, r 136.G 156.C 190.Q 192.R 201.H
 236.A, r 267 269.C 347.D 422.K 450.B, S App. A,
 Table 4
 Krull, Wolfgang (1899–1970) 12.B 67.D, E, J, r
 172.A, I 190.L 277.I 284.A, F, G 439.L
 Kruskal, Joseph Bernard (1928–) 346.E, r
 Kruskall, Martin David (1925–) 359.F 387.B
 Kruskal, William Henry (1919–) 371.D
 Krylov, Nikolaï Mitrofanovich (1879–1955) 290.A
 Krylov, N. S. 402.r
 Krylov, Nikolaï Vladimirovich (1941–) 115.r 136.H
 405.r
 Krylov, Vladimir Ivanovich (1902–) 46.r 217.r
 299.r 304.r
 Krzyżański, Mirosław 325.r
 Kshirsagar, Anant M. (1931–) 280.r
 Kubilius, Jonas P. (1921–) 295.r
 Kublanovskaya, Vera Nikolaevna (1920–) 298.F
 Kubo, Izumi (1939–) 136.F, G 395.r
 Kubo, Ryogo (1920–) 308.H 402.K
 Kubota, Tadahiko (1885–1952) 89.C
 Kubota, Tomio (1930–) 14.U 59.H 257.H 450.A,
 J, M
 Kudō, Hirokichi (1916–) 399.r 443.A

Kudryavtsev, Valerii Borisovich (1936–) 75.D
 Kuga, Ken'ichi (1956–) 114.K
 Kuga, Michio (1928–) 450.M, S
 Kugo, Taichiro (1949–) 150.G
 Kuhn, Harry Waldo (1874–) 108.r 173.B, r 255.r
 292.A, B
 Kuiper, Nicolaas H. (1920–) 105.r 114.B 126.G
 183.*, r 286.D 364.F 365.B, O
 Kuipers, Lauwerens 182.r 354.r
 Kulikov, Leonid Yakovlevich 2.D
 Kulk, W. V. D. 428.r
 Kulkarni, Ravi S. (1942–) 72.r
 Kullback, Solomon (1907–) 213.D, 398.G 403.C, r
 Kumano-go, Hitoshi (1935–82) 112.L 274.r 323.r
 345.A, B
 Kummer, Ernst Eduard (1810–93) 14.L, N, O, U
 15.H 145 167.A 172.F 206.A 236 267 379.I 450.J
 App. A, Tables 10.II, 19.I
 Kunen, Kenneth (1943–) 33.r
 Kunieda, Motoji (1879–1954) 121.B 240.B
 Kunita, Hiroshi (1937–) 86.r 115.D 260.J 261.r
 406.B, r
 Künnet, Hermann (1892–1975) 200.E, H 201.J
 450.Q
 Kuntzmann, Jean (1912–) 71.r 75.r
 Kunugui, Kinjiro (1903–75) 22.C, F 62.B, E, r
 100.A, r
 Künzi, Hans Paul (1924–) 292.F, r
 Kuo Yung-Huai 25.B
 Kupka, Ivan Adolf-Karl (1937–) 126.r
 Kupradze, Viktor Dmitrievich (1903–85) 188.r
 Kuramochi, Zenjiro (1920–) 207.C, D 367.E, G
 Kuranishi, Masatake (1926–) 72.G 249.D, V 286.J
 428.F, G, r
 Kurata, Masahiro (1943–) 126.J
 Kuratowski, Kazimierz (Casimir) (1896–1980)
 22.D, G, r 79.D, r 186.H 425.A, Q, r 426
 Kuroda, Shige Toshi (1932–) 331.E 375.C
 Kuroda, Sigekatu (1905–72) 411.J, r
 Kuroda, Tadashi (1926–) 367.E
 Kurosh, Aleksandr Gennadievich (1908–71) 2.E, r
 29.J 103.r 161.A, r 190.r 337.r
 Kurosu, Kōnosuke (1893–1970) 240.B
 Kürschák, József (1864–1933) 439.L
 Kurth, Rudolf (1917–) 116.r
 Kurusima, Yoshihiro (?–1757) 230
 Kusaka, Makoto (1764–1839) 230
 Kushner, Harold Joseph (1933–) 86.E 405.r
 Kushnirenko, Anatolii Georgievich 418.r
 Kusunoki, Yukio (1925–) 143.r 207.C, D, r 367.G, I
 Kutta, Wilhelm Martin (1867–1944) 301.D 303.D
 Kuwabara, Ruishi (1951–) 391.N
 Kuyk, Willem (1934–) 32.r

L

Lacey, Howard Elton (1937–) 37.r
 Lachlan, Alistair H. 276.F
 Lacroix, Sylvestre François (1765–1843) 181
 Ladyzhenskaya, Ol'ga Aleksandrovna (1922–)
 204.B–D, r 286.r 323.P
 Lafontaine, Jacques (1944–) 364.H
 Lagrange, Joseph Louis (1736–1813) 4.D 20 46.A,
 B 82.A 83.C, D 105.A 106.E, L 107.A, B 109 126.A,
 E, L 150.B 151.B 172.A, F 190.Q 205.A 223.A 238
 252.D, K 266 271.F 274.C 275.A 296.A 301.C
 322.B 324.D 336.G 342.A 420.B, D 428.C 442.C
 App. A, Tables 9.IV, 14.I, 15, 21.II
 Laguerre, Edmond Nicolas (1834–86) 76.B 137
 299.A 317.D 429.B App. A, Tables 14.II, 26.VI

Lainiotis, Dimitri G. 86.r
 Lakshmikantham, Vangipuram (1924–) 163.r
 Laksov, Dan (1940–) 9.E
 Lamb, Sir Horace (1849–1934) 205.r 446.r
 Lambert, Jack D. 303.r
 Lambert, Johann Heinrich (1728–77) 83.A, E 332
 339.C
 Lambert, Robert Joe (1921–) 303.r
 Lamé, Gabriel (1795–1870) 133.B, C 145 167.E
 Lamperti, John Williams (1932–) 44.E 342.r
 Lance, E. Christopher (1941–) 308.F
 Lanczos, Cornelius (1893–1974) 298.D, E 301.J, N
 302.r
 Land, A. H. 215.D
 Landau, Edmund Georg Herman (1877–1938)
 4.A, r 43.J, K, r 77.F 87.G, r 106.r 107.A 121.C–E
 123.B, F, r 131.r 160.G 216.r 240.B 242.A, r 294.r
 295.D 297.r 339.r 347.r 450.I, r
 Landau, Lev Davidovich (1908–68) 130.r 146.A,
 C, r 150.r 205.r 259.r 402.r 433.B
 Landau, Yoan D. 86.r
 Landen, John (1719–90) 134.B App. A, Table
 16.III
 Landkof, Naum Samoilovich (1915–) 338.r
 Landsberg, Georg (1865–1912) 11.r
 Landshoff, Peter Vincent (1937–) 146.r 386.r
 Lane, Jonathan Homer (1819–80) 291.F
 Landford, Oscar Erasmus, III (1940–) 126.K
 150.C 340.B, F 402.G
 Lang, Serge (1927–) 3.M, r 6.r 12.B 14.r 28.r 105.r
 118.D, F, r 134.r 172.r 182.r 198.r 200.r 256.r
 277.r 337.r 368.r 430.r 450.r
 Langevin, Paul (1872–1946) 45.I 402.K
 Langhaar, Henry L. 116.r
 Langlands, Robert Phelan (1936–) 32.H, r
 437.DD, r 450.A, G, N, O, S, T
 Lapidus, Leon (1910–75) 303.r
 Laplace, Pierre Simon (1749–1827) 30.B 103.D
 107.B 126.A 192.F 194.B 239 240.A 250.A 266
 306.A 323.A 342.A, r 401.E 442.D App. A, Tables
 12.I, 18.II
 Lappo-Danilevskii, Ivan Aleksandrovich 253.r
 LaSalle, Joseph Pierre (1916–83) 86.F
 Lascoux, Jean 146.A, C
 Lashnev, Nikolai Serafimovich 425.CC
 Lashof, Richard Kenneth (1922–) 279.C 365.O
 Lasker, Emanuel (1868–1941) 12.B
 Latter, Robert H. (1946–) 168.B
 Laufer, Henry B. (1945–) 418.r
 Laugwitz, Detlef (1932–) 111.r
 Laurent, Pierre Alphonse (1813–54) 198.D 339.A
 Laurent-Duhamel, Marie Jeanne (1797–1872)
 322.D
 Lauricella, G. 206.D
 Lavine, Richard B. (1938–) 375.C
 Lavita, James A. 375.r
 Lavrent'ev, Mikhail Alekseevich (1900–80) 336.F
 352.A, D, E 436.I
 Lavrik, Aleksandr Fëdorovich (1927–) 123.E
 Lawler, Eugene L. (1933–) 66.r 281.r 376.r
 Lawley, Derrick Norman 280.B, G, r 346.F, r
 Lawson, Charles L. 302.r
 Lawson, Herbert Blaine, Jr. (1942–) 80.r 154.r
 178.r 275.F, r 364.H 365.K
 Lax, Peter David (1926–) 112.J, P, S 204.r 274.r
 304.F 321.G 325.H 345.A, r 375.H 387.C, r
 Lazard, Daniel (1941–) 200.K
 Lazard, Michel Paul (1924–) 122.F
 Lazarov, Connor (1938–) 154.H
 Leadbetter, Malcolm Ross (1931–) 395.r

Lebesgue, Henri Léon

- Lebesgue, Henri Léon (1875–1941) 20.r 22.A 84.D 93.F 94.C, r 117.B, D, r 120.A, B, D 136.A, E 156.C 159.A–C, J 160.A 166.C 168.B 179.r 221.A–C 244 246.C 270.D, E, G, J, L, r 273.F 379.S 380.C, D, r 388.B
- Lebowitz, A. 134.r
- Lebowitz, Joel Louis (1930–) 136.G
- LeCam, Lucien (Marie) (1924–) 341.r 398.r 399.K, M, N, r
- Ledger, Arthur Johnson (1926–) 364.r
- Lê Dũng Tráng (1947–) 418.I
- Lee, Benjamin W. (1935–77) 132.r
- Lee, E. Bruce (1932–) 86.r
- Lee Tsung Dao (1926–) 359.C
- Lee, Y. W. 95.r
- Leech, John 151.I
- Leela, S. 163.r
- Lefébvre, Henri (1905–) 101.r
- Lefschetz, Solomon (1884–1972) 3.A 12.B 15.B 16.P, U, V 79.r 93.r 126.A, r 146.r 153.B, C, r 170.r 201.A, E, O, r 210.r 290.r 291.r 394.r 410.r 418.F, I 422.r 426.*, r 450.Q
- Legendre, Adrien Marie (1752–1833) 4.D 46.C 82.A 83.B 107.A 109 123.A 134.A, F 145 174.A 266 296.A, B 297.H, I 342.A 393.A–C 419.C App. A, Tables 14.II, 15.IV, 16.I, IV, 18.II, III
- Lehman, R. Sherman (1930–) 385.r 450.I
- Lehmann, Erich Leo (1917–) 371.A, C, H, r 396.r 399.C, E, H, P, r 400.B, r
- Lehmann, Harry Paul (1924–) 150.D
- Lehmer, Derrick Henry (1905–) 145 301.K 354.B
- Lehmer, Derrick Norman (1867–1938) 123.r NTR
- Lehmer, Emma (1906–) 145
- Lehner, Joseph (1912–) 32.r
- Lehrer, G. I. App. B, Table 5
- Lehto, Olli (1925–) 62.C 352.C
- Leibenzon (Leibenson), Zinovii Lazarevich (1931) 192.Q
- Leibler, Richard Arthur (1914–) 398.G
- Leibniz, Gottfried Wilhelm, Freiherr von (1646–1716) 20 38 75.A 106.D 107.A 156.B 165.A 245 265 283 293.A 332 379.C App. A, Tables 9, 10.III
- Leith, Cecil Eldon, Jr. (1923–) 433.C
- Leja, Franciszek (François) (1885–1979) 48.D
- Lelong, Pierre (1912–) 21.r
- Lelong-Ferrand, Jacqueline (1918–) 364.r
- Lemaire, Luc R. (1950–) 195.E, r
- Lenstra, J. K. 376.r
- Leon, Jeffrey S. 151.I
- Leonardo da Vinci (1452–1519) 360
- Leonardo Pisano → Fibonacci
- Leontief, Wassily W. (1906–) 255.E
- Leontovich, A. M. 420.G
- Leopoldt, Heinrich Wolfgang (1927–) 14.D, U 450.A, J
- Leray, Jean (1906–) 20 112.B 125.A 146.A 148.A, E 200.J 201.J 204.B, D, r 240.r 286.C, D 321.G 323.D 325.I, J, r 383.J, r 426
- Lerner, R. G. 414.r
- LeRoy, Edouard (1870–1954) 379.S
- Lesley, Frank David 275.C
- Lettenmeyer, Fritz (1891–1953) 254.D 289.D 314.A
- Levelt, Antonius H. M. 428.r
- LeVeque, William Judson (1923–) 118.r 295.r 296.r 297.r 430.r
- Levi, Eugenio Elia (1883–1917) 13.Q 21.F, I, Q 112.D 188.r 248.F 274.G 282 321.G 323.B 325.H 344.A
- Levi, Friedrich Wilhelm (1888–1966) 2.E 122.B
- Levi-Civita, Tullio (1873–1941) 80.A, K 109.*, r 364.B 420.F
- Levin, Viktor Isosifovich (1909–) 198.r 211.r
- Levine, Jerome Paul (1937–) 114.D 235.G
- Levinson, Norman (1912–75) 107.r 123.B 160.G 252.r 253.r 254.r 314.C, r 315.r 316.r 394.r 450
- Levitán, Boris Moiseevich (1914–) 112.O 287.C 315.r 375.G 387.D
- Lévy, Azriel (1934–) 22.F 33.F, r 356.G
- Lévy, Paul (1886–1971) 5.B, E, r 45.A, E, G, I, r 115.r 159.I 176.A, E, F 192.N 260.J 261.A 262.A 341.E–G 342.D 406.F 407.A, B
- Lewin, L. 167.r
- Lewis, Daniel Ralph (1944–) 443.A
- Lewis, Donald J. (1926–) 4.E 118.D, F
- Lewis, Richard M. 127.G
- Lewy, Hans (1904–) 112.C 274.G, I 275.B 300.r 304.F 320.I 323.I 334.F
- Li Chih (1192–1279) 57.B
- Li, Peter Wai-Kwong (1952–) 391.D, N
- Li Tien-Yien (1945–) 126.N 303.G
- Li Yen (1892–1963) 57.r
- Liao San Dao (1920–) 126.J
- Lichnérowicz, André (1915–) 80.r 152.C 359.r 364.F, H, r 391.D
- Lichtenstein, Leon (1878–) 217.r 222.r
- Lickorish, William Bernard Raymond 114.L 154.B
- Lie, Marius Sophus (1842–99) 13.C, F 76.B, C 105.O, Q 107.B 109.O, Q 137 139.B 183 190.Q 247 248.A, B, F, H, P, S, T, V, r 249.A–D, G, H, L, M, V, r 267 286.K 313.D 406.G 431.C, G 437.U
- Lieb, Elliott Hershel (1932–) 212.B, r 402.r
- Lieb, Ingo (1939–) 164.K
- Lieberman, David Ira (1941–) 16.R 23.G
- Lieberman, Gerald J. (1925–) STR
- Liebmann, Karl Otto Heinrich (1874–1939) 111.I 365.J
- Liénard, Alfred 290.C
- Liepmann, Hans Wolfgang (1914–) 205.r
- Lifshits, Evgenii Mikhaïlovich (1915–) 130.r 150.r 205.r 259.r 402.r
- Liggett, Thomas Milton (1944–) 162 286.X 340.r
- Lighthill, Michael James (1924–) 25.B, r 160.r 205.r 446.r
- Ligocka, Ewa (1947–) 344.D
- Lill 19.B
- Lin, C. C. 433.r
- Lin Jiguan 108.B
- Lin Shu 63.r
- Lind, Douglas A. (1946–) 136.E
- Lindeberg, J. W. 250.B
- Lindelöf, Ernst Leonhard (1870–1946) 43.C, H 123.C 425.S
- Lindemann, Carl Louis Ferdinand von (1852–1939) 179.A 332 430.A, D
- Lindenstrauss, Joram (1936–) 37.M, N, r 168.r 443.H
- Lindley, Dennis Viktor (1923–) 401.r
- Lindow, M. App. A, Table 21.r
- Linfoot, Edward Hubert (1905–82) 347.E
- Linnik, Yuriï Vladimirovich (1915–72) 4.A, C, E 123.D, E 136.H 250.r 341.E 374.H
- Lionnet, Eugène (1805–84) 297.D
- Lions, Jaques-Louis (1928–) 86.r 112.E, F 204.B 224.A, E, F, r 286.C 320.r 322.r 323.r 327.r 378.F, I, r 405.r 440.r
- Liouville, Joseph (1809–82) 107.A 112.I 126.L 131.A 134.E 171 182.G 219.A 252.C 272.A 315.B 402.C 430.B
- Lippmann, Bernard Abram (1914–) 375.C

Lipschitz, Rudolf Otto Sigismund (1832–1903)
84.A 107.A 159.B 168.B 279.C 286.B 316.D 406.D
Liptser, Roberg Shevilevich (1936–) 86.E 405.r
Listing, Johann Benedikt (1808–82) 426
Little, C. N. 235.A
Little, John A. 365.N
Little, John Dutton Conant (1928–) 133.r
Littlewood, Dudley Ernest (1903–79) App. B,
Table 5.r
Littlewood, John Edensor (1885–1977) 4.C, D
43.E, r 88.r 123.B, C 159.G 168.B 192.D, P 211.r
224 242.B, r 317.B 339.B, C 379.S 450.B, I App. A,
Table 8
Liu Chung Laung (1934–) 66.r
Liu Hui (fl. 260) 57.A 332
Liulevicius, Arunas L. (1934–) 202.S
Livesay, George Roger (1924–) 442.L
Livingood, John 328
Livshits, Mikhail Samulovich (1917–) 68.J
Lobachevskii, Nikolai Ivanovich (1793–1856) 35.A
181 267 285.A
Loday, Jean-Louis (1946–) 237.r
Loeb, Peter Albert (1937–) 293.D, r
Loève, Michel (1907–79) 341.r 342.r
Loewy, Alfred (1873–1935) 190.P
Löfström, Jörgen (1937–) 224.r
Logunov, Anatolii Alekseevich (1926–) 150.r
Lohwater, Arthur John (1922–82) 62.r
Łojasiewicz, Stanisław (1926–) 16.r 58.E
Lommel, Eugen Cornelius Joseph von (1837–99)
39.C App. A, Table 19.IV
Lomonosov, V. I. 251.L
Longley-Cook, Laurence H. 214.r
Longo, Giuseppe (1941–) 213.r
Lonsdale, Dame Kathleen Yardley (1903–71) 92.r
Looman, H. 198.A
Looman, M. 100.A
Loomis, Herschel H., Jr. 75.D, r
Loomis, Lynn H. (1915–) 36.r 126.r 192.r 225.r
Loos, Ottman 412.r
Lopatiniskii, Yaroslav Borisovich (1906–81) 323.H
325.K
López de Medrano, Santiago 114.r
Lorentz, George G. (1910–) 168.B
Lorentz, Hendrik Antoon (1853–1928) 60.J 150.B
258.A 359.B 391.I 402.H
Lorenz, Edward N. 126.N 433.B, r
Lorenz, Max O. 397.E
Lorenzen, Paul Peter Wilhelm (1915–) 243.G
Loria, Gino (1862–1954) 93.r
Łoś, Jerzy Maria (1920–) 276.F 293.C
Loschmidt, Joseph (1821–95) 41.A
Losik, Mark Vol'fovich (1935–) 105.r
Lotz, Heinrich P. 310.H
Louveau, Alain 22.G
Lovász, László (1948–) 186.r
Love, Augustus Edward Hough (1863–1940) 271.G
Love, Clyde Elton (1882–) 107.A
Low, Francis Eugene (1921–) 150.C 361.r
Lowdenslager, David B. (1930–) 164.G, H
Löwenheim, Leopold (1878–1940) 97.B 156.E, r
276.D
Löwner, Karl (Loewner, Charles) (1893–1968)
212.r 438.B, C
Lozinskii, Sergei Mikhailovich (1914–) 314.D
Lü Yinian 17.D
Lubański, J. K. 258.D
Lubin, Jonathan (1936–) 257.r
Lubkin, Saul (1939–) 450.Q
Lucas, William F. (1933–) 173.D, E, r

Luce, Robert Duncan (1925–) 96.r 173.C 346.G
Lüders, Gerhart Claus Friedrich (1920–) 150.D
386.B
Ludwig, Donald A. (1933–) 321.G 325.L, r 345.r
Luenberger, David G. (1937–) 86.E 264.r
Lukacs, Eugène (1906–) 341.r
Łukaszewicz, Jan (1878–1956) 411.L
Luke, Yudell L. 389.r App. A, Table 16 NTR
Lumer, Günter (1929–) 164.F, G
Lundberg, Filip 214.C
Luneburg, Rudolf Karl (1903–49) 180.A, r 325.L
Lunts, G. 198.r
Lüroth, Jakob (1844–1910) 16.J
Lustzig, G. App. B, Table 5
Luther, Herbert A. 304.r
Lutz, Elizabeth (1914–) 118.D, E
Luxemburg, Wilhelmus Anthonius Josephus
(1929–) 293.r
Luzin (Lusin), Nikolaï Nikolaevich (1883–1950)
22.A, C, F, G, I, r 100.A 156.C 159.I 270.J 425.CC
Lyapin, Evgenii Sergeevich (1914–) 190.r
Lyapunov, Aleksandr Mikhailovich (1857–1918)
107.A 120.A 126.A, F 163.G 250.B 286.V 314.A
394.A, C, r 398.C 443.G
Lyapunov, Aleksei Andreevich (1911–73) 22.r
Lyndon, Roger Conant (1917–) 97.r 200.M
Lyons, Richard Neil (1945–) 151.I
Lyusternik, Lazar' Aronovich (1899–1981) 279.G
286.Q, r NTR

M

Maak, Wilhelm (1912–) 18.r
Maass, Hans (1911–) 32.F, G, r 450.M
Macaulay, Francis Sowerby (1862–1937) 12.B
284.D
Mach, Ernst (1838–1916) 116.B 205.B 271.A
Machin, John (1680–1751) 332
Machover, Maurice (1931–) 356.G
Mack, C. 301.N
Mackay, Alan L. 92.F
MacKenzie, Robert E. (1920–) 279.C
Mackey, George Whitelaw (1916–) 36.G 424.M, N
437.EE
MacLane, Saunders (1909–) 8.r 52.r 70.F, r 91.r
103.r 200.M, r 201.G 202.T 277.r 305.A
Maclaurin, Colin (1698–1746) 20 266 379.J
MacMahon, Major Percy Alexander (1854–1929)
328 330.r
MacPherson, Robert Duncan (1944–) 366.E, r
MacRobert, Thomas Murray 393.r
Madansky, Albert 408.r
Maeda, Fumitomo (1897–1965) 162
Maeda, Fumi-Yuki (1935–) 207.D, r
Maeda, Yoshiaki (1948–) 364.G
Maehara, Shōji (1927–) 411.J, r
Magness, Enrico (1923–) 112.E 323.r
Magidor, Menachem 33.r
Magnus, Wilhelm (1907–) 161.B, r 389.r App. A,
Table 20.IV
Mahalanobis, Prasanta Chandra (1893–1972)
280.E
Mahler, Kurt (1903–) 182.r 430.B, C
Mahlo, P. 33.r
Mainardi, Gaspare (1800–79) 111.H App. A, Table
4.I
Maitra, Ashok P. 22.E 396.r
Majima, Hideyuki (1952–) 428.H
Makarov, Vitalii Sergeevich (1936–) 122.G
Malcolm, Donald G. 376.r

Malfatti, Gian Francesco

- Malfatti, Gian Francesco (1731–1807) 179.A
 Malgrange, Bernard (1928–) 58.C, E 68.F 112.B, C, R 125.W 320.H 418.r 428.H
 Malliavin, Paul (1925–) 115.D, r 192.M 406.E, r
 Malmquist, Johannes (1882–1952) 254.D 288.B, C, r 289.B–D 314.A
 Mal'tsev, Anatoliĭ Ivanovich (1909–67) 29.F 249.S 276.D
 Malus, Étienne Louis (1775–1812) 180.A
 Mandelbaum, Richard (1946–) 114.r
 Mandelbrojt, Szolem (1899–1983) 58.F 134.C, r 339.A, r
 Mandelbrot, Benoit B. (1924–) 246.K 433.r
 Mandelstam, Stanley (1928–) 132.C
 Mañé, Ricardo 126.J
 Mangasarian, Olvi L. (1934–) 292.D, r
 Mangoldt, Hans Carl Friedrich von (1854–1925) 123.B 450.B
 Manin, V. G. 80.r
 Manin, Yuriĭ Ivanovich (1937–) 16.J, V 80.r 118.E 387.r 450.J, M
 Mann, Henry Berthold (1905–) 4.A 371.A, C 421.r
 Mann, Larry N. (1934–) 364.F
 Manna, Zohar (1939–) 75.r
 Mannheim, Amédée (1831–1906) 111.F
 Manning, Anthony Kevin (1946–) 51.r 126.J, K
 Mansfield, Richard B. (1941–) 22.F
 Maranda, Jean-Marie A. (?–1971) 362.K
 Marchand, Jean-Paul 375.r
 Marchenko, Vladimir Aleksandrovich (1922–) 287.C 387.D
 Marchuk, Gurii Ivanovich (1925–) 304.r
 Marcinkiewicz, Józef (1910–) 159.H 224.A, E 336.E
 Marden, Morris (1905–) 10.r
 Mardešić, Sibe (1927–) 382.A
 Margulis, Gregorii A. (1946–) 122.G
 Marion, Jerry Baskerville (1929–) 271.r
 Markov, Andreĭ Andreevich (1856–1922) 5.H 44.D, E 126.F, J 127.E 136.B, D, G 150.F 176.F 182.G 260.A, H, J 261.A, B 336.C 340.C 379.I 403.E 405.C 406.D 407.B
 Markov, Andreĭ Andreevich (1903–) 31.B 161.B 356.r
 Markus, Lawrence J. (1922–) 86.r 126.A, H, L, r 291.r
 Markwald, Werner 81.A, r
 Marotto, Frederick Robert (1950–) 126.J
 Marsden, Jerrold E. (1942–) 126.r 183.r 271.r 286.r 316.r 364.H 420.r
 Marshall, Donald E. 164.I
 Marsten, Roy Earl (1942–) 215.r
 Martin, André (1931–) 150.D 386.B, r
 Martin, Donald A. 22.D, F, H, r 33.F, r
 Martin, Harold C. 304.r
 Martin, Paul C. 308.H
 Martin, Robert S. 207.C, D 260.I
 Martin, William Ted (1911–) 21.r
 Martineau, André (1930–72) 125.W, Y, r 162 168.B 424.X
 Martinet, Jean 110.E
 Martin-Löf, Per (1942–) 354.r
 Martio, Olli Tapani (1941–) 352.F
 Marty, F. (?–1939) 272.H 435.E
 Maruyama Gisiro (1916–86) 115.D 136.D, E 250.r 260.J 395.r
 Maruyama, Masaki (1944–) 16.Y
 Maruyama, Toru (1949–) 443.A
 Masani, Pesi R. (1919–) 395.r
 Mascheroni, Lorenzo (1750–1800) 179.B
 Maschke, Heinrich (1853–1908) 362.G
 Maschler, Michael (1927–) 173.D
 Maskit, Bernard (1935–) 122.I 234.D, r 416.r
 Masley, John M. (1947–) 14.L
 Maslov, Viktor Pavlovich (1930–) 30.r 274.C, I 345.r
 Massau, Junius (1852–1909) 19.B
 Masser, David W. 134.r 430.D, r
 Massey, William S. (1920–) 91.r 170.r 201.r 202.M, P 410.r
 Masuda, Kazuo (1946–) 72.r
 Masuda, Kyûya (1937–) 378.I, J
 Masuyama, Motosaburo (1912–) STR
 Matano, Hiroshi (1952–) 263.C 303.G, r
 Mather, John Norman (1942–) 51.C–E 126.J 154.E, r 286.J 418.r
 Mathews, George Ballard (1861–1922) 39.r
 Mathieu, Émile Léonard (1835–90) 151.H 268.A–D
 Matiyasevich, Yuriĭ Vladimirovich 97.*, r 118.A 196
 Matsuda, Michihiko (1938–) 428.G
 Matsumoto, Hideya (1939–) 13.R 122.F
 Matsumoto, Kazuo (1922–) 411.J
 Matsumoto, Kikuji (1931–) 62.B 124.C, r
 Matsumoto, Shigenori (1947–) 126.M
 Matsumoto, Yukio (1944–) 65.D 114.K
 Matsumura, Akitaka (1951–) 41.r 204.F
 Matsumura, Hideyuki (1930–) 284.r
 Matsusaka, Teruhisa (1926–) 12.B 16.P, W, r
 Matsushima, Yozo (1921–83) 32.r 122.F 199.r 249.r 384.r
 Matsushita, Shin-ichi (1922–) 338.L
 Matsuyama, Noboru (1916–) 310.r
 Mattis, Daniel Charles 402.r
 Mattuck, Arthur Paul (1930–) 118.E 450.P
 Matuda, Tizuko (1923–) 30.r 288.B, r 289.r
 Matumoto, Takao (1946–) 65.C 114.K
 Matunaga Yosisuke (1692?–1747) 230 332
 Matuzaka, Kazuo (1927–) 7.r 343.r
 Matveev, Vladimir Borisovich 387.r
 Maunder, Charles Richard Francis 201.r
 Maupertuis, Pierre Louis Moreau de (1698–1759) 180.A 441.B
 Maurer, Ludwig (1859–?) 249.R
 Maurus (c. 780–c. 856) 372
 Mautner, Friedrich Ignaz (1921–) 136.G 308.G 437.EE
 Mawhin, Jean 290.r
 Maxfield, John E. (1927–) NTR
 Maxwell, Albert Ernest 280.r 346.F, r
 Maxwell, George (1946–) 92.r
 Maxwell, James Clerk (1831–79) 51.F 130.A 150.A 180.A 393.D 402.B, H 419.B
 Maxwell, William L. (1934–) 376.r
 May, J. Peter (1939–) 70.r
 May, Kenneth Ownsworth (1915–77) 157.r
 May, Robert McCredie (1936–) 126.N 263.D, r
 Mayer, Dieter H. 402.G, r
 Mayer, Karl Heinz (1936–) 431.r
 Mayer, Walter 111.r 201.C, E, L
 Maynard, Hugh B. 443.H
 Mazet, Edmond 109.r 115.r 391.r
 Mazur, Barry C. (1937–) 16.r 37.C 65.C, G 114.C 126.K 426 450.J, r
 Mazur, Stanisław (1905–81) 36.E
 Mazurkiewicz, Stefan (1888–1945) 22.C 93.D 426
 McAndrew, Michael H. 126.K 136.H
 McAuley, Van A. 301.E
 McBride, Elna Browning 177.r
 McCarthy, John (1927–) 31.C

McCoy, Barry Malcolm (1940–) 402.r
 McCoy, Neal Henry (1905–) 368.r
 McCracken, Marsden 126.r
 McDuff, Dusa Waddington (1945–) 308.F
 McGregor, James 263.E
 McKay, John 151.I
 McKean, Henry P., Jr. (1930–) 41.C 45.I, r 115.A, r 176.F, K 260.J 261.A, r 323.M 340.F 387.E, r 391.B, C, K, r 406.r 407.B
 McLachlan, Norman William (1888–) 268.r
 McLaughlin, Jack (1923–) 151.I
 McMillan, Brockway (1915–) 136.E 213.D
 McShane, Edward James (1904–) 310.I, r
 Meeks, William Hamilton, III 235.E 275.C, D
 Mehler, Ferdinand Gustav (1835–95) App. A, Tables 18.II, 19.III
 Mehra, Raman K. 86.r
 Meinardus, Günter (1926–) 328
 Meinhardt, Hans 263.D
 Meixner, Josef (1908–) 268.r 389.r
 Melin, Anders (1943–) 345.A
 Mellin, Robert Hjalmar (1854–1933) 206.D 220.C
 Melrose, Richard B. (1949–) 325.M
 Menaechmus (375–325 B.C.) 187
 Mendelson, Elliot (1931–) 33.D, r 319.r
 Menelaus (of Alexandria) (fl. 98?) 7.A 187
 Menger, Karl (1902–) 93.D 117.A, B, D, r 426
 Menikoff, Arthur S. (1947–) 325.H
 Men'shov, Dmitrii Evgen'evich (1892–) 77.A 159.J 198.A, r 317.B
 Méray, Hugues Charles Robert (1835–1911) 267
 Mercer, J. 217.H
 Mergelyan, Sergei Nikitovich (1928–) 164.J 336.F 367.G
 Merluzzi, P. 303.r
 Merman, G. A. 420.D
 Mersenne, Marin (1588–1647) 297.E App. B, Table 1
 Mertens, Franz Carl Josef (1840–1927) 123.A 379.F
 Meschkowski, Herbert (1909–) 188.r
 Meshalkin, Lev Dmitrievich (1934–) 136.E
 Messiah, Albert M. L. (1921–) 351.r
 Messing, William (1945–) 450.Q
 Métivier, Michel (1931–) 443.H
 Meusnier, Jean Baptiste Marie Charles (1754–93) 109 111.H 275.A 334.B
 Meyer, Franz (1856–1934) 267
 Meyer, Kenneth R. (1937–) 126.K, I, M
 Meyer, Paul-André (1934–) 261.r 262.D, r 406.r 407.B, r
 Meyer, Wolfgang T. (1936–) 109.r 178.r 279.G, r
 Meyer, Yves 251.r
 Michael, Ernest Arthur (1925–) 425.X, Y, AA, CC
 Michel, René 178.r 304.r
 Michelson, Albert Abraham (1852–1931) 359.A
 Migdal, A. A. 361.C
 Mikami, Yoshio (1875–1950) 230.r
 Mikhailov, A. V. 387.G
 Mikhlin, Solomon Grigor'evich (1908–) 46.r 217.r
 Mikusiński, Jan G. (1913–) 306.A, B
 Miles, G. 136.E
 Milgram, Arthur N. (1912–) 112.J
 Milgram, R. James (1939–) 65.r
 Miller, Charles F., III (1941–) 97.r
 Miller, David Charles (1918–) 291.F
 Miller, K. S. 104.r
 Miller, Louis W. 376.r 389.r
 Millett, Kenneth Cary (1941–) 154.H
 Mills, Robert L. (1927–) 80.Q, r 150.G

Name Index

Moldestad, Johan

Mil'man, David Pinkhusovich (1913–) 37.G 424.U 443.H
 Milne, James Stuart (1942–) 450.r
 Milne, William Edmund (1890–) 303.E
 Milne-Thomson, Louis Melville (1891–1974) 104.r NTR
 Milnor, John Willard (1931–) 56.F, r 65.C, E 70.C 99.r 109.r 111.r 114.A–C, F–K, r 126.N 147.H, P 154.H 178.F 237.J, r 365.O 391.C 418.D, r 426 App. A, Table 5.V
 Mimura, Masayasu (1941–) 263.D
 Mimura, Yositaka (1898–1965) 434.C, r
 Mimura, Yukio (1904–84) 162
 Minakshisundaram, Subbaramiah (1913–68) 121.r 323.M 379.r 391.B, r
 Minemura, Katsuhiko (1945–) 437.r
 Minkowski, Hermann (1864–1909) 14.B, D, U 89.D, E 118.C 122.E, F 182.A, C–E, r 196 211.C 255.B 258.A 296.A 348.D, G, K 359.B App. A, Table 8
 Minlos, Robert Adol'fovich (1931–) 258.r 341.J 424.T
 Minorsky, Nicholas (1885–) 163.B
 Minsky, Marvin C. 385.r
 Minsky, M. L. 75.r
 Minty, George James (1929–) 281.r 286.C
 Miranda, Carlo (1912–82) 323.r
 Mirimanov, D. 145
 Mishchenko, Evgenii Frolovich (1922–) 86.r
 Mishkis, Anatolii Dmitrievich 163.B
 Misiurewicz, Michał (1948–) 126.K
 Misner, Charles William (1932–) 359.r
 Mitchell, Andrew Ronald 223.r
 Mitchell, Benjamin Evans (1920–) 52.N, r 200.I
 Mitome, Michiwo (1909–76) STR
 Mitropol'skii, Yurii Alekseevich (1917–) 290.D, F
 Mitsui, Takayoshi (1929–) 4.F, r 123.F 328
 Mittag-Leffler, Gustav Magnus (1846–1927) 47 267 272.A
 Mityagin, Boris Samuilovich (1937–) 424.S
 Miura, Robert Mitsuru (1938–) 387.B
 Miwa, Megumu (1934–) 118.E
 Miwa, Tetsuji (1949–) 112.R 253.E 387.C
 Miyadera, Isao (1925–) 162 378.B
 Miyajima, Kimio (1948–) 72.G
 Miyajima, Shizuo (1948–) 310.H
 Miyakawa, Tetsuro (1948–) 204.C
 Miyake, Katsuya (1941–) 16.Z
 Miyakoda, Tsuyako (1947–) 301.C
 Miyanishi, Masayoshi (1940–) 15.H, r
 Miyaoka, Yoichi (1949–) 72.K, r
 Miyata, Takehiko (1939–83) 226.r
 Miyoshi, Tetsuhiko (1938–) 304.r
 Mizohata, Sigeru (1924–) 112.B, D, P 274. B, G, I 320.I, r 321.F, G, r 323.M 325.G, H 345.A
 Mizukami, Masumi (1951–) 16.r
 Mizumoto, Hisao (1929–) 367.I
 Mizutani, Akira (1946–) 304.r
 Mizutani, Tadayoshi (1945–) 154.G
 Möbius, August Ferdinand (1790–1868) 66.C 74.E 76.A 267 295.C 410.B
 Moedomo, S. 443.H
 Mohr, Georg (1640–97) 179.B
 Moise, Edwin Evariste (1918–) 65.C 70.C 79.D 93.r 139.r 410.r
 Moiseiwitsch, Benjamin Lawrence (1927–) 441.r
 Moïshezon, Boris Gershevich (1937–) 16.E, U, W 72.r
 Molchanov, Stanislav Alekseevich 115.D 340.r
 Moldestad, Johan (1946–) 356.F, r

Moler, Cleve B. (1939–) 298.r 302.r
 Monge, Gaspard (1746–1818) 107.B 109 158 181
 255.E 266 267 278.A 324.F
 Monin, Andrei Sergeevich 433.r
 Montel, Paul Antoine Aristide (1876–1975) 272.F
 424.O 435.E, r
 Montgomery, Deane (1909–) 196 249.V, r 423.N
 431.r
 Montgomery, Hugh L. (1944–) 14.L 123.E, r
 Montucla, Jean Étienne (1725–99) 187.r
 Mook, Dent T. 290.r
 Moon, Philip Burton (1907–) 130.r
 Moore, Calvin C. (1936–) 122.F
 Moore, Eliakim Hastings (1862–1932) 87.H, K, r
 Moore, John Colemar (1923–) 147.r 200.r 203.r
 Moore, John Douglas 365.J
 Moore, Robert Lee (1882–1974) 65.F 273.K
 425.AA 426
 Moran, Patrick Alfred Pierce (1917–) 218.r
 Morawetz, Cathleen Synge (1923–) 112.S 345.A
 Mordell, Louis Joel (1888–1972) 118.A, E
 Morera, Giacinto (1856–1909) 198.A
 Morf, Martin (1944–) 86.r
 Morgan, Frank 275.C
 Morgenstern, Oskar (1902–77) 173.A, D 376.r
 Mori, Akira (1924–55) 352.B, C 367.E
 Mori, Hiroshi (1944–) 275.F
 Mori, Mitsuya (1927–) 59.H
 Mori, Shigefumi (1951–) 16.R, r 364.r
 Mori, Shin'ichi (1913–) 207.C, r
 Mori, Shinziro (1893–1979) 284.G
 Moriguti, Sigeiti (1916–) 299.B 389.r NTR
 Morimoto, Akihiko (1927–) 110.E 126.J 344.C
 Morimoto, Haruki (1930–) 399.r
 Morimoto, Hiroko (1941–) 224.F
 Morimoto, Mituo (1942–) 125.BB, DD 162
 Morimune, Kimio (1946–) 128.C
 Morishima, Taro (1903–) 145.*
 Morita, Kiiti (1915–) 8 117.A, C, E, r 273.K 425.S,
 X–Z, CC
 Morita, Masato (1927–) 353.r
 Morita, Reiko (1934–) 353.r
 Morita, Shigeyuki (1946–) 154.G
 Morita, Yasuo (1945–) 450.U
 Moriya, Mikao (1906–82) 59.G, H
 Morlet, Claude 147.Q
 Morley, Edward Williams (1838–1923) 359.A
 Morley, Michael 276.F, r
 Morrey, Charles Bradfield, Jr. (1907–84) 46.r 78.r
 112.D 125.A 194.F, r 195 246.C 275.A, C, r 323.r
 334.D 350.r 352.B
 Morris, Peter D. 443.H
 Morrow, James 72.K
 Morse, Harold Marston (1892–1977) 109 114.A,
 F 126.J 178.A 275.B 279.A–F 286.N, Q, r 418.F
 Morse, Philip McCord (1903–) 25.r 133.r 227.r
 Morton, Keith W. (1930–) 304.r
 Moschovakis, Yiannis Nicholas (1938–) 22.D, F,
 H, r 33.r 356.G, r
 Moser, Jürgen (Kurt) (1928–) 21.P 55.r 126.A, L, r
 136.r 286.J, r 323.L 344.B 420.C, G
 Moser, William O. J. (1927–) 92.r 122.r 151.r 161.r
 Mosher, Robert E. 64.r 70.r
 Mosteller, (Charles) Frederick (1916–) 346.C, G
 Mostow, George Daniel (1923–) 13.r 32.r 122.F, G
 249.r
 Mostowski, Andrzej (1913–75) 33.D, r 356.C, H
 Motohashi, Yoichi (1944–) 123.E
 Motoo, Minoru (1927–) 44.E 115.C, D, r 261.r
 Moulin, M. 375.F

Moulton, Forest Ray (1872–1952) 55.r 303.E
 Moussu, Robert (1941–) 154.H
 Moyal, José E. 44.r
 Muchnik, Al'bert Aramovich (1934–) 356.D
 Mugibayashi, Nobumichi (1926–) 125.BB
 Muhly, Paul Scott (1944–) 164.H
 Muirhead, Robb John (1946–) 280.r
 Mukherjee, Bishwa Nath 346.r
 Müller, Claus Ernst Friedrich (1920–) 323.J 393.r
 Muller, David Eugene (1924–) 301.C
 Müller, Werner (1949–) 391.M
 Müller-Breslau, Heinrich Franz Bernhard (1851–
 1925) 19.r
 Mullikin, Thomas Wilson (1928–) 44 r
 Mullis, Clifford T. (1943–) 86.D
 Mumford, David Bryant (1937–) 3.A, N, r 9.J, r
 12.B 15.E, F, r 16.R, W, Y, Z, r 32.r 72.G 226.r
 418.D
 Munkres, James Raymond (1930–) 70.r 105.r
 114.C, r
 Müntz, C. H. 336.A
 Münzner, Hans-Friedrich 365.I
 Murakami, Shingo (1927–) 32.r 122.F 384.r App. A,
 Table 5.I
 Muramatu, Toshinobu (1933–) 168.B 224.E
 251.O
 Murasugi, Kunio (1929–) 235.A, E, r
 Murata, Hiroshi (1945–) 75.r
 Murray, Francis Joseph (1911–) 136.F 308.F
 Murre, Jacob P. (1929–) 16.W
 Murthy, M. Pavaman 237.r
 Muskhelishvili, Nikolai Ivanovich (1891–1976)
 217.r 222.r 253.r
 Muto, Yosio (1912–) 364.F, r
 Mutou, Hideo (1953–) 391.E
 Mycielski, Jan (1932–) 22.H 33.F, r
 Myers, Sumner Byron (1910–55) 152.C 178.B
 Myrberg, Pekka Juhana (1892–1976) 367.E

N

Nachbin, Leopoldo (1922–) 21.r 37.M 425.BB
 Nagaev, Sergei Viktorovich (1932–) 250.r
 Nagamati, Sigeaki (1945–) 125.BB
 Nagami, Keiô (1925–) 117.A, C, r 273.K, r 425.Y,
 AA, CC, r
 Nagano, Tadashi (1930–) 191.r 275.F 279.C 344.C
 364.F 365.F, K
 Naganuma, Hidehisa (1941–) 450.L
 Nagao, Hiroshi (1925–) 151.H 200.L 362.I
 Nagasawa, Masao (1933–) 44.r
 Nagase, Michihiro (1944–) 251.O
 Nagata, Jun-iti (1925–) 117.C 273.K, r 425.r
 Nagata, Masayoshi (1927–) 8 12.B 13.I 15.r 16.D,
 T, V, AA, r 67.I, r 196 226.G, r 277.r 284.E, G
 369.r 370.r
 Nagell, Trygve (1895–) 118.D
 Nagumo, Mitio (1905–) 162 286.Z, r 316.E, r
 323.D
 Naïm, Linda 120.E 207.C
 Naïmark (Neumark), Mark Aronovich (1909–78)
 36.G, r 107.r 112.r 192.r 252.r 258.r 308.D 315.r
 437.W, EE
 Naitô, Hiroo (1950–) 365.F, N
 Nakada, Hitoshi (1951–) 136.C
 Nakagami, Yoshiomi (1940–) 308.r
 Nakagawa, Hisao (1933–) 365.L
 Nakai, Mitsuru (1933–) 169.r 207.C, D
 Nakai, Yoshikazu (1920–) 15.C, F 16.E, r
 Nakajima, Kazufumi (1948–) 384.r

- Nakamura, Iku (1947–) 72.K
 Nakamura, Kenjiro (1947–79) 310.r
 Nakamura, Michiko (1937–) 424.X
 Nakamura, Tokushi (1930–) 70.F, r
 Nakane, Genkei (1662–1733) 230
 Nakanishi, Noboru (1932–) 146.A C
 Nakanishi, Shizu (1924–) 100.A, r
 Nakano, Hidegorô (1909–74) 162 310.A 436.r
 Nakano, Shigeo (1923–) 21.L 72.H 147.O 232.r
 Nakano, Tadao (1926–) 132.A
 Nakao, Shintaro (1946–) 340.r
 Nakaoka, Minoru (1925–) 70.F, r 153.B 202.P 305.A
 Nakayama, Mikio (1947–) 173.E
 Nakayama, Tadaki (1912–64) 6.E 8 29.H, I 59.H 67.D 172.A 200.K–N 243.G
 Nakazi, Takahiko (1944–) 164.G
 Namba, Kanji (1939–) 33.r
 Namba, Makoto (1943–) 9.E 72.r
 Nambu, Yoichiro (1921–) 132.C
 Namikawa, Yukihiko (1945–) 16.Z
 Namioaka, Isaac (1928–) 310.r 424.r
 Napier, John (1550–1617) 131.D 265 432.C
 App A, Tables 2.II, III
 Narasimhan, Mudumbai S. (1932–) 112.D
 Narasimhan, Raghavan (1937–) 23.r 367.G
 Naruki, Isao (1944–) 21.P, Q 344.D
 Nash, John Forbes, Jr. (1928–) 173.A, C, r 204.F 286.J 323.L 327.G, r 365.B
 Navier, Louis Marie Henri (1785–1836) 204.B, C, F 205.C
 Nayfeh, Ali Hasan (1933–) 25.r 290.r
 Necăs, Jindřich (1929–) 304.r
 Nedoma, Jiří 213.F
 Ne'eman, Yuval (1925–) 132.D, r
 Nehari, Zeev (1915–78) 77.r 367.G 438.B
 Nelson, Joseph Edward (1932–) 115.D 150.F 176.F 293.E, r 341.r 437.S
 Nemytskiĭ, Viktor Vladimirovich (1900–) 126.E, r 394.r
 Nernst, Hermann Walter (1864–1941) 419.A
 Néron, André (1922–) 3. M, N, r 15.D 16.P
 Nersesyan, A. A. 164.J
 Nesbitt, Cecil James (1912–) 29.r 362.r 368.r
 Netto, Eugen (1846–1919) 177.r 330.r
 Neubüser, Joachim E. F. G. (1932–) 92.F
 Neugebauer, Otto (Eduard) (1899–) 24.r
 Neuhoﬀ, David L. 213.E, F
 Neukirch, Jürgen (1937–) 450.r
 Neumann, Bernhard Hermann (1909–) 161.C 190.M
 Neumann, Carl (Karl) Gottfried (1832–1925) 39.B 120.A 188.H 193.F 217.D 323.F App. A, Tables 19.III, IV
 Neustadt, L. W. 292.r
 Neuwirth, Lee P. (1933–) 235.r
 Nevanlinna, Frithiof (1894–1977) 272.K
 Nevanlinna, Rolf Herman (1895–1980) 21.N 43.r 109 124.B 164.G 198.r 272.B, D, E, K, r 367.E, I, r 429.B 438.B
 Neveu, Jacques (1932–) 136.C
 Neville, Charles William (1941–) 164.K
 Newcomb, Robert Wayne (1933–) 282.r
 Newcomb, Simon (1835–1909) 392.r
 Newell, Allen 385.r
 Newhauser, George L. 215.r
 Newhouse, Sheldon E. (1942–) 126.J, L, M
 Newlander, August, Jr. 72.r
 Newman, Charles Michael (1946–) 212.r
 Newman, Donald J. 328
 Newman, Maxwell Herman Alexander (1897–1984) 65.C, F 93.r 333.r
 Newton, Sir Isaac (1642–1727) 20 48.B, F, H 107.A 126.A 205.C 223.C 254.D 265 271.A–C 283 299.A 301.D 336.G 337.I 338.A 418.D
 App. A, Table 21
 Newton, Roger Gerhard (1924–) 375.G, r
 Ney, Peter E. (1930–) 44.C
 Neyman, Jerzy (1894–1981) 373.A, r 396.F 400.B, D 401.B, C, F, G, r
 Nicholson, John William (1881–1955) App. A, Tables 19.III, IV
 Nickel, Karl L. E. (1924–) 222.r 301.G
 Nickerson, Helen Kelsall (1918–) 94.r 442.r
 Nicolaenko, Basil 41.D
 Nicolaus Cusanus (1401–64) 360
 Nicolescu, Miron (1903–75) 193.r
 Nicomachus (50–150?) 187
 Nicomedes (fl. 250? B.C.) 93.H
 Niederreiter, Harald G. (1944–) 182.r 354.r
 Nielsen, Niels (1865–1931) 167.r 174.r
 Niino, Kiyoshi (1941–) 17.C
 Niino, Fumio (1923–) 310.H
 Nijenhuis, Albert (1926–) 72.B
 Nikaidô, Hukukane (1923–) 89.r
 Nikodým, Otto Martin (1878–) 270.L 323.E 380.C 443.H
 Nikolai, Paul John (1931–) 151.H
 Nikol'skiĭ, Nikolai Kapitonovich (1940–) 251.r
 Nikol'skiĭ, Sergei Mikhailovich (1905–) 168.B
 Nilson, Edwin Norman (1917–) 223.r
 Ninomiya, Nobuyuki (1924–) 338.C, D, J–M
 Nirenberg, Louis (1925–) 72.r 112.D, F, H 164.K 168.B 262.B 274.I 286.Z, r 304.F 320.I 323.H, r 345.A, B 365.J
 Nishi, Mieo (1924–) 12.B
 Nishida, Goro (1943–) 202.U
 Nishida, Takaaki (1942–) 41.D, E, r 204.F 263.D 286.Z
 Nishijima, Kazuhiko (1926–) 132.A 150.r
 Nishikawa, Seiki (1948–) 195.r
 Nishimori, Toshiyuki (1947–) 154.G, H
 Nishimura, Toshio (1926–) 156.E
 Nishina, Yoshio (1890–1951) 351.G
 Nishino, Toshio (1932–) 21.L, Q
 Nishiura, Yasumasa (1950–) 263.r
 Nisio, Makiko (1931–) 45.r 260.J 405.r
 Nitecki, Zbigniew 126.r
 Nitsche, Johannes C. C. (1925–) 275.C, r 334.F, r
 Niven, Ivan (1915–) 118.r
 Nöbeling, Georg 117.D 246.r
 Noether, Amalie Emmy (1882–1935) 8 12.B 16.D, X 27.D, E 29.F 150.B 277.I 284.A, D, G 368.F 450.L
 Noether, Max (1844–1921) 9.E, F, r 11.B, r 12.B 15.B, D 16.I 366.C
 Nogi, Tatsuo (1941–) 304.F
 Nohl, Craig R. 80.r
 Nomizu, Katsumi (1924–) 105.r 199.r 365.H, N, r 412.r 413.r 417.r
 Nordin, Clas 323.M
 Norguet, François (1932–) 21.I
 Norkin, Sim Borisovich (1918–) 163.r
 Nörlund (Nørlund), Niels Erik (1885–1981) 104.B, r 379.J, Q
 Northcott, Douglas Geoffrey 67.I 200.r 277.r 284.r
 Norton, Richard E. (1928–) 146.C
 Norton, Simon Phillips (1952–) 151.I
 Noshiro, Kiyoshi (1906–76) 62.B, C, E
 Nourein, Abdel Wahab M. 301.F

Novikov, Pëtr Sergeevich

- Novikov, Pëtr Sergeevich (1901–75) 22.D, F, H 97.*, r 161.B
 Novikov, Sergeĭ Petrovich (1937–) 56.F 114.J 126.N 154.B, D 387.C, r
 Nozaki, Akihiro (1936–) 31.r 75.D, r
 Nusselt, Ernst Kraft Wilhelm (1882–1957) 116.B
 Nyikos, Peter J. 273.K
 Nyquist, Harry (1889–1976) 86.A 402.K

O

- Obata, Morio (1926–) 364.F, G, r 391.D
 Oberhettinger, Fritz (1911–) 220.r 389.r App.A, Table 20.IV
 Ochan, Yuriĭ Semënovich (1913–) 100.r
 Ochiai, Takushiro (1943–) 21.N, O 191.r 384.r
 Oda, Tadao (1940–) 16.Z, r 72.K
 Oda, Takayuki (1950–) 450.S
 Odqvist, Folke K. G. 188.r
 Oenopides (c. 5th century B.C.) 187
 Ogasawara, Tōjiro (1910–78) 162
 Ogg, Andrew P. (1934–) 32.r
 Ogiue, Koichi (1941–) 110.E 365.L, r
 Ogus, Arthur E. (1946–) 450.r
 Oğuztöreli, Mehmet Namik (1923–) 163.r 222.r
 Ôharu, Shinnosuke (1941–) 162 286.X
 Ohm, Georg Simon (1787–1854) 130.B 259
 Ohnishi, Masao (1923–) 411.J
 Ohtsuka, Makoto (1922–) 62.C, r 77 120.A 143.B 193.r 207.C, r 246.A 338.C, D, M, r
 Ohya, Yujiro (1935–) 325.H, I 345.A
 Oikawa, Kōtaro (1928–) 48.r 77.E, r 367.r
 Ojanguren, Manuel (1940–) 29.r
 Ojima, Izumi (1949–) 150.G
 Oka, Kiyoshi (1901–78) 20 21.E, H, I, K, Q 23.D 72.E 147.O 383.J
 Oka, Yukimasa (1942–) 136.F
 Okabe, Yasunori (1943–) 176.F
 Okada, Norio (1947–) 173.E
 Okada, Yoshitomo (1892–1957) 379.P
 Okamoto, Kazuo (1948) 253.E
 Okamoto, Kiyosato (1935–) 437.AA
 Okamoto, Masashi (1923–) 280.r
 Okamoto, Shūichi (1951–) 306.A
 Okamura, Hiroshi (1905–48) 94.r 216.B 246.F 316.D, r
 Okano, Hatsuo (1932–) 100.r
 Okonek, C. 16.r
 Okubo, Kenjiro (1934–) 253.C
 Okugawa, Kōtaro (1913–) 113
 Okumura, Masafumi (1936–) 110.E
 Okuyama, Akihiro (1933–) 273.K 425.Y
 Oleinik, Ol'ga Arsen'evna (1925–) 112.D 323.r 325.H 327.r
 Olive, David Ian (1937–) 146.r 386.C, r
 Olivieri, E. 402.G
 Olkin, Ingram (1924–) 280.r
 Olmsted, John M. H. (1911–) 106.r 216.r
 Olum, Paul (1918–) 91.r 305.A, r
 Olver, Frank W. J. 30.r
 O'Meara, Onorato Timothy (1928–) 348.r
 Omnes, Roland Lucian (1931–) 150.r
 Omori, Hideki (1938–) 178.E 183 286.r
 Omura, Jim K. (1940–) 213.E
 O'Nan, Michael E. 151.H, I
 O'Neil, Richard 224.E
 O'Neill, Barrett (1924–) 111.r 178.r 365.B, G
 O'Neill, Bernard V., Jr. 164.F
 Ono, Harumi (1932–) 301.F
 Ono, Katuzi (1909–) 156.E, r
 Ono, Takashi (1928–) 13.P
 Onsager, Lars (1903–76) 340.B 402.K
 Oono, Yosiro (1920–) 282.r
 Oort, Frans (1935–) 9.J
 Oppenheim, Alexander 220.B 242.A
 Ord, J. Keith 374.r
 Ordeshook, Peter C. 173.r
 Ore, Oystein (1899–1968) 157.r 190.L
 Oresme, Nicole (c. 1320(30)–82) 372
 Orey, Steven (1928–) 260.J
 Orihara, Masae (1915–) 310.r
 Orlicz, Władysław (1903–) 168.B 443.D
 Ornstein, Donald S. (1934–) 5.G 136.B, C, E–G, r 162 213.E, F
 Ornstein, Leonard Salomon (1880–1941) 45.I
 Ortega, James McDonogh (1932–) 301.r
 Orzech, Morris (1942–) 29.r
 Oseen, William (1879–) 205.C
 Oseledets, Valerĭi Iustinovich (1940–) 136.B
 Osgood, William Fogg (1864–1943) 3.r 11.r 21.H, r 107.A
 Oshima, Toshio (1948) 274.r 437.CC, r
 Osikawa, Motosige (1939–) 136.F
 Osima, Masaru (1912–) 109.r 275.A–E, r 334.F, r 365.H 391.D
 Osterwalder, Konrad (1942–) 150.F
 Ostrogradskii, Mikhail Vasil'evich (1801–62) 94.F
 Ostrowski, Alexander (1893–) 14.F 58.F 88.A r 106.r 121.C 205.r 216.r 272.F 301.r 339.E 388.B 439.L
 Oswatitsch, Klaus (1910–) 207.C
 Ôtsuki, Nobukazu (1942–) 136.r
 Otsuki, Tominosuke (1917–) 275.A, F 365.B
 Ōuchi, Sunao (1945–) 378.F
 Ovsyannikov, Lev Vasil'evich (1919–) 286.Z
 Owen, Donald B. STR
 Oxtoby, John Corning (1910–) 136.H
 Ozawa, Mitsuru (1923–) 17.C 367.E 438.C
 Ozeki, Hideki (1931–) 365.I, r

P

- Paatero, Veikko (1903–) 198.r
 Pacioli, Luca (c. 1445–c. 1514) 360
 Padé, Henri Eugène (1863–1953) 142.E
 Page, Annie 123.D
 Paige, Christopher Conway (1939–) 241.C
 Painlevé, Paul (1863–1933) 198.G 288.A–D, r 420.C
 Pál, J. 89.C
 Palais, Richard Sheldon (1931–) 80.r 105.Z, r 183.*, r 191.G 279.A, E 286.Q, r 431.r
 Palamodov, Viktor Pavlovich (1938–) 112.R
 Paley, Raymond Edward Alan Christopher (1907–33) 45.r 58.r 125.O, BB 159.G 160.E, G, r 168.B 192.F, r 272.K 295.E 317.B
 Palis, Jacob, Jr. 126.C, J, M, r
 Pan, Viktor Yakovlevich (1939–) 71.D
 Panofsky, Wolfgang Kurt German (1919–) 130.r
 Papakyriakopoulos, Christos Dimitricu (1914) 65.E 235.A
 Papanicolaou, George C. (1943–) 115.D
 Papert, Seymour 385.r
 Pappus (of Alexandria) (fl. 320) 78.K 187 343.D
 Parasyuk (Parasiuk), Ostap Stepanovich (1921–) 146.A
 Paris, Jeffrey B. (1944–) 33.r
 Parker, Ernest Tilden (1926–) 151.H 241.B
 Parreau, Michel (1923–) 164.K 193.G 207.C 367.E
 Parry, William (1934–) 136.C, r

Parseval, Marc Antoine (1755–1836) 18.B 159.A 160.C, H 192.M 197.C 220.B, C, E
 Parshin, Aleksei Nikolaevich 118.E
 Parthasarathy, Kalyanapuram Rangachari 213.F 341.r 374.r
 Parzen, Emanuel (1929–) 421.D
 Pascal, Blaise (1623–62) 20 75.A 78.K 155.E 181 265 329 330 342.A 343.E
 Pascal, Ernesto (1865–?) 15.r
 Pascal, Étienne (1588–1651) 329
 Pasch, Moritz (1843–1930) 155.B
 Passman, Donald S. (1940–) 151.r
 Pasta, J. 287.r
 Pasternack, Joel 154.H
 Pastur, Leonid Andreevich 340.r
 Patil, Ganapati P. (1934–) 374.r
 Patodi, Vijay Kumar (1945–76) 237.r 366.r 391.C, K, L, N, r
 Pauli, Wolfgang Ernst (1900–58) 150.A 258.A, D 351.G, H, r 386.B
 Pawley, G. Stuart (1937–) 92.F
 Pazy, Amnon (1936–) 162 286.X 378.F
 Peano, Giuseppe (1858–1932) 93.D, J 106.H 107.A 117.A 156.B 246.F, G 267 294.A, B, r 316.E 411.A
 Pearcy, Carl Mark (1935–) 251.r
 Pearl, Raymond (1879–1940) 263.A
 Pears, Alan (1938–) 117.r
 Pearson, D. B. 331.E 375.B
 Pearson, Egon Sharpe (1895–1980) 400.B 401.B, F, r STR
 Pearson, Karl (1857–1936) 40.B 174.r 374.r 397.D 401.E 403.C STR
 Pécelet, Jean Claude Eugene 116.B
 Pedersen, Gert Kjærgård (1940–) 36.K, r 212.C 308.r
 Pederson, Roger N. (1930–) 438.C
 Pedoe, Daniel (1910–) 343.r
 Peetre, Jaak (1935–) 112.E, K 125.F 224.A, C, E, F, r
 Peierls, Rudolf Ernst (1907–) 212.B
 Peirce, Benjamin (1809–80) 231.B 368.F
 Peirce, Benjamin Osgood (1854–1914) App. A, Table 9.r
 Peirce, Charles Sanders (1839–1914) 156.B 411.A
 Peixoto, Mauricio Matos (1921–) 126.A, H, I, M
 Pełczyński, Aleksander 37.L, r 68.M 443.D
 Pell, John (1611–85) 118.A
 Pepis, Józef (c. 1922–c. 1942) 97.*
 Peressini, Anthony L. (1934–) 310.r
 Perko, Kenneth A., Jr. (1943–) 235.E
 Perlman, Michael David (1942–) 280.r
 Perron, Oskar (1880–1975) 83.r 100.A, F 107.A 120.C 121.C 123.B 254.D 269.N 280.F 289.D 294.r 310.H 314.A 316.E, r 379.L 394.r
 Pesin, Ya. B. 136.G
 Peter, F. 69.B 249.U, r 437.EE
 Péter, Rózsa (1905–77) 356.B, r
 Petermann, Abdeas (1922–) 361.r
 Peterson, Elmor Lee (1938–) 264.r
 Peterson, William Wesley (1924–) 63.r
 Petersson, Hans (1902–84) 32.B–D 328.r 450.Q
 Petkov, Vesselin Mihailev (1942–) 325.H
 Petrenko, Viktor Pavlovich (1936–) 272.K, r
 Petrie, Ted E. (1939–) 431.D
 Petrov, Valentin Vladimirovich (1931–) 250.r
 Petrovskii, Ivan Georgievich (1901–73) 107.r 112.D 196 320.r 321.E, r 323.I 324.r 325.F, G, J, r 327.H
 Pettis, Billy James (1913–79) 68.M 443.A, B, D, F–H

Petty, Sir William (1623–87) 40.A 401.E
 Petvyashvili, V. I. 387.F
 Peyret, Roger 304.r
 Pfaff, Johann Friedrich (1765–1825) 103.G 105.Q 107.B 428.A, B
 Pfanzagl, Johann (1928–) 399.M, O 400.r
 Pflüger, Albert (1907–) 143.A 272.K 367.E, r
 Pham, Frederic 146.A, C 386.C 418.r
 Phelps, Robert Ralph (1926–) 443.H
 Phillips, Aris (1915–1985) 154.F 279.C
 Phillips, E. 136.r
 Phillips, Melba N. (1907–) 130.r
 Phillips, Ralph Saul (1913–) 68.M 112.P, S 162 251.r 286.r 375.H 378.B, F, r 443.A, H
 Phragmén, Lars Edvard (1863–1937) 43.C
 Picard, Charles Émile (1856–1941) 3.A 11.r 12.B, r 15.B, D, r 16.P, U 20.*, r 107.A 113 124.B 206.D 232.C 237.J 253.r 272.E 288.C 289.r 316.D 323.D 367.D 388.r 418.F, r 429.B
 Pick, Georg 21.O
 Pielou, Evelyn C. 263.r
 Pietsch, Albrecht (1934–) 68.N, r 424.r
 Pillai, K. C. Sreedharan 280.B
 Pincherle, Salvatore (1853–1936) 217.F 240.B
 Pinchuk, S. I. 344.D, F
 Pinsker, Mark Shlemovich (1925–) 136.E 213.E
 Piper, Christopher J. 215.E
 Pitcher, Tom Stephen (1926–) 396.r
 Pitman, Edwin James George (1897–) 371.A, E 399.G, r 400.K
 Pitt, Harry Raymond (1914–) 160.G 192.r 339.r 379.r
 Pitt, Loren D. (1939–) 176.F
 Pitts, Jon T. 275.G
 Plancherel, Michel (1885–1967) 192.M 218.G 437.L
 Planck, Max Karl Ernst Ludwig (1858–1947) 115.A 351.A 402.I 419.r
 Plante, Joseph F. (1946–) 154.H
 Plateau, Joseph Antoine Ferdinand (1801–83) 109 275.C 334.A, B
 Platek, Richard A. 356.G
 Plato (427–347 B.C.) 187 357.B
 Platonov, Vladimir Petrovich (1939–) 13.Q
 Pleijel, Åke Vilhelm Carl (1913–) 391.B, C, r
 Plemelj, Josip 253.r
 Pliš, Andrzej (1929–) 321.F 323.J
 Pliss, Viktor Aleksandrovich (1932–) 126.J
 Plotkin, Morris 63.B
 Plücker, Julius (1801–68) 9.B 12.B 90.B 137 267
 Pochhammer, Leo (1841–1920) 206.C
 Pogorelov, Aleksei Vasil'evich (1919–) 365.J
 Pohlmann, Henry 450.S
 Poincaré, Henri (1854–1912) 3.A, C, D 11.B 12.B 16.E 20 21.Q 25.B 30.C, r 32.B, F, r 55.r 56.B, F 65.A, C 70.A 74.G 105.A 107.A 109 114.J, K 117.A 120.A, D 122.C, r 126.A, C, E, G, I, L, r 136.A, C 153.B 156.C 170 198.J 201.A, B, F, O 218.C, H 219.A, r 248.J 253.D 254.D 258.A 267 279.A 285.A, D 286.W 288.B 289.C 314.A 335 344.A 383.E 420.A, C 425.G 426 450.Q
 Poinot, Louis (1777–1859) 271.E
 Poisson, Siméon Denis (1781–1840) 5.D, F 82.B 105.M 126.E 159.C 168.B 192.C, L 193.G 198.B 260.H 266 271.F, G 323.A 324.C, D 325.D 338.A 341.D 391.J 397.F 407.D App. A, Tables 15.VI, 19.III
 Polit, Stephen H. 136.E
 Polkinghorne, John Charlton (1930–) 146.r 386.C, r

Pollaczek, Félix (1892–1981) 145 307.C
 Pollaczek-Geiringer, H. 298.r
 Polonsky, Ivan P. 223.r 299.r
 Pólya, George (1887–1985) 20.r 48.D, r 66.E 88.r
 121.C 211.r 228.B, r 272.K 339.D 374.J 429.B
 Polyakov, A. M. 80.r
 Pomeranchuk, Isaak Yakovlevich (1913–66) 386.B
 Pommerenke, Christian (1933–) 48.r 77.F 169.F
 438.r
 Poncelet, Jean-Victor (1788–1867) 179.B 181
 266 267
 Pong, D. H. 345.A
 Ponstein, J. 292.D
 Pontryagin, Lev Semënovich (1908–) 2.G 56.D,
 F, H 64.A, B 86.A, F 107.r 108.r 114.H 126.A, I, r
 192.K 201.A, r 202.B, U 203.D 249.r 305.A 318.r
 422.C, E, r 423.r
 Ponzano, Giorgio Enrico (1939–) 146.A
 Poor, Walter Andrew (1943–) 178.r
 Popov, M. V. 291.E
 Popov, Viktor Nikolaevich (1937–) 132.C 150.G
 Popp, Herbert (1936–) 16.W
 Port, Sidney Charles (1935–) 5.G
 Porter, Alfred William (1863–1939) 116.r
 Post, Emil Leon (1897–1954) 31.B 75.D 97.r 161.B
 240.D 356.A, D, H, r
 Postnikov, Aleksei Georgievich (1921–) 295.E
 328.*, r
 Postnikov, Mikhail Mikhailovich (1927–) 70.G
 148.D 172.r 305.A
 Poston, Tim 51.r
 Povzner, Aleksandr Yakovlevich (1915–) 375.A
 Powell, H. B. 151.r
 Powell, M. J. D. (1936–) 142.r
 Powers, Robert T. (1941–) 36.K 212.B 308.I, r
 Poynting, John Henry (1852–1914) 130.A
 Prabhu, Narahari Umanath (1924–) 260.J
 Prachar, Karl (1925–) 123.D, r 450.r
 Prandtl, Ludwig (1875–1953) 116.B 205.B–D 222.C
 Prasad, Gopal (1945–) 122.G
 Preissmann, Alexandre 178.B
 Presburger, M. 156.E, r
 Preston, Gordon Bamford (1925–) 190.r
 Price, Griffith Baley (1905–) 443.A
 Price, J. 423.r
 Priestley, Maurice Bertram (1933–) 421.r
 Prigogine, Ilya (1917–) 95
 Prikry, Karel L. (1944–) 33.F, r
 Pringsheim, Alfred (1850–1941) 58.E 83.E
 Proclus (410(411)–485) 187
 Prokhorov, Yuri Vasil'evich (1929–) 115.D
 250.E, r 341.F, r 374.r
 Protter, Murray H. (1918–) 78.r 106.r 216.r 323.r
 327.r 350.r
 Prüfer, Heinz (1896–1934) 2.D 200.K
 Prugovečki, Eduard (1937–) 375.r
 Przymusiński, Teodor C. 117.E
 Przytycki, Feliks 126.K
 Pták, Vlastimil (1925–) 424.X
 Ptolemy (Claudius Ptolemaeus) (c. 85–c. 165) 187
 432.C
 Pugh, Charles C. 126.J–L, r
 Puiseux, Victor Alexandre (1820–83) 339.A
 Pukanszky, Lajos 437.K, U
 Puppe, Dieter (1930–) 200.r 202.G
 Puri, Madan Lai (1929–) 280.r 371.r
 Pustyl'nik, Evgenii Izievich (1938–) 251.r
 Pusz, Wiesław 402.G
 Putnam, Calvin Richard (1924–) 251.K
 Putnam, Hilary Whitehall (1926–) 81.D, r 97.*, r

Pyatetskii-Shapiro, Il'ya Iosifovich (1929–) 32.H
 122.G 125.r 159.J 384.A, C, r 437.r 450.Q, S
 Pythagoras (572–492 B.C.) 60.O 118.A 139.B, D
 145 155.C 181 187

Q

Quenouille, Maurice Henri (1924–73) 421.D
 Quillen, Daniel G. (1940–) 12.r 16.Y 191.r 200.K
 237.A, I, J 369.F
 Quinn, Barbara Keyfitz 286.X
 Quinn, Frank S. (1946–) 114.K

R

Raabe, Joseph Ludwig App. A, Table 10.II
 Raanan, Joseph 173.E
 Rabie, M. 173.r
 Rabinowitz, Paul H. (1939–) 286.T, W, r
 Rabinowitz, Phillip (1926–) 223.r 299.r 301.r
 Racah, Giulio (1909–65) 353.A, B, r
 Rademacher, Hans Adolph (1892–1969) 4.A, C, D
 297.r 317.B, C 328.*, r 357.r
 Radjavi, Heydar (1935–) 251.r
 Radkevich, F. V. 112.D 323.r
 Radó, Tibor (1895–1965) 65.C 77.B 109 164.I
 193.r 246.r 275.A, C, D, r 323.E, I 334.C, r 367.A,
 F 410.B
 Radon, Johann (1887–1956) 94.C 125.CC 218.F
 270.I, L 380.C 443.H
 Rådström, Hans Vilhem (1919–70) 443.I
 Raghavarao, Damaraju (1938–) 102.r
 Raghunathan, Madabusi Santanam (1941–)
 122.G, r
 Raiffa, Howard 173.C 398.r
 Raikov, Dmitrii Abramovich (1905–) 192.G 256.r
 341.E 424.X 437.EE
 Rainville, Earl D. 389.r
 Rajchman, A. 159.J
 Rall, Louis B. (1930–) 138.r 301.r
 Ralston, Anthony (1930–) 142.r 223.r 303.D, r
 Ralston, James V. 345.A
 Ramachandra, Kanakanahalli (1933–) 123.E
 Ramamoorthi, R. V. 396.r
 Ramanathan, Kollagunta Gopalaiyer (1921–)
 118.D 450.K
 Ramanujam, Chidambaram Padmanabham (1938–
 74) 232.D, r
 Ramanujan, Srinivasa (1887–1920) 4.D 32.C, D
 295.D, E 328.*, r
 Ramis, Jean-Pierre (1943–) 68.F
 Ramsey, Frank Plumpton (1903–30) 156.B
 Ran, Ziv (1957–) 450.S
 Randles, Ronald Herman (1942–) 371.r 374.r
 Range, R. Michael (1944–) 164.K
 Rankin, Robert Alexander (1915–) 123.C
 Rankine, William John Macquorn (1820–72)
 204.G 205.B
 Rao, Calyampudi Radhakrishna (Radhakrishna
 Rao, Calyampudi) (1920–) 280.r 374.H 399.C,
 D, O, r 401.r
 Rao, Ranga R. (1935–) 374.r
 Raphson, Joseph (c. 1648–c. 1715) 301.D
 Rapoport, Michael 16.r
 Rasmussen, O. L. 301.r
 Rathbone, C. R. 332.r
 Ratner, Marina E. 126.J 136.F
 Rauch, A. 17.D
 Rauch, Harry Ernest (1925–79) 134.r 178.A, C
 Ray, Daniel Burrill (1928–79) 5.r 115.A

- Ray-Chaudhuri, Dwijendra K. 96.r
 Rayleigh, Lord (Strutt, John William) (1842–1919)
 46.F 68.H 228.B 298.C 304.B 318.r 331.A, D 446.r
 Raynaud, Michel (1938–) 3.N, r
 Razumikhin, B. S. 163.G, I
 Rebbi, Claudio 80.r
 Reckhow, Robert A. 71.r
 Ree, Rim Hak 151.I, J App. B, Table 5.III
 Reeb, Georges (1920–) 90.r 154.A, B, D 279.D
 Reed, George Michael (1945–) 273.K
 Reed, L. J. 263.A
 Reed, Michael (1942–) 331.r 375.r 390.r
 Reed, Myril Baird (1902–) 282.r
 Reeh, Helmut Rudolf (1932–) 150.E
 Rees, David (1918–) 67.I 284.A
 Regge, Tullio (1931–) 132.C 146.A, C 375.r 386.C
 Regiomontanus (Johann Müller) (1436–76) 360
 432.C
 Reich, Edgar (1927–) 352.C
 Reid, Constance 196.r
 Reid, John Ker (1938–) 302.r
 Reid, Miles A. (1948–) 16.r
 Reidemeister, Kurt Werner Friedrich (1893–1971)
 91.r 155.r 235.A, r
 Reif, Frederick (1927–) 402.r
 Reifenberg, E. R. 275.A, G 334.F
 Reilly, Robert C. 365.H
 Reiner, Irving (1924–) 29.r 92.r 151.r 277.r 362.r
 Reinhardt, Hans 59.F
 Reinhardt, Karl 21.B, Q
 Reinsch, C. (1934–) 298.r 300.r
 Rellich, Franz (1906–55) 68.C 188.D 323.G 331.A,
 B 351.C
 Remak, Robert (1888–?) 190.L 277.I
 Remes, E. 142.B
 Remmert, Reinhold (1930–) 20 21.M, Q, r 23.B–E,
 r 199.r
 Rémondos, Geörgios (1878–1928) 17.A, C, r
 Rengel, Ewald 77.E
 Rényi, Alfréd (1921–70) 4.C 123.E
 Rescboom, J. H. 376.r
 Resnikoff, George Joseph (1915–) STR
 Reuleaux, Franz (1829–1905) 89.E 111.E
 Revuz, Daniel Robert (1936–) 260.J 261.E
 Reynolds, Osborne (1842–1912) 116.B 205.C 259
 Rhaeticus, Georg Joachim (1514–74) 432.C
 Rheinboldt, Werner Carl (1927–) 301.r
 Rhodes, John L. (1937–) 31.r
 Ribenboim, Paulo (1928–) 145.r
 Ribet, Kenneth A. 14.L 450.J, r
 Riccati, Jacopo Francesco (1676–1754) 86.E 107.A
 405.G App. A, Table 14.I
 Ricci, Curbastro Gregorio (1853–1925) 109 364.D
 365.C 417.B, F App. A, Table 4.II
 Ricci, Matteo (1552–1610) 57.C
 Rice, John Richard (1934–) 299.r 336.r
 Richard, Jules Antoine (1862–1956) 319.B
 Richardson, C. H. 104.r
 Richardson, Lewis Fry (1881–1953) 302.C 304.E
 Richardson, Roger Wolcott, Jr. (1930–) 431.r
 Richert, Hans-Egon (1924–) 4.C 123.D, E, r
 Richter, Hans (1912–78) 443.A
 Richter, Wayne H. (1936–) 81.D, r
 Richtmyer, Robert Davis (1910–) 304.r
 Rickart, Charles Earl (1913–) 36.r 231.r 443.A
 Rickman, Seppo U. (1935–) 352.F
 Rieffel, Marc A. (1937–) 308.H 443.H
 Riemann, Georg Friedrich Bernhard (1826–66) 3.I,
 L, r 9.C, F, I 11.B–D, r 12.B 15.D 16.V 20 21.A,
 C, F 30.C 37.K 46.E 51.E 74.D 77.B 80.K 94.B
 105.A, P, W 107.A 109 110.E 123.A, B 137 152.A
 159.A 160.A 181 198.A, D, Q 199.A 216.A 217.J
 237.G 253.B, D 267 274.G 275.A 285.A 286.L
 323.E 325.D 334.C 344.A 363 364.A, B, D 365.A
 366.A–D 367.A, B, E 379.C, S 412.A–D, J 413.*
 416 426 447 450.A, B, I, Q App. A, Tables 4.II,
 14.II, 18.I
 Riemenschneider, Oswald W. (1941–) 232.r
 Riesz, Frigyes (Frédéric) (1880–1956) 43.D 68.A,
 E, r 77.B 136.B 162 164.G, I 168.B 193.S 197.A,
 F, r 251.O, r 260.D 310.A, B 317.A, B 390.r
 425.r
 Riesz, Marcel (1886–1969) 43.D 88.C 121.r
 125.A 164.G, I 224.A 338.B 379.R
 Riley, Robert Freed (1935–) 235.E
 Rim, Dock S. (1928–) 200.M
 Ringel, Gerhard (1919–) 157.E, r 186.r
 Ringrose, John Robert (1932–) 308.r
 Rinnooy Kan, Alexander H. G. (1949–) 376.r
 Rinow, Willi (1907–79) 178.A
 Riordan, John (1903–) 66.r 330.r
 Riquier, C. 428.B, r
 Rishel, Raymond W. 405.r
 Rissanen, Jorma (1932–) 86.D
 Ritt, Joseph Fels (1893–1951) 113.*, r 428.r
 Ritter, Klaus (1936–) 292.r
 Ritz, Walter (1878–1909) 46.F 303.I 304.B
 Rivière, Néstor Marcelo (1940–78) 224.E
 Roache, Patrick John (1938–) 300.r
 Robbin, Joel W. (1941–) 126.G, r 183
 Robbins, Herbert (Ellis) (1915–) 250.r 399.D
 Roberts, Joel L. (1940–) 16.I
 Roberts, John Elias (1939–) 150.E
 Roberts, John Henderson (1906–) 117.C
 Roberts, Richard A. (1935–) 86.D
 Robertson, Alex P. 424.r
 Robertson, Howard Percy (1903–61) 359.E
 Robertson, Wendy J. 424.X, r
 Robin, Gustave (1855–97) 48.B 323.F
 Robinson, Abraham (1918–74) 118.D 276.D, E, r
 293.A, D, r
 Robinson, Derek William (1935–) 36.K, r 308.r
 402.G, r
 Robinson, G. 301.r
 Robinson, Julia Bowman (1919–85) 97.*, r
 Robinson, R. Clark 77.F 126.H, J, L, r
 Robinson, Raphael Mitchel (1911–) 356.B
 Roch, Gustave (1839–66) 9.C, F 11.D 15.D 237.G
 366.A–D
 Roche, Edouard Albert (1820–83) App. A,
 Table 9.IV
 Rockafellar, R. Tyrrell (1935–) 89.r 292.D
 Rodin, Burton (1933–) 367.I, r
 Rodoskii, Kirill Andreevich (1913–) 123.E
 Rodrigues, Olinde (1794–1851) 393.B
 Roepstorff, Gert (1937–) 402.G
 Rogers, Claude Ambrose (1920–) 22.r 182.D
 443.D
 Rogers, Hartley, Jr. (1926–) 22.r 81.D, r 97.r
 356.r
 Rogers, William H. 371.r
 Roggenkamp, Klaus W. (1940–) 362.r
 Rogosinski, Werner Wolfgang (1894–1964) 159.H,
 r 242.A
 Röhrl, Helmut (1927–) 196 253.D
 Roitman, A. A. 16.R, r
 Rokhlin, Vladimir Abramovich (1919–84) 56.H
 114.H, K 136.E, H, r 213.r
 Rolfsen, Dale Preston Odin (1942–) 235.r
 Rolle, Michel (1652–1719) 106.E

Romanov, Vladimir Gabrilovich

Romanov, Vladimir Gabrilovich 218.H
 Romberg, W. 299.C
 Roquette, Peter Jaques (1927–) 118.D
 Rose, Milton Edward (1925–) 353.r
 Rosen, Judah Ben (1922–) 292.E
 Rosenberg, Alex (1926–) 29.r
 Rosenberg, Ivo G. (1939–) 75.D
 Rosenberg, Jonathan M. (1951–) 437.r
 Rosenblatt, Murray (1926–) 395.r 421.r
 Rosenbloom, Paul Charles (1920–) 255.D, E
 Rosenblueth, Arturo (1900–70) 95.r
 Rosenblum, Marvin (1926–) 331.E 421.r
 Rosenhain, Johann Georg (1816–87) 3.A
 Rosenhead, Louis (1906–84) 205.r
 Rosenlicht, Maxwell (1924–) 9.F 13.B, r
 Rosenthal, Peter (1941–) 251.r
 Roshko, Anatol (1923–) 205.r
 Ross, George G. 438.C
 Ross, Kenneth A. (1936–) 192.r
 Ross, Ronald (1857–1932) 263.A
 Rosser, John Barkley (1907–) 33.r 145 156.E, r
 185.r 356.D
 Rossetti, C. 132.r
 Rossi, Hugo 21.r 23.r 164.C, G 344.C 384.r
 Rota, Gian-Carlo (1932–) 66.r 203.r
 Roth, Klaus Friedrich (1925–) 118.D 182.G
 Roth, Leonard (1904–68) 12.r 16.r
 Rothstein, Wolfgang (1910–75) 21.M
 Rotman, Joseph J. (1934–) 2.E
 Rouché, Eugène (1832–1910) 10.E 99.D 198.F
 Rouche, Nicolas 290.r
 Rouet, A. 150.G
 Roumieu, Charles 125.A, U
 Rourke, Colin Patrik (1943–) 65.r 147.Q, r
 Roussarie, Robert 154.G, H
 Roussas, George Gregory (1933–) 399.N, r
 Roy, K. K. 396.r
 Roy, Prabir (1937–) 117.C
 Roy, Samarendra Nath 280.B
 Royden, Halsey Lawrence (1928–) 21.O 36.M
 164.K 166.r 207.C, D, r 221.r 270.r 367.E, I
 380.r 416
 Rozanov, Yurii Anatol'evich (1934–) 176.r 395.r
 Rozenfel'd (Rosenfel'd) B. I. 154.G
 Rozhdestvenskii, Boris Leonidovich (1928–) 204.r
 Rubel, Lee A. (1928–) 164.J
 Rückert, Walter 23.B
 Rudakov, Aleksei Nikolaevich 15.r
 Rudin, Mary Ellen (1924–) 425.Y
 Rudin, Walter (1921–) 20.r 36.r 84.r 87.r 106.r
 164.I, K, r 166.r 192.Q, r 198.r 216.r 221.r 270.r
 367 380.r 422.r
 Rudolph, Daniel Jay (1949–) 136.E, F
 Rudvalis, Arunas (1945–) 151.I
 Ruelle, David Pierre (1935–) 126.A, J, K, M, N, r
 136.C, G, H, r 150.D 154.H 340.B 402.G, r
 433.B, r
 Ruffini, Paolo (1765–1822) 172.A 190.Q
 Ruh, Ernst A. (1936–) 178.r
 Rund, Hanno (1925) 152.C, r
 Runge, Carl David Tolmé (1856–1927) 19.r 118.D
 223.A 301.D 303.D 416.F
 Running, Theodore Rudolph (1866–) 19.r
 Ruskai, Mary Beth (1944–) 212.B
 Russell, Bertrand Arthur William (1872–1970)
 156.A, B, r 319.B, r 411.A, K, r
 Rutman, Moisei Aronovich (1917–) 89.r 310.H
 Ryan, Patric J. 365.r
 Ryll-Nardzewski, Czesław (1926–) 22.E
 Ryser, Herbert John (1923–85) 66.r

S

Saaty, Thomas L. (1926–) 157.r 22.r 260.H 291.r
 Sabatier, Pierre Célestin (1935–) 375.r
 Saburi, Yutaka (1948–) 125.BB
 Saccheri, Girolamo (1667–1733) 285.A
 Sacker, R. S. 126.M
 Sacks, Gerald Enoch (1933–) 22.F 33.r 63.B 97.r
 276.r 356.r
 Sacks, Jerome (1931–) 195.E, r 275.D
 Sacksteder, Richard (1928–) 136.G 154.H 365.E
 Sadanaga, Ryoichi (1920–) 92.F
 Sadler, D. H. 291.F
 Sagher, Yoram 224.E
 Sah, Chih-Han (1934–) 151.r
 Sahni, Sartaj K. 71.r
 Sainouchi, Yoshikazu (1926–) 367.I
 Saint-Beuve, Marie-France 22.r
 Saint-Donat, Bernard 9.r 16.r
 Saint-Raymond, Jean 22.F
 Saito, Hiroshi (1947–) 450.G, r
 Saito, Kyoji (1944–) 418.D, r App. A, Table 5.r
 Saito, Masahiko (1931–) 122.F
 Saito, Tosiya (1920–) 126.E 289.E
 Saitō, Yoshimi (1939–) 375.C
 Sakai, Akira (1932–) 164.K
 Sakai, Fumio (1948–) 21.N
 Sakai, Makoto (1943–) 367.E, r
 Sakai, Shoichiro (1928–) 36.K, r 308.C, E, F, L
 Sakai, Takashi (1941–) 178.C 391.B 413.r
 Sakamoto, Heihachi (1914–) 374.H
 Sakamoto, Kunio (1948–) 365.N, r
 Sakamoto, Reiko (1939–) 323.M 325.K, r 327.H
 Sakane, Yusuke (1946–) 365.L
 Sakata, Shōichi (1911–70) 132.D, r
 Saks, Stanisław (1897–1942) 84.r 94.r 100.r 198.r
 221.r 246.r 270.r 380.r
 Salam, Abdus (1926–) 132.D
 Salem, Raphaël (1898–1963) 159.r 192.r
 Salmon, George (1819–1904) 78.r 350.r
 Salomaa, Arto Kustaa (1934–) 75.r
 Salzmann, Helmut Reinhard (1930–) 58.r
 Sambutsky, S. 353.r
 Samelson, Hans (1916–) 413.r 427.F
 Sampson, Joseph Harold (1925–) 183 195.E
 Samuel, Pierre (1921–) 12.B 67.r 284.G, r 370.r
 439.r
 Sanderson, Brian Joseph (1939–) 65.r 147.Q, r
 Sannami, Atsuro (1955–) 126.J
 Sanov, Ivan Nikolaevich (1919–) 161.C 403.r
 Sansone, Giovanni (1888–1979) 290.r 317.r
 Santaló, Luis Antonio (1911–) 218.C, E, H, r 228.A
 Šapiro → Shapiro
 Sapronov, Yu. I. 286.r
 Sarason, Donald Erik (1933–) 164.K, r
 Sard, Arthur (1909–80) 105.J 208.B 286.P
 Sargsyan (Sargsjan), L. S. 315.r
 Sarhan, Ahmed E. 374.r
 Saribekovich, Sargsyan Iskhan (1931–) 315.r
 Sario, Leo Reino (1916) 48.r 77.E, r 124.C, r 169.r
 207.r 367.E, G, r
 Sarton, George Alfred Léon (1884–1956) 26.r 209.r
 372.r
 Sasaki, Shigeo (1912–) 110.E 275.C 365.J
 Sasieni, Maurice W. 307.r
 Sataev, E. A. 136.F, r
 Satake, Ichiro (1927–) 13.r 16.Z 21.Q 32.F 59.H
 122.r 248.U, r 384.r 437.AA
 Sato, Atsushi (1954–) 154.H
 Sato, Fumihiko (1949–) 450.V

Sato, Ken-iti (1934–) 115.C, D 263.r
 Sato, Mikio (1928–) 20 112.D 125.A, V, W, BB, EE
 146.A, C 162.*., r 274.I 345.B 386.C 387.C 418.H
 450.A, M, Q, S, V
 Sato, Tokui (1906–83) 217.r 288.B
 Sattinger, David H. 126.M 286.r
 Savage, I. Richard (1925–) 371.A, C, r
 Savage, John E. 71.r
 Savage, Leonard Jimmie (1917–71) 342.G 399.F, r
 401.B, F
 Sawada, Ken (1953–) 126.J
 Sawashima, Ikuko (1929–) 310.H
 Saxer, Walter (1896–1974) 214.r
 Sazonov, Vyacheslav Vasil'evich (1935–) 341.J
 Scarf, Herbert Ely (1930–) 173.E 227.r
 Ščegol'kov → Shchegol'kov
 Schaaf, Manfred 258.r
 Schadé, J. P. 95.r
 Schaefer, Helmut (1925–) 217.r 310.A, H
 Schaeffer, Albert Charles (1907–) 438.B, C
 Schafheitlin, Paul (1861–) App. A, Table 19.III
 Schäfke, Friedrich Wilhelm (1922–) 268.r 389.r
 Schaible, Siegfried 264.r
 Schapira, Pierre M. (1943–) 112.D 125.Y 162
 Schark, I. J. 164.I
 Schatten, Robert (1911–1977) 68.I
 Schauder, Juliusz Pawel (1899–1943) 37.L 68.E
 153.D 286.D 323.C, D, r 325.C, r
 Schechter, Martin (1930–) 112, F, H 189.B 320.r
 323.H
 Scheffé, Henry (1907–77) 102.r 346.C 399.C, r
 Scheffers, Georg (1866–1945) 247.r
 Scheifele, Gerhard 55.r
 Scheinberg, Stephen 164.K
 Scheja, Günter (1932–) 21.M, r
 Scherk, Heinrich Ferdinand 275.A
 Scherk, John (1947–) 132.r
 Scherk, Peter (1910–85) 4.A
 Schetzen, Martin 95.r
 Schickard, Wilhelm (1592–1635) 75.A
 Schiffer, Menahem Max (1911–) 77.F, r 188.r 367.r
 438.B, C
 Schiffman, M. 275.B
 Schilling, Otto Franz Georg (1911–73) 257.r 439.r
 Schläfli, Ludwig (1814–95) 105.A 248.S 393.B
 App. A, Tables 19.III, IV
 Schlaifer, Robert 398.r
 Schlesinger, Ludwig (1864–1933) 253.E, r
 Schlessinger, Michael 16.r
 Schlichting, Hermann (1907–) 205.r 433.A
 Schlieder, Siegfried (1918–) 150.E
 Schlömilch, Otto 39.D App. A, Tables 9.IV, 10.II,
 19.III
 Schmeidler, David 173.D
 Schmetterer, Leopold (1919–) 399.N
 Schmid, Hermann Ludwig (1908–56) 59.H
 Schmid, Wilfried (1943–) 16.r 437.W
 Schmidt, Erhard (1876–1959) 68.C, I 139.G
 217.H, I 286.V 302.E 317.A 445
 Schmidt, Friedrich Karl (1901–77) 12.B 59.G
 450.P
 Schmidt, O. Y. → Shmidt
 Schmidt, Robert (1898–1964) 208.C 379.M
 Schmidt, Wolfgang M. (1933–) 83.r 118.B, D, r
 182.G, r 354.r 430.C 437.W
 Schnee, Walter 379.M
 Schneider, Michael (1942–) 16.r
 Schneider, Theodor (1911–) 182.r 196 430.A, B, r
 Schober, Glenn E. (1938–) 438.r
 Schoen, Richard M. (1950–) 275.D, F 364.r

Name Index

Seidenberg, Abraham

Schoenberg, Isaac Jacob (1903–) 178.A
 Schoenfeld, Lowell (1920–) 328
 Schoenflies, Arthur Moritz (1853–1928) 47.r 65.G
 92.E, F 93.D, K 122.H 381.r App. B, Table 5.IV
 Scholtz, Arnold 59.F
 Schönfinkel, M. 97.*
 Schönhage, Arnold 298.r
 Schopf, Andreas 200.I
 Schottky, Friedrich Hermann (1851–1935) 9.J
 43.J 234.B 367.C
 Schouten, Jan Arnoldus (1883–1971) 109.*, r 137
 417.r 428.r 434.C
 Schrader, Robert (1939–) 150.F
 Schreier, Otto (1901–29) 7.r 28 151.A, I 161.A
 172.F 190.G, N 200.M 256.r 343.r 350.r
 Schröder, A. 156.B
 Schröder, Friedrich Wilhelm Karl Ernst (1841–1902)
 44.B 388.D 411.A
 Schrödinger, Erwin (1887–1961) 331.A, D 340.E
 351.C, D 434.C
 Schubauer, G. B. 433.A
 Schubert, Hermann (1848–1911) 56.E 201.r
 Schubert, Horst (1919–) 235.A
 Schur, Friedrich Heinrich (1856–1932) 364.D
 Schur, Issai (1875–1941) 29.E 43.J 122.C, E, F, H, r
 151.E 226.r 277.H 295.E 368.G 379.L 437.D, EE
 App. B, Table 5.r
 Schütte, Kurt (1909–) 97.* 156.E, r
 Schuur, Jerry Dee (1936–) 290.r
 Schwank, Friedrich (1900–) 217.r
 Schwartz, Arthur J. (1932–) 126.I
 Schwartz, Jacob Theodore (1930–) 37.r 68.M
 112.I, O 136.B, r 162.r 168.r 240.r 251.r 279.r
 286.r 308.F, r 310.r 315.r 331.r 378.r 390.r 443.A,
 G, r
 Schwartz, Laurent (1915–) 20.*, r 68.r 94.r 112.D, r
 125.A, B, L, r 160.r 162.*, r 168.r 189.r 192.M
 240.r 262.r 270.I 306.A 322.r 424.R, S, X, r
 Schwartz, Richard 280.r
 Schwarz, Hermann Amandus (1843–1921) 11.D
 43.B 77.D 106.H 109 198.G 211.C 246.B 275.B, F
 334.C App. A, Tables 8, 9.III, 13.III
 Schwarzenberger, Rolf Ludwig Edward (1936–)
 92.r
 Schwarzschild, Karl (1873–1916) 359.E
 Schweber, Silvan Samuel (1928–) 150.r
 Schweitzer, Paul Alexander (1937–) 126.N 154.D
 Schwerdt, Hans 19.r
 Schwinger, Julian Seymour (1918–) 132.C 146.A
 150.A, F 308.H 361.A 375.C
 Seidmore, Allan K. (1927–) 96.r
 Scipione del Ferro (1465–1526) 360
 Scott, Dana S. 33.E, r
 Scott, William Raymond (1919–) 151.r
 Searle, Shayle R. (1928–) 403.r
 Sebastião e Silva, José (1914–) 125.BB
 Secrest, Don H. (1932–) 299.r
 Sedov, Leonid Ivanovich (1907–) 116.r
 Seebach, L. 425.r
 Seeley, Robert Thomas (1932–) 274.I 323.K
 Seelig, Carl (1894–1962) 129.r
 Segal, Graeme Bryce (1941–) 105.r 237.J 366.r
 Segal, Irving Ezra (1918–) 308.D 351.K
 Segal, Jack (1934–) 382.A, C
 Segre, Beniamino (1903–77) 366.r
 Segre, Corrado (1863–1924) 11.B
 Seibert, Peter 126.D
 Seidel, Philipp Ludwig von (1821–96) 302.C
 Seidel, Wladimir P. (1906–81) 62.C, D
 Seidenberg, Abraham (1916–) 9.r 343.r

- Seifert, Herbert (1907–) 65.r 91.r 99.r 126.N 154.D
170.r 201.r 235.A, C, r 410.r
- Seinfeld, John Hersch (1942–) 303.r
- Seitz, Gary M. 151.J
- Seki, Takakazu (Kowa) (c. 1642 (1639?)–1708)
230 332
- Sekiguchi, Jiro (1951–) 437.CC
- Selberg, Atle (1917–) 4.A 32.H, r 122.F, G 123.B,
D, E 412.K 437.X, CC, DD 450.A, I, K, T, r
- Selberg, Henrik Ludvig (1906–) 17.A, C, D, r 48.E
124.B 338.H
- Selfridge, R. G. NTR
- Sell, George R. (1937–) 126.M
- Selmer, Ernst Sejersted (1920–) 118.C
- Selten, Reinhard 173.B
- Semple, John Greenlees (1904–86) 12.r
- Sen, Pranab Kumar (1937–) 280.r 371.r
- Senior, James Kuhn 151.r
- Seregin, L. V. 115.r
- Sergner, J. A. 19.B
- Serre, Jean-Pierre (1926–) 3.N, r 9.r 12.B 13.r 15.E
16.C, E, T, r 20 21.L, Q 29.r 32.D 52.N 59.H, r
64.B, r 70.r 72.E, K, r 122.F 147.K, O 148.A
172.r 200.K, M, r 202.N, U, r 237.J 248.r 249.r
257.r 284.G 362.r 366.D 369.F, r 426 428.G
450.G, J, R, r
- Serret, Joseph Alfred (1819–85) 111.D 238.r
App. A, Table 4.I
- Serrin, James Burton (1926–) 275.A, D 323.D, E
- Servais, C. 297.D
- Seshadri, Conjeeveram Srirangachari (1932–)
16.Y, r
- Seshu, Sundaram (1926–) 282.r
- Sevast'yanov, Boris Aleksandrovich (1923–) 44.r
- Severi, Francesco (1879–1961) 9.F, r 11.B 12.B
15.B, D, F 16.P 232.C
- Sewell, Geoffrey Leon (1927–) 402.G
- Sewell, Walter Edwin (1904–) 336.H
- Sgarro, Andrea (1947–) 213.r
- Shabat, Aleksei Borisovich 387.F
- Shafarevich, Igor' Rostislavovich (1923–) 14.r 15.r
16.r 59.F, H 118.D, E 257.H 297.r 347.r 450.Q, S
- Shampine, Lawrence Fred (1939–) 303.r
- Shaneson, Julius L. 65.D 114.J, K, r
- Shanks, Daniel (1917–) 332.r
- Shanks, E. B. 109.r
- Shanks, William (1812–82) 332
- Shannon, Claude Elwood (1916–) 31.C 136.E
213.A, D–F 403.r
- Shannon, Robert E. 385.r
- Shapiro, Harold N. (1922–) 123.D
- Shapiro, Harold S. 43.r
- Shapiro, Harvey L. 425.r
- Shapiro, Jeremy F. 215.r 264.r
- Shapiro, Zoya Yakovlevna 258.r 323.H
- Shapley, Lloyd Stowell (1923–) 173.D, E
- Sharkovskii, Aleksandr Nikolaevich (1936–) 126.N
- Sharpe, Michael J. (1941–) 262.r
- Shaw, B. 251.K
- Shaw H. 75.r
- Shchegol'kov (Stschegolkow), Evgenii Alekseevich
(1917–) 22.r
- Shelah, Saharon 33.r 276.E, F, r
- Shelly, Maynard Wolfe 227.r
- Shelukhin, V. V. 204.F
- Shen Chao-Liang (1951–) 36.H
- Shenk, Norman A., II 112.P
- Shepard, Roger Newland (1929–) 346.E, r
- Sher, Richard B. (1939–) 382.D
- Sherman, Seymour (1917–77) 212.A, r
- Shewhart, Walter Andrew (1891–1967) 401.G
404.A, B
- Shiba, Masakazu (1944–) 367.I
- Shibagaki, Wasao (1906–) 174.r App. A, Table 20.r
NTR
- Shidlovskii, Andrei Borisovich (1915–) 430.D, r
- Shields, Allen Lowell (1927–) 43.G, r 164.J
- Shields, Paul C. (1933–) 136.E, r 213.F
- Shiga, Kiyoshi (1944–) 195.r
- Shiga, Kôji (1930–) 72.r 147.O
- Shige-eda, Shinsei (1945–) 96. r
- Shikata, Yoshihiro (1936–) 178.r
- Shilov, Georgii Evgen'evich (1917–75) 21.D 36.M
125.A, Q, S 160.r 162.r 164.C 384.D 424.r
- Shimada, Nobuo (1925–) 114.B 202.S
- Shimakura, Norio (1940–) 323.H, N
- Shimizu, Hideo (1935–) 32.H 450.L, r
- Shimizu, Ryoichi (1931–) 374.H
- Shimizu, Tatsujiro (1897–) 124.B 272.J
- Shimodaira, Kazuo (1928–) 230.r
- Shimura, Goro (1930–) 3.M, r 11.B 13.P 16.r 32.D,
F, H, r 59.A 73.B, r 122.F, r 450.A, L, M, S, U, r
- Shintani, Hisayoshi (1933–) 303.r
- Shintani, Takuro (1943–80) 450.A, E, G, V, r
- Shioda, Tetsuji (1940–) 450.Q, S
- Shiohama, Katsuhiko (1940–) 178.r
- Shiraiwa, Kenichi (1928–) 126.J
- Shirkov, Dmitrii Vasil'evich (1928–) 150.r 361.r
- Shiryaev, Al'bert Nikolaevich (1934–) 86.E 395.r
405.r
- Shisha, Oved (1932–) 211.r
- Shizuta, Yasushi (1936–) 41.D 112.F
- Shmidt (Schmidt), Otto Yul'evich (1891–1956)
190.L 277.I
- Shmul'yan, Yu. V. 37.E, G 162 424.O, V
- Shnider, Steven David (1945–) 344.C–E
- Shnirel'man, Lev Genrikhovich (1905–38) 4.A
279.G 286.Q, r
- Shoda, Kenjiro (1902–77) 8 29.F
- Shoenfield, Joseph Robert (1927–) 22. F, H, r 97.r
156.r 185.r 411.r
- Shohat, James Alexander (1886–1944) 240.r 341.r
- Shortley, George H. 353.r
- Shreider, Yulii Anatol'evich (1927–) 192.r
- Shrikhande, S. S. 102.K 241.B STR
- Shub, Michael (1943–) 126.J, K, r
- Shubik, Martin 173.r
- Shubnikov, Aleksei Vasil'evich (1887–) 92.F, r
- Shult, Ernest E. (1933–) 151.J
- Shultz, Frederic W. (1945–) 351.L
- Shvarts (Schwarz, Švarc), Al'bert Solomonovich
(1934–) 56.H 80.r 286.D
- Sibuya, Yasutaka (1930–) 289.D, E 428.H, r
- Sidák, Zbyněk (1933–) 371.r
- Sidon, S. 159.J 192.T
- Siebenmann, Laurence Carl (1939–) 65.A, C, r
70.C 114.J, K, r
- Siegel, Carl Ludwig (1896–1981) 3.A, r 4.F 11.B, r
14.E 21.Q 22.A, C, H, r 27.r 32.F, r 49.D, r 55.r
72.r 118.A, C, D 122.B, E, F, r 123.D 126.I 154.D
182.D, E, G, r 242.A 289.D 296.A 297.r 328 347.E
348.K.r 384.A, E, F 412.r 420.C, F 430.A, B, D, r
450.A, E, K, r
- Sierpiński, Wacław (1882–1969) 22.A, C, H 49.D
242.A 297.r 425.r 426
- Sigmund, Karl 136.r
- Sikonia, W. 331.E 390.I
- Sikorski, Roman (1920–83) 42.r
- Šilov → Shilov
- Silver, Jack H. 33.F, r

Silverman, Leonard M. (1939–) 86.D
 Silverstein, Martin Louis (1939–) 44.E 168.B
 Silvester II → Gerbert
 Silvet, S. D. 102.r
 Simart, Georges 11.r 12.r 15.r 418.r
 Simauti, Takakazu (1930–) 411.J
 Simon, Barry (1946–) 150.r 212.B, r 331.r 351.r 375.r 390.r
 Simon, Herbert Alexander (1916–) 385.r
 Simon, Leon M. (1945–) 275.C
 Simonis, Juriaan (1943–) 16.Y
 Simons, James H. (1938–) 275.A, F 364.r 365.G
 Simplicius (c. 6th century) 187
 Simpson, Thomas (1710–61) 299.A 303.E
 Sims, Charles C. (1937–) 14.L 151.A, I
 Sinaï, Yakov Grigor'evich (1935–) 126.A, J, N 136.C, E, G, r
 Singer, Isadore Manual (1924–) 20 68.F 80.r 91.r 109 153.C 183 191.r 303.H, r 323.K, M 366.A–C, r 390.J 391.B, C, K, L, r 428.r
 Singh, Avadhesh Narayan 209.r
 Sinha, Kalya B. 375.r
 Sinnott, W. 450.J
 Sirao, Tunekiti (1924–) 45.I, r
 Sitnikov, Kirill Aleksandrovich (1926–) 117.D
 Siu Yum-Tong (1943–) 195.r 232.C 364.r
 Sjölin, Per B. (1943–) 159.r
 Skibinsky, Morris (1925–) 396.J
 Skitovich, Viktor Pavlovich 374.H
 Skolem, Albert Thoralf (1887–1963) 97.B 118.C, D 156.E, r 276.D 293.A
 Skorniyakov, Lev Anatol'evich (1924–) 85.r
 Skorokhod, Anatolii Vladimirovich (1930–) 44.r 115.D, r 250.E, r 406.D, F, r
 Skramstad, H. K. 433.A
 Slater, Lucy Joan 167.r 206.r 292.B NTR
 Slodowy, Peter (1948–) 418.r
 Slowikowski, Wojciech (1932–) 424.X
 Smale, Stephen (1930–) 65.C 105.Z, r 114.A, B, D, F, r 126.A, J, K, r 136.G 183 279.D, E 286.P, Q 426
 Small, Charles (1943–) 29.r
 Smart, D. R. 153.r
 Smart, William Marshall (1889–) 55.r 392.r
 Smirnov, Modest Mikhaïlovich (1921–) 326.r
 Smirnov, Nikolaï Vasil'evich (1900–66) 250.F, r 374.E STR
 Smirnov, Vladimir Ivanovich (1887–1974) 20.r 106.r 216.r 371.F
 Smirnov, Yurii Mikhaïlovich (1921–) 273.K
 Smith, Brian T. (1942–) 298.r 301.O
 Smith, David Eugene (1860–1944) 187.r
 Smith, Gordon Dennis 304.r
 Smith, Guy Watson (1885–) 19.r
 Smith, H. L. 87.H, K, r
 Smith, Henry John Stephen (1826–83) 179.B
 Smith, J. M. 263.r
 Smith, Kennan Tayler (1926–) 276.E 338.E
 Smith, Paul Althaus (1900–80) 235.E 431.B
 Smith, Paul John (1943–) 151.I
 Smithies, Frank (1912–) 217.r
 Smorodinsky, Meir (1936–) 136.E
 Smullyan, Raymond M. 411.r
 Smyth, Brian 275.F 365.H, L
 Smythe, Robert T. (1941–) 340.r
 Snapper, Ernst (1913–) 16.E 200.M
 Sneddon, Ian Naismith (1919–) 389.r
 Sneddon, W. J. 336.r
 Snell, James Laurie (1925–) 260.J
 Snell (Snel van Roijen, Snellius), Willebrord

Name Index

Steffensen, John F.

(1580–1626) 180.A
 Sobolev, Sergei L'vovich (1908–) 20 46.r 125.A 162 168.B, r 224.E 320.r 323.G 325.r
 Sobolevskii, Pavel Evseevich (1930–) 251.r 286.r 378.I, J
 Sohncke, Leonhard (1842–97) 92.F
 Solitar, Donald Moiseevitch (1932–) 161.r
 Solovay, Robert M. (1938–) 22.F, H 33.E, F, r
 Sommer, Friedrich (1912–) 198.r 367.r
 Sommerfeld, Arnold Johannes Wilhelm (1868–1951) 130.r 188.D 271.r 274.r 402.H App. A, Table 19.III
 Sommerville, Duncan McLaren Young (1879–1934) 285.r
 Soms, A. 399.N
 Sonine (Sonin), Nikolaï Yakovlevich (1849–1915) 317.D App. A, Tables 19.III, 20.VI
 Sono Masazô (1886–1969) 8 284.G
 Soreau, R. (1865–?) 19.r
 Sotomayor, Jorge (1942–) 126.M
 Sova, Miroslav 378.D
 Sowe, E. R. 354.r
 Spanier, Edwin Henry (1921–) 64.r 70.r 148.r 170.r 201.M, r 202.I, r 305.r
 Späth, R. A. 274.F 314.A
 Spearman, Charles (1863–1945) 346.F, r 371.K
 Specht, Wilhelm (1907–85) 10.r 151.r 190.r
 Specker, W. H. 142.C
 Spector, Clifford (1930–) 81.r 156.E, r 356.H, r
 Speer, Eugene Richard (1943–) 146.A
 Speiser, Andreas (1885–1970) 151.r 172.J 190.r
 Spencer, Domina Eberle (1920–) 130.r
 Spencer, Donald Clayton (1912–) 12.B 15.F 72.G, r 232.r 367.r 428.E, r 438.B, C 442.r
 Spencer, Thomas 402.G
 Sperner, Emanuel (1905–80) 7.r 256.r 343.r 350.r
 Spindler, Heinz 16.r
 Spitzer, Frank Ludwig (1926–) 44.C 250.r 260.E, J 340.r
 Spivak, Michael D. (1940–) 114.J 191.r 365.r
 Sprindzhuk, Vladimir Gennadievich (1936–) 118.D 430.C
 Springer, George (1924–) 367.r
 Springer, Tonny Albert (1926–) 13.A, I, O, P, r
 Srinivasan, B. App. B, Table 5
 Srinivasan, T. P. 164.G
 Srivastava, Muni Shanker (1936–) 280.r
 Stallings, John Robert, Jr. (1935–) 65.A, C, E, F 235.G 426
 Stampacchia, Guido (1922–78) 440.r
 Stanasila (Stănășilă), Octavian (1939–) 23.r
 Stancu-Minasian, I. M. 408.r
 Stanley, Harry Eugene (1941–) 402.r
 Stanley, Richard Peter (1944–) 16.Z
 Stapp, Henry Pierce (1928–) 146.C 274.D, I 386.C
 Stark, Harold Mead (1939–) 83.r 118.D 182.G 347.E 450.E
 Stasheff, James Dillon (1936–) 56.r 201.r
 Staudt, Karl Georg Christian von (1798–1867) 267 343.C 450.J
 Stavroudis, Orestes Nicholas (1923–) 180.r
 Stearns, Richard Edwin (1936–) 75.r
 Stechkin, Sergei Borisovich (1920–) 211.r 336.C
 Steel, J. 22.F
 Steele, John Hyslop (1926–) 263.D
 Steen, Lynn Arthur (1941–) 425.r
 Steenbrink, Joseph H. M. (1947–) 9.J
 Steenrod, Norman Earl (1910–71) 52.r 56.r 64.A, B, r 70.F, r 91.r 147.r 148.D 201.A, C, Q, R 210.r 305.A, r 426.*, r 442.r
 Steffensen, John F. 223.r

- Stegall, Charles 443.H
 Stegun, Irene A. (1919-) NTR
 Stein, Charles M. 280.D, r 398.r 399.G, r 400.B, F
 Stein, Elias M. (1931-) 159.G 168.B, r 224.B, E, r 251.r 437.V, DD
 Stein, Karl (1913-) 20 21.H, L, M, Q 23.B, F, F 72.E 367.B, G, I
 Steinberg, Robert (1922-) 13.O 151.I 237.J 248.Z App. B, Table 5.r
 Steinbuch, Karl 95.r
 Steiner, Jakob (1796-1863) 78.K 89.C 179.A, B 181 228.B 267
 Steinhilber, Hugo (1887-1972) 37.H 317.r 424.J
 Steinitz, Ernst (1871-1928) 8 149.I, r 172.A 357.r
 Steinmann, Othmar Viktor (1932-) 150.D
 Stepanov, Sergei Aleksandrovich 450.P
 Stepanov, Vyacheslav Vasil'evich (1889-1950) 18.A, r 126.E, r 394.r
 Stephan, Frederick F. 280.J
 Stepin, Anatolii Mikhaïlovich (1940-) 136.E, G, H
 Stern, A. 297.D
 Sternberg, S. H. 346.r
 Sternberg, Shlomo (1936-) 105.r 111.r 126.G, r 132.r 191.r 274.r 325.L 428.F, G, r 431.r
 Stetter, Hans J. (1930-) 303.r
 Stevin, Simon (1548-1620) 360
 Stewart, F. M. 22.H
 Stewart, Gilbert W., III 298.r
 Stewart, Ian Nicholas (1945-) 51.r
 Stickelberger, Ludwig (1850-1936) 2.B
 Stiefel, Eduard Ludwig (1909-78) 55.r 56.A, B, F 65.B 147.A, I, M 199.B 302.D
 Stieltjes, Thomas Joannes (1856-94) 94.A-C 133.C 166.C 192.D, Q 220.D 240.A, K 270.L
 Stiemel, Erich 255.B, E
 Stigum, Bernt Petter (1931-) 44.r
 Stirling, James (1692-1770) 66.D 174.A 223.C App. A, Tables 17.I, 21.I
 Stoer, Josef (1934-) 303.F
 Stoilow, Simion 207.B, C 367.r
 Stoka, Marius Ion (1934-) 218.r
 Stoker, James Johnston (1905-) 111.r 205.r
 Stokes, George Gabriel (1819-1903) 94.F 105.U 167.E 188.E 204.B, C, F 205.C, F 254.D App. A, Table 3.III
 Stoll, Wilhelm Friedrich (1923-) 21.N 272.L
 Stolz, Otto (1842-1905) 106.G 333.B
 Stolzenberg, Gabriel 164.F
 Stone, Arthur Harold (1916-) 22.r 273.K 425.X, CC
 Stone, Charles J. 5.F
 Stone, Harold S. (1938-) 96.r
 Stone, Marshall Harvey (1903-) 42.D 112.O 162 168.B 197.r 207.c 251.r 310.I 378.C 390.r 425.T
 Stong, Robert Evert (1936-) 114.r 237.H
 Stora, Raymond Felix (1930-) 150.G
 Storer, James Edward (1927-) 282.r
 Størmer, Erling (1937-) 212.B
 Stout, Edgar Lee (1938-) 164.r
 Stracke, Gustav (1887-) 309.r
 Strang, William Gilbert (1934-) 300.r 304.r
 Strassen, Volker (1936-) 250.E
 Stratila (Strătilă), Serban 308.r
 Stratonovich, Ruslan Leont'evich (1930-) 115.D 406.C
 Stratton, Julius Adams (1901-) 130.r 133.r
 Strauss, Walter A. (1937-) 286.C 345.A
 Stray, Arne (1944-) 164.J
 Streater, Raymond F. (1936-) 150.r 386.r
 Strebel, Kurt O. (1921-) 352.C
 Street, Anne Penfold (1932-) 241.r
 Stroock, Daniel Wyler (1940-) 44.E 115.C, D, r 250.r 261.C 262.E 406.A, D, r
 Stroud, Arthur H. 299.r
 Stroyan, Keith Duncan (1944-) 293.r
 Struik, Dirk Jan (1894-) 187.r 266.r
 Strutt, Maximilian Julius Otto (1903-) 133.r 268.r
 Struve, Friedrich George Wilhelm von (1793-1864) 39.G App. A, Table 19.IV
 Stuart, Alan (1922-) 102.r 374.r 397.r 400.r
 Student (Gosset, William Sealy) (1876-1936) 374.B 400.G 401.F
 Stueckelberg, Ernst Carl Gerlach (1905-) 361.r
 Sturm, Jacques Charles François (1803-55) 10.E 107.A 112.I 301.C 315.B
 Subramanyam, K. 399.O
 Suetuna, Zyoiti (1898-1970) 242.B 295.D 450.E
 Sugawara, Masao (1902-70) 73.A
 Sugie, Toru (1952-) 15.H
 Sugimoto (Goto), Midori (1944-) 178.r
 Suita, Nobuyuki (1933-) 77.E
 Sukhatme, Balkrishna Vasudeo (1924-79) 373.r
 Sukhatme, Pandurang Vasudeo 373.r
 Sullivan, Dennis Parnell (1941-) 65.C 114.J, L 154.H, r 234.E
 Sumihiro, Hideyasu (1941-) 16.Z
 Sunada, Toshikazu (1948-) 195.r 391.C
 Sundman, Karl Frithiof (1873-1949) 420.C
 Sunouchi, Gen-ichiro (1911-) 159.G, H 310.r 336.D
 Sunzi (c. 3rd century) 57.A
 Suranyi, János (1918-) 97.B
 Suslin, Andrei Aleksandrovich 16.Y 200.K 369.F
 Suslin (Souslin), Mikhail Yakovlevich (1894-1919) 22.A-C, H, I 33.F 425.CC
 Süßmilch, Johann Peter (1707-67) 401.E
 Suzuki, Michio (1926-) 151.I, J, r 190.r App. B, Table 5.III
 Suzuki, Mitsuo (1928-) 173.E
 Švarc → Shvarts
 Swan, Richard G. (1933-) 200.M 237.r 362.r 383.r
 Swedler, Moss E. (1942-) 172.A, K 203.A
 Swierczkowski, S. 22.H
 Swinnerton-Dyer, Henry Peter Francis (1927-) 118.D, E 450.J, Q, S
 Switzer, Robert M. (1940-) 202.r
 Sylow, Peter Ludvig Mejdell (1832-1918) 151.B
 Sylvester, James Joseph (1814-97) 103.F 186.A 226.G 267 297.D 348.C 369.E, F
 Symanzik, Kurt (1923-83) 132.C 150.D, F 361.B, r 386.C
 Synge, John Lighton (1897-) 152.C 178.B, C
 Szabó, Árpád (1913-) 187.r
 Szankowski, Andrzej (1945-) 37.L
 Szász, Otto (1884-1952) 121.B
 Szczerba, Robert H. (1932-) 114.K
 Szebehely, Victor G. (1921-) 420.r
 Szegő, Gabor (1895-1985) 20.r 48.D, r 164.G 188.H 222.r 228.B, r 317.r 322.r 336.I 389.r
 Szegő, Giorgio P. 86.r 108.r 126.r
 Székredi, Endre 136.C
 Szmielew, Wanda (1918-76) 97.B
 Sz.-Nagy (Székefalvi-Nagy), Béla (1913-) 68.r 197.r 251.N, r 390.r
 Szpilrajn, Edward 117.G
 Szpiro, Lucien (1941-) 16.Y 118.E
T
 Tabata, Masahisa (1947-) 304.D
 Tachibana, Shun-ichi (1926-) 110.E

- Tai Yung-Sheng (1944–) 16.r
 Tait, Peter Guthrie (1831–1901) 157.C 235.A
 Takagi, Ryoichi (1943–) 365.I, K, L
 Takagi, Teiji (1875–1960) 14.L, O, R, U 59.A
 73.A, r 196 267 297.I 336.A 348.M 415 450.E
 App. B, Tables 4.I, II
 Takahara, Yositané (c. 17th century) 230
 Takahashi, Hidetoshi (1915–85) 142.D 299.r
 Takahashi, Moto-o (1941–) 156.E 411.J, r
 Takahashi, Reiji (1927–) 437.BB
 Takahashi, Shuichi (1928–) 362.K
 Takahashi, Tsunero (1933–) 275.F 365.I, K, L
 Takahasi, Yositoki (1764–1804) 230
 Takano, Kinsaku (1915–58) 213.F
 Takano, Kyoichi (1943–) 428.r
 Takasawa, Yoshimitsu (1942–) 299.B
 Takasu, Satoru (1931–) 200.K
 Takebe, Katahiro (1664–1739) 230 332
 Takeno, Hyôitirô (1910–) 434.r
 Takenouchi, Osamu (1925–) 437.E
 Takenouchi, Tanzo (1887–1945) 73.A 134.r
 Takens, Floris (1940–) 126.A, L, M, r 433.B, r
 Takesaki, Masamichi (1933–) 36.r 308.H, I, J
 Takeuchi, Kei (1933–) 128.C, r 346.r 371.A, H
 373.r 399.K, O 400.r
 Takeuchi, Masaru (1932–) 365.I, L, N, O 384.E, r
 Takeuchi, Mitsuhiro (1947–) 203.r
 Takeuti, Gaisi (1926–) 33.r 81.D 156.E, r 356.G, r
 411.J, r
 Takhtadzhian (Takhtajan), Leon Armenovich
 387.G
 Tall, Franklin D. (1944–) 273.K
 Talman, James Davis (1931–) 389.r
 Talmi, Igal 353.r
 Tamagawa, Tsunco (1925–) 6.F 13.P, r 59.H 118.C
 122.F 348.K 450.A, H, K, L
 Tamano, Hisahiro (1928–69) 425.X
 Tamarkin, Jacob David (1888–1945) 160.E 240.r
 341.r
 Tamura, Itiro (1926–) 114.B, F 154.B, H
 Tamura, Jirô (1920–) 124.C 367.F
 Tamura, Ryoji (1920–81) 371.C
 Tanabe, Hiroki (1932–) 378.I
 Tanabe, Kunio (1943–) 302.r
 Tanaka, Chuji (1916–) 121.B, C
 Tanaka, Hiroshi (1932–) 41.C 261.r 340.r 406.D
 Tanaka, Hisao (1928–) 22.C, F
 Tanaka, Jun-ichi (1949–) 164.H
 Tanaka, Makoto (1942–) 437.r
 Tanaka, Minoru (1909–) 295.E
 Tanaka, Minoru (1949–) 279.r
 Tanaka, Noboru (1930–) 21.P 80.r 191.r 344.B
 364.F 365.F 384.D, r
 Tanaka, Shigeru (1942–) 136.C
 Tanaka, Shunichi (1938–) 287.C
 Tanaka, Yosizane (1651–1719) 230
 Tandori, Károly (1925–) 317.B
 Tangora, Martin Charles (1936–) 64.r
 Tani, Atsusi (1946–) 204.F
 Taniyama, Yutaka (1927–58) 3.M 73.B 450.F, S
 Tannaka, Tadao (1908–86) 59.D 69.D 249.U
 Tannery, Paul (1843–1904) 144.r 187.r
 Tanno, Shukichi (1937–) 110.E 364.F, G 365.L
 391.C, E, N
 Tarski, Alfred (1902–83) 22.G 33.r 97.B, r 156.r
 185.D, r 276.D
 Tartaglia, Niccolò (1500?–57) 360
 Tartakovskii, Vladimir Abramovich 4.E 161.B
 Tashiro, Yoshihiro (1926–) 364.F, G
 Tate, John Torrence (1925–) 3.C, M, N, r 6.E, F, r
 28.r 59.H 118.D, E, r 200.K, N 257.r 450.F, G,
 L, N, P, Q, S, r
 Tatsuuma, Nobuhiko (1930–) 437.K
 Tatuzawa, Tikao (1915–) 4.F, r 123.E 450.r
 Tauber, Alfred (1866–1942?) 36.L 121.D 160.G
 192.F 339.B 379.N
 Taubes, Clifford Henry (1955–) 150.r
 Taylor, Angus Ellis (1911–) 106.r 216.r
 Taylor, Brook (1685–1731) 20 21.B 58.C 106.E, J
 286.F 339.A App. A, Table 9.IV
 Taylor, Sir Geoffrey Ingram (1886–1975) 205.E
 433.A, C
 Taylor, Howard Milton (1937–) 260.r
 Taylor, James Henry (1893–?) 152.C
 Taylor, John Clayton (1930–) 132.r
 Taylor, John R. 375.r
 Taylor, Joseph L. 36.M, r
 Taylor, Michael E. 345.A
 Taylor, Samuel James (1929–) 45.r
 Taylor, Thomas D. 304.r
 Teichmüller, Oswald (1913–43) 9.J 43.E 77.E
 352.A, C 416 438.B
 Teissier, Bernard (1945–) 16.Z 418.r
 Teixeira, Francisco Gomes 93.r
 Temam, Roger (1940–) 204.B, D 304.r
 Tennenbaum, Stanley (1927–) 33.F, r
 Teplitz, Vigdor L. 146.r
 Terada, Toshiaki (1941–) 206.D 428.H
 Terano, Takao (1952–) 301.F
 Terasaka, Hidetaka (1904–) 235.A, C
 ter Haar, Dick 402.r
 te Riele, H. J. J. 297.D
 Terjanian, Guy 118.F
 Terry, Milton Everett (1916–) 346.C 371.C
 Thales (c. 639–c. 546 B.C.) 35.A 181 187
 Theaitetus (415–369 B.C.) 187
 Theil, Henri (1924–) 128.r
 Theodorsen, Theodore (1897–) 39.F
 Theodorus (of Cyrene) (5th century B.C.) 187
 Theon (of Alexandria) (fl. 370) 187
 Theon (of Smyrna) (fl. 130) 187
 Thimm, Walter (1913–) 23.D
 Thirring, Walter Eduard (1927–) 212.B
 Thom, René F. (1923–) 12.B 51.A, B, E 56.E, F, r
 70.r 114.A, F–H 126.A, H, M 148.E 183 202.T
 263.D 418.G, r 426
 Thoma, Elmar Herbert (1926–) 437.E
 Thomas Aquinas (1225(27)–74) 372
 Thomas, J. 206.C
 Thomas, Lawrence E. 375.F
 Thomas, Paul Emery (1927–) 64.r
 Thomas, Richard Kenneth (1942–) 136.E
 Thomas, Tracy Yerkes (1899–1984) 152.C
 Thomason, Steven Karl (1940–) 22.F
 Thompson, Colin John (1941–) 212.B
 Thompson, J. F. 304.E
 Thompson, John Griggs (1932–) 151.D, H–J
 Thomsen, Gerhard (1899–) 155.H
 't Hooft, Gerald 132.C, D
 Thorin, G. O. 88.r 224.A
 Thorne, Kip S. 359.r
 Thorpe, J. A. 91.r
 Thrall, Robert McDowell (1914–) 29.r 173.D
 368.r
 Threlfall, William (1888–) 65.r 91.r 99.r 170.r
 201.r 235.r 410.r
 Thue, Axel (1863–1922) 31.B 118.D 182.G
 Thullen, Peter (1907–) 20 21.H, M, Q
 Thurston, William P. (1946–) 65.E 126.J, N
 154.A, D–H, r 234.A 235.B, E

Thurstone, Louis Leon

- Thurstone, Louis Leon (1887–1955) 346.C, F
 Tierney, Myles 200.r
 Tietäväinen, Aimo A. (1937–) 63.r
 Tietze, Heinrich (1880–1964) 425.Q
 Tikhonov, Andrei Nikolaevich (1906–) 153.D
 273.K 425.Q, S, T
 Timmesfeld, Franz-Georg (1943–) 151.J
 Timoshenko, Stephen P. (1878–1972) 271.r
 Tisserand, François Félix (1845–96) 55.r
 Tissot, Nicolas Auguste (1824–?) 206.C
 Titchmarsh, Edward Charles (1898–1963) 112.O
 123.B, D, r 160.C, r 192.r 198.r 220.C 242.A, r
 306.B 429.r 450.r
 Tits, Jacques Léon (1930–) 13.O, Q, R, r 151.I, J
 343.I
 Toda, Hirosi (1928–) 202.P, R, U
 Toda, Morikazu (1917–) 287.A, r
 Toda, Nobushige (1938–) 17.C
 Todd, John Arthur (1908–) 237.F 366.B, r
 Todhunter, Isaac (1820–84) 342.r
 Todorov, Andrei Nikolov (1948–) 232.C
 Todorov, Ivan T. (1933–) 146.r 150.r
 Toeplitz, Otto (1881–1940) 197.r 217.r 251.O
 379.L
 Tōki, Yukinari (1913–) 62.D 352.A 367.E
 Tollmien, Walter 433.A
 Tolman, Richard Chace (1881–1948) 402.r
 Tolstoy, I. 446.r
 tom Dieck, Tammo 431.E, r
 Tomi, Friedrich (1943–) 275.C
 Tomita, Minoru (1924–) 308.H
 Tomiyama, Jun (1931–) 36.K 164.E
 Tomonaga, Sin-itiirō (1906–79) 132.C 146.A 150.A
 359.C 361.A
 Tomotika, Susumu (1903–64) 134.r
 Tompkins, Charles Brown (1912–) 275.B 365.B
 Tondeur, Philippe Maurice (1932–) 154.G, H, r
 Tonelli, Leonida (1885–1946) 107.A 246.C
 Tonnelat-Baudot, Marie-Antoinette (1912–80)
 434.r
 Toponogov, Viktor Andreevich 178.A, F
 Topp, L. J. 304.r
 Topsøe, Flemming (1938–) 22.r
 Torelli, R. (1884–1915) 9.E, J 11.C
 Torgerson, Warren S. 346.E, r
 Torii, Tatsuo (1934–) 301.C
 Totoki, Haruo (1934–) 136.D 395.r
 Traub, Joe Fred (1932–) 71.r 301.r
 Trefftz, Erich Immanuel (1888–1937) 46.F
 Trémolières, Raymond (1941–) 440.r
 Treves, J. François (1930–) 112.D, L 125.r 274.I
 286.Z 320.I, r 321.r 345.A, B 424.r
 Tricomi, Francesco Giacomo (1897–1978) 217.N, r
 288.C 317.r 326.C, r
 Triebel, Hans (1936–) 168.r 224.r
 Trigg, G. L. 414.r
 Tristram, Andrew G. 114.K
 Trjitzinsky, Waldemar Joseph (1901–73) 254.D
 289.C, D 314.A
 Tromba, Anthony J. (1943–) 275.C, r 286.D
 Trotter, Hale Freeman (1931–) 235.C 351.F 378.E
 Trubowitz, Eugene B. (1951–) 387.E
 Trudinger, Neil Sidney (1942–) 364.H
 Truesdell, Clifford Ambrose T. (1919–) 389.B
 Trych-Pohlmeyer, E. B. 402.G
 Tschebyscheff → Chebyshev
 Tsen Chung-Tze 27.E 118.F
 Tsirel'son B. S. 406.D
 Tsu Ch'ung-Chih (429–500) 57.A 332
 Tsuboi, Takashi (1953–) 154.G
 Tsuchiya, Nobuo (1950–) 154.G, H
 Tsuda, Takao (1932–) 354.r
 Tsuji, Masatsugu (1894–1960) 48.r 62.B, D 124.C
 234.r 242.A 367.r 388.B
 Tsuji, Tadashi (1946–) 384.B
 Tsukada, Kazumi (1953–) 365.N 391.N
 Tsukamoto, Tatsuo (1940–) 353.r
 Tsukamoto, Yōtarō (1932–74) 178.B
 Tsushima, Ryuji (1952–) 32.r
 Tsuzuku, Tosihiro (1929–) 13.R
 Tucker, Albert William (1905–) 173.r 255.B, E
 292.A, B
 Tugué, Tosiuyuki (1926–) 22.C 81 356.G, r
 Tukey, John Wilder (1915–) 34.r 87.r 142.D, r
 304.r 371.A, r 397.r 421.C, r 425.X, r 436.r
 Tumarkin, Lev Abramovich (1904–74) 117.I
 Tumura, Yosiro (1912–) 17.C, D 62.D
 Tung Chuan 57.A
 Turán, Pál (Paul) (1910–76) 123.D, r
 Turing, Alan Mathison (1912–54) 22.G 31.B, C
 97.B 161.B 356.A
 Turner, M. J. 304.r
 Tushkanov, S. B. 17.C
 Tutte, William T. 186.r
 Tyupkin, Yu. S. 80.r
 Tzafiriri, Lior (1936–) 37.N, r 168.r
- U
- Uchida, Fuichi (1938–) 431.G
 Uchida, Kōji (1939–) 14.L
 Uchiyama, Akihito (1948–) 168.B
 Udagawa, Kanchisa (1920–65) 389.r
 Ueda, Tetsuo (1951–) 72.K
 Ueda, Yoshisuke (1936–) 126.N
 Uehara, Hiroshi (1923–) 202.P
 Ueno, Kenji (1945–) 16.r 72.I, r
 Ueno, Tadashi (1931–) 115.C
 Ugaheri, Tadashi (1915–) 240.B 338.C
 Uhl, J. Jerry, Jr. (1940–) 443.A, H
 Uhlenbeck, George Eugene (1900–) 41.C 45.I
 Uhlenbeck, Karen (1942–) 195.E, r 275.D
 Uhlhorn, Ulf 258.r
 Uhlmann, Armin 212.B
 Ukai, Seiji (1939–) 41.D
 Ulam, Stanislaw Marcin (1909–84) 33.F, r 153.B
 287.r 385.C
 Ullman, Jeffrey D. (1942–) 31.r 71.r 75.r 186.r
 Ullrich, Egon (1902–57) 17.A, C
 Ulm, Helmut (1908–75) 2.D
 Umemura, Hiroshi (1944–) 16.I
 Umemura (Yamasaki), Yasuo (1934–) 225.r 437.BB
 Umezawa, Hiroomi (1924–) 150.r
 Umezawa, Toshio (1928–) 411.F
 Uno, Toshio (1902–) NTR
 Ura, Taro (1920–) 126.D, F
 Urabe, Minoru (1912–75) 301.D
 Urakawa, Hajime (1946–) 391.E
 Ural'tseva, Nina Nikolaevna 286.r 323.D
 Urbanik, Kazimierz (1930–) 407.C
 Ursell, Harold Douglas 246.r
 Uryson, Pavel Samuilovich (1898–1924) 22.I 93.D
 117.A, r 273.K 425.Q, S, U, V, CC
 Ushiki, Shigehiro (1950–) 126.N
 Ushio, Kazuhiko (1946–) 96.r
 Utida, Itumi (1805–82) 230
 Utida, Shunro (1913–) 263.A
 Uzawa, Hirofumi (1928–) 292.A, E, r

V

Vahlen, Karl Theodor (1869–1945) 83.B
 Vaillancourt, Rémi (1934–) 304.F 345.A
 Vainberg, Boris Rufimovich (1938–) 323.K
 Vainshtein, Isaak Aronovich 273.K
 Väisälä, Jussi 143.r 352.F
 Vajda, Steven 408.r
 Valentine, F. A. 88.r
 Valiron, Georges (1884–1954) 17.A, C, D 43.K
 121.B, C, r 124.B 272.F, K 429.B 435.r
 van Beijeren, Henk 402.G
 van Ceulen, Ludolf (1540–1610) 332
 Van Daele, Alfons 308.H
 van Dantzig, D. 109 434.C
 van den Berg, Franciscus Johannes (1833–92) 19.B
 van der Corput, Johannes Gualtherus (1890–1975)
 4.C 182.H 242.A
 Vandermonde, Alexandre Théophile (1735–96)
 103.G 190.Q
 van der Pol, Balthasar (1889–1959) 240.r 290.C
 van der Waerden, Bartel Leendert (1903–) 8.*, r
 12.B 24.r 29.r 60.r 66.r 67.r 90.r 92.F, r 122.r 149.r
 172.r 187.r 190.r 196 284.r 337.r 351.r 362.r 368.r
 369.E 371.C 417.E
 Vandiver, Harry Shultz (1882–1973) 14.L 145.*, r
 van Hove, Leon Charles Prudent (1924–) 351.K
 402.G
 van Kampen, Egbertos R. (1908–42) 170
 Van Moerbeke, Pierre (1944–) 287.C
 van Roomen, Adriaan (1561–1615) 444
 van Schooten, Frans (1615–60) 444.r
 Varadarajan, Veeravalli Seshadri (1937–) 249.r
 Varadhan, Sathamangalam Ranga Ayyangar
 Srinivasa (1940–) 115.C, D, r 136.r 250.r 261.C
 262.E 340.r 406.A, D, r
 Varadier, M. 443.A
 Varaiya, Pravin P. 86.D 108.B 292.F
 Varchenko, A. N. 418.r
 Varga, Ottó (1909–69) 152.C
 Varga, Richard Steven (1928–) 302.r
 Varopoulos, Nicholas Theodoros 192.U
 Varopoulos, T. 17.C 267.r
 Varouchas, J. 232.C
 Varshamov, Rom Rubenovich (1927–) 63.B
 Vasilescu, Florin (1897–1958) 120.D
 Vaughan, Robert Charles 123.E
 Vaught, Robert L. 276.D, F
 Veblen, Oswald (1880–1960) 90.r 109.*, r 137
 152.C 201.r 343.r 434.C, r
 Vedesinov, N. 425.Q
 Veech, William Austin (1938–) 136.H
 Vekua, Il'ya Nestorovich (1907–77) 217.J 323.r
 Veldkamp, Ferdinando D. 13.R
 Velo, Giorgio 150.r
 Veneziano, Gabriele (1942–) 132.C 386.C
 Venkov, Boris Borisovich (1934–) 200.M
 Venttsel', Aleksandr Dmitrievich (1937–) 115.C
 261.r 406.F
 Verbeure, André (1940–) 402.G
 Verbiest, Ferdinand (1623–88) 57.C
 Verdier, Jean-Louis (1935–) 16.r 450.Q, r
 Ver Eecke, Paul (1867–1959) 187.r
 Vergne, Michèle 384.r
 Verhulst, Pierre François (1804–49) 263.A
 Verner, James Hamilton (1940–) 303.r
 Veronese, Giuseppe (1854–1917) 275.F
 Vershik, Anatolii Moiseevich (1933–) 136.D, r
 183.r

Name Index

Wagschal, Claude

Vesentini, Edoarc (1928–) 122.F
 Vessiot, Ernest (1865–1952) 107.A 113 249.V
 Vey, Jacques 154.G, r 384.r
 Vick, James Whitson (1942–) 201.r
 Viehweg, Eckart (1938–) 72.I, r 232.D, r
 Viète, François (1590–1603) 8 20 332 360 444
 Victoris, Leopold (1891–) 201.A, C, E, L 425.Q
 Vignéras, Marie-France (1946–) 391.C
 Vigué, Jean-Pierre (1948–) 384.r
 Vilenkin, Naum Yakovlevich (1920–) 112.r 125.r
 162.r 218.r 341.r 389.r 395.r 407.C 437.AA
 Villat, Henri René Pierre (1879–1972) App. A,
 Table 15.VI
 Ville, Jean A. 262.A
 Vinberg, Ernest Borisovich (1937–) 122.G 351.I
 384.C, r 385
 Vinogradov, Ivan M. (1891–1983) 4.C, E 123.B, E
 242.F 295.E
 Vinter, Richard B. 127.G
 Virtanen, Kaarlo I. 62.C 352.A, C 367.E, I
 Vishik, Mark Iosifovich (1921–) 112.E 323.N
 Vitali, Giuseppe (1875–1932) 270.G 380.D
 Viterbi, Andrew J. (1935–) 213.E
 Vitt, Aleksandr Adol'fovich 290.r
 Vitushkin, Anatolii Georgievich (1931–) 164.J
 169.E
 Vivanti, Giulio (1859–?) 217.r 339.A
 Vladimirova, S. M. 365. J
 Vogan, David Alexander, Jr. (1954–) 437.r
 Vogel, Kurt (1888–1985) 24.r
 Vogel, William R. 200.r
 Vogt, Dietmar (1941–) 168.B
 Voichick, Michael (1934–) 164.K
 Voiculescu, Dan Virgil (1949–) 36.J 331.E
 Voigt, Jürgen (1943–) 331.E
 Volder, J. E. 142.C
 Volk, Isaï Mikhaïlovich 289.E
 Volkov, Yurii Aleksandrovich (1930–) 365.J
 Volkovyskii, L. 198.r
 Volterra, Vito (1860–1940) 20 68.J 162 163.B
 198.J 217.A 222.A 263.B
 Voltyanskiĭ, V. G. 155.r
 von Eötvös, Roland (1848–1919) 359.D
 von Kármán, Theodore (1881–1963) 205.E 433.C
 von Koch, Helge (1870–1924) 246.K 450.I
 von Mises, Richard (1883–1953) 298.r 342.A
 354.E 399.K, r
 von Neumann, John (Johann) (1903–57) 18.A,
 E, r 20 22.F 33.A–C, r 36.G 68.I 69.B, C 75.B
 85.A 95 136.A, B, E, F 138.r 156.E, r 162 173.A, C,
 D, r 197.A, r 225.r 251.M 255.E 304.F 308.C, F,
 G, I, r 312.A 331.E 351.C, L, r 354.B 376 385.C
 390.I 445
 Vopěnka, Petr (1935–) 33.r
 Voronoĭ, Georgii Fedoseevich (1868–1908) 242.A
 Voss, Heinz-Jürgen 186.r
 Vranceanu, Gheorghe (1900–79) 434.C
 Vulikh, Boris Zakharovich (1913–78) 310.A

W

Wada, Junzo (1927–) 164.C
 Wada, Yasusi (Nei) (1787–1840) 230
 Waelbroeck, Lucien (1929–) 36.M
 Wage, M. L. 117.E
 Wagner, Harvey Maurice (1931–) 307.r 408.r
 Wagner, Herbert 39.F
 Wagner, S. W. 95.r
 Wagschal, Claude 321.G

Wagstaff, Samuel S.

- Wagstaff, Samuel S. 14.L 145
 Wahl, Jonathan Michael (1945-) 9.r
 Wahlen, G. E. 145.r
 Wait, R. 223.r 301.r 304.r
 Wakakuwa, Hidekiyo (1925-) 36.r
 Wald, Abraham (1902-50) 376.r 8.A 399.H, M, r
 400.r 401.F, r 421.r
 Waldhausen, Friedhelm (1938-) 65.E 235.B
 Waldschmidt, Michel (1946-) 430.D, r
 Wales, David B. (1939-) 151.I App. B, Table 6
 Walfisz (Val'fisz), Arnold Z. (1892-1962) 4.D 123.D
 220.B 242.r 295.D
 Walker, Arthur G. 359.E
 Walker, G. 421.D
 Walker, M. R. 376.r
 Walker, R. C. 425.r
 Walker, Robert John (1909-) 9.r 15.B
 Wall, Charles Terens Clegg (1936-) 114.B, F,
 H, J, K, r
 Wallace, Andrew Hugh (1926-) 114.F, L
 Wallach, Nolan R. 178.r 199.r 249.r 275.A, F
 364.r 365.G 437.W
 Wallis, Jennifer Seberry 241.r
 Wallis, John (1616-1703) 20 265 332 App. A,
 Table 10.VI
 Wallis, Walter Denis 241.r
 Wallis, Wilson Allen (1912-) 371.D
 Wallman, Henry (1915-) 117.r
 Walsh, John Joseph (1948-) 117.I
 Walsh, Joseph Leonard (1895-1973) 223.r 317.C
 336.F, I
 Walter, John H. (1927-) 151.J
 Walter, J. S. 142.C
 Walter, Wolfgang (1927-) 211.r
 Walters, Peter (1943-) 136.H
 Walther, Hansjoachim 186.r
 Wang Hsien-Chung (1918-78) 81 110.E 148.E
 152.r 199.r 413.r
 Wang, Ju-Kwei (1934-) 164.G
 Wang Xiaotong (c. early 7th century) 57.A
 Wantzel, Pierre-Laurent (1814-48) 179.A
 Ward, Harold Nathaniel (1936-) App. B, Table 5.r
 Ward, R. 16.r
 Waring, Edward (1736-98) 4.E
 Warner, Frank Wilson, III (1938-) 364.H
 Warner, Garth William, Jr. (1940-) 249.r 437.r
 Warning, E. 118.B
 Warschawski, Stefan Emanuel (1904-) 77.C
 Washington, A. 432.r
 Washington, Lawrence Clinton (1951-) 14.L 450.J
 Washio, Yasutoshi (1929-) 399.r
 Washizu, Kyuichiro (1921-81) 271.r
 Wasow, Wolfgang Richard (1909-) 25.B, r 30.r
 107.r 254.r 289.E 304.r
 Wassermann, Gordon 51.r
 Watabe, Tsuyoshi (1934-) 431.D
 Watanabe, Kinji (1946-) 323.J
 Watanabe, Shinzo (1935-) 44.E, r 45.r 115.C, r
 261.r 262.r 406.B, D, F
 Watanabe, Takeshi (1931-) 260.J
 Watari, Chinami (1932-) 336.D
 Waternaux, Christine M. 280.r
 Watson, George Leo (1909-) 4.E 348.r
 Watson, George Neville (1886-) 39.E, r 160.C
 174.r 220.B 268.r 389.r App. A, Tables 19.III, IV
 NTR
 Watson, H. W. 44.B, C, r
 Watt, J. M. 303.r
 Wayland, Harold (1909-) 298.r
 Weaver, W. 403.r
 Weber, Claude Alain (1937-) 65.r
 Weber, H. F. 39.G App. A, Table 19.IV
 Weber, Heinrich (1842-1913) 8.r 11.B, r 12.B 73.A
 98 167.C 236.r 363.*, r App. A, Tables 19.III, IV,
 20
 Weber, Wilhelm Eduard (1804-91) 363
 Webster, Arthur Gordon (1863-1923) 322.r
 Webster, Sidney M. (1945-) 344.F
 Wedderburn, Joseph Henry MacLagan (1882-1948)
 29.E, F 149.M 190.L 368.G
 Wehrl, Alfred (1941-) 212.r
 Weierstrass, Karl Theodor Wilhelm (1815-97)
 9.D 11.B, D 20 21.A, E 46.C 58.C 84.C 106.B 109
 120.A 134.F 140 168.B 174.A 198.D, I, N, Q
 229.r 236 267 272.A 273.F 274.F 275.A, B 294.A
 334.B, C 336.A, F 339.A, D 355.D, r 370.B 379.H
 429.B 430.D 435.A 447 App. A, Table 16.IV
 Weil, André (1906-) 3.C, E, M, r 4.D 6.E, r 9.E,
 H, r 12.B, r 13.M, r 14.r 16.A, C 20 21.G 27.r 28
 32.C, D 59.H, r 60.O 73.B 109.*, r 118.B, D, E
 122.F, G, r 182.E 192.r 196 225.G, r 232.B 422.r
 436.A, r 437.P 450.A, H, M, O-S, r
 Weinberg, B. L. 96.r
 Weinberg, Louis (1919-) 282.r
 Weinberg, N. 425.U
 Weinberg, Steven (1933-) 132.C, D, r
 Weinberger, Hans Felix (1928-) 323.r 327.r
 Weingarten, Leonhard Gottfried Johannes Julius
 (1836-1910) 111.H, I 365.C App. A, Table 4.I
 Weinstein, Alan David (1943-) 126.N 178.r
 Weinstock, Robert 441.r
 Weir, M. D. 425.r
 Weisberger, William I. (1937-) 132.C
 Weiss, Benjamin (1941-) 136.E-G
 Weiss, Edwin (1927-) 14.r 200.r
 Weiss, Guido Leopold (1928-) 168.B 224.r
 Weiss, Lionel (1923-) 398.r
 Weiss, Max L. (1933-) 43.r
 Weitsman, Allen W. 272.K, r
 Weitzenböck, Roland W. (1885-) 226.C
 Welch, Bernard Lewis 400.G
 Weldon, Edward J., Jr. 63.r
 Wells, Raymond O'Neil, Jr. (1940-) 164.K 232.r
 344.D, E
 Welsh, James Anthony Dominic 66.r
 Wendroff, Burton 304.F
 Wentzel, Gregor (1898-) 25.B
 Wentzell → Venttsel'
 Wermer, John (1927-) 164.F, G, I, K, r
 West, James Edward (1944-) 382.D
 Westlake, Joan R. 302.r
 Westwater, Michael John (1942-) 146.A
 Wets, M. J. 408.r
 Weyl, Claus Hugo Hermann (1885-1955) 7.r 9.r
 11.B 13.H, J, Q, R 14.r 18.A 20 21.N 60.r 69.B 109
 112.D, I, N, O 124.B, r 126.L 137 139.B, r 156.B
 182.F, H, r 190.r 196.r 197.A 198.r 225.I 226.r
 248.F, P, R, W, Z, r 249.U, V, r 255.E 272.L
 323.E, G, M 331.E 351.C, r 359.r 362.H 367.A, E, r
 377.A 390.I 391.I 413.G, J 434.B 437.DD 442.r 445
 448 App. A, Table 4.II
 Weyl, Fritz Joachim (1915-77) 21.N, r 124.B, r
 272.L 448
 Weyrich, Rudolf (1894-) 39.r App. A, Table 19.III
 Whaples, George William (1914-81) 14.F
 Wheeler, John Archibald (1911-) 359.r 386.C
 434.C
 White, Paul A. (1915-) 298.r
 Whitehead, Alfred North (1861-1947) 156.B 319.r
 411.A, r

**Encyclopedic
Dictionary
of
Mathematics**

Second Edition

Yamauchi, Kazunari (1945) 364.F
Yamauti, Ziro (1898–1984) 142.B NTR
Yamazaki, Keijiro (1932–) 362.J
Yamazato, Makoto (1949–) 341.H
Yamazi, Nusizumi (1704–72) 230
Yanagawa, Takashi (1940–) 371.A
Yanagihara, Hiroshi (1934–) 203.r
Yanai, Haruo (1940–) 346.r
Yanase, Mutsuo (1922–) 212.B
Yanenko, Nikolai Nikolaevich (1921–84) 204.r
Yang Chen Ning (1922–) 80.Q, r 132.D, r 150.G
359.C
Yang Chung-Tao (1923–) 178.r
Yang Hui (fl. 1261) 57.B
Yang, Paul C. P. (1947–) 391.E
Yano, Kentaro (1912–) 72.r 80.r 109.r 11 4.r
364.F 365.H 417.r
Yano, Shigeki (1922–) 159.G
Yanpol'skiĭ, Avraam Ruvimovich (1905–) NTR
Yaspan, Arthur J. 307.r
Yasugi, Mariko (1937–) 156.E
Yasuhara, Mitsuru (1932–) 411.J
Yasuura, Kamenosuke (1923–) 282.r
Yates, Frank (1902–) 19.r 371.C STR
Yau Shing-Tung (1949–) 72.G 115.D 183.r 195.r
232.C, r 235.E 275.C, D, H 364.r 365.H, L
391.D, E
Yen Chih-Ta 427.B
Yennie, Donald Robert (1924–) 146.B
Yin Wên-Lin 242.A
Yoneda, Nobuo (1930–) 52.r 200.K
Yoneyama, Kunizo (1877–1968) 79.D
Yor, Marc (1949–) 176.C
Yorke, James A. 126.N 303.G
Yoshida, Hiroyuki (1947–) 450.S
Yoshida, Masaaki (1948–) 428.H
Yoshida, Masao (1913–) NTR
Yoshida, Norio (1916–70) 367.I
Yoshida, Tomoyoshi (1946–) 431.D
Yoshikawa, Atsushi (1942–) 224.E
Yoshikawa, Zituo (1878–1915) 437.r
Yoshizawa, Taro (1919–) 163.r 290.r 394.r
Yosida, Kōsaku (1909–) 36.r 37.r 68.r 107.r 112.r
115.A 136.B, r 162 168.r 192.r 197.r 240.r 251.r
286.X 288.B 306.A 378.B, D, H, r 390.r
Yosida, Mituyosi (1598–1672) 230
Yosida, Yōiti (1898–) NTR
Youden, William John (1900–71) 102.K
Youla, Dante C. 86.D
Young, Alfred (1873–1940) 362.H
Young, David Monaghan (1923–) 301.r 302.r
Young, Gail Sellers (1915–) 79.r
Young, H. Peyton 173.E 201.r
Young, John Wesley (1879–1932) 343.r
Young, Richard Donald (1929–) 215.B
Young, Thomas (1773–1829) 271.G
Young, William Henry (1863–1942) 106.H 159.J
224.E 317.B App. A, Table 8
Youngs, J. W. T. 157.E
Yukawa, Hideki (1907–81) 132.A 150.A 338.M
Yule, George Udny (1871–1951) 421.D
Yuzvinskii, Sergei Aronovich (1936–) 136.r
Yvon, Jacques (1903–) 402.J

Z

Zabreiko, Pëtr Petrovich (1939–) 251.r
Zabusky, Norman J. (1929–) 387.B
Zacks, Shelemياهو (1932–) 399.H
Zagier, Don 15.H 72.r 450.T

Zak, F. L. 16.I
Zakharov, V. E. 80.r 387.F, G
Zaleman, Lawrence 169.r
Zamansky, Marc (1916–) 336.D
Zaremba, Stanislaw (1863–1942) 120.B
Zaring, Wilson B. 33.r
Zariski, Oscar (1899–1986) 12.B, r 15.B, D, E, H, r
16.A, I, J, L, X, r 67.r 284.C, D, G, r 370.r 418.r
439.r
Zarnke, Charles Robert 123.C
Zassenhaus, Hans J. (1912–) 92.A, F 151.E, H, J, r
190.r 362.K
Zeeman, Erik Christopher (1925–) 51.A, r 65.A,
C, D, r 235.G 426
Žegalov → Zhegalov
Zeigler, Bernard P. 385.r
Zeller, Karl (1924–) 43.G 58.r 379.r
Zeller-Meier, Georges 308.F
Zemansky, Mark Waldo (1900–) 419.r
Zener, Clarence (1905–) 264.r
Zeno (c. 490–c. 430 B.C.) 187 319.C
Zenor, Phillip L. 273.K
Zermelo, Ernst Friedrich Ferdinand (1871–1953)
33.A, B, r 34.B, r 41.A 381.F, G, r
Zerna, Wolfgang (1916–) 271.r
Zhang Qiujian (c. 5th century) 57.A
Zhegalov, Valentin Ivanovich 326.r
Zhelobenko, Dmitrii Petrovich (1934–) 437.EE, r
Zia-ud-Din, M. App. B, Table 5.r
Ziemer, William P. (1934–) 246.J
Zilber, Joseph Abraham (1923–) 70.E 201.J
Ziller, Wolfgang (1952–) 279.G 364.r
Zimmermann, Wolfhart (1928–) 146.A 150.D
Zippin, Leo (1905–) 2.D 196 249.V, r 423.N 431.r
Zobin, N. M. 424.S
Zoll, Otto 178.G
Zoretti, M. Ludovic 79.D
Zorn, Max A. (1906–) 34.C, r 54.r 450.L
Zoutendijk, Guus (1929–) 292.E
Zsidó, László (1946–) 308.r
Zuckerman, Herbert Samuel (1912–70) 118.r 192.P
Zumino, Bruno (1923–) 150.D 386.B
Zurmühl, Rudolf (1904–) 298.r
Zuse, Konrad (1910–) 75.A
Zverkin, Aleksandr Mikhaĭlovich (1927–) 163.r
Zvyagin, V. G. 286.r
Zwinggi, Ernst (1905–) 214.r
Zygmund, Antoni (1902–) 136.B 159.E, G, r 168.B,
r 192.r 198.r 217.J, r 224.r 251.O 274.B, I 336.C, r
Zykov, Aleksandr Aleksandrovich (1922–) 186.r

**Encyclopedic
Dictionary
of
Mathematics**

Second Edition

by the
**Mathematical Society
of Japan**

edited by
Kiyosi Itô

**Volume II
O–Z
Appendices and Indexes**

**The MIT Press
Cambridge, Massachusetts,
and London, England**

Subject Index

Note: Citation is to article and section or to a table in an appendix, not to page.

A

α (cardinal number of \mathbb{N}) 49.A
 $A(\Omega)$ (the totality of functions bounded and continuous on the closure of Ω and holomorphic in Ω) 168.B
 $A_p(\Omega)$ (the totality of functions f that are holomorphic in Ω and that satisfy $\int_{\Omega} |f(z)|^p dx dy < \infty$) 168.B
 α -capacity 169.C
 α -excessive function 261.D
 α -limit point 126.D
 α -limit set (of an orbit) 126.D
 α -perfect, α -perfectness 186.J
 α -point (of a meromorphic function) 272.B
 α -pseudo-orbit 126.J
 α -quartile 396.C
 α -string 248.L
 α -trimmed mean 371.H
 α -adic completion (of an R -module) 284.B
 α -adic topology (of an R -module) 284.B
 \mathcal{A} -characteristic class (of a real oriented vector bundle) 237.F
 A -balanced mapping 277.J
 A - B -bimodule 277.D
 A -homomorphism
 between A -modules 277.E
 of degree p (between two graded A -modules) 200.B
 A -linear mapping (between A -modules) 277.E
 A -module 277.C
 A -optimality 102.E
 A set 22.A 409.A
 A -stability 303.G
 $A(x)$ -stability 303.G
 A_0 -stability 303.G
 $A(0)$ -stable 303.G
 A -submodule 277.C
 A -summable 379.N
 A -number 430.C
abacus 75.A
Abel, N. H. 1
Abel continuity theorem 121.D 339.B
Abelian category 52.N
Abelian differential 11.C 367.H
Abelian equation 172.G
Abelian ergodic theorem 136.B
Abelian extension 172.B
Abelian function 3.J
 elementary 3.M
Abelian function field 3.J
Abelian group(s) 2 190.A
 category of 52.B
 class of 202.N
 dual topological 422.C
 elementary 2.B
 elementary topological 422.E
 free 2.C
 meta- 190.H
 mixed 2.A
 primary 2.A
 reduced 2.D
 topological 422.A

torsion 2.A
torsion-free 2.A
of type p^∞ 2.D
Abelian ideal (of a Lie algebra) 248.C
Abelian integral 11.C
Abelian Lie algebra 248.C
Abelian Lie group 249.D
Abelian linear group over K 60.L
Abelian p -group 2.A
 complete 2.D
 divisible 2.D
Abelian potential 402.G
Abelian projection operator 308.E
Abelian subvariety 3.B
Abelian surface 15.H
Abelian theorems 240.G
Abelian variety (varieties) 3
 isogenous 3.C
 polarized 3.G
 simple 3.B
Abel integral equation 217.L
Abel method, summable by 379.N
Abel method of summation 379.N
Abel partial summation 379.D
Abel problem 217.L
Abel test 379.D
Abel theorem
 (on the Cauchy product of two series) 379.F
 (in the theory of algebraic functions) 3.L 11.E
aberration 180.C
 annual 392
 diurnal 392
Aberth (DKA) method, Durand-Kerner- 301.F
Abramov's formula 136.E
abscissa
 of absolute convergence (of a Dirichlet series) 121.B
 of absolute convergence (of a Laplace transform) 240.B
 of boundedness (of a Dirichlet series) 121.B
 of convergence (of a Dirichlet series) 121.B
 of convergence (of a Laplace transform) 240.B.H
 of regularity (of a Dirichlet series) 121.B
 of regularity (of a Laplace transform) 240.C
 of simple convergence (of a Dirichlet series) 121.B
 of uniform convergence (of a Dirichlet series) 121.B
 of uniform convergence (of a Laplace transform) 240.B
absolute (for a quadric hypersurface) 285.C
absolute Borel summable 379.O
absolute class field 59.A
absolute continuity
 generalized 100.C
 generalized, in the restricted sense 100.C
 space of 390.E
absolute continuity (*), generalized 100.C
absolute convergence, abscissa of
 (of a Dirichlet series) 121.B
 (of a Laplace transform) 240.B
absolute covariant 226.D
absolute curvature (of a curve) 111.C
absolute figure (in the Erlangen program) 137
absolute homology group 201.L
absolute inequality 211.A
absolute integral invariant 219.A
absolute invariant 12.A 226.A
absolutely closed space 425.U

First MIT Press paperback edition, 1993

Originally published in Japanese in 1954 by Iwanami Shoten, Publishers, Tokyo, under the title *Iwanami Sūgaku Ziten*. Copyright © 1954, revised and augmented edition © 1960, second edition © 1968, third edition © 1985 by Nihon Sugakkai (Mathematical Society of Japan).

English translation of the third edition © 1987 by The Massachusetts Institute of Technology.

All rights reserved. No part of this book may be reproduced in any form or by any means, electronic or mechanical, including photocopying, recording, or by any information storage and retrieval system, without permission in writing from The MIT Press.

This book was set in Monolaser Times Roman by Asco Trade Typesetting Ltd., Hong Kong, and printed and bound by Arcata Graphics, Kingsport in the United States of America.

Library of Congress Cataloging-in-Publication Data

Iwanami sūgaku jiten (ziten). English.
Encyclopedic dictionary of mathematics.
Translation of: Iwanami sūgaku jiten.
Includes bibliographies and indexes.
I. Mathematics—Dictionaries. I. Itô, Kiyosi, 1915–. II. Nihon Sūgakkai. III. Title.
QA5.I8313 1986 510'.3'21 86-21092
ISBN 0-262-09026-0 (HB), 0-262-59020-4 (PB)

Whitehead, George William (1918–) 64.r 70.r 148.r
202.P, Q, T–V, r
Whitehead, John Henry Constantine (1904–60)
65.A, C, F 90.r 91.r 109.r 114.A, C 178.A 200.L
202.F, N, P 237.J 426
Whiteside, Derek Thomas 265.r 283.r
Whitham, Gerald Beresford (1927–) 205.r
Whitin, Thomson M. 227.r
Whitney, D. Ransom 371.A, C
Whitney, Hassler (1907–) 56.B, F 58.B–E 66.r
105.A, D, K, r 111, J 114.A, B, D, r 126.E 147.A,
F, M 168.B 186.H 201.A, J 418.G
Whittaker, Edmund Taylor (1873–1956) 167.B
174.r 268.r 271.r 301.C 389.r 420.r 450.O App. A,
Tables 14.II, 19, 20.r
Whittle, Peter (1927–) 421.r
Whyburn, Gordon Thomas (1904–69) 79.r 93.r 426
Wick, Gian Carlo (1909–) 351.K
Widder, David Vernon (1898–1973) 94.r 220.D
240.B, D, E 341.r
Widlund, Olof B. (1938–) 303.G
Widman, Kjell-Ove 195.E
Widom, Harold (1932–) 164.K
Wieferich, Arthur 145
Wielandt, Helmut (1910–) 151.B, E, H, r
Wiener, Norbert (1894–1964) 5.C 18.A, r 20
36.L 37.A 45.A, B, D, r 48.A, B 58.r 86.E 95.*, r
120.B–D 123.B 125.O, BB 136.B 159.I 160.B,
E, G, r 162 176.C, I 192.F, H, O, P, r 207.C, D
222.C 250.E 260.E 338.G 339.r 342.A 395.D, r
406.B 407.r
Wierman, John Charles (1949–) 340.r
Wigert, S. 295.E
Wightman, Arthur Strong (1922–) 150.C, D, r
212.B 258.r 351.K 386.r
Wigner, Eugene Paul (1902–) 150.A 212.B 258.C, r
351.H, J–L, r 353.B, r 377.B 437.DD
Wijsman, Robert Arthur (1920–) 396.r
Wilcox, Calvin Hayden (1924–) 375.B
Wilcoxon, Frank 371.A–C
Wilczynski, Ernst Julius (1876–1932) 109 110.B
Wilder, Raymond Louis (1896–1982) 79.r 93.r
Wiles, Andrew (1953–) 14.L 257.H 450.J
Wilkes, James Oscroft (1932–) 304.r
Wilkinson, James Hardy (1919–) 138.C 298.r
300.r 301.r 302.r
Wilks, Samuel Stanley (1906–64) 280.B 374.r
396.r 399.P
Willers, Friedrich-Adolf (1883–) 19.r
Williams, David (1938–) 260.P
Williams, H. C. 123.C
Williams, Robert F. (1928–) 126.J, K, N
Williamson, Jack (1940–) 272.K, r
Williamson, Robert E. (1937–) 114.H
Willmore, Thomas James (1919–) 111.r 365.O
Wilson, B. M. 295.E
Wilson, Edwin Bidwell (1879–1964) 374.F
Wilson, John (1741–93) 297.G
Wilson, K. B. 102.r 301.L
Wilson, Kenneth G. (1936–) 361.r
Wiman, Anders (1865–1959) 429.B
Winnink, Marinus 308.H
Winter, David John (1939–) 172.r
Winters, Gayn B. 9.r
Wintner, Aurel Frederick (1903–58) 55.r 420.r
435.E
Wirtinger, Wilhelm (1865–?) 3.r 235.B, D 365.L
App. A, Table 8
Wishart, John 374.C
Witt, Ernst (1914–) 9.E 59.H 60.O 149.M 151.H

Name

Yamato, Kenji

161.B 248.J 348.E 449.A, B
Witten, Louis (1921–) 359.r
Wold, Herman O. A. 395.D
Wolf, Emil (1922–) 446.r
Wolf, Joseph Albert (1936–) 178.r 364.D 365.F
412.r 413.r
Wolfe, Douglas A. 371.r 374.r
Wolfe, Philip (1927–) 292.D 349.C
Wolff, Julius 743.K 77.C
Wolff, Thomas H. 164.I
Wolfowitz, Jacob (1910–81) 63.r 213.F 399.J, N, r
400.r
Wolovich, William A. (1937–) 86.D
Wolratschek, Hans 92.F
Wong, E. (1934–) 403.C
Wong, Sai-Wing (1940–) 314.r
Wong, Yui (1935–) 21.N
Wong, W. Muray (1934–) 86.E
Wood, John William (1941–) 154.B, H
Wood, Rex Chester (1920–) 19.r
Woodcock, Alexander E. R. 51.r
Woods, E. J. (1936–) 308.I 377.r
Woods, H. J. 92.F
Woolard, Edgar William (1899–) 392.r
Woronowicz, Stanisław Lech (1941–) 402.G
Wrench, John William, Jr. (1911–) 332.r
Wright, D. App. B, Table 5
Wright, Elisabeth Maitland 4.D, r 83.r 291.F 295.r
328.*, r
Wright, Sewall (1889–) 263.E
Wronski, Höené Joseph Maria (1776–1853) 208.E
Wu Chien-Shiung (1913–) 359.C
Wu Hung-Hsi (1940–) 178.r
Wu Tai Tsun (1933–) 402.r
Wu Wen-Tsün (1919–) 56.F, r 90.r App. A,
Table 6.V
Wulf, W. A. 75.r
Wylie, Alexander (1815–87) 57.C
Wylie, Shaun (1913–) 70.r 201.r

X

Xavier, Frederico José de Vasconcelos (1951–)
275.E
Xiahou Yang (c. 5th century) 57.A
Xu Yue (fl. 200) 57.A

Y

Yabuta, Kōzō (1940–) 164.G
Yaglom, Akiva Moiseevich (1921–) 44.B, C 395.r
407.C 433.r
Yaglom, Isaak Moiseevich (1921–) 89.r
Yamabe, Hidehiko (1923–60) 183 196 249.D, V
364.H
Yamada, Masami (1926–) 353.r
Yamada, Toshihiko (1939–) 362.r 450.T
Yamada, Toshio (1937–) 406.D
Yamagami, Shigeru (1955–) 212.B
Yamaguchi, Keizo (1951–) 344.C
Yamaguti, Masaya (1925–) 126.N 303.G, r 304.F
325.H
Yamamoto, Koichi (1921–) 241.E
Yamamoto, Sumiyasu (1917–) 96.F
Yamamoto, Yoshihiko (1941–) 450.S
Yamanaka, Takesi (1931–) 286.Z
Yamanoshita, Tsuneyo (1929–) 202.S
Yamashita, Hiroshi (1946–) 301.F
Yamasuge, Hiroshi (1926–60) 114.F
Yamato, Kenji (1948–) 154.G

Affine minimal surface

- affine minimal surface 110.C
- affine normal 110.C
- affine principal normal 110.C
- affine ring 16.A
- affine scheme 16.D
- affine space 7.A
- affine symmetric space 80.J
- affine torsion 110.C
- affine transformation(s) 7.E 364.F
 - group of 7.E
 - of a manifold with an affine connection 80.J
 - proper 7.E
 - regular 7.E
- affine variety 16.A
- affine Weyl group (of a symmetric Riemann space) 413.G
- affinity 7.E
 - equivalent 7.E
- age-dependent branching process 44.E
- Agnesi, witch of 93.H
- Ahlfors finiteness theorem 234.D
- Ahlfors five-disk theorem 272.J
- Ahlfors function 43.G 77.E
- Ahlfors principal theorem 367.B
- Ahlfors theory of covering surfaces 367.B
- Airy integral App. A, Table 19.IV
- Aitken interpolation scheme 223.B, App. A, Table 21
- Akizuki theorem, Krull- 284.F
- Alaoglu theorem, Banach-
 - (in a Banach space) 37.E
 - (in a topological linear space) 424.H
- Albanese variety 16.P
 - of a compact Kähler manifold 232.C
 - strict 16.P
- Aleksandrov compactification 207.B
- aleph 49.E
 - α (\aleph_α) 312.D
 - zero (\aleph_0) 49.E
- Alexander cohomology group 201.M
 - relative 201.M
- Alexander duality theorem 210.O
- Alexander horned sphere 65.G
- Alexander ideal (of a knot) 235.C
- Alexander-Kolmogorov-Spanier cohomology theory 201.M
- Alexander matrix (of a knot) 235.C
- Alexander polynomial (of a knot) 235.C
- Alexander polynomial (of a link) 235.D
- Alexander trick 65.D
- Alexander-Whitney mapping (map) 201.J
- Alfvén wave 259
- algebra 8
 - (over a field) 203.F
 - (of sets) 270.B
 - algebraic 29.J
 - alternative 231.A
 - AF- 36.H
 - approximately finite 36.H
 - association 102.J
 - associative 29 231.A
 - augmented 200.L
 - AW^* - 36.H
 - Azumaya 29.K
 - Banach 36.A
 - Banach *- 36.F
 - Boolean 42.A 243.A
 - Boolean, generalized 42.B
 - C^* - 36.G
 - C^* -, of type I 430.J
 - C^* -group (of a locally compact Hausdorff group) 36.L
 - Calkin 36.J 390.I
 - Cayley 54
 - Cayley, general 54
 - central separable 29.K
 - central simple 29.E
 - Clifford 61.A
 - over a commutative ring 29.A
 - composition 231.B
 - current 132.C
 - cyclic 29.G
 - derived (of a Lie algebra) 248.C
 - Dirichlet 164.B
 - disk 164.B
 - distributive 231.A
 - division 29.A
 - Douglas 164.I
 - dual 203.F
 - enveloping 200.L 231.A
 - enveloping, universal (of a Lie algebra) 248.J
 - enveloping von Neumann 36.G
 - exterior (of a linear space) 256.O
 - Frobenius 29.H
 - full matrix 269.B
 - function 164.A
 - graded 203.B
 - Grassmann (of a linear space) 255.O
 - group 29.C 38.L 192.H
 - Hecke 29.C 32.D
 - homological 200.A
 - Hopf 203.H
 - Hopf, dual 203.C
 - Hopf, elementary 203.D
 - Hopf, graded 203.C
 - increasing family of σ - 407.B
 - j - 384.C
 - Jordan 231
 - L_1 - (of a locally compact Hausdorff group) 36.L
 - Lie 248.A
 - liminal C^* - 36.H
 - linear 8
 - of logic 411.A
 - logmodular 164.B
 - multiplier 36.K
 - nonassociative 231.A
 - normal j - 384.C
 - normal simple 29.E
 - operator 308.A
 - optional σ - 407.B
 - PI - 29.J
 - postliminal C^* - 36.H
 - power associative 231.A
 - predictable σ - 407.B
 - quasi-Frobenius 29.H
 - quaternion 29.D
 - quaternion, generalized 29.D
 - quaternion, Hamilton 29.B
 - quaternion, totally definite 27.D
 - quotient 29.A
 - Racah 353.A
 - reduced 231.B
 - relationship 102.J
 - residue class 29.A
 - σ - 270.B
 - semigroup 29.C
 - semigroup, large 29.C
 - (semi)simple 29.A
 - separable 29.F,K 200.L

- additive functor 52.N
- additive group 2.E 190.A
 - complete 2.E
 - divisible 2.E
 - free 2.E
 - ordered 439.B
 - totally ordered 439.B
- additive interval function 380.B
 - continuous 380.B
- additive measure
 - completely 270.D
 - finitely 270.D
 - σ - 270.D
- additive number theory 4
- additive operator 251.A
- additive processes 5 342.A
 - temporally homogeneous 5.B
- additive set function 380.C
 - completely 380.C
 - finitely 380.B
 - μ -absolutely continuous 380.C
- additive valuation 439.B
- additivity
 - (in the theory of local observables) 150.E
 - complete (of the integral) 221.C
 - complete (of a measure) 270.D
 - for the contours (in the curvilinear integral) 94.D
 - countable 270.D
 - of probability 342.B
 - σ - 270.D
- address 75.B
 - single-, instructions 75.C
- adele 6.C
 - and idele 6
 - principal 6.C
- adele group
 - of an algebraic group 13.P
 - of a linear algebraic group 6.C
- adele ring (of an algebraic number field) 6.C
- adeles and ideles 6
- Adem formula App. A, Table 6.II
- Adem relation 64.B
- adequacy 396.J
- adherent point 425.B
- ADI (alternating direction implicit) method 304.F
- adiabatic law 205.B
- adiabatic process, quasistatic 419.B
- adiabatic wall 419.A
- adjacent
 - (chamber) 13.R
 - (edges) 186.B
 - (germ) 418.E
 - (vertices) 186.B
- adjacement matrix 186.G
- adjoin
 - a set to a field 149.D
 - a variable to a commutative ring 337.A
- adjoint
 - left (linear mapping) 52.K 256.Q
 - right (linear mapping) 52.K 256.Q
 - self- \rightarrow self-adjoint
- adjoint boundary condition 315.B
- adjoint boundary value problem 315.B
- adjoint differential equations 252.K
- adjoint differential expression 252.K
- adjoint functor 52.K
- adjoint group
 - isogenous to an algebraic group 13.N
 - of a Lie algebra 248.H
 - of a Lie group 249.P
- adjoint Hilbert space 251.E
- adjoint kernel (of a kernel of a potential) 338.B
- adjoint Lie algebra 248.B
- adjoint matrix 269.I
- adjoint operator
 - (on Banach spaces) 37.D 251.D
 - (on Hilbert spaces) 251.E
 - (of a linear partial differential operator) 322.E
 - (of a microdifferential operator) 274.F
 - (of a microlocal operator) 274.F
- adjoint representation
 - of a Lie algebra 248.B
 - of a Lie group 249.P
 - of a linear representation 362.E
- adjoint space (of a linear topological space) 424.D
- adjoint system
 - (of a complete linear system on an algebraic surface) 15.D
 - of differential equations 252.K
- adjunction formula 15.D
- adjustment
 - sampling inspection with 404.C
 - seasonal 397.N
- Adler-Weisberger sum rule 132.C
- admissible
 - (decision function) 398.B
 - (estimator) 399.G
 - (extremal length) 143.A
- admissible automorphic representations 450.N
- admissible control 405.A
- admissible function 46.A 304.B
- admissible homomorphism (between Ω -groups) 190.E
- admissible isomorphism (between Ω -groups) 190.E
- admissible lattice (in \mathbf{R}^n), S - 182.B
- admissible monomial (in Steenrod algebra) 64.B,
- admissible normal subgroup 190.E
- admissible ordinal 356.G
- admissible sequence (in Steenrod algebra) 64.B,
- App. A, Table 6.III
- admissible subgroup (of an Ω -group) 190.E
- Ado theorem 248.F
- advanced type (of functional differential equation) 163.A
- a.e. (almost everywhere) 270.D
- AF-algebra 36.H
- affect (of an algebraic equation) 172.G
- affectless algebraic equation 172.G
- affine (morphism) 16.D
- affine algebraic group 13.A
- affine algebraic variety 16.A
 - quasi- 16.C
- affine arc element 110.C
- affine arc length 110.C
- affine binormal 110.C
- affine connection 80.H 286.L
 - canonical (on \mathbf{R}^n) 80.J
 - coefficients of 80.L
- affine coordinates 7.C
- affine curvature 110.C
- affine differential geometry 110.C
- affine frame (of an affine space) 7.C
- affine geometry 7
 - in the narrower sense 7.E
- affine length 110.C
- affine locally symmetric space 80.J
- affinely congruent 7.E
- affine mapping 7.E

Absolutely continuous

- absolutely continuous
 - (function) 100.C
 - (mapping in the plane) 246.H
 - (measure) 270.L
 - (set function) 380.C
 - (vector measure) 443.G
 - generalized 100.C
 - μ - 380.C
 - in the restricted sense 100.C
 - in the sense of Tonelli 246.C
- absolutely continuous (*) 100.C
- absolutely continuous spectrum 390.E
- absolutely convergent
 - (double series) 379.E
 - (infinite product) 379.G
 - (Laplace-Stieltjes integral) 240.B
 - (power series) 21.B
 - (series) 379.C
 - (series in a Banach space) 443.D
 - uniformly 435.A
- absolutely convex set (in a topological linear space) 424.E
- absolutely integrable function 216.E
- absolutely irreducible (representation) 362.F
- absolutely irreducible character 362.E
- absolutely measurable 270.L
- absolutely p -valent 438.E
 - locally 438.E
- absolutely simple algebraic group 13.L
- absolutely stable
 - (linear k -step method) 303.G
 - (system of differential equations) 291.E
- absolutely summing (operator) 68.N
- absolutely uniserial algebra 29.I
- absolute minimality 16.I
- absolute moment (k th) 341.B
- absolute multiple covariant 226.E
- absolute neighborhood retract 202.D
 - fundamental (FANR) 382.C
- absolute norm (of an integral ideal) 14.C
- absolute parallelism 191.B
- absolute retract 202.D
 - fundamental (FAR) 382.C
- absolute stability 303.G
 - interval of 303.G
 - region of 303.G
 - region of, of the Runge-Kutta (P, p) method 303.G
- absolute temperature 419.A
- absolute value
 - (of a complex number) 74.B
 - (of an element of an ordered field) 149.N
 - (of an element of a vector lattice) 310.B
 - (of a real number) 355.A
 - (of a vector) 442.B
- absorb (a subset, topological linear space) 424.E
- absorbing barrier 115.B
- absorbing set (in a topological linear space) 424.E
- absorption cross section 375.A
- absorption law
 - in the algebra of sets 381.B
 - in a lattice 243.A
- absorption principle, limiting 375.C
- abstract algebraic variety 16.C
- abstract L space 310.G
- abstract L_p space 310.G
- abstract M space 310.G
- abstract Riemann surface 367.A
- abstract simplicial complex 70.C
- abstract space 381.B
- abstract variety 16.C
- abundant number 297.D
- acceleration parameter 302.C
- acceptance 400.A
- acceptance region 400.A
- accepted 31.D
- accessible (from a region) 93.K
- accessible boundary point 333.B
- accretive operator (in a Hilbert space) 251.J 286.C
- accumulated error 138.C
- accumulation point 87.C 425.O
 - complete 425.O
- acoustic problem 325.L
- act
 - on a commutative ring 226.A
 - freely (on a topological space) 122.A
- action 431.A
 - R - 126.B
 - rational 226.B
 - reductive 226.B
 - Z - 126.B
- action and reaction, law of 271.A
- action integral 80.Q
- action space 398.A
- activity 281.D
- activity analysis 376
- actuarial mathematics 214.A
- acute angle 139.D
- acute type 304.C
 - strongly 304.C
- acyclic complex 200.C 200.H
- Adams-Bashforth method 303.E
- Adams conjecture 237.I
- Adams-Moulton method 303.E
- Adams operation 237.E
- adapted (stochastic process) 407.B
- adaptive scheme 299.C
- addition
 - (in a commutative group) 190.A
 - (of natural numbers) 294.B
 - (in a ring) 368.A
 - (of unfoldings) 51.D
- addition formula
 - algebraic 3.M
 - for e^z 131.G
 - for sine and cosine 432.A
 - of trigonometric functions App. A, Table 2.1
- addition theorem
 - of Bessel functions 39.B
 - of cylindrical functions App. A, Table 19.III
 - of Legendre functions 393.C
 - of the \wp -function App. A, Table 16.IV
 - of sn, cn, dn App. A, Table 16.III
 - of the ζ -function App. A, Table 16.IV
- additive
 - completely \rightarrow completely additive
 - countably \rightarrow countably additive
 - finitely \rightarrow finitely additive
 - σ - \rightarrow σ -additive
 - totally \rightarrow totally additive
- additive category 52.N
- additive class
 - completely 270.B
 - countably 270.B
 - finitely 270.B
- additive functional
 - (of a Markov process) 261.E
- martingale 261.E
- natural 261.E
- perfect 261.E

- σ - 270.B
- σ , tail 342.G
- σ -, topological 270.C
- σ -, well-measurable 407.B
- simple 29.A
- solvable 231.A
- Staudt 343.C
- Steenrod 64.B
- supplemented 200.L
- symmetric 29.H
- tensor (on a linear space) 256.K
- Thom 114.H
- total matrix 269.B
- uniform 164.A
- uniserial 29.I
- uniserial, absolutely 29.I
- uniserial, generalized 29.I
- unitary 29.A
- universal enveloping (of a Lie algebra) 248.J
- vector App. A, Table 3.I
- von Neumann 308.C
- von Neumann, induced 308.C
- von Neumann, reduced 308.C
- W^* - 308.C
- weak* Dirichlet 164.G
- zero 29.A
- algebra class (of central simple algebras) 29.E
- algebra class group 29.E
- algebra extension 29.D 200.L
- algebra homomorphism 29.A
- algebraic addition formula 3.M
- algebraic algebra 29.J
- algebraic analysis 125.A
- algebraically closed (in a field) 149.I
- algebraically closed field 149.I
 - quasi- 118.F
- algebraically dependent (on a family of elements of a field) 149.K
- algebraically dependent elements (of a ring) 369.A
- algebraically equivalent cycles 16.R
- algebraically equivalent to 0 (a divisor on an algebraic variety) 16.P
- algebraically independent (over a field) 149.K
- algebraically independent elements (of a ring) 369.A
- algebraically simple eigenvalue 390.E
- algebraic branch point (of a Riemann surface) 367.B
- algebraic closure
 - of a field 149.I
 - separable 257.E
- algebraic correspondence 9.H 16.I
 - group of classes of 9.H
- algebraic curves 11.A
 - irreducible 11.B
 - plane 9.B
- algebraic cycles 450.Q
- algebraic differential equation 113 288.A
- algebraic dimension (of a compact complex manifold) 74.F
- algebraic element (of a field) 149.E
- algebraic equations 10, App. A, Table 1
 - in m unknowns 10.A
- algebraic extension 149.E
- algebraic family (of cycles on an algebraic variety) 16.R
- algebraic fiber space 72.I
- algebraic function 11.A
- algebraic function field
 - over k of dimension l 9.D
 - over k of transcendence degree 1 9.D
 - in n variable 149.K
- ζ -function of 450.P
- algebraic fundamental group 16.U
- algebraic geometry 12.A
- algebraic groups 13
 - absolutely simple 13.U
 - affine 13.A
 - almost simple 13.U
 - connected 13.A
 - isogenous 13.A
 - k -almost simple 15.O
 - k -anisotropic 13.G
 - k -compact 13.G
 - k -isotropic 13.G
 - k -quasisplit 13.O
 - k -simple 13.O
 - k -solvable 13.F
 - k -split 13.N
 - linear 13.A
 - nilpotent 13.F
 - reductive 13.I
 - semisimple 13.I
 - simple 13.L
 - solvable 13.F
 - unipotent 13.E
- algebraic group variety 13.B
- algebraic homotopy group 16.U
- algebraic integer 14.A
- algebraic K -theory 237.J
 - higher 237.J
- algebraic Lie algebra 13.C
- algebraic linear functional 424.B
- algebraic multiplicity (of an eigenvalue) 309.B
- algebraic number 14.A
- algebraic number fields 14
 - relative 14.I
- algebraic pencil 15.C
- algebraic point (over a field) 369.C
- algebraic scheme 16.D
- algebraic sheaf, coherent 16.E 72.F
- algebraic singularity (of an analytic function) 198.M
- algebraic solution (of an algebraic equation) 10.D
- algebraic space 16.W
- algebraic subgroup 13.A
- algebraic surfaces 15
- algebraic system
 - of r equations 10
 - in the wider sense 409.B
- algebraic topology 426
- algebraic torus 13.D
- algebraic varieties 16 16.C
 - abstract 16.C
 - affine 16.A
 - complex 16.T
 - normal 16.F
 - product 16.A
 - projective 16.A
 - quasi-affine 16.C
 - quasiprojective 16.C
- algebra isomorphism 29.A
- algebroidal function(s) 17
 - entire 17.B
 - k -valued 17.A
- algorithm 97 356.C
 - composite simplex 255.F
 - division 297.A
 - division (of polynomials) 337.C
 - dual simplex 255.F

- Euclidean 297.A
- Euclidean (of polynomials) 337.D
- fractional cutting plane 215.B
- greedy 66.G
- heuristic 215.E
- partitioning 215.E
- primal-dual 255.F
- variable-step variable-order (VSVO) 303.E
- VSVO 303.E
- aliases 102.I
- alignment chart 19.D
- allied series (of a trigonometric series) 159.A
- all-integer 215.B
- all-integer algorithm 215.B
- all-integer programming problem 215.A
- allocation, optimum 373.E
- allocation process, multistage 127.A
- allowed homomorphism (between A -modules) 277.E
- allowed submodule 277.C
- almost all 342.B
- almost all points of a variety 16.A
- almost certainly converge 342.D
- almost certainly occur 342.B
- almost complex manifold 72.B
 - stably 114.H
 - weakly 114.H
- almost complex structure 72.B
 - tensor field of (induced by a complex structure) 72.B
- almost conformal 275.C
- almost contact manifold 110.E
- almost contact metric structure 110.E
- almost contact structure 110.E
- almost effective action (on a set) 415.B
- almost effectively (act on a G -space) 431.A
- almost everywhere converge 342.D
- almost everywhere hold (in a measure space) 270.D
- almost finite memory channels 213.F
- almost G -invariant statistic 396.I
- almost invariant test 400.E
- almost parallelizable manifold 114.I
- almost periodic (motion) 126.D
- almost periodic differential equation 290.A
- almost periodic functions 18
 - analytic 18.D
 - on a group 18.F
 - with respect to ρ 18.C
 - in the sense of Bohr 18.B
 - uniformly 18.B
- almost periodic group, maximally 18.I
- almost periodic group, minimally 18.I
- almost simple algebraic group 13.L
 - k - 13.O
- almost subharmonic 193.T
- almost surely converge 342.D
- almost surely occur 342.B
- almost symplectic structure 191.B
- alphabet 31.B 63.A 213.B
- alternate angles 139.D
- alternating contravariant tensor 256.N
- alternating covariant tensor 256.N
- alternating direction implicit (ADI) method 304.F
- alternating function 337.I
- alternating group of degree n 151.G
- alternating knot 235.A
- alternating law (in a Lie algebra) 248.A
- alternating matrix 269.B
- alternating multilinear form 256.H
- alternating multilinear mapping 256.H
- alternating polynomial 337.I
 - simplest 337.I
- alternating series 379.C
- alternating tensor field 105.O
- alternative (in game theory) 173.B
- alternative algebra 231.A
- alternative field 231.A
- alternative hypothesis 400.A
- alternative theorem, Fredholm 68.E 217.F
- alternizer 256.N
- altitude (of a commutative ring) 67.E
- altitude theorem, Krull 284.A
- amalgamated product (of a family of groups) 190.M
- amalgamated sum 52.G
- ambient isotropic 65.D
- ambient isotropy 65.D
- ambig class (of a quadratic field) 347.F
- ambig ideal (of a quadratic field) 347.F
- ambiguous point 62.D
- amicable number 297.D
- Amitsur cohomology group 200.P
- Amitsur complex 200.P
- amount insured 214.A
- amount of inspection, expected 404.C
- amount of insurance 214.A
- Ampère equations, Monge- 278, App. A, Table 15.III
- Ampère transformation 82.A
- amphicheiral knot 235.A
- ample divisor 16.N
 - very 16.N
- ample linear system 16.N
 - very 16.N
- ample over S (of a sheaf on a scheme) 16.E
 - relatively 16.E
 - very 16.E
- ample vector bundle 16.Y
- amplification matrix 304.F
- amplification operator of a scheme 304.F
- amplitude
 - (of a complex number) 74.C
 - (function) App. A, Table 16.III
 - (of an oscillation) 318.A
 - (of time series data) 397.N
 - (of a wave) 446
 - Feynman 146.B
 - partial wave scattering 375.E
 - probability 351.D
 - scattering 375.C,E 386.B
- amplitude function (of a Fourier integral operator) 274.C 345.B
- AMU estimator, k th-order 399.O
- analog, difference 304.E
- analog computation 19
- analog computers 19.E
 - electronic 19.E
- analog of de Rham's theorem 21.L
- analog quantity 138.B
- analog simulation 385.A
- analysis 20
 - activity 376
 - algebraic 125.A
 - backward 138.C
 - backward error 302.B
 - combinatorial 66.A
 - consistency of 156.E
 - convex 88
 - design-of-experiment 403.D
 - dimensional 116

Diophantine 296.A
 experimental 385.A
 factor 280.G
 forward 138.C
 function 20
 functional 162
 functional, nonlinear 286
 harmonic (on locally compact Abelian groups) 192.G
 intrablock 102.D
 microlocal 274.A 345.A
 multivariate 280
 of a network 282.C
 numerical 300
 principal component 280.F
 regression 403.D
 spectral 390.A
 analysis of variance 400.H 403.D
 multivariate 280.B
 table 400.H
 analytic
 (function) 21.B,C 198.A,H
 (predicate) 356.H
 complex, structure 72.A
 micro- (hyperfunction) 125.CC
 pseudo- (function) 352.B
 quasi- (function) 352.B
 quasi- (in the generalized sense) 58.F
 real 106.K 198.H
 analytical dynamics 271.F
 analytically continuable 198.I
 analytically hypoelliptic 112.D 323.I
 analytically independent elements 370.A
 analytically normal local ring 284.D
 analytically thin set 23.D
 analytically uniform spaces 125.S
 analytically unramified semilocal ring 284.D
 analytic almost periodic function 18.D
 analytic automorphism 21.J
 analytic capacity 169.F
 analytic continuation 198.G
 along a curve 198.I
 direct 198.G
 uniqueness theorem of 198.I
 in the wider sense 198.O
 analytic covering space 23.E
 analytic curve
 in an analytic manifold 93.B
 in a Euclidean plane 93.B
 analytic differential (on a Riemann surface) 367.H
 analytic fiber bundle
 complex 147.O
 real 147.O
 analytic functions 198.A,H
 complex 198.H
 inverse 198.L
 many-valued 198.J
 multiple-valued 198.J
 n -valued 198.J
 real 106.K 198.H
 in the sense of Weierstrass 198.I
 of several complex variables 21.B,C
 in the wider sense 198.O
 analytic geometry 181
 analytic hierarchy 356.H
 analytic homomorphism (between Lie groups) 249.N
 analytic index
 of an elliptic complex 237.H
 of an elliptic differential operator 237.H

Subject Index

Angle

analytic isomorphism 21.J
 between Lie groups 249.N
 analyticity, set of 192.N
 analytic manifold
 complex 72.A
 real 105.C
 analytic mapping 21.J
 analytic measurable space 270.C
 analytic neighborhood
 of a function element in the wider sense 198.O
 of a Riemann surface 367.A
 analytic number theory 296.B
 analytic operation 22.B
 analytic operation function 37.K
 analytic perturbation 331.D
 analytic polyhedron 21.G
 analytic prolongation 198.G
 analytic relations, invariance theorem of 198.K
 analytic representation (of $GL(V)$) 60.D
 analytic set
 (in set theory) 22
 (in the theory of analytic spaces) 23.B
 co- 22.A
 complementary (in set theory) 22.A
 germ of 23.B
 irreducible (at a point) 23.B
 principal 23.B
 purely d -dimensional 23.B
 purely d -dimensional (at a point) 23.B
 analytic sheaf 72.E
 coherent 72.E
 analytic spaces 23
 Banach 23.G
 C - 23.E
 general 23.G
 K -complete 23.F
 normal 23.D
 in the sense of Behnke-Stein 23.E
 analytic structure
 complex 72.A
 real 105.D
 on a Riemann surface 367.A
 analytic submanifold, complex 72.A
 analytic subset (of a complex manifold) 72.E
 analytic subspace 23.C,G
 analytic torsion 391.M
 analytic vector (with respect to a unitary representation of a Lie group) 437.S
 analytic wave front set 274.D
 analyzer
 differential 19.E
 harmonic 19.E
 anchor ring 410.B
 ancient mathematics 24
 ancillary statistic 396.H 401.C
 Anger function 39.G, App. A, Table 19.IV
 angle 139.D 155.B
 (of a geodesic triangle) 178.A,H
 (of hyperspheres) 76.A
 (of a spherical triangle) 432.B
 acute 139.D
 alternate 139.D
 corresponding 139.D
 eccentric (of a point on a hyperbola) 78.E
 eccentric (of a point on an ellipse) 78.D
 Euler 90.C
 general 139.D
 non-Euclidean (in a Klein model) 285.C
 obtuse 139.D

Angular derivative (of a holomorphic function)

- regular polyhedral 357.B
- right 139.D
- straight 139.D
- straightening of 114.F
- supplementary 139.D
- trisection of 179.A
- vertical 139.D
- angular derivative (of a holomorphic function) 43.K
- angular domain 333.A
- angular frequency (of a sine wave) 446
- angular momentum 258.D 271.E
 - integrals of 420.A
 - intrinsic 415.G
 - orbital 315.E
 - theorem of 271.E
- angular momentum density 150.B
- angular transformation 374.D
- anharmonic ratio 343.D
- anisotropic
 - (quadratic form) 13.G
 - k - (algebraic group) 13.G
- annihilation operator 377.A
- annihilator 422.D
 - left 29.H
 - reciprocity of (in topological Abelian groups) 422.E
 - right 29.H
- annual aberration 392
- annual parallax 392
- annuity contract 214.B
- annular domain 333.A
- annulator 422.D
- annulus conjecture 65.C
- anomaly
 - eccentric 309.B
 - mean 309.B
 - true 309.B
- Anosov diffeomorphism 126.J 136.G
- Anosov flow 126.J 136.G
- Anosov foliations 126.J
- Anosov vector field 126.J
- ANR (absolute neighborhood retract) 202.D,E
- antiautomorphism
 - (of a group) 190.D
 - (of a ring) 368.D
 - principal (of a Clifford algebra) 61.B
- antiendomorphism
 - (of a group) 190.D
 - (of a ring) 368.D
- antiequivalence (between categories) 52.H
- anti-Hermitian form 256.Q
- anti-Hermitian matrix 269.I
- antiholomorphic 195.B 275.B
- antihomomorphism
 - of groups 190.D
 - of lattices 243.C
 - of rings 368.D
- anti-isomorphic lattices 243.C
- anti-isomorphism
 - of groups 190.D
 - of lattices 243.C
 - of ordered sets 311.F
 - of rings 368.D
- antinomy 319.A
- antiparticle 132.A 386.B
- antipodal points (on a sphere) 140
- antipode 203.H
- anti-self-dual (G -connection) 80.Q
- antisymmetric
 - (Fock space) 377.A
 - (multilinear form) 256.H
 - (multilinear mapping) 256.H
 - (relation) 358.A
 - (tensor) 256.N
 - law 311.A
 - matrix 269.B
- antisymmetry, set of 164.E
- Antoine's necklace 65.G
- apartment 13.R
- aperiodic 136.E 260.B
- Apollonius problem (in geometric construction) 179.A
- a posteriori distribution 388.B
- a posteriori probability 342.F
- apparent force 271.D
- apparent singular point 254.L
- Appell hypergeometric functions of two variables 206.D, App. A, Table 18.I
- application 31.B
- approach
 - Bayesian 401.B
 - group-theoretic 215.C
 - non-Bayesian 401.B
 - S -matrix 132.C
 - state-space 86.A
- approximate derivative (of a measurable function) 100.B
- approximate functional equation (for zeta function) 450.B
- approximately derivable (measurable function) 100.B
- approximately finite (von Neumann algebra) 308.I
- approximately finite algebra 36.H
- approximately finite-dimensional 308.I
- approximation(s)
 - best (of a continuous function) 336.B
 - best (in evaluation of functions) 142.B
 - best (of an irrational number) 83.B
 - best polynomial, in the sense of Chebyshev 336.H
 - Diophantine 182.F
 - full discrete 304.B
 - least square 336.D
 - of linear type 142.B
 - method of successive (for an elliptic partial differential equation) 323.D
 - method of successive (for Fredholm integral equations of the second kind) 217.D
 - method of successive (for ordinary differential equations) 316.D
 - n th (of a differentiable function) 106.E
 - Oseen 205.C
 - overall, formula 303.C
 - Padé 142.E
 - Pauli 415.G
 - Prandtl-Glauert 205.B
 - polynomial, 336
 - scmidiscrete 304.B
 - simplicial (to a continuous mapping) 70.C
 - Stokes 205.C
 - Wilson-Hilferty 374.F
 - Yosida 286.X
- approximation method
 - in physics 25
 - projective 304.B
- approximation property
 - (of a Banach space) 37.L

bounded 37.L
 approximation theorem
 (on functions on a compact group) 69.B
 (on valuations) 439.G
 cellular 70.D
 Eichler's 27.D
 Kronecker's 422.K
 polynomial (for C^∞ -functions) 58.E
 simplicial 70.C
 Weierstrass 336.A
 a priori distribution 388.B
 least favorable 388.H
 a priori estimate 323.C
 in L^2 sense 323.H
 a priori probability 342.F
 AR (absolute retract) 202.D
 Arabic numerals 26
 Arab mathematics 26
 Araki axioms, Haag- 150.E
 Araki-Sewell inequality, Roepstorff- 402.G
 arbitrary constant 313.A
 arbitrary set 381.G
 arc(s) 93.B 186.B
 continuous 93.B
 Farey 4.B
 geodesic 178.H 364.B
 joined by an 79.B
 Jordan 81.D 93.B
 major 4.B
 minor 4.B
 open 93.B
 pseudo- 79.B
 simple 93.B
 Arccos 131.E
 arc cos (arc cosine) 131.E
 arc element
 affine 110.C
 conformal 110.D
 Archimedean lattice-ordered group 243.G
 Archimedean ordered field 149.N
 Archimedean unit (of a vector lattice) 310.B
 Archimedean valuation 14.F 439.C
 Archimedean vector lattice 310.C
 Archimedes axiom
 in geometry 155.B
 for real numbers 355.B
 Archimedes spiral 93.H
 arc length 111.D
 affine 110.C
 representation in terms of (for a continuous arc) 246.A
 Arcsin 131.E
 arcsin (arcsine) 131.E
 arcsine law
 for Brownian motion 45.E
 for distribution function 250.D
 for random walk 260.E
 arcsine transformation 374.D
 Arctan 131.E
 arctan (arctangent) 131.E
 arcwise connected component 79.B
 arcwise connected space, locally 79.B
 area 246
 (Euclidean) 139.F
 (of a polygon) 155.F
 (of a set in \mathbf{R}^2) 216.F
 Banach (of a surface) 246.G
 of concentration 397.E
 definite, set of 216.F
 Geöcze (of a surface) 246.E

Subject Index

Ascending chain

Gross (of a Borel set) 246.G
 inner 216.F 270.G
 Janzen (of a Borel set) 246.G
 Lebesgue (of a surface) 246.C
 mixed (of two ovals) 89.D
 outer 216.F 270.G
 Peano (of a surface) 246.F
 surface, of unit hypersphere App. A, Table 9.V
 areal element (in a Cartan space) 152.C
 areal functional 334.B
 areally mean p -valent 438.E
 area theorem 438.B
 Bers 234.D
 Arens-Royden theorem 36.M
 Arens theorem, Mackey- 424.N
 Arf-Kervaire invariant 114.J
 Argand plane, Gauss- 74.C
 argument
 (of a complex number) 74.C
 behind-the-moon 351.K
 argument function 46.A
 argument principle 198.F
 arithmetical (predicate) 356.H
 arithmetical hierarchy 356.H
 of degrees of recursive unsolvability 356.H
 arithmetically equivalent
 (lattices) 92.B
 (pairs) 92.B
 (structures) 276.D
 arithmetic crystal classes 92.B
 arithmetic function 295.A
 arithmetic genus
 (of an algebraic curve) 9.F
 (of an algebraic surface) 15.C
 (of a complete variety) 16.E
 (of a divisor) 15.C
 virtual (of a divisor) 16.E
 arithmetic mean 211.C 397.C
 arithmetic of associative algebras 27
 arithmetico-geometric mean 134.B
 arithmetic operations 294.A
 arithmetic progression 379.I, App. A, Table 10.I
 prime number theorem for 123.D
 arithmetic subgroup 13.P 122.F, G
 arithmetic unit 75.B
 arithmetization (of metamathematics) 185.C
 array 96.C
 balanced 102.L
 k - 330
 orthogonal 102.L
 arrow diagram 281.D
 Arrow-Hurwicz-Uzawa gradient method 292.E
 artificial variables 255.C
 Artin, E. 28
 Artin conjecture 450.G
 Artin general law of reciprocity 59.C
 Artin-Hasse function 257.H
 Artinian module 277.I
 Artinian ring 284.A
 left 368.F
 right 368.F
 Artin L -function 450.G, R
 Artin-Rees lemma 284.A
 Artin-Schreier extension (of a field) 172.F
 Artin symbol 14.K
 Arzelà theorem, Ascoli- 168.B 435.D
 ascending central series (of a Lie algebra) 248.C
 ascending chain
 in an ordered set 311.C
 of subgroups of a group 190.F

Ascending chain condition

- ascending chain condition
 - in an ordered set 311.C
 - for subgroups of a group 190.F
- Ascoli-Arzelà theorem 168.B 435.D
- Ascoli theorem 435.D
- a.s. consistent 399.K
- assembler 75.C
- associate (of an element of a ring) 67.H
- associated convergence radii 21.B
- associated diagrams (in irreducible representations of orthogonal groups) 60.J
- associated differential equation, Legendre's 393.A
- associated factor sets (of crossed products) 29.F
- associated factor sets (for extension of groups) 190.N
- associated fiber bundle 147.D
- associated flow 136.F
- associated form (of a projective variety) 16.S
- associated graded ring 284.D
- associated integral equation (of a homogeneous) integral equation) 217.F
- associated Laguerre polynomials 317.D
- associated Legendre functions 393.C, App. A, Table 18.III
- associated prime ideal 67.F
- associated principal bundle 147.D
- association, measure of 397.K
- association algebra 102.J
- association matrix 102.J
- associative, homotopy 203.D
- associative algebra(s) 102.J 231.A
 - power 231.A
- associative law
 - for the addition and multiplication of natural numbers 294.B
 - in the algebra of sets 381.B
 - for cardinal numbers 49.C
 - for the composite of correspondences 358.B
 - general (for group composition) 190.C
 - for group composition 190.A
 - in a lattice 243.A
 - in a ring 368.A
- associative multiplication of a graded algebra 203.B
- assumed rate of interest 214.A
- assumption
 - inverse 304.D
 - Stokes 205.C
- asteroid 93.H
- astronomy, spherical 392
- asymmetric (factorial experiment) 102.H
- asymmetric Cauchy process 5.F
- asymptote (of an infinite branch) 93.G
- asymptotically developable (function) 30.A
- asymptotically distributed 374.D
- asymptotically efficient estimator 399.N
 - first-order 399.O
 - k th-order 399.O
- asymptotically mean unbiased 399.K
- asymptotically median unbiased estimator (AMU) k th-order 399.O
- asymptotically normal estimator
 - best (BAN) 399.K
 - consistent and (CAN) 399.K
- asymptotically normally distributed 399.K
- asymptotically optimal 354.D
- asymptotically stable 126.F 286.S 394.B
 - globally 126.F
 - uniformly 163.G
- asymptotically unbiased 399.K
- asymptotic bias 399.K
- asymptotic completeness 150.D
- asymptotic concentration 399.N
- asymptotic condition, LSZ 150.D
- asymptotic cone 350.B
- asymptotic convergence 168.B
- asymptotic covariance matrix 399.K
- asymptotic curve 110.B 111.H
- asymptotic direction 111.H
- asymptotic distribution, k th-order 399.O
- asymptotic efficiency 399.N
 - second-order 399.O
 - higher-order 399.O
- asymptotic expansion 30.A, App. A, Table 17.I (of a pseudodifferential operator) 345.A
 - method of matched 112.B
 - Mirakshisundaram-Pleijel 391.B
- asymptotic fields 150.D
- asymptotic freedom 361.B
- asymptotic method 290.D
- asymptotic normality 399.K
- asymptotic path (for a meromorphic function) 272.H
- asymptotic perturbation theory 331.D
- asymptotic power series 30.A
- asymptotic property (of solutions of a system of linear ordinary differential equations) 314.A
- asymptotic ratio set 308.I
- asymptotic ray 178.F
- asymptotic representation
 - Debye 39.D, App. A, Table 19.III
 - Hankel App. A, Table 19.III
- asymptotic sequence 30.A
- asymptotic series 30.A
- asymptotic set 62.A
- asymptotic solution 325.M
- asymptotic tangent 110.B
- asymptotic value of a meromorphic function 62.A 272.H
- asymptotic value theorem, Lindelöf 43.C
- asynchronous system (of circuits) 75.B
- Atiyah-Bott fixed point theorem 153.C
- Atiyah-Singer fixed point theorem 153.C
- Atiyah-Singer index theorem 237.H
 - equivariant 237.H
- atlas 105.C
 - of class C^r 105.D
 - of class C^∞ 105.D
 - oriented 105.F
- atled (nabla) 442.D
- atmospheric refraction 392
- at most (for cardinal numbers) 49.B
- atomic
 - (measurable set) 270.D
 - at 0 163.H
- atomic element (in a complemented modular lattice) 243.F
- atomic formula 276.A 411.D
- atomless 398.C
- at random 401.F
- attaching a handle 114.F
- attaching space 202.E
- attraction, domain of 374.G
- attractor 126.F
 - strange 126.N
- attribute, sampling inspection by 404.C
- augmentation
 - (of an algebra) 200.M
 - (of a chain complex) 200.C
 - (of a coalgebra) 203.F

(of a cochain complex) 200.F
 (of a complex in an Abelian category) 200.H
 augmented algebra 200.M
 augmented chain complex 200.C
 autocorrelation 421.B
 autocorrelation coefficient 397.N
 autocovariance, sample 421.B
 automata 31
 automatic integration scheme 299.C
 automaton 31.A
 deterministic linear bounded 31.D
 finite 31.D
 nondeterministic linear bounded 31.D
 push-down 31.D
 automorphic form 450.O
 of dimension $\sim k$ 32.B
 of type U 437.DD
 of weight m 32.A
 of weight k 32.B
 automorphic function(s) 32
 multiplicative 32.A
 with respect to Γ 32.A
 automorphism
 (of an algebraic system) 409.C
 (of a field) 149.B
 (of a group) 190.D
 (of an object in a category) 52.D
 (of a polarized Abelian variety) 3.G
 (of a probability space) 136.E
 (of a ring) 368.D
 analytic 21.J
 anti- (of a group) 190.D
 anti- (of a ring) 368.D
 differential 113
 Frobenius (of a prime ideal) 14.K
 group of (of a group) 190.D
 holomorphic 21.J
 inner (of a group) 190.D
 inner (of a ring) 368.D
 inner, group of (of a group) 190.D
 inner, group of (of a Lie algebra) 248.H
 involutive (of a Lie group) 412.B
 k -fold mixing 136.E
 Kolmogorov 136.E
 metrically isomorphic (on a measure space) 136.E
 modular 308.H
 outer, group of (of a group) 190.D
 outer, group of (of a Lie algebra) 248.H
 principal (of a Clifford algebra) 61.B
 shift 126.J
 spatially isomorphic (on a measure space) 136.E
 spectrally isomorphic (on a measure space) 136.E
 strongly mixing 136.E
 weakly isomorphic 136.E
 weakly mixing 136.E
 automorphism group (of a Lie algebra) 248.A
 automorphy, factor of 32.A
 autonomous 163.D 290.A
 autoregressive Gaussian source 213.E
 autoregressive integrated moving average process 421.G
 autoregressive moving average process 421.D
 autoregressive process 421.D
 auxiliary circle 78.D
 auxiliary equation, Charpit 320.D
 auxiliary units 414.A

Subject Index

Axiom of the power set

auxiliary variable 373.C
 average 211.C
 moving 397.N
 moving, process 421.D
 phase 402.C
 weighted moving 397.N
 average complexity 71.A
 average outgoing quality level 404.C
 average sample number 404.C
 averaging, method of 290.D
 AW^* -algebra 36.H
 axial-vector currents, partially conserved 132.C
 axial visibility manifold 178.F
 axiom(s) 35.A 411.I
 Archimedes (in geometry) 155.B
 Archimedes (for real numbers) 355.B
 congruence (in geometry) 155.B
 Eilenberg-Steenrod 201.Q
 Euclid 139.A
 first countability 425.P
 the first separation 425.Q
 the fourth separation 425.Q
 Fréchet 425.Q
 Haag-Araki 150.E
 Haag-Keslev 150.E
 Hausdorff 425.Q
 Kolmogorov 425.Q
 logical 337.C 411.I
 Martin 33.F
 mathematical 337.C 411.I
 Osterwalder-Schrader 150.F
 Pasch 155.B
 second countability 425.P
 the second separation 425.Q
 system of 35.B
 the third separation 425.Q
 Tietze's first 425.Q
 Tietze's second 425.Q
 Tikhonov's separation 425.Q
 Vietoris 425.Q
 Wightman 150.D
 axiom A diffeomorphism 126.J
 axiom A flow 126.J
 axiomatic quantum field theory 150.D
 axiomatic set theory 36 156.E
 axiom A vector field 126.J
 axiomatization 35.A
 axiomatize (by specifying a system of axioms) 35.B
 axiom of choice 33.B 34.A
 and continuum hypothesis, consistency of 33.D
 and continuum hypothesis, independence of 33.D
 axiom of comprehension 33.B 381.G
 axiom of constructibility 33.D
 axiom of determinacy 22.H
 axiom of determinateness 33.F
 axiom of ε -induction 33.B
 axiom of extensionality 33.B
 axiom of foundation 33.B
 axiom of free mobility (in Euclidean geometry) 139.B
 axiom of infinity 33.B 381.G
 axiom of linear completeness (in geometry) 155.B
 axiom of mathematical induction 294.B
 axiom of pairing 381.G
 axiom of parallels (in Euclidean geometry) 139.A 155.B
 axiom of the power set 33.B 381.G

Axiom of reducibility (in symbolic logic)

- axiom of reducibility (in symbolic logic) 156.B
 - 411.K
- axiom of regularity 33.B
- axiom of replacement 33.B 381.G
- axiom of separation 33.B
- axiom of strong infinity 33.E
- axiom of subsets 33.B 381.G
- axiom of substitution 381.G
- axiom of the empty set 33.B
- axiom of the sum set 33.B
- axiom of the unordered pair 33.B
- axiom of union 381.G
- axioms of continuity
 - Dedekind's 355.A
- axiom system(s) 35
 - of a structure 409.B
 - of a theory 411.I
- axis (axes)
 - of a circular cone 78.A
 - conjugate (of a hyperbola) 78.C
 - of convergence 240.B
 - coordinate (of an affine frame) 7.C
 - coordinate (of a Euclidean space) 140
 - imaginary 74.C
 - major (of an ellipse) 78.C
 - minor (of an ellipse) 78.C
 - optical 180.B
 - of a parabola 78.C
 - principal (of a central conic) 78.C
 - principal (of a parabola) 78.C
 - principal (of a quadric surface) 350.B
 - principal, of inertia 271.E
 - principal, transformation to 390.B
 - real 74.C
 - of rotation (of a surface of revolution) 111.I
 - transverse (of a hyperbola) 78.C
 - x_i - (of a Euclidean space) 140
- Ax-Kochen isomorphism theorem 276.E
- azimuth App. A, Table 3.V
- azimuthal quantum number 315.E
- Azumaya algebra 29.K
- Azumaya lemma, Krull- 67.D

B

- $\beta \rightarrow$ beta
- $\mathcal{B}(\Omega)$
 - $(=\mathcal{D}_{L,\infty}(\Omega))$ 168.B
 - (the space of hyperfunctions) 125.V
- $\mathbf{B}_{p,q}^s$ (Besov spaces) 168.B
- β -KMS state 402.G
- β -shadowed 126.J
- β -traced 126.J
- \mathfrak{B} -measurable function 270.J
- \mathfrak{B} -measurable set 270.C
- \mathfrak{B} -regular measure 270.F
- \mathfrak{B} -summable series 379.O
- $|\mathfrak{B}|$ -summable series 379.O
- b -function 125.EE 418.H
- BN -pair 13.R 343.I
- (B, N) -pair 151.J
- B_n set 22.D
- B**-complete (locally convex space) 424.X
- BA 102.L
- back substitution 302.B
- backward analysis 138.C
- backward difference 223.C, App. A, Table 21
- backward emission 320.A
- backward equation, Kolmogorov 115.A 260.F
- backward error analysis 302.B

- backward interpolation formula
 - Gauss 223.C
 - Newton 223.C
- backward moving average representation 395.D
 - canonical 395.D
- backward type 304.D,F
- badly approximable 83.B
- Baer sum (of extensions) 200.K
- Bahadur efficiency 400.K
- Baire condition 425.L
- Baire function 84.D
- Baire-Hausdorff theorem 273.J 425.N
- Baire measurable 270.L
- Baire property 425.L
 - Lebesgue measurability and 33.F
- Baire set 126.H 270.C
- Baire space 425.L
- Baire zero-dimensional space 273.B
- Bairstow method 301.E
- balanced array 102.L
- balanced fractional factorial design 102.I
- balanced incomplete block design 102.E
 - partially 102.J
- balanced mapping, A - 277.J
- balayage 338.L
- balayage principle 338.L
- ball 140
 - n - 140
 - open 140
 - open n - 140
 - spin 351.L
 - unit 140
 - unit (of a Banach space) 37.B
- ball knot, (p, q) - 235.G
- ball pair 235.G
- BAN (best asymptotically normal) 399.K
- Banach-Alaoglu theorem
 - (in a Banach space) 37.E
 - (in a topological linear space) 424.H
- Banach algebra(s) 36.A
- Banach analytic space 23.G
- Banach area (of a surface) 246.G
- Banach (extension) theorem, Hahn-
 - (in a normed space) 37.F
 - (in a topological linear space) 424.C
- Banach integral 310.I
- Banach lattice 310.F
- Banach Lie group 286.K
- Banach limit 37.F
- Banach manifold 105.Z
- Banach space(s) 37.A,B
 - reflexive 37.G
 - regular 37.G
- Banach star algebra 36.F
- Banach-Steinhaus theorem
 - (in a Banach space) 37.H
 - (in a topological linear space) 424.J
- Banach theorem 37.I
- band, Möbius 410.B
- bang-bang control 405.C
- Barankin theorem 399.D
- bar construction (of an Eilenberg-MacLane complex) 70.F
- bargaining set 173.D
- bargaining solution, Nash 173.C
- Barnes extended hypergeometric function 206.C,
 - App. A, Table 18.I
- barrel (in a locally convex space) 424.I
- barreled (locally convex space) 424.I
 - quasi- 424.I

- barrier 120.D
 - absorbing 115.B
 - reflecting 115.B,C
- Bartle-Dunford-Schwartz integral 443.G
- barycenter
 - (of points of an affine space) 7.C
 - (of a rigid body) 271.E
- barycentric coordinates
 - (in an affine space) 7.C 90.B
 - (in a Euclidean complex) 70.B
 - (in the polyhedron of a simplicial complex) 70.C
- barycentric derived neighborhood, second 65.C
- barycentric refinement 425.R
- barycentric subdivision
 - (of a Euclidean complex) 70.B
 - (of a simplicial complex) 70.C
- baryons 132.B
- base
 - (in a Banach space) 37.L
 - (curve of a roulette) 93.H
 - (of a logarithmic function) 131.B
 - (of a point range) 343.B
 - (of a polymatroid) 66.F,G
 - data 96.B
 - filter 87.I
 - local 425.E
 - for the neighborhood system 425.F
 - normal 172.E
 - open 425.F
 - for the space 425.E
 - for the topology 425.F
 - for the uniformity 436.B
- base functions 304.B
- base point
 - of a linear system 16.N
 - of a loop 170
 - of a topological space 202.B
- base space
 - of a fiber bundle 147.B
 - of a fiber space 148.B
 - of a Riemann surface 367.A
- base term (of a spectral sequence) 200.J
- base units 414.A
- Bashforth method, Adams- 303.E
- basic components (of an m -dimensional surface) 110.A
- basic concept (of a structure) 409.B
- basic equation 320.E
- basic feasible solution 255.A
- basic field (of linear space) 256.A
- basic form 255.A
- basic interval 4.B
- basic invariant 226.B
- basic limit theorem 260.C
- basic open set 425.F
- basic optimal solution 255.A
- basic property (of a structure) 409.B
- basic ring (of a module) 277.D
- basic set (for an Axiom A flow) 126.J
- basic set (of a structure) 409.B
- basic solution 255.A
 - feasible 255.A
 - optimal 255.A
- basic space (of a probability space) 342.B
- basic surface (of a covering surface) 367.B
- basic variable 255.A
- basic vector field 80.H
- basic Z_I -extension 14.L
- basin 126.F
- basis
 - (of an Abelian group) 2.B
 - (in a Banach space) 37.L
 - (of a homogeneous lattice) 182.B
 - (of an ideal) 67.B
 - (of a linear space) 256.E
 - (of a module) 277.G
 - canonical 201.B
 - canonical homology 11.C
 - Chevalley canonical 248.Q
 - dual 256.G
 - minimal 14.B
 - normal 172.E
 - of order r in N 4.A
 - orthonormal 197.C
 - Schauder 37.L
 - strongly distinguished 418.F
 - transcendence 149.K
 - Weyl canonical 248.P
- basis theorem
 - Hilbert (on Noetherian rings) 284.A
 - Ritt (on differential polynomials) 113
- bath, heat 419.B
- Bayes estimator 399.G
- Bayes formula 342.F 405.I
- Bayesian approach 401.B
- Bayesian model 403.G
- Bayes risk 398.B
- Bayes solution 398.B
 - generalized 398.B
 - in the wider sense 398.B
- Bayes sufficient σ -field 396.J
- BCH (Base-Chaudhuri-Hooquenghem) code 63.D
- BDI, type (symmetric Riemannian spaces) 412.G
- BDII, type (symmetric Riemannian spaces) 412.G
- BDH (Brown-Douglas-Fillmore) theory 36.J 390.J
- behavior, Regge 386.C
- behavior strategy 173.B
- behind-the-moon argument 351.K
- Behnke-Stein, analytic space in the sense of 23.E
- Behnke-Stein theorem 21.H
- Behrens-Fisher problem 400.G
- Bellman equation 405.B
- Bellman function 127.G
- Bellman principle 405.B
- Bell inequality 351.L
- Bell number 177.D
- belong
 - (to a set) 381.A
 - to the lower class with respect to local continuity 45.F
 - to the lower class with respect to uniform continuity 45.F
 - to the upper class with respect to local continuity 45.F
 - to the upper class with respect to uniform continuity 45.F
- Beltrami differential equation 352.B
- Beltrami differential operator
 - of the first kind App. A, Table 4.II
 - of the second kind App. A, Table 4.II
- Beltrami operator, Laplace- 194.B
- Bergman kernel function 188.G
- Bergman metric 188.G
- Bernays-Gödel set theory 33.A
- Bernoulli
 - loosely 136.F
 - monotonely very weak 136.F

Bernoulli differential equation

- Bernoulli differential equation App. A, Table 14.I
 Bernoulli family 38
 Bernoulli lemniscate 93.H
 Bernoulli method 301.J
 Bernoulli number 177.B
 Bernoulli polynomial 177.B
 Bernoulli process 136.E
 very weak 136.E
 weak 136.E
 Bernoulli sample 396.B
 Bernoulli shift 136.D
 generalized 136.D
 Bernoulli spiral 93.H
 Bernoulli theorem 205.B
 Bernoulli trials, sequence of 396.B
 Bernshtein inequality (for trigonometric polynomials) 336.C
 Bernshtein polynomial 366.A 418.H
 Sato- 125.EE
 Bernshtein problem 275.F
 generalized 275.F
 Bernshtein theorem
 (on cardinal numbers) 49.B
 (on the Laplace transform) 240.E
 (on minimal surfaces) 275.F
 Bers area theorem 234.D
 Bertini theorems 15.C
 Bertrand conjecture 123.A
 Bertrand curve 111.F
 Besov embedding theorem, Sobolev- 168.B
 Besov space 168.B
 Bessaga-Pelczyński theorem 443.D
 Bessel differential equation 39.B, App. A, Table 14.II
 Bessel formula, Hansen- App. A, Table 19.III
 Bessel function(s) 39, App. A, Table 19.III
 half 39.B
 modified 39.G
 spherical 39.B
 Bessel inequality 197.C
 Bessel integral 39.B
 Bessel interpolation formula App. A, Table 21
 Bessel series, Fourier- 39.D
 Bessel transform, Fourier- 39.D
 best (statistical decision function) 398.B
 best approximation
 (of a continuous function) 336.B
 (in evaluation of functions) 142.B
 (of an irrational number) 83.B
 in the sense of Chebyshev 336.H
 best asymptotically normal estimator 399.K
 best invariant estimator 399.I
 best linear unbiased estimator (b.l.u.e.) 403.E
 best polynomial approximation (in the sense of Chebyshev) 336.H
 beta density 397.D
 beta distribution 341.D, App. A, Table 22
 beta function 174.C, App. A, Table 17.I
 incomplete 174.C, App. A, Table 17.I
 beta-model, Luce 346.G
 better, uniformly (statistical decision function) 398.B
 Betti group (of a complex) 201.B
 Betti number
 of a commutative Noetherian ring 200.K
 of a complex 201.B
 between (two points in an ordered set) 311.B
 between-group variance 397.L
 Beurling generalized distribution 125.U
 Beurling-Kunugui, theorem, Iversen- 62.B
 Bezout theorem 12.B
 BG (= Bernays-Gödel set theory) 33.A
 Bhattacharyya inequality 399.D
 biadditive mapping 277.J
 bialgebra 203.G
 quotient 203.G
 semigroup 203.G
 universal enveloping 203.G
 bialgebra homomorphism 203.G
 Bianchi identities 80.J 417.B, App. A, Table 4.II
 bias 399.C
 biaxial spherical harmonics 393.D
 BIBD (balanced incomplete block design) 102.E
 bicharacteristic curve 325.A
 bicharacteristic strip 320.B
 bicomact 425.S
 bicomplex 200.H
 Bieberbach conjecture 438.C
 Biehler equality, Jacobi- 328
 biequicontinuous convergence, topology of 424.R
 bifurcation, Hopf 126.M
 bifurcation equation 286.V
 bifurcation method 290.D
 bifurcation point
 (in bifurcation theory) 126.M 286.R
 (in nonlinear integral equations) 217.M
 bifurcation set 51.F 418.F
 bifurcation theorem, Hopf 286.U
 bifurcation theory 286.R
 biharmonic (function) 193.O
 biholomorphic mapping 21.J
 biideal 203.G
 bijection
 (in a category) 52.D
 (of sets) 381.C
 bijective mapping 381.C
 bilateral network 382.C
 bilinear form
 (on linear spaces) 256.H
 (on modules) 277.J
 (on topological linear spaces) 424.G
 associated with a quadratic form 256.H
 matrix of 256.H
 nondegenerate 256.H
 symmetric (associated with a quadratic form) 348.A
 bilinear functional 424.G
 integral 424.R
 bilinear Hamiltonian 377.A
 bilinear mapping
 (of a linear space) 256.H
 (of a module) 277.J
 canonical (on tensor products of linear spaces) 256.I
 bilinear programming 264.D
 bilinear relations, Hodge-Riemann 16.V
 bimatrix game 173.C
 bimeasurable transformation 136.B
 bimodular germ (of an analytic function) 418.E
 bimodule 277.D
 A-B- 277.D
 binary quadratic form(s) 348.M
 primitive 348.M
 properly equivalent 348.M
 binary relation 358.A 411.G
 binding energy 351.D
 Binet formula
 (on Fibonacci sequence) 295.A
 (on gamma function) 174.A
 binomial coefficient 330, App. A, Table 17.II

- binomial coefficient series 121.E
- binomial distribution 341.D 397.F, App. A, Table 22
 - negative 341.D 397.F, App. A, Table 22
- binomial equation 10.C
- binomial probability paper 19.B
- binomial series App. A, Table 10.IV
- binomial theorem 330, App. A, Table 17.II
- binormal 111.F
 - affine 110.C
- bioassay 40.C
- biology, mathematical models in 263
- biometrics 40
- bipartite graph 186.C
 - complete 186.C
- bipolar (relative to a pairing) 424.H
- bipolar coordinates 90.C
- bipolar cylindrical coordinates App. A, Table 3.V
- bipolar theorem 37.F 424.H
- biprojective space 343.H
- biquadratic equation App. A, Table 1
- birational correspondence 16.I
- birational invariant 12.A
- birational isomorphism
 - between Abelian varieties 3.C
 - between algebraic groups 13.A
- birational mapping 16.I
- birational transformation 16.I
- Birch–Swinnerton-Dyer conjecture 118.D 450.S
- biregular mapping (between prealgebraic varieties) 16.C
- Birkhoff integrable (function) 443.E
- Birkhoff integral 443.E
- Birkhoff fixed-point theorem, Poincaré- 153.B
- Birkhoff-Witt theorem, Poincaré- (on Lie algebras) 248.J
- Birnbaum theorem 399.C
- birth and death process 260.G
- birth process 260.G
- birth rate, infinitesimal 260.G
- bispectral density function 421.C
- bispinor of rank (k, n) 258.B
- bit 75.B 213.B
 - check 63.C
 - information 63.C
- bivariate data 397.H
- bivariate distribution 397.H
- bivariate moments 397.H
- bivariate normal density 397.I
- Blackwell-Rao theorem 399.C
- Blakers-Massey theorem 202.M
- Blaschke manifold 178.G
 - at a point p 178.G
- Blaschke product 43.F
- Blaschke sequence 43.F
- Bleuler formalism, Gupta- 105.G
- Bloch constant 77.F
 - schlicht 77.F
- Bloch theorem 77.F
- block
 - (bundle) 147.Q
 - (of irreducible modular representations) 362.I
 - (of a permutation group) 151.H
 - (of plots) 102.B
 - complete 102.B
 - incomplete 102.B
 - initial 102.E
- block bundle 147.Q
 - normal 147.Q
 - q - 147.Q
- block code 63.A 213.F
 - sliding 213.E
- block design 102.B
 - balanced incomplete 102.E
 - efficiency-balanced 102.E
 - optimal 102.E
 - randomized 102.B
 - variance-balanced 102.E
- block effect 102.B
- block size 102.B
- block structure, q - 147.Q
- blowing up
 - (of an analytic space) 23.D
 - (of a complex manifold) 72.H
 - (by an ideal sheaf) 16.K
 - b.l.u.e (best linear unbiased estimator) 403.E
- Blumenthal zero-one law 261.B
- BMO (bounded mean oscillation) 168.B
- Bochner integrable 443.C
- Bochner integral 443.C
- Bochner theorem 36.L 192.B
- body
 - bounded star 182.C
 - rigid 271.E
- body forces 271.G
- Bogolyubov inequality, Peierls- 212.B
- Bohr, almost periodic function in the sense of 18.B
- Bohr compactification 18.H
- Bokshtein homomorphism 64.B
- Bokshtein operation 64.B
- Boltzmann constant 402.B
- Boltzmann distribution law, Maxwell- 402.B
- Boltzmann equation 41.A 402.B
- Bolzano-Weierstrass theorem 140 273.F
- bond percolation process 340.D
- Bonnet formula, Gauss- 111.H 364.D, App. A, Table 4.I
- Bonnet fundamental theorem 111.H
- Bonnet-Sasaki-Nitsche formula, Gauss- 275.C
- Boolean algebra 243.E
 - generalized 42.B
- Boolean lattice 42.A 243.E
 - of sets 243.E
- Boolean operations 42.A
- Boolean ring 42.C
 - generalized 42.C
- Boolean space 42.D
- Boolean-valued set theory 33.E
- Borcher theorem 150.E
- bord (for a G -manifold) 431.E
- bordant 431.E
- border set 425.N
- Borel-Cantelli lemma 342.B
- Borel direction (of a meromorphic function) 272.F
- Borel embedding, generalized 384.D
- Borel exceptional value 272.E
- Borel exponential method, summable by 379.O
- Borel field 270.B,C
- Borel integral method, summable by 379.O
- Borel isomorphic 270.C
- Borel-Lebesgue theorem 273.H
 - mapping 270.C
- Borel measurable function 270.J
- Borel measure 270.G
- Borel method of summation 379.O
- Borel set(s)
 - (in a Euclidean space) 270.C
 - (in the strict sense) 270.C
 - (in a topological space) 270.C

Borel space

- nearly 261.D
- Borel space 270.C
 - standard 270.C
- Borel subalgebra (of a semisimple Lie algebra) 248.O
- Borel subgroup
 - of an algebraic group 13.G
 - k - (of an algebraic group) 13.G
 - of a Lie group 249.J
- Borel subset 270.C
- Borel summable, absolute 379.O
- Borel theorem
 - (on classifying spaces) App. A, Table 6.V
 - (on meromorphic functions) 272.E
 - Heine 273.F
- Borel-Weil theorem 437.Q
- bornological
 - locally convex space 424.I
 - ultra- (locally convex space) 424.W
- Borsuk-Ulam theorem 153.B
- Bortolotti covariant derivative, van der Waerden- 417.E
- Bose particle 132.A
- Bose statistics 377.B 402.E
- boson 132.A 351.H
 - Nambu-Goldstone 132.C
- Bott fixed point theorem, Atiyah- 153.C
- Bott generator 237.D
- Bott isomorphism 237.D
- Bott periodicity theorem
 - on homotopy groups 202.V, App. A, Table 6.VII
 - in K -theory 237.D
- bound
 - Froissart 386.B
 - greatest lower (of a subset in an ordered set) 311.B
 - greatest lower (of a subset of a vector lattice) 310.C
 - Hamming 63.B
 - least upper (of a subset in an ordered set) 311.B
 - least upper (of a subset of a vector lattice) 310.C
 - lower (of a subset in an ordered set) 311.B
 - Plotkin 63.B
 - upper (of a subset in an ordered set) 311.B
 - Varshamov-Gilbert-Sacks 63.B
- boundary (boundaries)
 - (of a convex cell) 7.D
 - (cycle) 200.H
 - (of a function algebra) 164.C
 - (of a manifold) 65.B 105.B
 - (of a topological space) 425.N
 - Choquet (for a function algebra) 164.C
 - closed (for a function algebra) 164.C
 - C^* -manifold with 105.E
 - C^* -manifold without 105.E
 - differential manifold with, of class C^* 105.E
 - domain with regular 105.U
 - domain with smooth 105.U
 - dual Martin 260.I
 - entrance (of a diffusion process) 115.B
 - exit (of a diffusion process) 115.B
 - harmonic 207.B
 - ideal 207.A
 - Martin 207.C 260.I
 - module of 200.C
 - natural (of an analytic function) 198.N
 - natural (of a diffusion process) 115.B
 - Newton, of f in the coordinate 418.D
 - nondegenerate Newton 418.D
 - of null (open Riemann surface) 367.E
 - pasting together 114.F
 - of positive (open Riemann surface) 367.E
 - regular (of a diffusion process) 115.B
 - relative 367.B
 - Shilov (for a function algebra) 21.D 164.C
 - Shilov (of a Siegel domain) 384.D
 - surface with 410.B
 - topological manifold with 105.B
 - topological manifold without 105.B
- boundary cluster set 62.A
- boundary condition 315.A 323.F
 - adjoint 315.B
 - operator with 112.F
- boundary element (in a simply connected domain) 333.B
- boundary function 160.E
- boundary group 234.B
- boundary homomorphism
 - of homology exact sequence 201.L
 - in homotopy exact sequences 202.L
- boundary layer 205.C
- boundary layer equation, Prandtl 205.C
- boundary operator 200.C
 - between chain groups 201.B
 - linear 315.B
 - partial 200.E
 - total 200.E
- boundary point
 - dual passive 260.H
 - entrance 260.H
 - exit 260.H
 - irregular 120.D
 - passive 260.H
 - regular 120.D
 - of a subset 425.N
- boundary set 425.N
- boundary space 112.E
- boundary value
 - (of a conformal mapping) 77.B
 - (of a hyperfunction) 125.V
 - (relative to a differential operator) 112.E
- boundary value problem 315
 - (for harmonic functions) 193.F
 - (in numerical solution of ordinary differential equations) 303.H
 - adjoint 315.B
 - first (for elliptic differential equations) 323.C
 - first (for harmonic functions) 193.F
 - general 323.H
 - homogeneous (of ordinary differential equations) 315.B
 - inhomogeneous (of ordinary differential equations) 315.B
 - of ordinary differential equations 315
 - second (elliptic differential equations) 323.F
 - second (for harmonic functions) 193.F
 - self-adjoint 315.B
 - solution of App. A, Table 15.VI
 - third (for elliptic differential equations) 323.F
 - third (for harmonic functions) 193.F
 - two-point (of ordinary differential equations) 315.A
 - weak form of the (of partial differential equations) 304.B
- bounded
 - (in an affine space) 7.D
 - (half-plane) 155.B

(metric space) 273.B
 (ordered set) 311.B
 (set in a topological linear space) 424.F
 (set of real numbers) 87.B
 (torsion group) 2.F
 (vector lattice) 310.B
 (vector measure) 443.G
 order 310.B
 relatively 331.B
 T - 311.B
 totally 273.B
 bounded, essentially (measurable function) 168.B
 bounded approximation property (of a Banach space) 37.J
 bounded automaton, linear
 deterministic 31.D
 nondeterministic 31.D
 bounded domain
 divisible 284.F
 homogeneous 384.A 412.F
 irreducible symmetric 412.F
 sweepable 284.F
 symmetric 412.F
 bounded from above
 (in an ordered set) 311.B
 (for real numbers) 87.B
 bounded from below
 (for a filtration) 200.J
 (in an ordered set) 311.B
 (for real numbers) 87.B
 (for a spectral sequence) 200.J
 bounded functions 43.A
 bounded linear operator 37.C
 boundedly complete σ -field 396.E
 bounded matrix 269.K
 bounded mean oscillation (*BMO*) 168.B
 bounded metric space, totally 273.B
 bounded motion 420.D
 bounded μ -operator 356.B
 boundedness, abscissa of (of a Dirichlet series) 121.B
 boundedness principle, upper (in potential theory) 388.C
 boundedness theorem, uniform 37.H
 bounded quantifier 356.B
 bounded set
 in an affine space 7.D
 in a locally convex space 424.F
 in a metric space 273.B
 totally (in a metric space) 273.B
 bounded star body 182.C
 bounded uniform space
 locally totally 436.H
 totally 436.H
 bounded variation
 function of 166.B
 integrable process of 406.B
 mapping of 246.H
 in the sense of Tonelli 246.C
 set function of 380.B
 vector measure of 443.G
 bound state 351.D
 bound variable 411.C
 bouquet 202.F
 Bouquet differential equation, Briot- 288.B 289.B
 Bouquet formulas (on space curves) 111.F
 Bourbaki, Fréchet space in the sense of 424.I
 Boussinesq equation 387.F
 boxes 140

Subject Index

Brouwer mapping theorem

box topology 425.K
 brachistochrone 93.H
 bracket 105.M
 Lagrange 84.A 324.D
 Poisson (of two functions) 105.M
 Poisson (of two vector fields) 271.F 324.C.D
 Toda 202.R
 bracket product (in a Lie algebra) 248.A
 Bradley-Terry model 346.C
 braid(s) 235.F
 closed 235.F
 braid group 235.F
 branch
 (of an analytic function) 198.J
 (of a curve of class C^k) 93.G
 (of a graph) 282.A
 finite (of a curve of class C^k) 93.G
 infinite (of a curve of class C^k) 93.G
 branch and bound methods 215.D
 branch divisor (in a covering of a nonsingular curve) 9.I
 branched minimal immersion 275.B
 branched minimal surface 275.B
 branching Markov process 44.E
 branching processes 44 342.A
 age-dependent 44.E
 continuous state 44.E
 Galton-Watson 44.B
 Markov 44.D
 multitype Markov 44.E
 branch point
 (of a covering surface) 367.B
 (of an ordinary curve) 93.C
 algebraic (of a Riemann surface) 367.B
 fixed (of an algebraic differential equation) 288.A
 logarithmic (of a Riemann surface) 367.B
 movable (of an algebraic differential equation) 288.A
 branch source 282.C
 Brandt law 241.C
 Brauer character (of a modular representation) 362.I
 Brauer group
 (of algebra classes) 29.E
 (of a commutative ring) 29.K
 Brauer theorem 450.G
 Bravais class 92.B
 Bravais group 92.B
 Bravais lattice 92.B
 Bravais type 92.B
 simple 92.E
 breadth (of an oval) 89.C
 Brelot solution, Perron- (of Dirichlet problem) 120.C
 Brelot solution, Perron-Wiener- (of Dirichlet problem) 120.C
 Brianchon theorem
 on conic sections 78.K
 in projective geometry 343.E
 bridge, Brownian 250.F 374.E
 Brieskorn variety 418.D
 Brill-Noether number 9.E
 Briot-Bouquet differential equation 288.B 289.B
 broken line 155.F
 broken symmetry 132.C
 Bromwich integral 240.D 322.D, App. A, Table 12.I
 Brouwer fixed-point theorem 153.B
 Brouwer mapping theorem 99.A

Brouwer theorem on the invariance of domain

Brouwer theorem on the invariance of domain
117.D

Browder-Livesay invariant 114.L

Brownian bridge 250.F 374.E

Brownian functional 176.I

Brownian motion 5.D 45 342.A

d -dimensional 45.C

$\{F_t\}$ - 45.B 406.B

on Lie groups 406.G

with an N -dimensional time parameter 45.I

Ornstein-Uhlenbeck 45.I

right invariant 406.G

space-time 45.F

Brown-Shield-Zeller theorem 43.C

BRS transformation 150.G

Bruck-Ryser-Chowla theorem 102.E

Bruhat decomposition (of an algebraic group) 13.K
relative 13.Q

Brun theorems, Poincaré- 420.A

Brun-Titchmarsh theorem 123.D

Bucy filter, Kalman- 86.E 405.G

building 130.R 343.I

bulk viscosity, coefficients of 205.C

bundle(s)

canonical 147.F

complex conjugate 147.F

complex line 72.F

complex line, determined by a divisor 72.F

conormal 274.E

coordinate 147.B

coordinate, equivalent 147.B

cotangent 147.F

cotangential sphere 274.E

dual 147.F

fiber 147.B

fiber, associated 147.D

fiber, of class C^r 147.O

fiber, complex analytic 147.O

fiber, orientable 147.L

fiber, real analytic 147.O

flat F - 154.B

foliated 154.B,H

frame, orthogonal 364.A

frame, tangent orthogonal n - 364.A

G - 147.B

Hopf 147.E

induced 147.G

line 147.F

Maslov 274.C

normal 105.C 114.B 154.B,E 364.C 274.E

normal block 147.Q

normal k -vector 114.J

normal sphere 274.E

n -sphere 147.K

n -universal 147.G

principal 147.C

principal, associated 147.D

principal fiber 147.C

product 147.E

q -block 147.Q

quotient 16.Y 147.B

reduced 147.J

spin 237.F

Spin^c 237.F

sub- 16.Y 147.F

tangent 105.H 147.F 154.B 286.K

tangential sphere 274.E

tangent r -frame 108.H 147.F

tautological line 16.E

tensor 147.F

trivial 147.E

unit tangent sphere 126.L

universal 147.G,H

vector 16.Y 147.F

vector, ample 16.Y

vector, complex 147.F

vector, cotangent 147.F

vector, dual 147.F

vector, indecomposable 16.Y

vector, normal 105.L

vector, quaternion 147.F

vector, quotient 16.Y

vector, semistable 16.Y

vector, stable (on algebraic varieties) 16.Y

vector, stable (on topological spaces) 237.B

vector, stably equivalent 237.B

vector, tangent 108.H 147.F

bundle group (of a fiber bundle) 147.B

bundle mapping (map) 147.B

bundle of homomorphisms 147.F

bundle of p -vectors 147.F

bundle space (of a fiber bundle) 147.B

Bunyakovskii inequality 211.C, App. A, Table 8

Burali-Forti paradox 319.B

Burnside conjecture 151.D

Burnside problem 161.C

restricted 161.C

Burnside ring 431.F

Burnside theorem 151.D

burst error 63.E

Bush-Mosteller model 346.G

C

c (cardinal number of \mathbf{R}) 49.A

$C^l(\Omega)$ (the totality of l times continuously differentiable functions in Ω) 168.B

$C_0^l(\Omega)$ (the totality of functions in $C^l(\Omega)$ whose supports are compact subsets of Ω) 168.B

c (a sequence space) 168.B

\mathbf{C} (complex numbers) 74.A 294.A

χ -equivalent (closed on G -manifolds) 431.F

\mathcal{C} -group 52.M

\mathcal{C} -theory, Serre 202.N

C -analytic hierarchy 356.H

C -arithmetical hierarchy 356.H

C -equivalent almost complex manifolds 114.H

C_r -field 118.F

$C_i(d)$ -field 118.F

C_n set 22.D

$\{c_n\}$ -consistency, $\{c_n\}$ -consistent 399.K

C^* -algebra 36.G

liminal 36.E

postliminal 36.E

of type I 308.L

C^* -cross norm 36.H

C^* -dynamical system 36.K

C^* -group algebra (of a locally compact Hausdorff space) 36.L

C^* -tensor product, projective 36.H

(C, α) -summation 379.M

c_1 -bundle 237.F

c_1 -mapping 237.G

C -analytic space 23.E

C -covering space 23.E

C^r -conjugacy, C^r -conjugate 126.B

C^r -equivalence, C^r -equivalent 126.B

C^r -flow 126.B

C^r -foliation 154.G

C^r -function in a C^∞ -manifold 105.G

- C^r -manifold 105.D
 - with boundary 105.E
 - without boundary 105.E
 - compact 105.D
 - paracompact - 105.D
- C^r -mapping 105.J
- C^r -norm 126.H
- C^r -structure 108.D
 - subordinate to (for a C^s -structure) 108.D
 - on a topological manifold 114.B
- C^r - Ω -stable 126.H
- C^r -structurally stable 126.H
- C^r -triangulation 114.C
- C^ω -homomorphism (between Lie groups) 249.N
- C^ω -isomorphism (between Lie groups) 249.N
- C^ω -function (of many variables) 58.B
 - germ of (at the origin) 58.C
 - preparation theorem for 58.C
 - rapidly decreasing 168.B
 - slowly increasing 125.O
- C^ω -functions and quasi-analytic functions 58
- C^ω topology, weak 401.C
- CA set (in set theory) 22.A
- Caianiello differential equation 291.F
- calculable function, effective 356.C
- calculable number 22.G
- calculation, graphical 19.B
- calculator 75.A
- calculus
 - differential 106
 - fundamental theorem of the infinitesimal 216.C
 - Heaviside 306.A
 - holomorphic functional 36.M
 - infinitesimal (in nonstandard analysis) 273.D
 - Kirby 114.L
 - operational 251.G 306, App. A, Table 12.II
 - predicate 411.J
 - predicate, with equality 411.J
 - propositional 411.F
 - of residue 198.F
 - stochastic 406.A
 - tensor 417.A, App. A, Table 4.II
- calculus of variations 46
 - classical theory of 46.C
 - conditional problems in 46.A
 - fundamental lemma in 46.B
- Calderón-Zygmund singular integral operator 217.J 251.O
- Calderón-Zygmund type, kernel of 217.J
- Calkin algebra 36.J
- Callan-Symanzik equation 361.B
- Campbell-Hausdorff formula 249.R
- CAN estimator 399.K
- canceling 138.B
- cancellation law
 - on the addition of natural numbers 294.B
 - in a commutative semigroup 190.P
 - on the multiplication of natural numbers 294.B
- canonical affine connection (on \mathbf{R}^n) 80.J
- canonical anticommutation relation 277.A
- canonical backward moving average representation 395.D
- canonical basis (of a chain group of a finite simplicial complex) 201.B
 - Chevalley (of a complex semisimple Lie algebra) 248.Q
 - Weyl (of a semisimple Lie algebra) 248.P
- canonical bilinear mapping (on tensor products of linear spaces) 256.I
- canonical bundle (of a differentiable manifold) 147.F
- canonical class (of an algebraic curve) 9.C
- canonical cohomology class (in Galois cohomology in class field theory) 59.H
- canonical commutation relations 351.C 377.A,C
- canonical coordinates (of a Lie group)
 - of the first kind 249.Q
 - of the second kind 249.Q
- canonical coordinate system (for a conic section) 78.C
- canonical correlation coefficient 280.E 374.C
- canonical decomposition (of a closed operator) 251.E
- canonical decomposition theorem 86.C
- canonical divisor
 - (of an algebraic curve) 9.C
 - (of an algebraic variety) 16.O
 - (of a Jacobian variety) 9.E
 - (of a Riemann surface) 11.D
- canonical element (in the representation of a functor) 52.L
- canonical ensemble 402.D
 - grand 402.D
- canonical equation 324.E
 - Hamilton 271.F
- canonical field 377.C
- canonical form
 - (of $F(M)$) 191.A
 - (of a linear hypothesis) 400.H
 - (of a regular submanifold of $F(M)$) 191.A
 - of the equation (of a quadric surface) 350.B
 - Khinchin (of an infinitely divisible probability distribution) 341.G
 - Kolmogorov (of an infinitely divisible probability distribution) 341.G
 - Lévy (of an infinitely divisible probability distribution) 341.G
 - Weierstrass (for an elliptic curve) 9.D
 - Weierstrass (of the gamma function) 174.A
- canonical function (on a nonsingular curve) 9.E
- canonical homology basis 11.C
- canonical homomorphism
 - (on direct products of rings) 368.E
 - (on tensor products of algebras) 29.A
- canonical injection
 - (from a direct summand) 381.E
 - (in direct sums of modules) 277.F
 - (in free products of groups) 190.M
 - (from a subgroup) 190.D
 - (from a subset) 381.C
- canonically bounded 200.J
- canonically polarized Jacobian variety 3.G 9.E
- canonical measure
 - (in a birth and death chain) 260.G
 - (in a diffusion process) 115.B
 - (in a Markov chain) 260.I
- canonical model 251.N
- canonical 1-form (of the bundle of tangent n -frames) 80.H
- canonical parameter
 - of an arc 111.D
 - local (for power series) 339.A
- canonical product, Weierstrass 429.B
- canonical projection
 - (on direct products of modules) 277.F
 - (onto a quotient set) 135.B
- canonical representation
 - (of a Gaussian process) 176.E
 - generalized 176.E

- canonical scale 115.B
- canonical scores 397.M
- canonical surjection
 - (on direct products of groups) 190.L
 - (to a factor group) 190.D
 - (onto a quotient set) 135.B
- canonical transformation 271.F
 - (concerning contact transformations) 82.B
 - group of 271.F
 - homogeneous 82.B
- canonical variables (in analytical dynamics) 271.F
- canonical variates 280.E
- canonical vectorial form 417.C
- Cantelli lemma, Borel- 342.B
- Cantelli theorem, Glivenko- 374.E
- Cantor, G. 47
- Cantor discontinuum 79.D
- Cantor intersection theorem 273.F
- Cantor-Lebesgue theorem 159.J
- Cantor normal form (for an ordinal number) 312.C
- Cantor set 79.D
 - general 79.D
- Cantor's theory of real numbers 244.E
- capability, error-correcting 63.B
- capacitable set 48.H
- capacitary dimension 48.G
- capacitary mass distribution 338.K
- capacitated network 281.C
- capacity
 - (of discrete memoryless channel) 213.F
 - (of a prime ideal) 27.A
 - (of a set) 48 260.D
 - (transportation and scheduling) 281.D
 - α - 169.C
 - analytic 169.F
 - continuous analytic 164.J
 - ergodic 213.F
 - logarithmic 48.B
 - Newtonian 48.B
 - Newtonian exterior 48.H
 - Newtonian inner 48.F
 - Newtonian interior 48.F
 - Newtonian outer 48.H
 - of order α 169.C
 - stationary 213.F
- capacity constraint 281.D
- capillary wave 205.F
- cap product
 - (of a cochain and a chain) 200.K
 - (of a cohomology class and a homology class) 201.K
- capture
 - complete 420.D
 - partial 420.D
- CAR 377.A
- Carathéodory measure 270.E
- Carathéodory outer measure 270.E
- Carathéodory pseudodistance 21.D
- Cardano formula (on a cubic equation) 10.D, App. A, Table 1
- cardinality 33.F
 - (of an ordinal number) 49.E
 - (of a set) 49.A 312.D
- cardinal number(s) 49.A 312.D
 - comparability theorem for 49.B
 - of continuum 49.A
 - corresponding to an ordinal number 49.E
 - finite 49.A
 - infinite 49.A
 - measurable 33.E
 - of \mathbf{N} 49.A
 - of \mathbf{R} 49.A
 - of all real-valued functions on $[0, 1]$ 49.A
 - of a set 49.A 312.D
 - strongly compact 33.E
 - strongly inaccessible 33.E
 - transfinite 49.A
 - weakly compact 33.E
 - weakly inaccessible 33.E
- cardinal product (of a family of ordered sets) 311.F
- cardinal sum (of a family of ordered sets) 311.F
- cardioid 93.H
- Carleman condition, Denjoy- 168.B
- Carleman inequality App. A, Table 8
- Carleman theorem
 - (on asymptotic expansions) 30.A
 - (on bounded functions) 43.F
- Carleman type, kernel of 217.J
- carrier
 - (of a differential form) 108.Q
 - (of a distribution) 125.D
 - (of a function) 125.B 168.B
 - (of a real-valued function) 425.R
- Carson integral App. A, Table 12.II
- Cartan, E. 50
- Cartan, differential form of Maurer- 249.R
- Cartan, system of differential equations of Maurer- 249.R
- Cartan connection 80.M
- Cartan criterion of semisimplicity (on Lie algebras) 248.F
- Cartan criterion of solvability (on Lie algebras) 248.F
- Cartan formula
 - for Steenrod p th power operations 64.B
 - for Steenrod square operations 64.B
- Cartan integer (of a semisimple Lie algebra) 248.N
- Cartan invariant (of a finite group) 362.I
- Cartan involution 437.X
- Cartan-Kähler existence theorem 428.E
- Cartan-Mal'tsev-Iwasawa theorem (on maximal compact subgroups) 249.S
- Cartan maximum principle 338.L
- Cartan pseudoconvex domain 21.I
 - locally 21.I
- Cartan relative integral invariant 219.B
- Cartan subalgebra
 - (of a Lie algebra) 248.I
 - (of a symmetric Riemannian space) 413.F
- Cartan subgroup
 - (of an algebraic group) 13.H
 - (of a group) 249.I
- Cartan space 152.C
- Cartan theorem
 - on analytic sheaves (H. Cartan) 72.E
 - on representations of Lie algebras (E. Cartan) 248.W
- Cartan-Thullen theorem 21.H
- Cartan-Weyl theorem 248.W
- Carter subgroup 151.D
- Cartesian coordinates (in an affine space) 7.C
- Cartesian product
 - (of complexes) 70.C.E
 - (of mappings) 381.C
 - (of sets) 381.B.E
- Cartesian space 140
- Cartier divisor 16.M
- Cartier operator 9.E
- CA set 22.A
- case complexity, worst 71.A

- Casimir element (of a Lie algebra) 248.J
- Casorati determinant 104.D
- Casorati-Weierstrass theorem (on essential singularities) 198.D
- Cassini oval 93.H
- Casson handle 114.M
- Castelnuovo criterion 15.E
- Castelnuovo lemma 3.E 9.H
- casus irreducibilis 10.D, App. A, Table 1
- Catalan constant App. A, Table 10.III
- catastrophe, elementary 51.E
- catastrophe point 51.F
- catastrophe set 51.F
- catastrophe theory 51
- categorical
 - (data) 397.B
 - (set of closed formulas) 276.F
- categorical system (of axioms) 35.B
- categoricity in powers 276.F
- categories and functors 52
- category 52.A
 - Abelian 52.N
 - of Abelian groups 52.B
 - additive 52.N
 - of analytic manifolds 52.B
 - cohomology theory on the 261.Q
 - of commutative rings 52.B
 - diagram in the 52.C
 - of differentiable manifolds 52.B
 - dual 52.F
 - exact 237.J
 - Grothendieck 200.I
 - of groups 52.B
 - homotopy, of topological spaces 52.B
 - of left (right) R -modules 52.B
 - of linear spaces over R 52.B
 - of pointed topological spaces 202.B
 - PL 65.A
 - product 52.A
 - quotient 52.N
 - of rings 52.B
 - set of the first 425.N
 - set of the second 425.N
 - of sets 52.B
 - shape 382.A
 - of S -objects 52.G
 - of topological spaces 52.B
- catenary 93.H
- catenoid 111.I
- Cauchy, A. L. 53
- Cauchy condensation test 379.B
- Cauchy condition (on D-integral and $D^{(*)}$ -integral) 100.E
- Cauchy criterion (on the convergence of a sequence of real numbers) 87.C, App. A, Table 10.II
- Cauchy data 321.A
- Cauchy distribution 341.D, App. A, Table 22
- Cauchy existence theorem (for partial differential equations) 320.B
- Cauchy filter (on a uniform space) 436.G
- Cauchy-Hadamard formula 339.A
- Cauchy inequality 211.C, App. A, Table 8
- Cauchy integral formula 198.B
- Cauchy integral representation 21.C
- Cauchy integral test (for convergence) 379.B
- Cauchy integral theorem 198.A,B
 - stronger form of 198.B
- Cauchy-Kovalevskaya existence theorem 321.A
- Cauchy-Kovalevskaya theorem, abstract 286.Z
- Cauchy net (in a uniform space) 436.G
- Cauchy polygon 316.C
- Cauchy principal value
 - of an improper integral 216.D
 - of the integral on infinite intervals 216.E
- Cauchy problem
 - (of ordinary differential equations) 316.A
 - (for partial differential equations) 315.A 320.B 321.A 325.B
 - abstract 286.X
- Cauchy process 5.F
 - asymmetric 5.F
 - symmetric 5.F
- Cauchy product (of two series) 379.F
- Cauchy remainder App. A, Table 9.IV
- Cauchy-Riemann (differential) equation 198.A 274.G
 - (for a holomorphic function of several complex variables) 21.C
 - (for a holomorphic function of two complex variables) 320.F
- Cauchy-Riemann structure 344.A
- Cauchy-Schwarz inequality 211.C, App. A, Table 8
- Cauchy sequence
 - (in an α -adic topology) 284.B
 - (in a metric space) 273.J
 - (of rational numbers) 294.E
 - (of real numbers) 355.B
 - (in a uniform space) 436.G
- Cauchy sum (of a series) 379.A
- Cauchy theorem 379.F
- Cauchy transform 164.J
- causality, macroscopic 386.C
- cause, most probable 401.E
- caustic 325.L
- Cayley algebras 54
 - general 54
- Cayley number 54
- Cayley projective plane 54
- Cayley theorem (in group theory) 151.H
- Cayley theorem, Hamilton- 269.F
- Cayley transform (of a closed symmetric operator in a Hilbert space) 251.I
- Cayley transformation 269.J
- CCP (chance-constrained programming) 408.B
- CCR 377.A
- CCR algebra 36.H
- Čech cohomology group (for topological spaces) 201.L,P
 - relative 201.M
- Čech cohomology group with coefficient sheaf \mathcal{F} 383.F
- Čech compactification, Stone- 207.C
- Čech complete space 425.T 436.I
- Čech homology group (for topological spaces) 201.M
 - relative 201.M
- ceiling function 136.D
- ceiling states 402.G
- celestial mechanics 55
- cell 70.D
 - convex (in an affine space) 7.D
 - fundamental (of a symmetric Riemann space) 413.F
 - n - (in a Hausdorff space) 70.D
 - $(n - q)$ -dual 65.B
 - topological n - 140
 - unit 140
- cell complex 70.D
 - closure finite 70.D
 - countable 70.D

Euclidean 70.B
finite 70.D
locally countable 70.D
locally finite 70.D
regular 70.D
cell-like (CE) 382.D
cellular approximation theorem 70.D
cellular cohomology group 201.H
cellular decomposition (of a Hausdorff space) 70.D
cellular homology group 201.F,G
cellular mapping (between cell complexes) 70.D
CE mapping 382.D
center
 (of a central symmetry) 139.B
 (of a continuous geometry) 85.A
 (of gravity) 271.E
 (of a group) 190.C
 (of a hyperbola or ellipse) 78.C
 (of a lattice) 243.E
 (of a Lie algebra) 248.C
 (of mass) 271.E
 (of a nonassociative algebra) 231.A
 (of a pencil of hyperplanes) 343.B
 (of a quadric hypersurface) 7.G
 (of a quadric surface) 350.A
 (of a regular polygon) 357.A
 (of a regular polyhedron) 357.B
 (of a ring) 368.F
 (of a solid sphere) 140
 (of a sphere) 139.I
 (of a von Neumann algebra) 308.C
centered process 5.B
centering 5.B
center manifold theorem 286.V
center of curvature 111.E
center of projection 343.B
center surface 111.I
central composite design 102.M
central configuration 420.B
central conic(s) 78.C
central difference 223.C 304.E, App. A, Table 21
central element (in a lattice) 243.E
central extension (of a group) 190.N
central figure 420.B
centralizer
 (of a subset of a group) 190.C
 (of a subset of a ring) 368.F
central limit theorem 250.B
central moment 397.C
central motion 126.E
central potential 315.E
central processor 75.B
central quadric hypersurface 7.F 350.G
central quadric surface 350.B
central series
 ascending (of a Lie algebra) 248.C
 descending (of a Lie algebra) 248.C
 lower (of a group) 190.J
 upper (of a group) 190.J
central simple algebra 29.E
 similar 29.E
central symmetry (of an affine space) 139.B
centrifugal force 271.D
certainly, almost 342.B,D
certainty equivalent 408.B
Cesàro method of summation of order α 379.M
 summable by 379.M
Ceva theorem 7.A
CFL condition 304.F
CG method 302.D

chain 200.H
 ascending (of normal subgroups of a group) 190.F
 ascending (in an ordered set) 311.C
 conservative 260.A
 descending (in a lattice) 243.F
 descending (of (normal) subgroups of a group) 190.F
 descending (in an ordered set) 311.C
 equivalence 200.H
 general Markov 260.J
 homotopy 200.H
 irreducible (a Markov chain) 260.B
 Markov 260.A
 minimal 260.F
 normal (in a group) 190.G
 normal (in Markov chains) 260.D
 q - (of a chain complex) 201.B
 recurrent 260.B
 regular (of integral elements) 428.E
 u - 260.I
chain complex(cs) 200.C,H 201.B
 in an Abelian category 201.B
 of A -modules 200.C
 augmented 200.C
 double 200.E
 oriented simplicial 201.C
 positive 200.H
 product double 200.E
 quotient 200.C
 relative 200.C
 singular (of a topological space) 201.E
chain condition (in an ordered set) 311.C
 ascending (in an ordered set) 311.C
 descending (in an ordered set) 311.C
chained metric space, well- 79.D
chain equivalence 200.C
chain homotopy 200.C
chain mapping 200.C 201.B
 over an A -homomorphism 200.C
chain recurrent 126.E
chain recurrent set 126.E
chain rule 106.C
chain subcomplex 200.C
chain theorem 14.J
chain transformation (between complexes) 200.H
Chaitin complexity, Kolmogorov- 354.D
chamber
 positive Weyl 248.R
 Weyl 13.J 248.R
chamber complex 13.R
chance-constrained programming 408.A
chance constraint 408.A
chance move 173.B
change
 scalar (of a B -module) 277.L
 time 261.F 406.B
 of variables (in integral calculus) 216.C
channel 375.F
 almost finite memory 213.F
 d -continuous 213.F
 discrete memoryless 213.F
 finite memory 213.F
 noisy 213.A
 test 213.E
channel coding theory 213.A
channel Hilbert space 375.F
channel wave operators 375.F
chaos 126.N 303.G 433.B
 propagation of 340.F

- Chaplygin's differential equation 326.B
 Chapman complement theorem 382.B
 Chapman-Kolmogorov equality 261.A
 Chapman-Kolmogorov equation 260.A
 Chapman-Robbins-Kiefer inequality 399.D
 Chapman theorem (on (C, α) -summation) 379.M
 character
 (of an Abelian group) 2.G
 (of an algebraic group) 13.D
 (irreducible unitary representation) 437.V
 (of a linear representation) 362.E
 (of a regular chain) 428.E
 (of a representation of a Lie group) 249.O
 (of a semi-invariant) 226.A
 (of a topological Abelian group) 422.B
 absolutely irreducible 362.E
 Brauer (of a modular representation) 362.I
 Chern (of a complex vector bundle) 237.B
 Dirichlet 295.D
 Hecke 6.D
 identity (of an Abelian group) 2.G
 integral (on the 1-dimensional homology group of a Riemann surface) 11.E
 irreducible (of an irreducible representation) 362.E
 Minkowski-Hasse (of a nondegenerate quadratic form) 348.D
 modular (of a modular representation) 362.I
 nonprimitive 450.C,E
 planar 367.G
 primitive 295.D 450.C,E
 principal (of an Abelian group) 2.G
 principal Dirichlet 295.D
 reduced (of an algebra) 362.E
 residue 295.D
 simple (of an irreducible representation) 362.E
 character formula, Weyl 248.Z
 character group
 (of an Abelian group) 2.G
 (of a topological Abelian group) 422.B
 characteristic(s)
 (of a common logarithm) 131.C
 (of a field) 149.B
 Euler (of a finite Euclidean cellular complex) 201.B
 Euler-Poincaré (of a finite Euclidean complex) 16.E 201.B
 operating 404.C
 population 396.C
 quality 404.A
 sample 396.C
 Todd 366.B
 two-terminal 281.C
 characteristic class(es) 56
 (of an extension of a module) 200.K
 (of a fiber bundle) 147.K
 (of foliations) 154.G
 (of a vector bundle) 56.D
 \mathcal{A} (of a real oriented vector bundle) 237.F
 of codimension q 154.G
 of a manifold 56.F
 smooth, of foliations 154.G
 characteristic curve
 (of a network) 281.B
 (of a one-parameter family of surfaces) 111.I
 (of a partial differential equation) 320.B 324.A,B
 characteristic equation
 (of a differential-difference equation) 163.F
 (for a homogeneous system of linear ordinary differential equations) 252.J
 (of a linear difference equation) 104.E
 (of a linear ordinary differential equation) 252.E
 (of a matrix) 269.F
 (of a partial differential equation) 320.D
 (of a partial differential equation of hyperbolic type) 325.A,F
 characteristic exponent
 (of an autonomous linear system) 163.F
 (of the Hill differential equation) 268.B
 (of a variational equation) 394.C
 characteristic function(s)
 (of a density function) 397.G
 (of a graded R -module) 369.F
 (of a meromorphic function) 272.B
 (of an n -person cooperative game) 173.D
 (for an optical system) 180.C
 (of probability distributions) 341.C
 (of a subset) 381.C
 empirical 396.C
 Hilbert (of a coherent sheaf) 16.E
 Hilbert (of a graded module) 369.F
 characteristic functional (of a probability distribution) 407.C
 characteristic hyperplane (of a partial differential equation of hyperbolic type) 325.A
 characteristic hypersurface (of a partial differential equation of hyperbolic type) 325.A
 characteristic linear system (of an algebraic family) 15.F
 characteristic line element 82.C
 characteristic manifold (of a partial differential equation) 320.B
 characteristic mapping (map) (in the classification theorem of fiber bundles) 147.G
 characteristic measure 407.D
 characteristic multiplier
 (of a closed orbit) 126.G
 (of a periodic linear system) 163.F
 characteristic number
 (of a compact operator) 68.I
 (of a manifold) 56.F
 Lyapunov 314.A
 characteristic operator function 251.N
 characteristic polynomial
 (of a differential operator) 112.A
 (of a linear mapping) 269.L
 (of a matrix) 269.F
 (of a partial differential operator) 321.A
 characteristic root
 (of a differential-difference equation) 163.A
 (of a linear mapping) 269.L
 (of a linear partial differential equation) 325.F
 (of a matrix) 269.F
 characteristic series (in a group) 190.G
 characteristic set
 (of an algebraic family on a generic component) 15.F
 (of a partial differential operator) 320.B
 characteristic strip 320.D 324.B
 characteristic surface 320.B
 characteristic value
 (of a linear operator) 390.A
 sample 396.C
 characteristic variety (of a system of microdifferential equations) 274.G
 characteristic vector
 (of a linear mapping) 269.L
 (of a linear operator) 390.A

Character module (of an algebraic group)

- character module (of an algebraic group) 13.D
- character system (of a genus of a quadratic field) 347.F
- charge 150.B
- charge symmetry 415.J
- Charpit method, Lagrange- 322.B, App. A, Table 15.II
- Charpit subsidiary equation 82.C 320.D
- chart
 - alignment 19.D
 - control 404.B
 - intersection 19.D
- Châtelet group, Weil- 118.D
- Chebotarev density theorem 14.S
- Chebyshev formula 299.A
 - Gauss- (in numerical integration) 299.A
- Chebyshev function App. A, Table 20.II
- Chebyshev interpolation 233.A 366.J
- Chebyshev orthogonal polynomial 19.G
- Chebyshev polynomial 336.H
- Chebyshev q -function 19.G, App. A, Table 20.VII
 - simplest 19.G
- Chebyshev system 336.B
- Chebyshev theorem 336.B
- check bits 63.C
- check matrix, parity 63.C
- Cheng domain theorem, Courant- 391.H
- Chern character (of a complex vector bundle) 237.B
- Chern class
 - of a C^n -bundle 56.C
 - of a manifold 56.F
 - of a real $2n$ -dimensional almost complex manifold 147.N
 - total 56.C
 - of a $U(n)$ -bundle 147.N
 - universal 56.C
- Chern formula (in integral geometry) 218.D
- Chern number 56.F
- Cherwell-Wright differential equation 291.F
- Chevalley canonical basis (of a complex semisimple Lie algebra) 248.Q
- Chevalley complexification (of a compact Lie group) 249.U
- Chevalley decomposition (on algebraic groups) 13.I
- Chevalley group 151.I
- Chevalley theorem (forms over finite fields) 118.B
- Chevalley theorem (on algebraic groups) 13.B
- Chevalley type (algebraic group) 13.N
- Chinese mathematics 57
- Chinese remainder theorem 297.G
- chi-square distribution 374.B, App. A, Table 22
 - noncentral 374.B
- chi-square method, modified minimum 400.K
- chi-square test 400.G
 - of goodness of fit 400.K
- choice, axiom of 33.B 34.A
- choice function 33.B 34.A
- choice process, multistage 127.A
- choice set 33.B 34.A
- Cholesky method 302.B
- Chomsky grammar 31.D
- Chomsky hierarchy 31.D
- Choquest boundary 164.C
- Chow coordinates (of a positive cycle) 16.S
- Chow-Kodaira theorem 72.F
- Chowla theorem, Bruck-Ryser- 102.E
- Chow lemma 72.H
- Chow ring (of a projective variety) 16.R
- Chow theorem
 - on an analytic submanifold of P^N 72.F
 - on the field of rational functions of an analytic space 23.D
- Chow variety 16.S
- Christoffel-Darboux formula 317.D
- Christoffel symbol 80.L 111.H 417.D
- Christoffel transformation, Schwarz- 77.D, App. A, Table 13
- Christoffel transformation formula, Schwarz- 77.D
- chromatic number 157 186.I
- chromodynamics, quantum 132.C
- Chung-Erdős theorem 342.B
- Church's thesis 356.C
- circle(s) 140
 - Apollonius App. A, Table 3.V
 - auxiliary (of an ellipse) 78.D
 - of convergence (of a power series) 339.A
 - of curvature 111.E
 - director (of an ellipse) 78.D
 - great (of a sphere) 140
 - inscribed (of a regular polygon) 357.A
 - isometric 234.C
 - of meromorphy (of a power series) 339.D
 - open 140
 - oscillating 111.E
 - quadrature of 179.A
 - unit 74.C 140
- circled subset (of a topological linear space) 424.E
- circle geometry 76.C
- circle method 4.B
- circle problem, Gauss 242.A
- circle type, limit 112.I
- circuit 66.G
- circuit matrix 254.B
- circular cone 78.A 111.I
 - oblique 350.B
 - right 350.B
- circular cylinder 111.I 350.B
- circular cylindrical coordinates App. A, Table 3.V
- circular cylindrical surface 350.B
- circular disk 140
- circular domain 333.A
- circular frequency (of a simple harmonic motion) 318.B
- circular function 131.F 432.A
- circular section 350.F
- circular unit 14.L
- circulation (of a vector field) 205.B 442.D
- circumference 140
- circumferentially mean p -valent 438.E
- circumscribed circle (of a regular polygon) 357.A
- circumscribing sphere (of a simplex) 139.I
- cissoidal curve 93.H
- cissoid of Diocles 93.H
- Clairaut differential equation App. A, Table 14.I
- Clairaut partial differential equation App. A, Table 15.II
- class
 - (in axiomatic set theory) 33.C 381.G
 - (of a lattice group) 13.P
 - (of a nilpotent group) 190.J
 - (of a plane algebraic curve) 9.B
 - (of a quadratic form) 348.H.I
 - \mathcal{A} -characteristic (of a real oriented vector bundle) 237.F
 - algebra (of central simple algebras) 29.E
 - ambig (of a quadratic field) 347.F

- Bravais 92.B
 canonical (of an algebraic curve) 9.C
 canonical cohomology 59.H
 canonical divisor 11.D
 characteristic (of an extension of module) 200.K
 characteristic (of a fiber bundle) 147.K
 characteristic (of foliations) 154.G
 characteristic (of a vector bundle) 56
 characteristic, of codimension q 154.G
 characteristic, of a manifold 56.F
 Chern (of a C^n -bundle) 56.C
 Chern (of a manifold) 56.F
 Chern (of a real $2n$ -dimensional almost complex manifold) 147.N
 Chern (of a $U(n)$ -bundle) 147.N
 cohomology 200.H
 combinational Pontryagin 56.H
 complete 398.B
 completely additive 270.B
 conjugacy (of an element of a group) 190.C
 countably additive 270.B
 crystal 92.B
 curve of the second 78.K
 differential divisor (of a Riemann surface) 11.D
 divisor (on a Riemann surface) 11.D
 Dynkin 270.B
 the Dynkin, theorem 270.B
 equivalence 135.B
 ergodic 260.B
 essentially complete 398.B
 Euler-Poincaré (of a manifold) 56.F
 Euler-Poincaré (of an oriented R^n -bundle) 56.B
 finitely additive 270.B
 fundamental (of an Eilenberg-MacLane space) 70.F
 fundamental (of a Poincaré pair) 114.J
 fundamental (of the Thom complex **MG**) 114.G
 fundamental, with coefficient Z_2 65.B
 generalized Hardy 164.G
 Gevrey 58.G 125.U
 group of congruence 14.H
 Hardy 43.F 159.G
 Hilbert-Schmidt 68.I
 holosymmetric 92.B
 homology 200.H 201.B
 homotopy 202.B
 ideal (of an algebraic number field) 14.E
 ideal (of a Dedekind domain) 67.K
 ideal, in the narrow sense 14.G 343.F
 idele 6.D
 idele, group 6.D
 linear equivalence (of divisors) 16.M
 main 241.A
 mapping 202.B
 minimal complete 398.B
 monotone 270.B
 the monotone, theorem 270.B
 multiplicative 270.B
 nuclear 68.I
 oriented cobordism 114.H
 Pontryagin (of a manifold) 56.F
 Pontryagin (of an R^n -bundle) 56.D
 proper 381.G
 q -dimensional homology 201.B
 of a quadratic form over an algebraic number field 348.H
 residue (modulo an ideal in a ring) 368.F
 Steifel-Whitney (of a differentiable manifold) 147.M
 Stiefel-Whitney (of a manifold) 56.F
 Stiefel-Whitney (of an $O(n)$ -bundle) 147.M
 Stiefel-Whitney (of an R^n -bundle) 56.B
 Stiefel-Whitney (of a topological manifold) 56.F
 surface of the second 350.D
 Todd 237.F
 total Chern 56.C
 total Pontryagin 56.D
 total Stiefel-Whitney 56.B
 trace 68.I
 universal Chern 56.C
 universal Euler-Poincaré 56.B
 universal Stiefel-Whitney 56.B
 unoriented cobordism 114.H
 Wu (of a topological manifold) 56.F
 Zygmund 159.E
 class C^∞
 function of 106.K
 function of (of many variables) 58.B
 mapping of 286.E
 oriented singular r -simplex of 108.T
 partition of unity of 108.S
 singular r -chain of 108.T
 singular r -cochain of 108.T
 class C^k
 curve of (in a differentiable manifold) 93.B
 curve of (in a Euclidean plane) 93.B
 class C^n , function of 106.K
 class C^ω , function of 106.K
 class C^1
 function of 106.K
 mapping of 286.E
 class C'
 atlas of 105.D
 coordinate neighborhood of 105.D
 diffeomorphism of 105.J
 differentiable manifold of 105.D
 differentiable manifold with boundary of 105.E
 differentiable mapping of 105.J
 differentiable structure of 105.D
 fiber bundle of 147.O
 function of (in a C^∞ -manifold) 105.G
 function of (at a point) 105.G
 functionally dependent of (components of a mapping) 208.C
 functional relation of 208.C
 mapping of 208.B 286.E
 nonsingular mapping of 208.B
 regular mapping of 208.B
 vector field of 105.M
 Z -action of 126.B
 class C' , tensor field of 108.O
 class C^0 , mapping into Banach space of 286.E
 class (C^0), semigroup of 378.B
 class D^∞ , curve of 364.A
 class field 59.B
 absolute 59.A
 class field theory 59
 local 59.G
 class field tower problem 59.F
 class formation 59.H
 class function (on a compact group) 69.B
 class group divisor 11.D
 classical (potential) 402.G
 classical (state) 402.G

Classical compact real simple Lie algebra

- classical compact real simple Lie algebra 248.T
- classical compact simple Lie group 249.L
- classical complex simple Lie algebra 248.S
- classical complex simple Lie group 249.M
- classical descriptive set theory 356.H
- classical dynamical system 126.L 136.G
- classical group(s) 60.A
 - infinite 147.I 202.V
- classical logic 411.L
- classical mechanics 271.A
- classical risk theory 214.C
- classical solution (to Plateau's problem) 275.C
- classical statistical mechanics 402.A
- classical theory of the calculus of variations 46.C
- classification (with respect to an equivalence relation) 135.B
- classification theorem
 - on a fiber bundle 147.G
 - first (in the theory of obstructions) 305.B
 - Hopf 202.I
 - second (in the theory of obstructions) 305.C
 - third (in the theory of obstructions) 305.C
- classification theory of Riemann surfaces 367.E
- classificatory procedure 280.I
- classifying mapping (map) (in the classification theorem of fiber bundles) 147.G
- classifying space
 - (of a topological group) 174.G,H
 - cohomology rings of App. A, Table 6.V
 - for Γ_n^* -structures 154.E
 - n - (of a topological group) 147.G
- class n
 - function of 84.D
 - projective set of 22.D
- class $N_{\bar{\alpha}}$, null set of 169.E
- class number
 - (of an algebraic number field) 14.E
 - (of a Dedekind domain) 67.K
 - (of a simple algebra) 27.D
- class of Abelian groups 202.N
- class ω , function of 84.D
- class 1
 - function of 84.D
 - function of at most 84.D
- class theorems, complete 398.D
- class ξ , function of 84.D
- class 0, function of 84.D
- Clebsch-Gordan coefficient 258.B 353.B
- Clenshaw-Curtis formulas 299.A
- Clifford algebras 61
- Clifford group 61.D
 - reduced 61.D
 - special 61.D
- Clifford number 61.A
- Clifford torus 275.F
- Clifford torus, generalized 275.F
- clinical trials 40.F
- closable operator 251.D
- closed
 - absolutely (space) 425.U
 - algebraically (in a field) 149.I
 - algebraically (field) 149.I
 - boundary 164.C
 - H- (space) 425.U
 - hyperbolic, orbit 126.G
 - integrally (ring) 67.I
 - k - (algebraic set) 13.A
 - multiplicatively, subset (of a ring) 67.I
 - quasi-algebraically (field) 118.F
 - r - (space) 425.U
 - real, field 149.N
 - Zariski 16.A
- closed arc 93.B
- closed braid 235.F
- closed convex curve 111.E
- closed convex hull 424.H
- closed convex surface 111.I
- closed covering 425.R
- closed curve, simple 93.B
- closed differential 367.H
- closed differential form 105.Q
- closed formula 276.A 299.A
 - in predicate logic 411.J
- closed geodesic 178.G
- closed graph theorem 37.I 251.D 424.X
- closed group 362.J
- closed half-line (in affine geometry) 7.D
- closed half-space (of an affine space) 7.D
- closed ideals in $L_1(G)$ 192.M
- closed image (of a variety) 16.I
- closed interval 140
 - in \mathbf{R} 355.C
- closed linear subspace (of a Hilbert space) 197.E
- closed manifold 105.B
- closed mapping 425.G
- closed operator (on a Banach space) 251.D
- closed orbit 126.D,G
 - hyperbolic 126.G
- closed plane domain 333.A
- closed path
 - (in a graph) 186.F
 - (in a topological space) 170
 - direct 186.F
 - space of 202.C
- closed range theorem 37.J
- closed Riemann surface 367.A
- closed set 425.B
 - locally 425.J
 - relative 425.J
 - system of 425.B
 - Zariski 16.A
- closed subalgebra 36.B
- closed subgroup (of a topological group) 423.D
- closed submanifold (of a C^∞ -manifold) 105.L
- closed subsystem (of a root system) 13.L
- closed surface 410.B
 - in a 3-dimensional Euclidean space 111.I
- closed system 419.A
- closed system entropy 402.G
- closed term (of a language) 276.A
- closure 425.B
 - (in a matroid) 66.G
 - (of an operator) 251.D
 - algebraic (of a field) 149.I
 - convex (in an affine space) 7.D
 - integral (of a ring) 67.I
 - Pythagorean (of a field) 155.C
- closure finite (cell complex) 70.D
- closure operator 425.B
- closure-preserving covering 425.X
- clothoid 93.H
- cloverleaf knot 235.C
- cluster 375.F
- cluster decomposition Hamiltonian 375.F
- clustering property 402.G
- cluster point 425.O
- cluster set(s) 62.A
 - boundary 62.A

curvilinear 62.C
 interior 62.A
 cluster value 62.A
 cluster value theorem 43.G
 cn App. A, Table 16.III
 coalgebra 203.F
 cocommutative 203.F
 dual 203.F
 graded 203.B
 quotient 203.F
 coalgebra homomorphism 203.F
 coanalytic set 22.A
 coarse moduli scheme 16.W
 coarse moduli space of curves of genus g 9.J
 coarser relation 135.C
 coarser topology 425.H
 cobordant 114.H
 foliated 154.H
 h - 114.I
 mod 2 114.H
 normally 114.J
 cobordism, knot 235.G
 cobordism class 114.H
 oriented 114.H
 unoriented 114.H
 cobordism group
 complex 114.H
 oriented 114.H
 unoriented 114.H
 cobordism group of homotopy n -spheres, h - 114.I
 cobordism ring 114.H
 complex 114.H
 cobordism theorem, h - 114.F
 coboundary (coboundaries) 200.H
 (in a cochain complex) 201.H
 (in the theory of generalized analytic functions) 164.H
 module of 200.F
 coboundary homomorphism (on cohomology groups) 201.L
 coboundary operator 200.F
 cobounded 201.P
 cochain(s) 200.H 201.H
 (products of) 201.K
 deformation 305.B
 finite (of a locally finite simplicial complex) 201.P
 n - (for an associative algebra) 200.L
 separation 305.B
 cochain complex 200.F 201.H
 singular 201.H
 cochain equivalence 200.F
 cochain homotopy 200.F
 cochain mapping 200.F 201.H
 cochain subcomplex 200.F
 Cochran theorem 374.B
 cocommutative coalgebra 203.F
 cocycle(s) 200.H 201.H
 (in the theory of generalized analytic functions) 164.H
 continuous 200.N
 difference 305.B
 module of 200.F
 obstruction 147.L 305.B
 separation 305.B
 vanishing (on an algebraic variety) 16.U
 Codazzi, equation of 365.C
 Codazzi-Mainardi equations 111.H, App. A, Table 4.I
 code(s) 63.A 213.D

BHC (Bose-Chaundhuri-Hocquenghem) 63.D
 block 63.A 213.F
 convolutional 63.E
 cyclic 63.D
 Goppa 63.E
 group 63.C
 Hamming 63.C
 linear 63.C
 perfect 63.B
 sliding block 213.E
 tree 213.E
 trellis 213.E
 code word 63.A
 codimension
 (of an algebraic subvariety) 16.A
 (of a C^r -foliation) 154.B
 (of an element in a complex) 13.R
 (of the germ of a singularity) 51.C
 (of a linear space) 256.F
 (of a PL embedding) 65.D
 codimension q , characteristic class of 154.G
 coding rate 213.D
 coding theorem
 block 213.D
 noiseless source 213.D
 source 213.D,E
 coding theory 63
 channel 213.A
 source 213.A
 codomain (of a mapping) 381.C
 co-echelon space 168.B
 coefficient(s)
 (of a linear representation) 362.E
 (of a system of algebraic equations) 10.A
 (of a term of a polynomial) 337.B
 of an affine connection 80.L
 autocorrelation 397.N
 binomial 330, App. A, Table 17.II
 of bulk viscosity 205.C
 canonical correlation 280.E 374.C
 Clebsch-Gordan 258.A 353.B
 confidence 399.Q
 correlation (of two random variables) 342.C 397.H
 of determination 397.H,J
 differential 106.A
 of excess 341.H 396.C
 expansion 317.A
 Fourier 159.A 197.C 317.A, App. A, Table 11.I
 Fourier (of an almost periodic function) 18.B
 Gini, of concentration 397.D
 indeterminate, Lagrange method of 106.L
 Lagrange interpolation 223.A
 Legendre 393.B
 multiple correlation 397.J
 of order p 110.A
 partial correlation 397.J
 partial differential 106.F
 population correlation 396.D
 Racah 353.B
 reflection 387.D
 regression 397.H,J 403.D
 of a Riemannian connection 80.L
 sample correlation 396.D
 sample multiple correlation 397.J
 sample partial correlation 280.E
 of shear viscosity 205.C
 of skewness 341.H
 of thermal expansion 419.B
 torsion (of a complex) 201.B

- transmission 387.D
- transport 402.K
- universal, theorem 200.D.G.H 201.G.H
- of variation 397.C
- of viscosity 205.C
- Wigner 353.B
- coefficient field
 - (of an affine space) 7.A
 - (of a projective space) 343.C
 - (of a semilocal ring) 284.D
- coefficient group (of the cohomology theory) 201.Q
- coefficient module 200.L
- coefficient problem 438.C
- coefficient ring
 - of a semilocal ring 284.D
- coefficient sheaf \mathcal{F}
 - Čech cohomology with 383.F
 - cohomology group with 383.E
- coercive (boundary condition) 112.H 323.H
- cofactor (of a minor) 103.D
- cofibring 202.G
- cofinal
 - (ordinal numbers) 312.C
 - (subset of a directed set) 311.D
- cofinality 312.C
- cofinality, cardinality and 33.F
- cofinal object 52.D
- cofinal subnet 87.H
- cogenerator (of an Abelian category) 200.H
- cographic 66.H
- Cohen theorem (on Noetherian rings) 284.A
- coherence 397.N 421.E
- coherent algebraic sheaf 16.E 72.F
- coherent analytic sheaf 72.E
- coherently oriented (pseudomanifold) 65.B
- coherent \mathcal{O} -module 16.E
 - quasi- 16.E
- coherent sheaf of rings 16.E
- coherent vector 277.D
- Cohn-Vossen theorem 111.I
- cohomological dimension
 - (of an associative algebra) 200.L
 - (of a scheme) 16.E
 - (of a topological space) 117.F
- cohomological functor 200.I
- cohomology 6.E 200.H
 - equivariant 431.D
 - exact sequence of 200.F
 - Galois 200.N
 - Gel'fand-Fuks 105.AA
 - l -adic 450.Q
 - non-Abelian 200.M
 - Tate 200.N
 - Weil 440.Q
- cohomology class 200.H
 - canonical (in Galois cohomology in the class field theory) 59.H
 - orientation 201.N
- cohomology exact sequence (for simplicial complexes) 201.L
- cohomology group(s) 201.H
 - \tilde{H} - 72.D
 - Alexander 201.M
 - Amitsur 200.P
 - Čech (for topological spaces) 201.M,P
 - Čech, with coefficient sheaf \mathcal{F} 383.F
 - cellular 201.H
 - with coefficient sheaf \mathcal{F} 383.E
 - de Rham 201.H
 - Dolbeault 72.D
 - generalized (for CW complexes) 201.H
 - Hochschild 200.L
 - integral (of a topological space) 201.H
 - local 125.W
 - rational 200.O
 - reduced (of a topological space) 201.H
 - relative (sheaf cohomology) 125.W
 - relative Alexander 201.M
 - relative Čech 201.M
 - singular, with compact supports 201.P
 - singular, of X with coefficients in G 201.H,R
- cohomology module 200.F
- cohomology operation(s) 64
 - functional 202.S
 - primary 64.B
 - stable 64.B
 - stable primary 64.B
 - stable secondary 64.C
- cohomology ring
 - of a compact connected Lie Group App. A, Table 6.IV
 - de Rham (of a differential manifold) 105.R
 - de Rham (of a topological space) 201.I
 - of an Eilenberg-MacLane complex App. A, Table 6.III
 - singular (of a topological space) 201.I
- cohomology sequence, exact 200.H
- cohomology set 172.K
- cohomology spectral sequence 200.J
- cohomology theory (theories)
 - Alexander-Kolmogorov-Spanier 201.M
 - on the category of topological pairs 201.Q
 - complete 200.N
 - with E-coefficient, generalized 202.T
 - generalized 201.Q
- cohomology vanishing theorems 194.G
- cohomotopy group 202.I
- coideal 203.F
- coimage
 - (of an A -homomorphism) 277.E
 - (of a homomorphism of presheaves of sheaves) 383.D
 - (of a morphism) 52.N
- coincidence number (of mappings) 153.B
- coincidence point (of mappings) 153.B
- coindex (of C^∞ -function) 279.E
- cokernel
 - (of an A -homomorphism) 277.E
 - (of a homomorphism of presheaves or sheaves) 383.D
 - (of a morphism) 52.N
- collapsing 65.C
 - elementary 65.C
- collectionwise Hausdorff space 425.AA
- collectionwise normal space 425.AA
- collective 342.A
 - ψ - 354.E
- collective risk theory 214.C
- collinear points 343.B
- collinear vectors 442.A
- collineation(s) 343.D
 - group of 343.D
 - projective 343.D
 - projective, in the wider sense 343.D
 - in the wider sense 343.D
- collocation method 303.I
- colocal (coalgebra) 203.I
- color conjecture, four 186.I
- colored symmetric group 92.D
- coloring (of a graph) 186.I

- coloring, Tait 157.C
- color lattice 92.D
- color point group 92.D
- colors, number of 93.D
- color symmetric group 92.D
- column(s) (of a matrix) 269.A
 - iterated series by (of a double series) 379.E
 - repeated series by (of a double series) 379.E
- column finite matrix 269.K
- column nullity (of a matrix) 269.D
- column vector 269.A
- comass (on k -forms) 275.G
- Combesure, correspondence of 111.F
- combination
 - k - 330
 - linear (of elements in a linear space) 256.C
 - linear (of ovals) 89.D
 - multiple App. A, Table 17.II
 - number of treatment 102.L
- combination theorem, Klein 234.D
- combinatorial analysis 66.A
- combinatorial equivalence 65.A
- combinatorially equivalent 65.A
- combinatorial manifold 65.C
- combinatorial mathematics 66.A
- combinatorial Pontryagin class 56.H
- combinatorial problems App. A, Table 17.II
- combinatorial property 65.A
- combinatorial sphere, group of oriented differentiable structures on 114.I
- combinatorial theory 66.A
- combinatorial topology 426
- combinatorial triangulation 65.C
- combinatorial triangulation problem 65.C
- combinatorics 66
- comb space 89.A
- commensurable 122.F
- common divisor (of elements of a ring) 67.H
 - greatest 67.H 297.A
- common logarithm 131.C
- common multiple (of elements of a ring) 67.H
 - least 67.H 297.A
- common notion 35.A
- communality 346.F
- commutant 308.C
- commutation relations
 - canonical 351.C 377.A
 - normal 150.D
- commutative algebra 203.F
- commutative diagram 52.C
- commutative field 368.B
- commutative group 2.A 190.A
- commutative law
 - of addition (in a ring) 368.A
 - on the addition of natural numbers 294.B
 - in the algebra of sets 381.B
 - on cardinal numbers 49.C
 - for group composition 190.A
 - in a lattice 243.A
 - for multiplication (in a commutative ring) 368.A
 - on the multiplication of natural numbers 294.B
- commutative Lie group 249.D
- commutatively convergent series 379.C
- commutative multiplication (of a graded algebra) 203.B
 - homotopy- 203.D
- commutative ring 368.A
 - category of 52.B
- commutativity, event 346.G
- commutator
 - (of differential operators) 324.C
 - (of two elements of a group) 190.H
 - self- 251.K
- commutator group 190.H
- commutator subgroup 190.H
- commutor 368.E
- comonad 200.Q
- compact
 - (continuous mapping) 286.D
 - (kernel distribution) 125.L
 - (linear operator) 68.B
 - (topological space) 425.S
 - linearly 422.L
 - locally 425.V
 - locally linearly 422.L
 - relatively (linear operator) 331.B
 - relatively (maximum likelihood method) 399.M
 - relatively (subset) 273.F 425.S
 - real- 425.BB
 - sequentially 425.S
 - σ - 425.V
 - T - 68.F 331.B
 - uniformly locally 425.V
 - weakly (linear operator) 68.M
- compact algebraic group, k - 13.G
- compact cardinal number
 - strongly 33.E
 - weakly 33.E
- compact complex manifolds, family of 72.G
- compact C^r -manifold 105.D
- compact element (of a topological Abelian group) 422.F
- compact foliation 154.H
- compact form (of a complex semisimple Lie algebra) 248.P
- compact group 69.A
- compact homotopy class 286.D
- compactification
 - (of a complex manifold) 72.K
 - (of a Hausdorff space) 207.A
 - (of a topological space) 425.T
- compactifying (kernel) 125.L
 - Aleksandrov 207.C
 - Bohr 18.H
 - F - 207.C
 - Kerékjártó-Stoïlow 207.C
 - Kuramochi 207.C
 - Martin 207.C
 - one-point (of a topological space) 425.T
 - resolutive 207.B
 - Royden 207.C
 - Stone-Čech 207.C 425.T
 - Wiener 207.C
- compact leaf 154.D
- compact metric space 273.F
- compactness theorem (in model theory) 276.E
- compact open C^∞ topology 279.C
- compact-open topology 279.C 435.D
- compact operators 68
- compact real Lie algebra 248.P
- compact real simple Lie algebra
 - classical 248.T
 - exceptional 248.T
- compact set 425.S
 - in a metric space 273.F
 - relatively 399.M 425.S
 - uniform convergence on 435.C

- compact simple Lie group
 - classical 249.L
 - exceptional 249.L
- compact space 425.S
 - countably 425.S
 - locally 425.V
 - real- 425.BB
 - sequentially 425.S
 - σ - 425.V
 - uniformly locally 425.V
- compact support (singular q -cochain) 201.P
- compact type (symmetric Riemannian homogeneous space) 412.D
- compactum, dyadic 79.D
- companion matrix 301.I
- comparability theorem for cardinal numbers 49.B
- comparison, paired 346.C
- comparison test (for convergence) 379.B
- comparison theorem (in the theory of differential equations) 316.E
 - metric 178.A
 - triangle 178.A
- compass 179.A
- compatible
 - with composition 409.C
 - with C^* -structure 114.B
 - with the multiplication of a group 190.C
 - with operation (of an operator domain) 277.C
 - with operations in a linear space 256.F
 - with topology 436.H
 - with a triangulation (a C^* -structure) 114.C
- compiler 75.C
- complement
 - (of a decision problem) 71.B
 - (in lattice theory) 42.A 243.E
 - (in set theory) 381.B
 - orthogonal (of a subset of a Hilbert space) 197.E
 - relative (of two sets) 381.B
- complementary analytic set 22.A
- complementary degenerate series 437.W
- complementary degree 200.J
- complementary event 342.B
- complementary law of reciprocity 14.O
- complementary law of the Jacobi symbol 297.I
- complementary law of the Legendre symbol
 - first 297.I
 - second 297.I
- complementary modulus (in Jacobi elliptic functions) 134.J, App. A, Table 16.I
- complementary series 437.W
- complementary set 381.B
- complementary slackness, Tucker theorem on 255.B
- complementary submodule 277.H
- complementary subspace (of a linear subspace) 256.F
- complementation, law of (in a Boolean algebra) 42.A
- complement conjecture, knot 235.B
- complemented (Banach space) 37.N
- complemented lattice 243.E
- complement theorem 382.B
 - Chapman's 382.B
- completable topological group 423.H
- complete
 - (Abelian p -group) 2.D
 - (algebraic variety) 16.D
 - (increasing family of σ -algebras) 407.B
 - (logical system) 276.D
 - (metric space) 273.J
 - (ordinary differential equation) 126.C
 - (predicate) 356.H
 - (recursively enumerable set) 356.D
 - (set of closed formulas) 276.F
 - (statistics) 396.E
 - (system of axioms) 35.B
 - (system of orthogonal functions) 317.A
 - (topological group) 423.H
 - (uniform space) 436.G
 - (valuation) 439.D
 - (vector lattice) 310.C
 - (wave operator) 375.B,H
 - (Zariski ring) 284.C
 - B - (locally convex space) 424.X
 - fully (locally convex space) 424.X
 - holomorphically (domain) 21.F
 - NP- 71.E
 - at o (in the theory of deformation) 72.G
 - quasi- (locally convex space) 424.F
 - σ - (vector lattice) 310.C
 - weakly 1- , manifold 21.L
- complete accumulation point 425.O
- complete additive group 2.E
- complete additivity (of the Lebesgue integral) 221.C
- complete additivity (of a measure) 270.D
- complete additivity theorem, Pettis 443.G
- complete analytic space, K - 23.F
- complete bipartite graph 186.C
- complete blocks 102.B
- complete capture 420.D
- complete class 398.B
 - essentially 398.B
 - minimal 398.B
- complete class theorems 398.D
- complete cohomology theory 200.N
- complete distributive law (in a lattice-ordered group) 243.G
- complete elliptic integral App. A, Table 16.I
 - of the first kind 134.B
 - of the second kind 134.C
- complete form, theorem on 356.H
- complete free resolution (of \mathbf{Z}) 200.N
- complete graph 186.C
- complete hyperbolic manifolds 21.O
- complete induction 294.B
- complete integrability condition 428.C
- complete intersection 16.A
- complete lattice 243.D
 - conditionally 243.D
 - σ - 243.D
- complete linear system
 - on an algebraic curve 9.C
 - on an algebraic variety 16.N
 - defined by a divisor 16.N
- complete local ring 284.D
 - structure theorem of 284.D
- completely additive
 - (arithmetic function) 295.B
 - (measure) 270.D,E
 - (vector measure) 443.G
- completely additive class 270.B
- completely additive set function 380.C
- completely continuous operator 68.B
- completely integrable 154.B
- completely integrable (system of independent 1-forms) 428.D
- completely integrally closed (ring) 67.I
- completely monotonic function 240.E,K

- completely multiplicative number-theoretic function 295.B
 completely nonunitary 251.M
 completely normal space 425.Q
 completely passive 402.G
 completely positive 36.H
 completely positive entropy 136.E
 completely primary ring 368.H
 completely randomized design 102.A
 completely reducible A -module 277.H
 completely reducible group 190.L
 completely reducible linear representation 362.C
 completely regular space 425.Q
 completely unstable flow 126.E
 complete manifold, weakly 1- 21.L
 complete mapping 241.B
 complete maximum principle 338.M
 complete measure 270.D
 complete measure space 270.D
 complete metric space 273.J
 completeness
 (for a Cartan connection) 80.N
 (of a logical system) 276.D
 (of the predicate calculus) 411.J
 asymptotic 150.D
 NP- 71.E
 theorem of (in geometry) 155.B
 completeness of real numbers 294.E 355.B
 completeness of scattering states 150.D 386.B
 completeness theorem, Gödel 411.J
 complete observation 405.C,D
 complete orthogonal system 217.G
 complete orthonormal set (Hilbert space) 197.C
 complete orthonormal system 217.G
 of fundamental functions 217.G
 complete pivoting 302.B
 complete predicate 356.H
 complete product measure space 270.H
 complete quadrangle 343.C
 complete reducibility theorem, Poincaré 3.C
 complete Reinhardt domain 21.B
 complete residue system modulo m 297.G
 complete ring (with respect to an ideal I) 16.X
 complete scheme, k - 16.E
 complete set 241.B
 complete σ -field 396.E
 boundedly 396.E
 complete solution (of partial differential equations) 320.C
 complete space 436.G
 Čech 425.T 436.I
 Dieudonné 436.I
 holomorphically 23.F
 topologically 436.I
 complete system
 of axioms 35.B
 of independent linear partial differential equations 324.C
 of inhomogeneous partial differential equations 428.C
 of nonlinear partial differential equations 428.C
 complete valuation 439.D
 complete vector lattice 310.C
 σ - 310.C
 complete Zariski ring 284.C
 completion
 α -adic (of an R -module) 284.B
 of a field (with respect to a valuation) 439.D
 of a measure space 270.D
 of a metric space 273.J
 μ - 270.D
 of an ordered set 243.D
 of a ring along an ideal 16.X
 of $\text{Spec}(A)$ along $V(I)$ 16.X
 of a T_2 -topological group 423.H
 of a uniform space 436.G
 of a valuation 439.D
 of a valuation ring (of a valuation) 439.D
 complex(es) 70
 (in an Abelian category) 201.B
 (in buildings and BN pairs) 13.R
 (over an object of an Abelian category) 200.H
 \tilde{c} - 72.D
 abstract simplicial 70.C
 acyclic 201.B
 Amitsur 200.P
 cell 70.D
 chain 200.C
 chain (in an Abelian category) 201.B
 chain, over C 201.C
 chain, over Λ 201.G
 chamber 13.R
 closure finite cell 70.D
 cochain, over Λ 201.H
 cochain (in an Abelian category) 200.F,H
 cochain (of a simplicial complex) 201.H
 countable cell 70.D
 countable simplicial 70.C
 Coxeter 13.R
 CW 70.D
 de Rham 201.H
 de Rham (as an elliptic complex) 237.H
 Dolbeault 72.D
 double 200.H
 double chain 200.E
 dual 65.B
 Eilenberg-MacLane 70.F
 elliptic (on a compact C^∞ -manifold) 237.H
 Euclidean 70.B
 Euclidean cell 70.B
 Euclidean simplicial 70.B
 finite cell 70.D
 finite simplicial 70.C
 geometric 70.B
 isomorphic simplicial 70.C
 isomorphic s.s. 70.E
 Kan 70.E
 linear (in projective geometry) 343.E
 linear line 343.E
 locally countable cell 70.D
 locally countable simplicial 70.C
 locally finite cell 70.D
 locally finite simplicial 70.C
 minimal 70.E
 multiple 200.H
 negative 200.H
 ordered (of a simplicial complex) 70.E
 ordered simplicial 70.C
 oriented simplicial chain 201.C
 Poincaré 114.J
 positive 200.H
 positive chain 200.C
 Postnikov 70.G
 product 200.H
 product (of cell complexes) 70.D
 product double chain 200.E
 quotient 201.L

Complex algebraic variety

- quotient chain 200.C
- rectilinear 70.B
- regular cell 70.D
- relative chain 200.C
- relative cochain 200.F
- semisimplicial (s.s.) 70.E
- simplicial 65.A 70.C
- singular (of a topological space) 70.E
- singular chain (of a topological space) 201.E
- singular cochain 201.H
- space form 365.L
- s.s. 70.E
- s.s., realization of 70.E
- standard (of a Lie algebra) 200.O
- Thom 114.G
- Thom, associated with (G, n) 114.G
- Thom, fundamental cohomology class of the 114.G
- complex algebraic variety 16.T
- complex analytic fiber bundle 147.O
- complex analytic function 198.H
- complex analytic manifold 72.A
- complex analytic structure (in a complex manifold) 72.A
- complex analytic submanifold 72.A
- complex cobordism group 114.H
- complex cobordism ring 114.H
- complex conjugate bundle 147.F
- complex conjugate representation 362.F
- complex dimension (of a complex manifold) 72.A
- complex form (on a Fourier series) 159.A
- complex form (of a real Lie algebra) 248.P
- complex function 165.B
- complex Gaussian 176.B
- complex Gaussian process 176.C
- complex Gaussian random variable 176.B
- complex Gaussian system 176.B
- complex Grassmann manifold 199.B
- complex group (over a field) 60.L
- complex Hermitian homogeneous space 199.A
- complex Hilbert space 197.B
- complexification
 - Chevalley 249.U
 - of a real Lie algebra 248.P
- complex interpolation space 224.C
- complexity
 - average 71.A
 - of computation 71
 - Kolmogorov-Chaitin 354.D
 - space 71.A
 - time 71.A
 - worst-case 71.A
- complex Lie algebra 248.A
 - of a complex Lie group 249.M
- complex Lie group 249.A
- complex linear space 256.A
- complex line bundle 72.F
 - determined by a divisor 72.F
- complex manifold(s) 72
 - almost 72.B
 - compact, family of 72.G
 - isomorphic 72.A
 - stably almost 114.H
 - weakly almost 114.H
- complex multiplication 73
- complex number plane 74.C
- complex numbers 74.A 294.F
 - conjugate 74.A
- complex of lines 110.B
- complex orthogonal group 60.I
- complex orthogonal matrix 269.J
- complex plane 74.C
- complex projective space 343.E
 - infinite-dimensional 56
- complex quadratic form 348.A,B
- complex representation (of a Lie group) 249.O
- complex simple Lie algebra
 - classical 248.S
 - exceptional 248.S
- complex simple Lie group
 - classical 249.M
 - exceptional 249.M
- complex space form 365.L
- complex special orthogonal group 60.I
- complex spectral measure 390.D
- complex spectral representation 390.E
- complex spectral resolution 390.E
- complex sphere 74.D
- complex spinor group 61.E
- complex Stiefel manifold 199.B
- complex structure
 - (in a complex manifold) 72.A
 - (pseudogroup structure) 105.Y
 - (on \mathbb{R}^{2n}) 3.H
 - (on a Riemann surface) 367.A
 - almost 72.B
 - deformation of 72.G
 - tensor field of almost (induced by a complex structure) 72.B
- complex symplectic group 60.L
- complex topological linear space 424.A
- complex torus 3.H
- complex-valued function 165.B
- complex variable 165.C
 - theory of functions of 198.Q
- complex vector bundle 147.F
- component(s)
 - (of a direct product set) 381.E
 - (in graph theory) 186.F
 - (of a matrix) 269.A
 - (of a point in a projective space) 343.C
 - (of a tensor of type (p, q)) 256.J
 - (of a vector) 256.A 442.A
 - (of a vector field) 105.M
 - arcwise connected 79.B
 - basic (of an m -dimensional surface) 110.A
 - connected 79.A 186.F
 - of degree n (of a graded A -module) 200.B
 - embedded primary (of an ideal) 67.F
 - fixed (of a linear system) 16.N
 - ghost (of an infinite-dimensional vector) 449.A
 - horizontal (of a homogeneous space) 110.A
 - horizontal (of a vector field) 80.C
 - identity (of a topological group) 423.F
 - irreducible (of an algebraic variety) 16.A
 - irreducible (of an analytic space) 23.C
 - irreducible (of a linear representation) 362.D
 - isolated primary (of an ideal) 67.F
 - i th (of an element relative to a basis) 256.C
 - i th (of an n -tuple) 256.A
 - nilpotent (of a linear mapping) 269.L
 - orthogonal (of an element of a linear space) 139.G
 - path- 79.B
 - primary (of an ideal) 67.F
 - principal 280.F
 - principal, analysis 280.F
 - principal, of order p 110.A
 - proper (of an intersection of subvarieties) 16.G
 - Reeb 154.B

- relative (of a Lie transformation group) 110.A
- secondary (of a homogeneous space) 110.A
- semisimple 269.L
- simple (of a semisimple ring) 368.G
- strongly connected 186.F
- unipotent (of a linear mapping) 269.L
- variable (of a linear system) 15.C 16.N
- vertical (of a vector field) 80.C
- component model 403.F
- components-of-variance model 403.C
- composite
 - of cohomology operations 64.B
 - of correspondences 358.B
 - of homotopy classes 202.B
 - of mappings 381.C
 - of morphisms 52.A
 - of subsets 436.A
 - of valuations 439.F
- composite designs, central 102.M
- composite field 149.D
- composite function 106.I
- composite hypothesis 400.A
- composite number 297.B
- composite particles 132.A
- composition
 - (of knots) 235.A
 - (of probability distributions) 341.E
 - external law of (of a set of another set) 409.A
 - internal law of (of a set) 409.A
 - law of (on a set) 409.A
 - secondary 202.R
- composition algebra 231.B
- composition factor 190.G
- composition factor series 190.G
- composition product (of functions) 192.H
- composition series
 - (in a group) 190.G
 - (in a lattice) 243.F
- composition theorem (in class field theory) 59.C
- compound Poisson process 5.F
- comprehension, axiom of 33.B 381.G
- compressibility, isothermal 419.B
- compressible fluid 205.B
- compression, information 96.B
- computable (partial function) 31.B
- computation
 - analog 19.A
 - complexity of 71
 - high-precision 138.B
- compute 31.B 71.B
- computers 75
 - analog 19.E
 - digital 75.B
 - electronic 75.A
 - electronic analog 19.E
 - hybrid 19.E
- comultiplication 203.B,F
 - Hopf 203.D
- concatenation (of paths) 170
- concave function 88.A
 - strictly 88.A
- concave programming problem 292.A
- concentration
 - area of 397.E
 - asymptotic 399.L
 - Gini coefficient of 397.E
 - measure of 397.E
 - spectral 331.F
- concentration function
 - maximal 341.E
 - mean 341.E
- concept
 - basic (of structure) 409.B
 - global (in differential geometry) 109
 - in the large (in differential geometry) 109
 - local (in differential geometry) 109
 - in the small (in differential geometry) 109
- conchoidal curve 93.H
- conchoid of Nicomedes 93.H
- concircularly flat space App. A, Table 4.II
- concordant 154.F
- concurrent
 - (in nonstandard analysis) 293.B
 - (in projective geometry) 343.B
- condensation of singularities, principle of 37.H
- condensation point 425.O
- condensation test, Cauchy 379.B
- condition(s)
 - adjoint boundary 315.B
 - ascending chain (for (normal) subgroups of a group) 190.F
 - ascending chain (in an ordered set) 311.C
 - Baire 425.N
 - boundary (for an ordinary differential equation) 315.A
 - boundary (for partial differential equations of elliptic type) 323.F
 - boundary, operator with 112.F
 - Cauchy (on the D-integral and the D(*)-integral) 100.E
 - CFL (Courant-Friedrichs-Lewy) 304.F
 - chain (in an ordered set) 311.C
 - complete integrability 428.C
 - consistency 341.I
 - Denjoy-Carleman 168.B
 - descending chain (for (normal) subgroups of a group) 190.F
 - descending chain (in an ordered set) 311.C
 - entropy 204.G
 - of finite character (for functions) 34.C
 - of finite character (for sets) 34.C
 - finiteness, for integral extension 284.F
 - Frobenius integrability 154.B
 - Haar (on best approximation) 336.B
 - Harnack (on the D-integral and the D(*)-integral) 100.E
 - Hölder, of order α 84.A
 - initial (for ordinary differential equations) 316.A
 - initial (for partial differential equations) 321.A
 - Jacobi 46.C
 - KMS 308.H
 - Levi 325.H
 - Lindeberg 250.B
 - Lipschitz 84.A 163.D 286.B 316.D
 - Lipschitz, of order α 84.A
 - Lorentz 130.A
 - LSZ asymptotic 150.D
 - Lyapunov 250.B
 - maximal (in an ordered set) 311.C
 - minimal (in an ordered set) 311.C
 - no cycle 126.J
 - Palais-Smale 279.E 286.Q
 - Poincaré (in the Dirichlet problem) 120.A
 - restricted minimal (in a commutative ring) 284.A
 - of R. Schmidt (in (C, α) -summation) 379.M
 - Sommerfeld radiation 188.D
 - spectrum 150.D
 - strip 320.D

- strong transversality 126.J
- transversality 108.B
- of transversality (in the calculus of variations) 46.B
- uniqueness (for solution of an ordinary differential equation) 316.D
- von Neumann 304.F
- Whitney, *(b)* 418.G
- Whitney, *(b)* at a point 418.G
- conditional density 397.I
- conditional distribution 397.I
- conditional entropy 213.B
- conditional expectation (of a random variable) 342.E
- conditional inequality 211.A
- conditionality, principle of 401.C
- conditionally complete lattice 243.D
- conditionally convergent 379.C,E
- conditionally σ -complete lattice 243.D
- conditionally stable 394.D
- conditional mean (of a random variable) 342.E 397.I
- conditional moments 397.J
- conditional probability 342.E
 - regular 342.E
- conditional probability distribution 342.E
- conditional problems in the calculus of variations 46.A
- conditional relative extremum (of a function) 106.L
- conditional self-intersection 213.B
- conditional stability 394.D
- condition number 302.A
- conductivity 130.B
- conductor
 - (of an Abelian extension) 14.Q
 - (of a class field) 59.B
 - (of Dirichlet L -functions) 450.C
 - (of an ideal group) 14.H
 - (of a Grössencharakter) 450.F
 - (of Hecke L -functions) 450.E
 - (of a nonprimitive character or a primitive character) 450.C
 - (of a quadratic field) 347.G
 - (of a residue character) 295.D
 - (of a subring of a principal order) 14.B
 - p - (of norm-residue) 14.P
- conductor-ramification theorem (in class field theory) 59.C
- conductor with a group character 450.G
- cone
 - (of a PL embedding) 65.D
 - (in a projective space) 343.E
 - (of a simplicial complex) 70.C
 - (over a space) 202.E
 - asymptotic 350.B
 - circular 78.A 111.I
 - conjugate convex 89.F
 - convex 89.F
 - convex polyhedral 89.F
 - dual convex 89.F
 - extension (of a PL embedding) 65.D
 - future 258.A
 - light 258.A
 - Mach 205.B
 - mapping 202.E
 - natural positive 308.K
 - oblique circular 350.B
 - past 258.A
 - quadratic 350.B
 - reduced (of a topological space) 202.F
 - reduced mapping 202.F
 - regular 384.A
 - right circular 350.B
 - self-dual regular 384.E
 - side 258.A
- confidence coefficient 399.Q
- confidence interval 399.Q
- confidence level 399.Q
- confidence limits 399.Q
- confidence region 399.Q
 - invariance of 399.Q
 - unbiased 399.Q
 - uniformly most powerful 399.Q
 - uniformly most powerful unbiased 399.Q
- configuration
 - central 420.B
 - Pascal 78.K
- configuration space 126.L 402.G
- confluent differential equation 167.A
- confluent hypergeometric differential equation 167.A, App. A, Table 14.II 19.I
- confluent type
 - function of 167.A
 - hypergeometric function of 167.A, App. A, Table 19.I
- confocal central conics, family of 78.H
- confocal parabolas, family of 78.H
- confocal quadrics, family of 350.E
- conformal 77.A
 - almost 275.C
- conformal arc element 110.D
- conformal connection 80.P
- conformal correspondence (between surfaces) 111.I
- conformal curvature 110.D
- conformal curvature tensor, Weyl 80.P, App. A, Table 4.II
- conformal differential geometry 110.D
- conformal function, μ - 352.B
- conformal geometry 76.A
- conformal invariant 77.E
- conformally equivalent 77.A 367.A 191.B
- conformally flat 191.B
- conformally flat space App. A, Table 4.II
- conformal mapping 198.A, App. A, Table 13
 - generalized 246.I
 - extremal quasi- 352.C
 - quasi- 352
- conformal space 76.A
- conformal structure 191.B
- conformal structure (on a Riemann surface) 367.A
- conformal torsion 110.D
- conformal transformation 80.P 364.F
- confounded
 - with blocks 102.J
 - partially (with blocks) 102.J
- congruence
 - (in geometry) 155.B
 - (in number theory) 297.G
 - linear (in projective geometry) 343.E
 - of lines 110.B
 - multiplicative 14.H
- congruence axiom (of geometry) 155.B
- congruence classes modulo m^* , group of 14.H
- congruence subgroup (of a modular group) 122.D
 - principal, of level N 122.D
- congruence zeta function 450.P
- congruent
 - (figures) 139.C
 - (segment) 155.B
 - affinely 7.E

- in the Erlangen program 137
- congruent modulo m 297.G
- congruent transformation(s) 139.B
 - group of 285.C
- conic(s) 78.A
 - central 78.C
 - focal (of a quadric) 350.E
 - pencil of 343.E
- conical function App. A, Table 18.II
- conical hypersurface, quadric 350.G
- conical surface 111.I
 - quadric 350.B
- conic Lagrange manifold 274.C 345.B
- conic section(s) 78.A
 - equation of 78.C
 - canonical form of the equation of 78.C
- conjecture
 - Adams (on J -homomorphisms) 237.I
 - annulus (on combinatorial manifolds) 65.C
 - Artin (on Artin L -functions) 450.G
 - Bieberbach (on univalent functions) 438.C
 - Birch–Swinnerton-Dyer (on L -functions of elliptic curves) 118.D 450.S
 - Burnside (on finite groups) 151.D
 - C_n (on Kodaira dimension) 72.H
 - entropy 126.K
 - four color 186.I
 - fundamental (in topology) 70.C
 - generalized Poincaré 65.C
 - general knot 235.B
 - Hasse (on Hasse zeta function) 450.S
 - Hodge (on cycles on algebraic varieties) 450.S
 - Iwasawa main (on p -adic L -functions) 450.J
 - knot complement 235.B
 - Leopoldt (on p -adic L -functions) 450.J
 - Mordell (on Diophantine equations) 118.E
 - Poincaré (on a characterization of spheres) 65.C
 - property P - (on knot groups) 235.B
 - Ramanujan (on automorphic functions) 32.D
 - Ramanujan-Petersson (on Hecke operators) 32.D
 - Sato (on Hasse zeta functions) 450.S
 - Schreier (on simple groups) 151.I
 - Seifert (on vector fields) 126.K 154.D
 - Smith (on knot theory) 235.E
 - stability 126.J
 - Taniyama-Weil (on L -functions of elliptic curves) 450.S
 - Tate (on Hasse ζ -functions) 450.S
 - unknotting 235.E
 - Vandiver (on the class number of cyclotomic fields) 14.L
 - Weil (on congruence zeta functions) 450.Q
- conjugacy
 - C^* - 126.B
 - topological 126.B
- conjugacy class (of an element of a group) 190.C
- conjugate
 - (CG method) 302.D
 - (diameter) 78.G
 - (element) 149.J
 - (point in a geodesic) 178.A
 - (point in a projective space) 343.E
 - (with respect to a quadric surface) 350.C
 - (quaternion) 29.D
 - (subset) 190.C
 - C^* - 126.B
 - harmonic (in projective geometry) 343.D
 - Ω - 126.H
 - topological 126.B
- conjugate axis (of a hyperbola) 78.C
- conjugate complex number 74.A
- conjugate convex cone 89.F
- conjugate differential (on Riemann surface) 367.H
- conjugate exponent 168.C
- conjugate field 149.J 377.C
- conjugate Fourier integral 160.D
- conjugate function 159.E 160.D
- conjugate gradient (CG) method 302.D
- conjugate harmonic function 193.C
- conjugate hyperbola 78.E
- conjugate ideal (of a fractional ideal) 14.I
- conjugate operator
 - (in Banach spaces) 37.D
 - (of a differential operator) 125.F
 - (of a linear operator) 251.D
- conjugate planes (with respect to a quadric surface) 350.C
- conjugate point(s)
 - (in the calculus of variations) 46.C
 - (in a Riemannian manifold) 364.C
- conjugate pole 350.C
- conjugate Radon transform 218.F
- conjugate representation 362.F
 - complex 362.F
- conjugate series (of a trigonometric series) 159.A
- conjugate space
 - (of a linear topological space) 424.D
 - (of a normed linear space) 37.D
- conjugation mapping (of a Hopf algebra) 203.E
- conjugation operator 164.K
- conjunction (of propositions) 411.B
- connected
 - (affine algebraic group) 13.A
 - (design) 102.K
 - (graded module) 203.B
 - (graph) 186.F
 - (topological space) 79.A
 - (treatment) 102.B
 - arcwise (space) 79.B
 - K - 186.F
 - locally (at a point) 79.B
 - locally (space) 79.A
 - locally arcwise (at a point) 79.B
 - locally arcwise (space) 79.B
 - locally n - (at a point) 79.C
 - locally n - (space) 79.C
 - locally ω - (space) 79.C
 - multiply (plane domain or space) 333.A
 - n - (pair of topological spaces) 202.L
 - n - (space) 79.C 202.L
 - n -ply (plane domain) 333.A
 - ω - (space) 79.C
 - path- (space) 79.B
 - simply (covering Lie group) 249.C
 - simply (space) 79.C 170
 - simply, group 13.N
 - strongly (components) 186.F
- connected component 79.A 186.F
 - arcwise- 79.B
 - strongly 186.F
- connected Lie subgroup 249.D
- connectedness 79 186.F
 - of real numbers 294.E
- connectedness theorem
 - general, due to W. Fulton and J. Hansen 16.I
 - Zariski 16.X
- connected part 150.D
- connected sequences of functors 200.I

- connected set 79.A
- connected space 79.A
- connected sum
 - (of oriented compact C^∞ -manifolds) 114.F
 - (of 3-manifolds) 65.E
- connecting homomorphism
 - in cohomology 200.F
 - in homology 200.C
 - on homology groups 201.C,L
- connecting morphism 200.H,I
- connection(s) 80
 - affine 80.H 286.L
 - affine, coefficients of 80.L
 - canonical affine (on \mathbf{R}^n) 80.J
 - Cartan 80.M
 - conformal 80.P
 - Euclidean 364.B
 - Euclidean, manifold with 109
 - flat 80.E
 - Gauss-Manin (of a variety) 16.V
 - Levi-Civita 364.B
 - linear 80.H
 - locally flat 80.E
 - metric 80.K
 - normal 365.C
 - projective 80.O
 - Riemannian 80.K 364.B
 - Riemannian, coefficients of 80.L
- connection form 80.E 417.B
- connection formula
 - for the solutions of a differential equation 253.A
- connection of spin and statistics 132.A 150.D
- connection problem 253.A
- connective fiber space, n - 148.D
- connectives, propositional 411.E
- connectivity (of a space) 201.A
- conoid, right 111.I
- conoidal neighborhood 274.E
- conormal 323.F
- conormal bundle 274.E
- conormal sphere bundle 274.E
- co-NP 71.E
- conservation laws, even-oddness 150.D
- conservative (measurable transformation) 136.C
- conservative chain 260.A
- conservative process 261.B
- conserved axial-vector currents, partially 132.C
- consistency
 - (condition in the multistep method) 303.E
 - (of an estimator) 399.K
 - (of a logical system) 276.D
 - of analysis 156.E
 - of the axiom of choice and the continuum hypothesis 33.D
 - $\{c_n\}$ - 399.K
 - relative 156.D
- consistency condition 341.I
- consistency proof 156.D
 - of pure number theory 156.E
- consistent
 - (finite difference scheme) 304.F
 - (formal system) 411.I
 - (system of axioms) 35.B
 - a.s. 399.K
 - $\{c_n\}$ - 399.K
 - Fisher 399.K
 - ω - 156.E
- consistent and asymptotically normal (CAN) estimator 399.K
- consistent estimator 399.K
- consistent kernel (in potential theory) 338.E
- consistent-mass scheme 304.D
- consistent test 400.K
 - uniformly 400.K
- constant(s) 165.C
 - arbitrary (in a general solution of a differential equation) 313.A
 - Bloch 77.F
 - Boltzmann 402.B
 - dielectric 130.B
 - empirical 19.F
 - error 303.E
 - Euler 174.A
 - integral 216.C
 - integration (in a general solution of a differential equation) 313.A
 - isoperimetric 391.D
 - Lagrange method of variation of 252.D
 - Landau 77.F
 - method of variation of 55.B 252.I
 - phase (of a sine wave) 446
 - Planck 115
 - renormalization 150.D
 - Robin 48.B
 - schlicht Bloch 77.F
 - structural (of a Lie algebra) 248.C
 - universal (in the theory of conformal mapping) 77.F
- constant breadth, curve of 89.C
- constant curvature
 - space of 364.D, App. A, Table 4.II
 - surface of 111.I
- constant function 381.C
- constant inclination, curve of 111.F
- constant mapping 381.C
- constant pressure, specific heat at 419.B
- constant sheaf 383.D
 - locally constructible 16.AA
- constant stratum, μ - 418.E
- constant-sum game 173.A
- constant term
 - of a formal power series 376.A
 - of a polynomial 337.B
 - unfolding 51.D
- constant variational formula 163.E
- constant volume, specific heat at 419.B
- constant width, curve of 111.E
- constituent (of an analytic or coanalytic set) 22.C
- constraint 102.L 264.B
 - capacity 281.D
 - chance 408.B
 - unilateral 440.A
- constraint qualification
 - Guignard 292.B
 - Slater 292.B
- constraint set (of a minimization problem) 292.A
- constructibility, axiom of (in axiomatic set theory) 33.D
- constructible (set in axiomatic set theory) 33.D
- constructible sheaf 16.AA
 - locally, constant 16.AA
- construction
 - bar (of an Eilenberg-MacLane complex) 70.F
 - geometric, problem 179.A
 - GNS 308.D
 - group measure space 136.F
 - impossible, problem 179.A
 - possible, problem 179.A
 - W - (of an Eilenberg-MacLane complex) 70.F

construction problem (of class field tower) 59.F
 constructive field theory 150.F
 constructive method 156.D
 constructive ordinal numbers 81.B
 consumer's risk 404.C
 contact, thermal 419.A
 contact element 428.E
 in a space with a Lie transformation group 110.A
 contact form 110.E
 contact manifold 110.E
 contact metric structure 110.E
 contact network 282.B
 contact pair (in circle geometry) 76.C
 contact process 340.C
 contact structure 105.Y
 contact transformations 82, App. A, Table 15.IV
 quantized 274.F
 contain 381.A
 physically 351.K
 content (of a tolerance region) 399.R
 Jordan 270.G
 mean 399.R
 context-free grammar 31.D
 context-sensitive grammar 31.D
 contiguous 399.M
 contingency table 397.K 400.K
 continuous, analytically 198.I
 continuation
 analytic 198.G
 analytic, along a curve 198.I
 analytic, in the wider sense 198.O
 direct analytic 198.G
 harmonic 193.M 198.G
 continuation method 301.M
 continuation theorem
 Hartogs 21.F
 Remmert-Stein 23.B
 Riemann 21.F
 unique 323.J
 continued fractions 83.A
 finite 83.A
 infinite 83.A
 mixed periodic 83.C
 normal 83.E
 pure periodic 83.C
 recurring 83.B
 simple 83.A
 continuity
 absolute, space of 390.E
 axioms of (in geometry) 155.B
 Dedekind axiom of (for real numbers) 355.A
 equation of (for a fluid) 205.A
 equation of (for electromagnetics) 130.A 204.B
 Hartogs theorem of 21.H
 interval of (for a probability distribution) 341.C
 local 45.F
 modulus of (of a function) 84.A
 modulus of, of k th order (of a continuous function) 336.C
 properties of 85.A
 of real numbers 294.E
 uniform 45.F
 continuity (*), generalized absolute 100.C
 in the restricted sense 100.C
 continuity principle
 for analytic functions of several complex variables 21.H
 in potential theory 338.C

Subject Index

Continuous geometry

quasi- (in potential theory) 338.I
 continuity property for Čech theory 201.M
 continuity requirement, variational principles with relaxed 271.G
 continuity theorem
 Abel (for Dirichlet series) 339.B
 Abel (for power series) 121.D
 Lévy 341.F
 continuous
 (additive interval function) 380.B
 (flow) 136.D
 (function of ordinal numbers) 312.C
 (mapping) 84.A 425.G
 absolutely (function) 100.C
 absolutely (mapping in the plane) 246.H
 absolutely (measure) 270.L
 absolutely (set function) 380.C
 absolutely (vector measure) 443.G
 absolutely, in the restricted sense 100.C
 absolutely, in the sense of Tonelli 246.C
 absolutely, (*) 100.C
 completely (operator) 68.D
 equi- 435.D
 equi-, semigroup of class (C^0) 378.B
 generalized absolutely 100.C
 hypo- 424.Q
 left 84.B
 from the left 84.B
 in the mean 217.M
 in the mean (stochastic process) 407.A
 μ -absolutely 380.C
 with respect to the parameter (a distribution) 125.H
 piecewise, function 84.B
 in probability 407.A
 right 84.B
 from the right 84.B
 separately (bilinear mapping) 424.Q
 strongly (function with values in a Banach space) 37.K
 uniformly 84.A 273.I 436.E
 uniformly, on a subset 436.G
 weakly (function with values in a Banach space) 37.K
 continuous action (in topological dynamics) 126.B
 continuous additive interval function 380.B
 continuous analytic capacity 164.J
 continuous arc(s) 93.B
 continuous cocycle 200.N
 continuous distribution (probability theory) 341.D
 continuous dynamical system 126.B
 continuous flow
 (in ergodic theory) 136.D
 (on a topological space) 126.B
 continuous functions 84
 absolutely 100.C
 generalized absolutely 100.C
 lower semi- 84.C
 lower semi- (at a point) 84.C
 on a metric space 84.C
 piecewise 84.B
 quasi- 338.I
 right 84.B
 semi- (at a point) 84.C
 uniformly (in a metric space) 84.A
 upper semi- 84.C
 upper semi- (at a point) 84.C
 continuous geometry 85
 irreducible 85.A
 reducible 85.A

- continuous homomorphism (between topological groups) 423.J
- open 423.J
- continuous image 425.G
- continuously differentiable function, n -times 106.K
- continuous mapping 425.G
 - space of 435.D
 - strongly 437.A
 - uniformly (of metric spaces) 273.I
 - uniformly (of uniform spaces) 436.E
- continuous plane curve 93.B
- continuous representation
 - strongly (of a topological group) 69.B
 - weakly (of a topological group) 69.B
- continuous semiflow 126.B
- continuous semimartingale 406.B
- continuous spectrum 390.A
 - absolutely 390.E
 - of an integral equation 217.J
- continuous spin 258.C
- continuous state branching process 44.E
- continuous tensor product 377.D
- continuum 79.D
 - cardinal number of 49.A
 - indecomposable 79.D
 - irreducible 79.D
 - Peano 93.D
- continuum hypothesis 49.D
 - consistency of the axiom of choice and 33.D
 - generalized 49.D
 - independence of the axiom of choice and 33.D
- contour(s)
 - additivity of (in the curvilinear integral) 94.D
 - of an integration 94.D
- contract, annuity 214.B
- contracted tensor 256.L
- contractible space 79.C 202.D
 - locally 79.C 202.D
 - locally, at a point 79.C
- contraction
 - (of a graph) 186.E
 - (linear operator) 37.C
 - (of a mapping) 381.C
 - (of a matroid) 66.H
 - (of a tensor) 256.L
 - sub- 186.E
- contraction principle 286.B
- contractive 251.N
 - purely 251.N
 - purely, part 251.N
- contradiction 411.I
- contradictory formal system 411.I
- contragredient (of a linear mapping) 256.G
- contragredient representation 362.E
- contrast
 - elementary 102.C
 - normalized 102.C
 - treatment 102.C
- contravariant functor 52.H
- contravariant index (of a component of a tensor) 256.J
- contravariant of order r and covariant of order s 108.D
- contravariant spinor 258.B
- contravariant tensor
 - alternating 256.N
 - of degree p 256.J
 - symmetric 256.N
- contravariant tensor algebra 256.K
- contravariant tensor field of order r 105.O
- contravariant vector 256.J
- contravariant vector field 105.O
- control
 - admissible 405.A
 - bang-bang 405.C
 - feedback 405.C
 - impulse 405.E
 - inventory 227
 - local 102.A
 - optimal 46.D 86.B,C 405.A
 - quality 404.A
 - stochastic 342.A 405
 - time-optimal 86.F
- control chart 404.B
- controllability 86.C
- controlled stochastic differential equation 405.A
- controlled tubular neighborhood system 418.G
- control limit
 - lower 404.F
 - upper 404.B
- control problem, time optimal 86.F
- control space (in static model in catastrophe theory) 51.B
- control theory 86
- control unit 75.B
- convention
 - Einstein 256.J
 - Einstein summation 417.B
 - Maxwell 51.F
 - perfect delay 51.F
- converge
 - (filter) 87.I
 - (infinite product) 379.G
 - (in a metric space) 273.D
 - (net) 87.H
 - (sequence of lattices) 182.B
 - (sequence of numbers) 87.B 355.B
 - (series) 379.A
 - (in a topological space) 87.E
 - almost certainly 342.D
 - almost everywhere 342.D
 - almost surely 342.D
 - in distribution 168.B 342.D
 - in the mean of order p 342.D
 - in the mean of power p 168.B
 - in probability 342.D
 - with probability 342.D
 - strongly 37.B
 - uniformly (in a uniform space) 435.A
 - weakly (in a normed linear space) 37.E
 - weakly (in a topological linear space) 424.H
- convergence 87
 - (of a filter) 87.I
 - (of a net) 87.H
 - (of probability measures) 341.F
 - (of truncation errors) 303.B
 - abscissa of (of a Dirichlet series) 121.B
 - abscissa of (of a Laplace transform) 240.B,H
 - absolute, abscissa of (Dirichlet series) 121.B
 - absolute, abscissa of (of a Laplace transform) 240.B
 - associated, radii 21.B
 - asymptotic 168.B
 - axis of 240.B
 - circle of (of a power series) 339.A
 - exponent of 429.B
 - generalized 331.C
 - norm resolvent 331.C

- radius of (of a power series) 339.A
- relative uniform star 310.F
- simple, abscissa of (of a Dirichlet series) 121.B
- star 87.K
- strong (of operators) 251.C
- strong resolvent 331.C
- uniform 435
- uniform (of a series) 435.A
- uniform (of operators) 251.C
- uniform, abscissa of (of a Dirichlet series) 121.B
- uniform, abscissa of (of a Laplace transform) 240.B
- uniform, on compact sets 435.C
- weak (of operators) 251.C
- weak (of probability measures) 341.F
- weak (of a sequence of submodules) 200.J
- Weierstrass criterion for uniform 435.A
- convergence criterion for positive series App. A, Table 10.II
- convergence domain (of a power series) 21.B
- convergence in measure 168.B
- convergence method 354.B
- convergence theorem
 - on distributions 125.G
 - Lebesgue 221.C
 - of martingales 262.B
- convergent
 - (continued fraction) 83.A
 - (double series) 379.E
 - (filtration) 200.J
 - (infinite integral) 216.E
 - (sequence) 87.B 355.B
 - (series) 379.A
 - absolutely (double series) 379.E
 - absolutely (infinite product) 379.G
 - absolutely (Laplace-Stieltjes integral) 240.B
 - absolutely (power series) 21.B
 - absolutely (series) 379.C
 - absolutely (series in a Banach space) 443.D
 - commutatively 379.C
 - conditionally 379.C.E
 - intermediate 83.B
 - (α)- 87.I.
 - (α)-star 87.L
 - order (in a vector lattice) 310.C
 - pointwise 435.B
 - principal 83.B
 - simply 435.B
 - unconditionally 379.C
 - uniformly (on a family of sets) 435.C
 - uniformly (sequence, series, or infinite product) 435.A
 - uniformly, in the wider sense 435.C
 - uniformly absolutely 435.A
- convergent power series 370.B
- convergent power series ring 370.B
- convergent sequence 355.B
- convex
 - (function on a G -space) 178.H
 - (function on a Riemannian manifold) 178.B
 - (subset of a sphere) 274.E
 - (subset of a sphere bundle) 274.E
 - absolutely 424.E
 - holomorphically, domain 21.H
 - locally (linear topological space) 424.E
 - logarithmically (domain) 21.B
 - matrix (of order m) 212.C
 - operator 212.C
 - properly 274.E
 - uniformly (normed linear space) 37.G
- convex analysis 88
- convex body 89.A
- convex cell (in an affine space) 7.D
- convex closure (in an affine space) 7.D
- convex cone
 - conjugate 89.F
 - dual 89.F
- convex curve, closed 111.E
- convex functions 88.A
 - proper 88.D
 - strictly 88.A
- convex hull 89.A
 - (in an affine space) 7.D
 - (of a boundary curve) 275.B
 - (in linear programming) 255.D
 - closed 424.H
- convexity theorem
 - Lyapunov 443.G
 - M. Riesz 88.C
- convex neighborhood 364.C
- convex polyhedral cone 89.F
- convex polyhedron 89.A
- convex programming 264.C
- convex programming problem 292.A
- convex rational polyhedral 16.Z
- convex set(s) 89
 - absolutely (in a linear topological space) 424.E
 - in an affine space 7.D
 - P - (for a differential operator 112.C
 - regularly 89.G
 - strongly P - 112.C
 - strongly separated 89.A
- convex surface, closed 111.I
- convolution
 - (of arithmetic functions) 295.C
 - (of distributions) 125.M
 - (of functions) 159.A 192.H
 - (of hyperfunctions) 125.X
 - (of probability distributions) 341.E
 - (in the theory of Hopf algebra) 203.H
 - generalized (of distributions) 125.M
- convolutional code 63.E
- cooperative game 173.A.D
- coordinate(s) 90
 - (of an element of a direct product of sets) 381.E
 - (in the real line) 355.E
 - affine 7.C
 - barycentric (in an affine space) 7.C 90.B
 - barycentric (in a Euclidean simplicial complex) 70.B
 - barycentric (in the polyhedron of a simplicial complex) 70.C
 - bipolar 90.C, App. A, Table 3.V
 - bipolar cylindrical App. A, Table 3.V
 - canonical (of a Lie group) 249.Q
 - Cartesian (in an affine space) 7.C
 - Chow (of a positive cycle) 16.S
 - circular cylindrical App. A, Table 3.V
 - curvilinear 90.C, App. A, Table 3.V
 - cylindrical 90.C, App. A, Table 3.V
 - ellipsoidal 90.C 133.A, App. A, Table 3.V
 - elliptic 90.C 350.E, App. A, Table 3.V
 - elliptic cylindrical App. A, Table 3.V
 - equilateral hyperbolic 90.C, App. A, Table 3.V
 - generalized (in analytical dynamics) 271.F
 - generalized cylindrical App. A, Table 3.V
 - geodesic 80.J
 - geodesic polar 90.C

- Grassmann (in a Grassmann manifold) 90.B
- homogeneous (of a point in a projective space) 343.C
- hyperbolic cylindrical App. A, Table 3.V
- hyperplane (of a hyperplane in a projective space) 343.C
- inhomogeneous (of a point with respect to a frame) 343.C
- isothermal 90.C
- i*th (of an element relative to a basis) 256.C
- Klein line 90.B
- Kruskal 359.D
- line (of a line) 343.C
- local (on an algebraic variety) 16.O
- local (on a topological manifold) 105.C
- local, transformation of 90.D
- moving App. A, Table 3.IV
- multiplanar 90.C
- multipolar 90.C
- $(n + 2)$ -hyperspherical 76.A 90.B
- normal 90.C
- oblique (in a Euclidean space) 90.B
- orthogonal curvilinear 90.C
- parabolic 90.C
- parabolic cylindrical App. A, Table 3.V
- parallel (in an affine space) 7.C
- pentaspherical 90.B
- plane (of a plane) 343.C
- Plücker (in a Grassmann manifold) 90.B
- polar 90.C, App. A, Table 3.V
- projective 343.C
- rectangular (in a Euclidean space) 90.B
- rectangular hyperbolic 90.C
- rotational App. A, Table 3.V
- rotational hyperbolic App. A, Table 3.V
- rotational parabolic App. A, Table 3.V
- spherical 90.C 133.D
- tangential polar 90.C
- tetracyclic 90.B
- trilinear 90.C
- tripolar 90.C
- coordinate axis
 - of an affine frame 7.C
 - i*th (of a Euclidean space) 140
- coordinate bundle(s) 147.B
 - equivalent 147.B
- coordinate curve (in a Euclidean space) 90.C
- coordinate function
 - (of a fiber bundle) 147.B
 - (in the Ritz method) 304.B
- coordinate hyperplane (of an affine frame) 7.C
- coordinate hypersurface (in a Euclidean space) 90.C
- coordinate neighborhood
 - of class C^r 105.D
 - of a fiber bundle 147.B
 - of a manifold 105.C
- coordinate ring (of an affine variety) 16.A
 - homogeneous 16.A
- coordinate system 90.A
 - (of a line in a projective space) 343.C
 - geodesic, in the weak sense 232.A
 - holomorphic local 72.A
 - isothermal curvilinear App. A, Table 3.V
 - l*-adic 3.E
 - local (of a topological space) 90.D 105.C
 - moving 90.B
 - orthogonal, adapted to a flag 139.E
 - orthogonal curvilinear App. A, Table 3.V
 - projective 343.C
- coordinate transformation (of a fiber bundle) 147.B
 - (of a locally free \mathcal{O}_X -Module) 16.E
- coplanar vectors 442.A
- coproduct
 - of commutative algebras 29.A
 - of an element in a graded coalgebra 203.B
 - Hopf 203.D
 - of two objects 52.E
- coradical 293.F
- CORDIC 142.A
- core 173.D
- coregular representation (of an algebra) 362.E
- corestriction (homomorphism of cohomology groups) 200.M
- Corioli force 271.D
- corner polyhedron 215.C
- Cornish-Fisher expansions 374.F
- Cornu spiral 93.H 167.D
- Corona problem 43.G
- Corona theorem 164.I
- coroot 13.J
- correcting, error- 63.A
- correcting capability, error- 63.B
- correctly posed
 - (initial value problem) 321.E
 - (problems for partial differential equations) 322.A
- corrector (in a multistep method) 303.E
 - Milne 303.E
- correlation 343.D
 - involutive 343.D
 - Kendall rank 371.K
 - serial 397.N
 - serial cross 397.N
 - Spearman rank 371.K
- correlation coefficient
 - (of two random variables) 342.C 397.H
 - canonical 280.E 374.C
 - multiple 397.J
 - partial 397.J
 - population 396.D
 - sample 396.D
 - sample multiple 280.E
 - sample partial 280.E
 - serial 421.B
- correlation inequalities 212.A
- correlation matrix 397.J
- correlation ratios 397.L
- correlation tensor 433.C
- correlogram 397.N
- correspond 358.B
- correspondence 358.B
 - algebraic (of an algebraic variety) 16.I
 - algebraic (of a nonsingular curve) 9.H
 - algebraic, group of classes of 9.H
 - birational 16.I
 - of Combescur 111.F
 - conformal (between surfaces) 111.I
 - geodesic (between surfaces) 111.I
 - homothetic (between surfaces) 111.I
 - inverse 358.B
 - one-to-one 358.B
 - similar (between surfaces) 111.I
 - univalent 358.B
- correspondence principle 351.D
- correspondence ring (of a nonsingular curve) 9.H
- corresponding angles 139.D
- corresponding points (with respect to confocal quadrics) 350.E
- cos (cosine) 131.E 432.A

\cos^{-1} 131.E
 cosec (cosecant) 131.E 432.A
 cosech (hyperbolic cosecant) 131.F
 cosemisimple 203.F
 coset
 double (of two subgroups of a group) 190.C
 left (of a subgroup of a group) 190.C
 right (of a subgroup of a group) 190.C
 coset space (of a topological group)
 left 423.E
 right 423.E
 cosh (hyperbolic cosine) 131.F
 cosigma functions 134.H, App. A, Table 16.IV
 cosine(s) 432.A
 first law of 432.A, App. A, Table 2.II
 hyperbolic 131.F
 integral 167.D
 law of (on spherical triangles) 432.B, App. A, Table 2.III
 optical direction 180.A
 second law of 432.A, App. A, Table 2.II
 cosine integral 167.D, App. A, Table 19.II
 cosine series, Fourier, App. A, Table 11.I
 cosine transform, Fourier 160.C, App. A, Table 11.II
 cospecialization (in étale topology) 16.AA
 cospectral density 397.N
 cost 281.D
 imputed 255.B
 shadow 292.C
 unit 281.D
 cost of insurance 214.B
 cost of observation 398.F
 cot (cotangent) 131.E
 cotangent(s) 432.A
 hyperbolic 131.F
 law of App. A, Table 2.III
 cotangent bundle 147.F
 cotangential sphere bundle 274.E
 cotangent vector bundle 147.F
 Cotes formula, Newton- (in numerical integration) 299.A
 coth (hyperbolic cotangent) 131.F
 cotree 186.G
 cotriple 200.Q
 counit 203.F
 counity (of a coalgebra) 203.B
 countability axioms 425.P
 countable additivity 270.B
 countable cell complex 70.D
 countable Lebesgue spectrum 136.E
 countable model (of axiomatic set theory) 156.E
 countable ordinal number 49.F
 countable set 49.A
 countable simplicial complex 70.C
 locally 70.C
 countably additive class 270.B
 countably compact space 425.S
 countably equivalent sets 136.C
 countably Hilbertian space 424.W
 countably infinite set 49.A
 countably normed space 424.W
 countably paracompact space 425.Y
 countably productive property 425.Y
 counting constants, principle of 16.S
 counting function (of a meromorphic function) 272.B
 Courant-Cheng domain theorem 391.H
 Cousin problem 21.K
 first 21.K

Subject Index

Covering Lie group, simply connected

second 21.K
 covariance (of two random variables) 342.C 397.H
 population 396.D
 sample 396.D
 covariance distribution 395.C
 covariance function 395.A,B
 sample 395.G
 covariance matrix 341.B 397.J
 asymptotic 399.K
 variance 341.B 397.J
 covariant 226.D
 absolute 226.D
 absolute multiple 226.E
 with ground forms 226.D
 multiple 226.E
 of an n -ary form of degree d 226.D
 relativistically 150.D
 covariant derivative
 (of a tensor field) 80.I 417.B, App. A, Table 4.II
 (of a vector field) 80.I
 (of a vector field along a curve) 80.I
 van der Waerden–Bortolotti 417.E
 covariant differential
 (of a differential form) 80.G
 (of a tensor field) 80.I, L 417.B
 (of a vector field) 80.I
 covariant functor 52.H
 covariant index (of a component of a tensor) 256.J
 covariant spinor 258.B
 covariant tensor
 alternating 256.N
 of degree q 256.J
 symmetric 256.N
 covariant tensor field of order s 105.O
 covariant vector 256.J
 covariant vector field 105.O
 covector, p - 256.O
 cover (a set) 381.D
 covering(s)
 (covering space) 91.A
 (curve) 9.I
 (of a set) 381.D 425.R
 closed (of a set) 425.R
 closure-preserving 425.X
 countable (of a set) 425.R
 degree of (of a nonsingular curve) 9.I
 discrete (of a set) 425.R
 ε - (of a metric space) 273.B
 finite (of a set) 425.R
 locally finite (of a set) 425.R
 mesh of (in a metric space) 273.B
 n -fold (space) 91.A
 normal (of a set) 425.R
 open (of a set) 425.R
 point-finite (of a set) 425.R
 regular (space) 91.A
 σ -discrete (of a set) 425.R
 σ -locally finite (of a set) 425.R
 star-finite (of a set) 425.R
 unramified (of a nonsingular curve) 9.I
 covering curve 9.I
 covering differentiable manifold 91.A
 covering dimension (of a normal space) 117.B
 covering family (in Grothendieck topology) 16.AA
 covering group (of a topological group) 91.A 423.O
 universal (of a topological group) 91.B 423.O
 covering homotopy property 148.B
 covering law 381.D 425.L
 covering Lie group, simply connected (of a Lie algebra) 249.C

Covering linkage invariant(s)

- covering linkage invariant(s) 235.E
- covering manifold 91.A
- covering mapping (map) 91.A
- covering space(s) 91
 - analytic 23.E
 - C- 23.E
 - in the sense of Cartan 23.E
 - ramified 23.B
 - universal 91.B
- covering surface(s) 367.B
 - Ahlfors theory of 367.B
 - with relative boundary 367.B
 - unbounded 267.B
 - universal 367.B
 - unramified 367.B
- covering system, uniform 436.D
- covering theorem, Vitali 380.D
- covering transformation 91.A
 - of an unramified unbounded covering surface 367.B
- covering transformation group 91.A
- Coxeter complex 13.R
- Coxeter diagram (of a complex semisimple Lie algebra) 248.S
- Coxeter group 13.R
- CPM 307.C 376
- Cramer-Castillon problem (in geometric construction) 179.A
- Cramér-Rao inequality 399.D
- Cramer rule 269.M
- Cramér theorem 399.M
- crash duration 281.D
- creation operator 377.A
- Cremona transformation 16.I
- CR-equivalence 344.A
- criterion
 - Cartan, of semisimplicity (of Lie algebras) 248.F
 - Cartan, of solvability (of Lie algebras) 248.F
 - Castelnuovo 15.E
 - Cauchy 87.C, App. A, Table 10.II
 - convergence, for positive series App. A, Table 10.II
 - d'Alembert App. A, Table 10.II
 - Euler 297.H
 - Gauss App. A, Table 10.II
 - Jacobian (on regularity of local rings) 370.B
 - Kummer 145, App. A, Table 10.II
 - logarithmic App. A, Table 10.II
 - Nakai-Moishezon (of ampleness) 16.E
 - Nyquist's 86.A
 - Raabe App. A, Table 10.II
 - of ruled surfaces 15.E
 - Schlomilch App. A, Table 10.II
 - simplex 255.D
 - Weierstrass, for uniform convergence 435.A
 - Weyl 182.H
- criterion function 127.A
- critical (Galton-Watson process) 44.B
- critical determinant 182.B
- critical exponent 111.C
- critical inclination problem 55.C
- critical lattice in M with respect to S 182.B
- critical manifold, nondegenerate 279.D,E
- critical path 376
- critical percolation probability 340.D
- critical point
 - (of a C^x -function on a manifold) 279.B
 - (of a C^x -mapping $\varphi: M \rightarrow M'$) 105.J
 - (of a flow) 126.D
 - (of a function on \mathbf{R}^1) 106.L
 - (of a harmonic function) 193.J
 - (of a mapping $u: \mathbf{R}^n \rightarrow \mathbf{R}^m$) 208.B
 - degenerate 106.L 279.B
 - nondegenerate 106.L 279.B 286.N
- critical region 400.A
- critical value
 - (in bifurcation theory) 286.R
 - (of a C^x -function on a manifold) 279.B
 - (of a C^x -mapping $\varphi: M \rightarrow M'$) 105.J
 - (of a contact process) 340.C
 - (of an external magnetic field) 340.B
 - (of a mapping $u: \mathbf{R}^n \rightarrow \mathbf{R}^m$) 208.B
- Crofton formula (in integral geometry) 218.B
- cross cap (a surface) 410.B
- crosscut(s) (of a plane domain) 333.A
 - fundamental sequence of (in a simply connected domain) 333.B
- crossed homomorphism (of an associative algebra) 200.L
- crossed product
 - (in C^* -algebra theory) 36.I
 - (of a commutative ring and a group) 29.D
 - (in von Neumann algebra theory) 308.J
- crossings
 - normal 16.L
 - only normal 16.L
- crossing symmetry 132.C 386.B
- cross norm, C^* - 36.H
- cross product
 - (of cohomology groups) 201.J
 - (of homology groups) 201.J
 - (of vector bundles) 237.C
- cross ratio 343.D
- cross section
 - (of a fiber bundle) 147.L 286.H
 - (of a fiber space) 148.D
 - (of a flow) 126.C
 - absorption 375.A
 - differential 375.A 386.B
 - local (in a topological group) 147.E
 - scattering 375.A
 - total 386.B
 - total elastic 386.B
- cross-sectional data 128.A 397.A
- cross spectral density function 421.E
- Croout method 302.B
- CR structure 344.A
- crystal class 92.B
 - arithmetic 92.B
 - geometric 92.B
 - 3-dimensional App. B, Table 5.IV
- crystal family 92.B
- crystallographic group 92
- crystallographic restriction 92.A
- crystallographic space group 92.A
- crystal system 92.B
- cube 357.B
 - duplication of 179.A
 - Hilbert 382.B
 - unit 139.F 140
 - unit n - 140
- cubic equation 10.D, App. A, Table 1
- cubic map 157.B
- cubic P 92.E
- cubic resolvent App. A, Table 1
- cumulant 397.G
 - factorial 397.G
 - joint 397.I
- cumulative distribution 397.B

- cumulative distribution curve 397.B
- cumulative distribution function 341.B 342.C
- cumulative distribution polygon 397.B
- cup product
 - (of cohomology classes) 201.I
 - (of derived functors) 200.K
 - (in K -theory) 237.C
- cup product reduction theorem (on cohomology or homology groups) 200.M
- curl (of a differentiable vector field) 442.D
- current 125.R
 - 4-, density 150.B
 - integral 275.G
 - partially conserved axial-vector 132.C
 - random 395.I
- current algebra 132.C
- Curtis formulas, Clenshaw- 299.A
- curvature
 - (of a curve of class C^n) 111.D
 - (of a plane curve) 111.E
 - absolute (of a curve) 111.C
 - affine 110.C
 - circle of 111.E
 - conformal 110.D
 - constant, space of 364.D, App. A, Table 4.II
 - constant, surface of 111.I
 - Gaussian (of a surface) 111.H, App. A, Table 4.I
 - geodesic 111.H, App. A, Table 4.I
 - S. Germain (of a surface) 111.H, App. A, Table 4.I
 - holomorphic sectional 364.D
 - integral (of a surface) 111.H
 - line of (on a surface) 111.H
 - Lipschitz-Killing 279.C
 - mean 364.D
 - mean (of a surface) 111.H 365.D, App. A, Table 4.I
 - mean, vector 365.D
 - minimum, property 223.F
 - negative 178.H
 - nonpositive, G -space with 178.H
 - normal (of a surface) 111.H
 - principal (of a surface) 111.H 365.C
 - radius of (of a plane curve) 111.E
 - radius of (of a space curve) 111.F
 - radius of principal (of a surface) 111.H
 - Ricci 364.D
 - Riemannian 364.D
 - scalar 364.D, App. A, Table 4.II
 - sectional 364.D
 - total (of an immersion) 365.O
 - total (of a surface) 111.F, H, App. A, Table 4.I
 - total Gaussian (of a surface) 111.H
 - total mean 365.O
- curvature form 80.G 364.D
- curvature tensor
 - (of an affine connection) 80.J, L 417.B
 - (of a Fréchet manifold) 286.L
 - (of a Riemannian manifold) 364.D
 - projective App. A, Table 4.II
 - Weyl conformal 80.P, App. A, Table 4.II
- curve(s) 93 111.A
 - algebraic 9.A
 - analytic (in an analytic manifold) 93.B
 - analytic (in a Euclidean plane) 93.B
 - asymptotic 110.B 111.H
 - Bertrand 111.F
 - bicharacteristic 325.A
 - characteristic (network flow problem) 281.B
 - characteristic (of a 1-parameter family of surfaces) 111.I
 - characteristic (of a partial differential equation) 320.B 324.A, B
 - cissoidal 93.H
 - of class C^k (in a differentiable manifold) 93.B
 - of class C^k (in a Euclidean plane) 93.B
 - closed convex 111.E
 - of constant breadth 89.C
 - of constant inclination 111.F
 - of constant width 111.E
 - continuous plane 93.B
 - coordinate (in a Euclidean space) 90.C
 - covering 9.I
 - Darboux 110.B
 - Delaunay 93.H
 - dual (of a plane algebraic curve) 9.B
 - elliptic 9.C
 - exceptional 15.G
 - exponential 93.H
 - of the first kind 15.G
 - Fréchet 246.A
 - fundamental (with respect to a birational mapping) 16.I
 - fundamental theorem of the theory of 111.D
 - general 93.D
 - generating 111.I
 - hyperelliptic 9.D
 - influence 371.I
 - integral (of a Monge equation) 324.F
 - integral (of ordinary differential equations) 316.A
 - Jordan 93.B
 - logarithmic 93.H
 - Lorentz 397.F
 - Mannheim 111.F
 - meromorphic 272.L
 - nodal 391.H
 - OC- 404.C
 - ordinary 93.C
 - Peano 93.J
 - pedal 93.H
 - plane App. A, Table 4.I
 - plane algebraic 9.B
 - of pursuit 93.H
 - rational 9.C
 - rational (in a Euclidean plane) 93.H
 - rectifiable 93.F
 - rolling (of a roulette) 93.H
 - of the second class 78.K
 - of the second order 78.I
 - simple closed 93.B
 - sine 93.H
 - solution (of ordinary differential equations) 316.A
 - stable 9.K
 - stationary 46.B
 - stationary (of the Euler-Lagrange differential equations) 324.E
 - of steepest descent 46.A
 - timelike 324.A
 - tooth 181.E
 - in a topological space 93.B
 - transcendental 93.H
 - u - 111.H
 - unicursal 9.C 93.H
 - unicursal ordinary 93.C
 - universal 93.E
 - v - 111.H
 - variation 178.A

Curve fitting

- curve fitting 19.F
 - curve tracing 93.G
 - curvilinear cluster set 62.C
 - curvilinear coordinates 90.C, App. A, Table 3.V
 - orthogonal 90.C
 - planar App. A, Table 3.V
 - in 3-dimensional space App. A, Table 3.V
 - curvilinear coordinate system
 - isothermal App. A, Table 3.V
 - orthogonal App. A, Table 3.V
 - curvilinear integrals 94.A
 - with respect to a line element 94.D
 - with respect to a variable 94.D
 - cushioned refinement 425.X
 - cusp
 - of a curve 93.G
 - of a Fuchsian group 122.C
 - parabolic (of a Fuchsian group) 122.C
 - of a plane algebraic curve 9.B
 - cusp form 450.O
 - in the case of one variable 32.B
 - in Siegel half-space 32.F
 - cuspidal parabolic subgroup 437.X
 - cusp singularity 418.C
 - cut
 - (in a projective space) 343.B
 - (of \mathbf{Q}) 294.E
 - (of \mathbf{R}) 355.A
 - disjunctive 215.C
 - Gomory 215.B
 - subadditive 215.C
 - cut locus 178.A
 - cutoff 150.C
 - cut point (on a geodesic) 178.A
 - cutset (in a graph) 186.G
 - cutset matrix (of a graph), fundamental 186.G
 - cutting (\mathbf{P}^n by \mathbf{P}^r) 343.B
 - cutting plane 215.B
 - fractional, algorithm 215.B
 - CW complex 70.D
 - CW decomposition 70.D
 - CW pair 201.L
 - cybernetics 95
 - cycle
 - (on an algebraic variety) 16.M
 - (of basic sets) 126.J
 - (of a chain complex) 200.H
 - (= cyclic permutation) 151.G
 - (of time series data) 397.N
 - algebraic 450.Q
 - algebraically equivalent 16.R
 - dividing (on an open Riemann surface) 367.I
 - foliation 154.H
 - fundamental (of an oriented pseudomanifold) 65.A
 - fundamental (in a resolution of a singular point) 418.C
 - limit 126.I
 - module of 200.C
 - no, condition 126.J
 - numerically equivalent 16.Q
 - one 16.R
 - positive (on an algebraic variety) 16.M
 - rationally equivalent 16.R
 - Schubert 56.E
 - vanishing 418.F
 - zero 16.R
 - cycle index 66.E
 - cyclic algebra 29.G
 - cyclic code 63.D
 - cyclic determinant 103.G
 - cyclic element 251.K
 - cyclic equation 172.G
 - cyclic extension 172.B
 - cyclic group 190.C
 - cyclic Jacobi method 298.B
 - cyclic part (of an ergodic class) 260.B
 - cyclic representation (Banach algebra) 36.E
 - cyclic representation (topological groups) 437.A
 - cyclic subgroup (of a group) 190.C
 - cyclic vector (of a representation space of a unitary representation) 437.A
 - cyclide 90.B
 - cyclide of Dupin 111.H
 - cycloid 93.H
 - cyclomatic number 186.G
 - cyclotomic field 14.L
 - cyclotomic polynomial 14.L
 - cyclotomic \mathbf{Z}_p -extension 14.L
 - cyclotomy 296.A
 - cylinder
 - circular 111.I 350.B
 - elliptic 350.B
 - hyperbolic 350.B
 - mapping 202.E
 - parabolic 350.B
 - cylinder function
 - elliptic 268.B
 - parabolic 167.C
 - cylinder set 270.H
 - n - 270.G
 - cylindrical coordinates 90.C, App. A, Table 3.V
 - bipolar App. A, Table 3.V
 - circular App. A, Table 3.V
 - elliptic App. A, Table 3.V
 - generalized App. A, Table 3.V
 - hyperbolic App. A, Table 3.V
 - parabolic 167.C, App. A, Table 3.V
 - cylindrical equation, parabolic App. A, Table 14.II
 - cylindrical functions 39.B, App. A, Table 19.III
 - cylindrical hypersurface, quadric 350.G
 - cylindrical surface 111.I
 - circular 350.B
 - elliptic 350.B
 - hyperbolic 350.B
 - parabolic 350.B
- D**
- $\delta \rightarrow$ delta
 - $\mathcal{D}(\Omega)$ 125.B 168.B
 - $\mathcal{D}'(\Omega)$ 125.B
 - $\mathcal{D}_{\{M_p\}}, \mathcal{D}_{(M_p)}$ 168.B
 - $\mathcal{D}'_{\{M_p\}}, \mathcal{D}'_{(M_p)}$ 125.U
 - $\mathcal{D}_{L_p}(\Omega)$ (the totality of functions $f(x)$ in $C^\infty(\Omega)$ such that for all α , $D^\alpha f(x)$ belongs to $L_p(\Omega)$ with respect to Lebesgue measure) 168.B
 - δ -measure 270.D
 - Δ -refinement (of a covering) 425.R
 - Δ_n^1 -set 22.D
 - ∂ -functor 200.I
 - universal 200.I
 - ∂^* -functor 200.I
 - $\bar{\partial}$ -complex 72.D
 - $\bar{\partial}$ -cohomology groups 72.D
 - d -continuous channels 213.F
 - d -dimensional analytic set, purely 23.B
 - d -trial path dependent 346.G
 - d'' -cohomology group 72.D
 - D -sufficient σ -field 396.J

- D*-wave 315.E
D-integrable function 100.D
D-integral
 definite 100.D
 indefinite 100.D
D(*)-integral 100.D
d'Alembert criterion App. A, Table 10.II
d'Alembertian 130.A
d'Alembert method of reduction of order 252.F
d'Alembert paradox 205.C
d'Alembert solution 325.D
damped oscillation 318.B
damping ratio (of a damped oscillation) 318.B
Daniell-Stone integrable function 310.I
Daniell-Stone integral 310.I
Danilevskii method 298.D
Darboux curve 110.B
Darboux formula, Christoffel- 317.D
Darboux frame 110.B
Darboux quadric 110.B
Darboux sum 216.A
Darboux tangent 110.B
Darboux theorem 216.A 428.A
Darmois theorem, Skitovich- 374.H
data 96.B
 cross-sectional 128.A
 macroeconomic 128.A
 microeconomic 128.A
 scattering 287.C 387.C
data analysis, statistical 397.A
data base 96.B
data processing 96
data structures 96.B
Davidenko's method of differentiation with respect
 to a parameter 301.M
death insurance 214.B
death process 260.G
 birth and 260.G
death rate, infinitesimal 260.G
Debye asymptotic representation 39.D, App. A,
 Table 19.III
decidable number-theoretic predicate 356.C
decision 127.A
decision function(s) 398.A
 invariant 398.E
 minimax 398.B
 sequential 398.F
 space of 398.A
 statistical 398.A
decision problem 71.B 97 186.J
 n- 398.A
 sequential 398.F
 statistical 398.A
decision procedure, statistical 398.A
decision process, Markov 127.E
decision rule
 sequential 398.F
 terminal 398.F
decision space 398.A
decision theoretically sufficient σ -field 396.J
decoder 213.D
decoding 63.A
decomposable operator (on a Hilbert space) 308.G
decompose (a polygon) 155.F
decomposed into the direct sum of irreducible
 representations 437.G
decomposition
 (of a set) 381.D
 Bruhat (of an algebraic group) 13.K
 canonical (of a closed operator) 251.E
 cellular (of a Hausdorff space) 70.D
 Chevalley (on algebraic groups) 13.I
 cluster, Hamiltonian 375.F
 CW 70.D
 de Rham (of a Riemannian manifold) 364.E
 direct (of a group) 190.L
 Doob-Meyer 262.C
 D-optimality 102.E
 dual direct product (of a decomposition of a
 compact or discrete Abelian group) 422.H
 ergodic (of a Lebesgue measure space) 136.H
 Fefferman-Stein 168.B
 formula of Radon 125.CC
 Heegurard 65.C
 Iwasawa (of a connected semisimple Lie group)
 249.T
 Iwasawa (of a real semisimple Lie algebra)
 248.V
 Jordan (of an additive set function) 380.C
 Jordan (of a function of bounded variation)
 166.B
 Jordan (of a linear mapping) 269.L
 Jordan (in an ordered linear space) 310.B
 Khinchin 395.B
 Lebesgue, theorem 270.L
 Levi (on algebraic groups) 13.Q
 Levi (on Lie algebras) 248.F
 multiplicative Jordan (of a linear transforma-
 tion) 269.L
 Peirce (of a Jordan algebra) 231.B
 Peirce left (in a unitary ring) 368.F
 Peirce right (in a unitary ring) 368.F
 plane wave 125.CC
 polar 251.E
 relative Bruhat 13.Q
 Riesz (in Markov process) 260.D
 Riesz (in martingale) 262.C
 Riesz (of a superharmonic or subharmonic
 function) 193.S
 semimartingale 406.B
 simplicial (of a topological space) 79.C
 singular value (SVD) 302.E
 spectral 126.J
 Wiener-Itô 176.I
 Witt (of a quadratic form) 348.F
 Wold 395.D
 Zariski 15.D
decomposition-equal polygons 155.F
decomposition field (of a prime ideal) 14.K
decomposition group (of a prime ideal) 14.K
decomposition number (of a finite group) 362.I
 generalized (of a finite group) 362.I
decomposition theorem
 canonical 86.C
 in class field theory 59.C
 for dimension 117.C
 Lebesgue (on a completely additive set function)
 380.C
 unique (for a 3-manifold) 65.E
decreasing, monotone 380.B
decreasing C^∞ -function, rapidly 168.B
decreasing distribution, rapidly 125.O
decreasing Fourier hyperfunction, exponentially
 125.BB
decreasing function
 monotone 166.A
 strictly 166.A
 strictly monotone 166.A
decreasing real analytic function, exponentially
 125.BB

Decreasing sequence, monotonically

- decreasing sequence, monotonically (of real numbers) 87.B
- rapidly 168.B
- decrement, logarithmic (of a damped oscillation) 318.B
- Dedekind, J. W. R. 98
- Dedekind, test of du Bois-Reymond and 379.D
- Dedekind axiom of continuity (for real numbers) 355.A
- Dedekind discriminant theorem 14.J
- Dedekind eta function 328.A
- Dedekind principle (in a modular lattice) 243.F
- Dedekind sum 328.A
 - reciprocity law for 328.A
- Dedekind theory of real numbers 294.E
- Dedekind zeta function 14.C 450.D
- deep water wave 205.F
- defect
 - (of a block of representations) 362.I
 - (of a conjugate class in a group) 362.I
 - (of a meromorphic function) 272.E
- defect group
 - of a block of representations 362.I
 - (of a conjugate class in a group) 362.I
- deficiency
 - (of an algebroidal function) 17.C
 - (of a closed operator) 251.D
 - (of a linear system on a surface) 15.C
 - maximal (of an algebraic surface) 15.E
- deficiency index
 - (of a closed symmetric operator) 251.I
 - (of a differential operator) 112.I
- deficient number (in elementary theory of numbers) 297.D
- defined along V' (for a rational mapping) 16.I
- defined over k' (for an algebraic variety) 16.A
- define recursively 356.C
- defining functions (of a hyperfunction) 125.V
 - standard 125.Z
- defining ideal (of a formal spectrum) 16.X
- defining module (of a linear system) 16.N
- defining relations (among the generators of a group) 161.A
- definite
 - negative (function) 394.C
 - negative (Hermitian form) 348.F
 - negative (quadratic form) 348.B
 - positive (function) 36.L 192.B.J 394.C 437.B
 - positive (Hermitian form) 348.F
 - positive (kernel) 217.H
 - positive (matrix) 269.I
 - positive (potential) 338.D
 - positive (quadratic form) 348.B
 - positive (sequence) 192.B
 - semi- (Hermitian form) 348.F
 - semi- (kernel) 217.H
 - totally (quaternion algebra) 27.D
- definite D-integral 100.D
- definite integral 216.C, App. A, Table 9.V
 - (of a hyperfunction) 125.X
- definite quadratic form 348.C
- definition
 - field of 16.A
 - first (of algebraic K -group) 237.J
 - second (of algebraic K -group) 237.J
 - truth 185.D
- definition by mathematical induction 294.B
- definition by transfinite induction 311.C
- deflation
 - in homological algebra 200.M
 - method for an eigenvalue problem 298.C
- deformation
 - (of complex structures) 72.G
 - (of a graph) 186.E
 - infinitesimal, to the direction $\partial/\partial s$ 72.G
 - isomonodromic 253.E
 - isospectral 387.C
 - projective (between surfaces) 110.B
 - of a scheme over a connected scheme 16.W
 - of a surface 110.A
- deformation cochain 305.B
- deformation retract 202.D
 - neighborhood 202.D
 - strong 202.D
- degeneracy (of energy eigenvalues) 351.H
 - set of (of a holomorphic mapping between analytic spaces) 23.C
- degeneracy index 17.C
- degeneracy operator (in a semisimplicial complex) 70.E
- degenerate
 - (critical point) 106.L 279.B
 - (eigenvalue) 390.A.B
 - (mapping) 208.B
 - (quadratic surface) 350.B
 - (simplex) 70.E
 - totally 234.B
- degenerate kernel 217.F
- degenerate module 118.D
- degenerate series
 - (of unitary representations of a complex semisimple Lie group) 437.W
 - complementary (of unitary representations of a complex semisimple Lie group) 437.W
- degree
 - (of an algebraic element) 149.F
 - (of an algebraic equation) 10.A
 - (of an algebraic variety) 16.G
 - (of an angle) 139.D
 - (of a central simple algebra) 29.E
 - (of a divisor class) 11.D
 - (of a divisor of an algebraic curve) 9.C
 - (of an element with respect to a prime ideal of a Dedekind domain) 439.F
 - (of an extension) 149.F
 - (of a graph) 186.B
 - (of a Jordan algebra) 231.B
 - (of a linear representation) 362.D
 - (of a matrix representation) 362.D
 - (of an ordinary differential equation) 313.A
 - (of a permutation representation) 362.B
 - (of a polynomial) 337.A
 - (of a prime divisor) 9.D
 - (of a rational homomorphism) 3.C
 - (of a representation of a Lie algebra) 248.B
 - (of a representation of a Lie group) 249.O
 - (of a square matrix) 269.A
 - (of a term of a polynomial) 337.B
 - (of a valuation) 439.I
 - (of a 0-cycle on an algebraic variety) 16.M
 - complementary (of a spectral sequence) 200.J
 - of covering (of a nonsingular curve) 9.I
 - filtration 200.J
 - formal (of a unitary representation) 437.M
 - of freedom (of the dynamical system) 271.F
 - of freedom (of error sum of squares) 403.E
 - of freedom (of sampling distributions) 374.B
 - in- 186.B
 - Leray-Schauder 286.D
 - local, of mapping 99.B

- mapping 99.A
- of mapping 99.A
- out- 186.B
- of the point 99.D
- of a prime divisor of an algebraic function field
 - of dimension 1 9.D
- of ramification (of a branch point) 367.B
- of recursive unsolvability 97
- relative (of a finite extension) 257.D
- relative (of a prime ideal over a field) 14.I
- of symmetry 431.D
- total (of a spectral sequence) 200.J
- transcendence (of a field extension) 149.K
- of transcendency (of a field extension) 149.K
- of unsolvability 97
- degree k
 - holomorphic differential forms of 72.A
 - tensor space of 256.J
- degree n
 - alternating group of 151.G
 - component of 200.B
 - general linear group of 60.B
 - projective general linear group of 60.B
 - Siegel modular function of 32.F
 - Siegel modular group of 32.F
 - Siegel space of 32.F
 - Siegel upper half-space of 32.F
 - special linear group of 60.B
 - symmetric group of 151.G
- degree p , contravariant tensor of 256.J
- degree q , covariant tensor of 256.J
- degree r
 - differential form of 105.Q
 - differential form of (on an algebraic variety) 16.O
 - mean of (of a function with respect to a weight function) 211.C
- Dehn lemma (on 3-manifolds) 65.E
- Dejon-Nickel method 301.G
- Delaunay curve 93.H
- delay convention, perfect 51.F
- delay-differential equation 163.A
- delayed recurrent event 260.C
- Delos problem (in geometric construction) 179.A
- delta, Kronecker 269.A, App. A, Table 4.II
- delta function, Dirac App. A, Table 12.II
- demography 40.D
- de Moivre formula 74.C
- de Moivre-Laplace theorem 250.B
- de Morgan law 381.B
 - in a Boolean algebra 42.A
- Denjoy-Carleman condition 168.B
- Denjoy integrable in the wider sense 100.D
- Denjoy integrals 100
 - in the restricted sense 100.D
- Denjoy-Luzin theorem 159.I
- denominator, partial (of an infinite continued fraction) 83.A
- dense
 - (set) 425.N
 - (totally ordered set) 311.B
 - locally 154.D
 - nowhere 425.N
 - relatively 126.E
 - Zariski 16.A
- dense in itself 425.O
- denseness of rational numbers 355.B
- density
 - (on a maximal torus) 248.Y
 - (of a set of prime ideals) 14.S
 - (of a subset of integers) 4.A
 - angular momentum 150.B
 - beta 397.D
 - bivariate normal 397.I
 - conditional 397.I
 - cospectral 397.N
 - electric flux 130.A
 - energy 195.B
 - 4-current 150.B
 - free Lagrangian 150.B
 - gamma 397.D
 - joint 397.I
 - kinetic 218.A
 - Lagrangian 150.B
 - magnetic flux 130.A
 - point of (of a measurable set of the real line) 100.B
 - posterior 401.B
 - prior 401.B
 - probability 341.D
 - sojourn time 45.G
- density function 397.D
 - bispectral 421.C
 - marginal 397.I
 - normal 397.D
 - rational spectral 176.F
- density matrix 351.B
- density theorem
 - (on discrete subgroups of a Lie group) 122.F
 - Chebotarev 14.S
 - Kaplansky 308.C
 - Lebesgue 100.B
 - von Neumann 308.C
- dependence, domain of 325.B
- dependent
 - algebraically (elements of a ring) 369.A
 - algebraically (on a family of elements of a field) 149.K
 - functionally (components of a mapping) 208.C
 - functionally, of class C^r (components of a mapping) 208.C
 - linearly (elements in a linear space) 256.C
 - linearly (elements in an additive group) 2.E
 - linearly (with respect to a difference equation) 104.D
 - path, d -trial 346.G
- dependent points
 - (in an affine space) 7.A
 - (in a projective space) 343.B
- dependent set 66.G
- dependent variable 165.C
- depending choice, principle of 33.F
- depth (of an ideal) 67.E
- de Rham cohomology group 201.H
- de Rham cohomology group (of a differentiable manifold) 105.R
- de Rham cohomology ring (of a differentiable manifold) 105.R
- de Rham cohomology ring (of a topological space) 201.I
- de Rham complex (as an elliptic complex) 237.H
- de Rham decomposition (of a Riemannian manifold) 364.E
- de Rham equations 274.G
- de Rham homology theory 114.L
- de Rham system, partial 274.G
- de Rham theorem (on a C^∞ -manifold) 105.V 201.H
 - analog of 21.L

derivable

- approximately (measurable function) 100.B
- in the general sense (a set function) 380.D
- in the ordinary sense (a set function) 380.D

derivation

- (of an algebra) 200.I
- (of an algebraic function field) 16.O
- (of a commutative ring) 113
- (of a field) 149.L
- (of a Lie algebra) 248.H
- (of a linear operator in a C^* -algebra) 36.K
- $*$ - 36.K
- inner (in an associative algebra) 200.L
- inner (in a Lie algebra) 248.H
- invariant (on an Abelian variety) 3.F
- over k 149.L
- Lie algebra of 248.H

derivation tree 31.E

derivative

- (of a distribution) 125.E
- (of a function) 106.A
- (of a holomorphic function) 198.A
- (of a hyperfunction) 125.X
- (of a polynomial) 337.G
- (of an element in a differential ring) 113
- angular (of a holomorphic function) 43.K
- approximate (of a measurable function) 100.B
- covariant (of a tensor field) 80.I, App. A, Table 4.II
- covariant (of a tensor field in the direction of a tangent vector) 417.B
- covariant (of a vector field) 80.I
- covariant (of a vector field along a curve) 80.I
- directional 106.G
- distribution 125.E
- exterior (of a differential form) 105.Q
- first-order 106.A
- Fréchet 286.E
- free 235.C
- Gâteaux 286.E
- general (of a set function) 380.D
- generalized 125.E
- general lower (of a set function) 380.D
- general upper (of a set function) 380.D
- higher-order (of a differentiable function) 106.D, App. A, Table 9.III
- higher-order partial 106.H
- Lagrangian 205.A
- left (on the left) 106.A
- Lie (of a differential form) 105.Q
- Lie (of a tensor field) 105.O
- normal 106.G
- n th (of a differentiable function) 106.D
- ordinary (of a set function) 380.D
- ordinary lower (of a set function) 380.D
- ordinary upper (of a set function) 380.D
- partial 106.F,K
- partial, n th order 106.H
- at a point 106.A
- Radon-Nikodým 380.C
- right (on the right) 106.A
- Schwarzian App. A, Table 9.III
- spherical (for an analytic or meromorphic function) 435.E
- variational 46.B
- weak 125.E

derivatives and primitive functions App. A, Table 9.I

derived (language) 31.D

derived algebra (of a Lie algebra) 248.C

derived function 106.A

n th 106.D

derived group (of a group) 190.H

derived neighborhood 65.C

second barycentric 65.C

derived normal model (of a variety) 16.F

derived series (of a Lie algebra) 248.C

derived set (of a set) 425.O

derived sheaf 125.W

derived unit 414.P

Desarguesian geometry, non- 155.E 343.C

Desargues theorem 155.E 343.C

Descartes, R. 101

folium of 93.H

Descartes theorem 10.E

descending central series (of a Lie algebra) 248.C

descending chain

(in a lattice) 243.F

(of (normal) subgroups of a group) 190.F

(in an ordered set) 311.C

descending chain condition

(for a (normal) subgroup of a group) 190.F

(in an ordered set) 311.C

descent

curve of steepest 46.A

line of swiftest 93.H

method of steepest 212.C

descriptive set theory

classical 356.H

effective 356.H

design

block (\rightarrow block design)

central composite 102.M

completely randomized 102.A

factorial 102.H

first-order 102.M

fractional factorial 102.I

second-order 102.M

Youden square 102.K

design matrix 102.A 403.D

design-of-experiment analysis 403.D

design of experiments 102

designs for estimating parameters 102.M

designs for exploring a response surface 102.M

designs for two-way elimination of heterogeneity 102.K

desingularization (of an analytic space) 23.D

de Sitter space 355.D

desuspend 114.L

detecting, error- 63.A

determinacy

axiom of 22.H

projective 22.H

determinant(s) 103

(of an element of the general linear group over a noncommutative field) 60.O

(of a nuclear operator) 68.L

Casorati 104.D

critical 182.B

cyclic 103.G

Fredholm 217.E

Gramian 103.G 208.E

Hankel 142.E

Hill 268.B

infinite (in Hill's method of solution) 268.B

Jacobian 208.B

of a lattice 182.B

Lopatinski 325.K

Pfaffian 103.G

Vandermonde 103.G

Wronskian 208.E

- determinantal equation, Hill 268.B
- determinant factor (of a matrix) 269.E
- determinateness, axiom of 33.F
- determination, coefficient of 397.H,J
- determination, orbit 309.A
- determined system
 - of differential operators 112.R
 - of partial differential equations 320.F
- determining set (of a domain in C^n) 21.C
- deterministic
 - (in prediction theory) 395.D
 - (Turing machine) 31.B
- deterministic linear bounded automaton 31.D
- deterministic process 127.B
- Deuring-Heilbronn phenomenon 123.D
- developable function, asymptotically 30.A
- developable space 425.AA
- developable surface 111.I
- development
 - along a curve 364.B
 - of a curve 80.N 111.H 364.B
- deviation
 - large 250.B
 - mean absolute 397.C
 - standard 341.B 342.C 397.C
- deviation point 335.B
- devices, peripheral 75.B
- de Vries equation, Korteweg- 387.A
- (DF)-space 424.P
- DFT (Discrete Fourier Transform) 142.D
- diagonal (of a Cartesian product of sets) 381.B 436.A
- diagonalizable (linear transformation) 269.L
- diagonalizable operator (in an Abelian von Neumann algebra) 308.G
- diagonal mapping (of a graded coalgebra) 203.B,F
- diagonal matrix 269.A
- diagonal morphism (in a category) 52.E
- diagonal partial sum (of a double series) 379.E
- diagonal sum (of a matrix) 269.F
- diagram 52.C
 - (of a symmetric Riemann space) 413.F
 - arrow 281.D
 - associated (in irreducible representations of orthogonal groups) 60.J
 - in a category 52.C
 - commutative 52.C
 - Coxeter (of a complex semisimple Lie algebra) 248.S
 - Dynkin (of a complex semisimple Lie algebra) 248.S, App. A, Table 5.I
 - extended Dynkin App. A, Table 5.I
 - Feynman 146.B
 - mutually associated 60.J
 - Newton 254.D
 - Satake (of a compact symmetric Riemannian space) 437.AA
 - Satake (of a real semisimple Lie algebra) 248.U, App. A, Table 5.II
 - scatter 397.H
 - Schläfli (of a complex semisimple Lie algebra) 248.S
 - Young 362.H
- diameter
 - (of a central conic) 78.G
 - (of a solid sphere) 140
 - (of a subset in a metric space) 273.B
 - conjugate (of a diameter of a central conic) 78.G
 - transfinite 48.D
- diathermal wall 419.A
- dichotomy 398.C
- Dido's problem 228.A
- dielectric constant 130.B
- Dieudonné complete 436.I
- Dieudonné theorem 425.X
- diffeomorphic C^∞ -manifolds 105.J
- diffeomorphism(s)
 - Anosov 126.J 136.G
 - Axiom A 126.J
 - C^r - (in nonlinear functional analysis) 286.E
 - of class C^r 105.J
 - group of orientation-preserving 114.I
 - horse-shoe 126.J
 - minimal 126.N
 - Morse-Smale 126.J
 - Y- 136.G
- diffeotopy theorem 178.E
- difference 102.E 104.A
 - backward 223.C, App. A, Table 21
 - central 223.C 304.E, App. A, Table 21
 - divided 223.D
 - finite 223.C
 - forward 304.E
 - of the n th order 104.A
 - primary 305.C
 - second 104.A
 - symmetric 304.E
 - of two sets 381.B
- difference analog 304.E
- difference cocycle 305.B
- difference-differential equation 163.A
- difference equation 104
 - geometric 104.G
 - homogeneous 104.C
 - inhomogeneous 104.C
 - linear 104.C
 - nonhomogeneous 104.C
- difference group (of an additive group) 190.C
- difference method 303.A
- difference product 337.I
- difference quotient 104.A
- difference scheme 304.F
 - of backward type 304.F
 - of forward type 304.F
- difference set 102.E
- difference table 223.C
- different
 - (of an algebraic number field) 14.J
 - (of a maximal order) 27.B
 - relative 14.J
- differentiable
 - complex function 21.C
 - Fréchet 286.E
 - Gâteaux 286.E
 - infinitely 106.K
 - left (on the left) 106.A
 - n -times 106.D
 - n -times continuously 106.K
 - with respect to the parameter (a distribution) 125.H
 - partially 106.F
 - at a point (a complex function) 198.A
 - at a point (a real function) 106.A
 - right (on the right) 106.A
 - in the sense of Stolz 106.G
 - on a set 106.A
 - termwise (infinite series with function terms) 379.H
 - totally 21.C 106.G

Differentiable dynamical system of class C^r

- differentiable dynamical system of class C^r 126.B
- differentiable manifold(s) 105
 - with boundary of class C^r 105.E
 - of class C^r 105.D
 - Riemann-Roch theorem for 237.G
- differentiable mapping of class C^r 105.J
 - differential of (at a point) 105.J
- differentiable pinching problem 178.E
- differentiable semigroup 378.F
- differentiable slice theorem 431.C
- differentiable structure(s) 114.B
 - of class C^r 105.D
 - group of oriented (on a combinatorial sphere) 114.I
- differentiable transformation group 431.C
- differential
 - (= boundary operator) 200.H
 - (= coboundary operator) 200.F
 - (of a differentiable function) 106.B
 - (of a differentiable mapping at a point) 105.J
 - (of a function on a differentiable manifold) 105.I
 - (Fréchet derivative) 286.E
 - Abelian (of the first, second, third kind) 11.C 367.H
 - analytic (on a Riemann surface) 367.H
 - conjugate (on a Riemann surface) 367.H
 - covariant (of a differential form) 80.G
 - covariant (of a tensor field) 80.I, L 417.B
 - covariant (of a vector field) 80.I
 - exterior (of a differential form) 105.Q
 - harmonic (on a Riemann surface) 367.H
 - holomorphic (on a Riemann surface) 367.H
 - kernel 188.G
 - meromorphic (on a Riemann surface) 367.H
 - n th (of a differentiable function) 106.D
 - of n th order (of a differentiable function) 106.D
 - \mathfrak{D} - (on an algebraic curve) 9.F
 - partial 200.H
 - quadratic (on a Riemann surface) 11.D
 - r th (Fréchet derivative) 286.E
 - stochastic 406.C
 - total 106.G 200.H 367.H
- differential analyzer 19.E
- differential and integral calculus App. A, Table 9
- differential automorphism 113
- differential calculus 106
- differential coefficient 106.A
 - partial 106.A
- differential cross section 375.A 386.B
- differential divisor (of an algebraic curve) 9.C
- differential divisor class (of a Riemann surface) 11.D
- differential equation(s) 313.A
 - adjoint 252.K
 - algebraic 113 288.A
 - almost periodic 290.A
 - Beltrami 352.B
 - Bernoulli App. A, Table 14.I
 - Bessel 39.B, App. A, Table 14.II
 - Briot-Bouquet 288.B 289.B
 - Caianiello 291.F
 - Cauchy-Riemann 198.A
 - Cauchy-Riemann (for a holomorphic function of several complex variables) 21.C
 - Cauchy-Riemann (for a holomorphic function of two complex variables) 320.F
 - Chaplygin 326.B
 - Chebyshev App. A, Table 14.I
 - Cherwell-Wright 291.F
 - Clairaut App. A, Table 14.I
 - Clairaut partial App. A, Table 15.II
 - confluent 167.A
 - confluent hypergeometric 167.A, App. A, Table 14.II
 - delay 163.A
 - with deviating argument 163.A
 - difference- 163.A
 - Duffing 290.C
 - elliptic partial App. A, Table 15.VI
 - Emden 291.F
 - Euler (in dynamics of rigid bodies) 271.E
 - Euler-Lagrange 46.B
 - Euler linear ordinary App. A, Table 14.I
 - exact App. A, Table 14.I
 - Fokker-Planck partial 115.A
 - functional- 163.A
 - Gauss hypergeometric App. A, Table 14.II
 - Gaussian 206.A
 - generalized Lamé 167.E
 - generalized Riccati App. A, Table 14.I
 - Hamilton 324.E
 - Hamilton-Jacobi 23.B 324.E
 - Helmholtz 188.D, App. A, Table 15.VI
 - Hermite App. A, Tables 14.II 20.III
 - Hill 268.B
 - Hodgkin-Huxley 291.F
 - hyperbolic 325
 - hyperbolic partial 325
 - hypergeometric 206.A, App. A, Table 18.I
 - hyperspherical 393.E
 - integro- 163.A 222
 - integro-, of Fredholm type 222.A
 - integro-, of Volterra type 222.A
 - Jacobi App. A, Tables 14.II 20.V
 - Killing 364.F
 - Kummer App. A, Table 19.I
 - with lag 163.A
 - Lagrange 320.A, App. A, Table 14.I
 - Lagrange partial App. A, Table 15.II
 - Laguerre App. A, Tables 14.II 20.VI
 - Lame 113.B
 - Laplace 323.A, App. A, Table 15.III
 - Laplace, in the 2-dimensional case App. A, Table 15.VI
 - Laplace, in the 3-dimensional case App. A, Table 15.VI
 - Legendre 393.B, App. A, Table 14.II
 - Legendre associated 393.A
 - Liénard 290.C
 - linear ordinary 252.A 313.A
 - linear partial 320.A
 - Löwner 438.B
 - Mathieu 268.A
 - matrix Riccati 86.E
 - modified Mathieu 268.A
 - Monge 324.F
 - Monge-Ampère 278.A, App. A, Table 15.III
 - nonlinear ordinary 313.A
 - nonlinear partial 320.A
 - ordinary 313
 - partial 313.A 320
 - partial, of elliptic type 323
 - partial, of hyperbolic type 325
 - partial, of mixed type 326
 - partial, of parabolic type 327
 - Poisson 323.A, App. A, Table 15.III
 - polytropic 291.F
 - rational 288.A

- related 254.F
- with retardation 163.A
- retarded 163.A
- Riccati App. A, Table 14.I
- Riemann App. A, Table 18.I
- self-adjoint 252.K
- self-adjoint system of 252.K
- stochastic 342.A 406
- Stokes 167.E 188.E
- strongly nonlinear 290.C
- system of, of Maurer-Cartan 249.R
- system of hyperbolic (in the sense of Petrovskii) 325.G
- system of linear, of the first order 252.G
- system of ordinary 313.B
- system of partial, of order l (on a differentiable manifold) 428.F
- system of total 428.A
- Tissot-Pochhammer 206.C
- total 428.A, App. A, Table 15.I
- Tricomi 326.C
- van der Pol 290.C
- weakly nonlinear 290.C
- Weber 167.C, App. A, Table 20.III
- Weber-Hermite 167.C
- Whittaker 167.B, App. A, Tables 14.II 19.II
- differential extension ring 113
- differential field 113
 - Galois theory of 113
- differential form 105.Q
 - closed 105.Q
 - of degree l 105.O
 - of degree r 105.Q
 - of degree r (on an algebraic variety) 16.O
 - divisor of (on an algebraic variety) 16.O
 - exact 105.Q
 - exterior, of degree r 105.Q
 - of the first kind (on an algebraic variety) 16.O
 - of the first kind (on a nonsingular curve) 9.E
 - harmonic 194.B
 - holomorphic, of degree k 72.A
 - invariant (on an Abelian variety) 3.F
 - of Maurer-Cartan 249.R
 - primitive 232.B
 - of the second kind (on a nonsingular curve) 9.E
 - of the third kind (on a nonsingular curve) 9.E
 - of type (r, s) 72.C
- differential geometry 109, App. A, Table 4
 - affine 110.C
 - conformal 110.D
 - projective 110.B
- differential geometry in specific spaces 110
- differential geometry of curves and surfaces 111
- differential ideal 113
 - of a differential ring 113
 - involutive 428.E
 - prime (of a differential ring) 113
 - semiprime (of a differential ring) 113
 - sheaf on a real analytic manifold 428.E
- differential index (in a covering of a nonsingular curve) 9.I
- differential invariant
 - fundamental (of a surface) 110.B
 - on an m -dimensional surface 110.A
 - Poincaré 74.G
- differential law 107.A
- differential operator(s) 112 223.C 306.B
 - Beltrami, of the first kind App. A, Table 4.II
 - Beltrami, of the second kind App. A, Table 4.II
 - elliptic 112.A
 - of the k th order 237.H
 - ordinary 112.A
 - partial 112.A
 - pseudo- 345
 - strongly elliptic 112.G 323.H
 - system of 112.R
- differential polynomial(s) 113
- ring of 113
- differential quotient (at a point) 106.A
- differential representation (of a unitary representation of a Lie group) 439.S
- differential rings 113
- differential subring 113
- differential system 191.I
 - restricted 191.I
- differential topology 114
- differential variable 113
- differentiation
 - (in a commutative ring) 113
 - (of a differential function) 106.A
 - graphical 19.B
 - higher (in a commutative ring) 113
 - logarithmic App. A, Table 9.I
 - numerical 299.E
 - partial 106.F
 - theorem of termwise (on distribution) 125.G
 - of a vector field App. A, Table 3.II
- diffraction (of waves) 446
- diffusion, Ehrenfest model of 260.A
- diffusion-convection equation 304.B
- diffusion kernel 338.N
- diffusion process 115
 - on manifolds 115.D
 - multidimensional 115.C
- digamma function 174.B
- digital computer 75.B
- digital quantity 138.B
- dihedral group 151.G
- dilatation
 - in Laguerre geometry 76.B
 - maximal 352.B
- dilated maximum principle (in potential theory) 338.C
- dilation (of a linear operator) 251.M
 - power 251.M
 - strong 251.M
 - unitary 251.M
- dilation theorem 251.M
- dimension
 - (of an affine space) 7.A
 - (of an algebraic variety) 16.A
 - (of an analytic set) 23.B
 - (of an automorphic form) 32.B
 - (of a cell complex) 70.D
 - (of a convex cell in an affine space) 7.D
 - (of a divisor class on a Riemann surface) 11.D
 - (of a Euclidean simplicial complex) 70.B
 - (of a free module) 277.G
 - (of a Hilbert space) 197.C
 - (of a linear space) 256.C
 - (of a linear system of divisors) 9.C 16.N
 - (of a physical quantity) 116
 - (of a projective space) 343.B
 - (of a simplicial complex) 70.C
 - (of a topological space) 117
 - algebraic (of an algebraic surface) 72.F
 - capacity 48.G

Dimension — k

- cohomological (of an associative algebra) 200.L
- cohomological (of a scheme) 16.E
- cohomological (of a topological space) 117.F
- complex (of a complex manifold) 72.A
- covering (of a normal space) 117.B
- decomposition theorem for 117.C
- geometric (of a vector bundle) 114.D
- global (of an analytic set) 23.B
- global (of a ring) 200.K
- harmonic (of a Heins end) 367.E
- Hausdorff 117.G 234.E 246.K
- homological (of a module) 200.K
- homological (of a topological space) 117.F
- injective (of a module) 200.K
- Kodaira (of a compact complex manifold) 72.I
- Krull 67.E
- large inductive 117.B
- Lebesgue (of a normal space) 117.B
- left global (of a ring) 200.K
- local (of an analytic set at a point) 23.B
- product theorem for 117.C
- projective (of a module) 200.K
- right global (of a ring) 200.K
- small inductive 117.B
- sum theorem for 117.C
- theorem on invariance of, of Euclidean spaces 117.D
- weak (of a module) 200.K
- weak global (of a ring) 200.K
- dimension — k
 - automorphic form of 32.B
 - Fuchsian form of 32.B
 - Hilbert modular form of 32.G
 - Siegel modular form of 32.F
- dimensional analysis 116
- dimensional formula 116
- dimension function (on a continuous geometry) 85.A
- dimension theorem
 - (of affine geometry) 7.A
 - (on modular lattice) 243.F
 - (of projective geometry) 343.B
- dimension theory 117
- dimension type (of a topological space) 117.H
- Dini-Hukuhara theorem 314.D
- Dini-Lipschitz test (on the convergence of Fourier series) 159.B
- Dini series 39.C
- Dini surface 111.I
- Dini test (on the convergence of Fourier series) 159.B
- Dini theorem (on uniform convergence) 435.B
- Diocles, cissoid of 93.H
- Diophantine (relation) 97
- Diophantine analysis 296.A
- Diophantine approximation 182.F
- Diophantine equations 118
- Dirac delta function App. A, Table 12.II
- Dirac distribution 125.C
- Dirac equation 377.C 415.G
- Dirac field, free 377.C
- Dirac γ -matrix 415.G
- Dirac hole theory 415.G
- Dirac matrix 377.C
- direct analytic continuation 198.G
- direct circle 78.D
- direct closed path 186.F
- direct decomposition (on a group) 190.L
- directed family 165.D
- directed graph 186.B
- directed set 311.D
- direct factor
 - (of a direct product of sets) 381.E
 - (of a group) 190.L
- direct image (of a sheaf) 383.G
- direct integral 308.G
 - of unitary representations 437.H
- direction
 - asymptotic 111.H
 - Borel (of a meromorphic function) 272.F
 - Julia (of a transcendental entire function) 429.C
 - positive (in a curvilinear integral) 198.B
 - principal (of a surface) 111.H
- directional derivative 106.G
- direction ratio (of a line in an affine space) 7.F
- direct limit (of a direct system of sets) 210.B
- direct method 46.E 302.B
- direct path 186.F
- direct product
 - (of algebras) 29.A
 - (of distributions) 125.K
 - (of a family of lattices) 243.C
 - (of a family of ordered sets) 311.F
 - (of a family of sets) 381.E
 - (of a family of topological groups) 423.C
 - (of a family of topological spaces) 425.K
 - (of groups) 190.L
 - (of G -sets) 362.B
 - (of Lie groups) 249.H
 - (of mappings) 381.C
 - (of measurable transformations) 136.D
 - (of modules) 277.F
 - (of objects of a category) 52.E
 - (of rings) 368.E
 - (of sets) 381.B
 - (of sheaves) 383.I
 - restricted (of an infinite number of groups) 190.L
 - restricted (of locally compact groups) 6.B
 - semi- (of groups) 190.N
- direct product decomposition 190.L
 - dual 422.H
- directrix (of an ellipse) 78.B
 - of Wilczynski 110.B
- direct set, increasing 308.A
- direct sum
 - (of a family of ordered sets) 311.F
 - (of a family of sets) 381.E
 - (of G -sets) 362.B
 - (of Hilbert spaces) 197.E
 - (of ideals of a ring) 368.F
 - (of an infinite number of groups) 190.L
 - (of Lie algebras) 248.A
 - (of linear representations) 362.C
 - (of linear spaces) 256.F
 - (of modules) 277.B,F
 - (of a mutually disjoint family of sets) 381.D
 - (of quadratic forms) 348.E
 - (of sheaves) 383.I
 - (of topological groups) 423.C
 - (of two objects) 52.E
 - (of unitary representations) 437.G
 - integral 308.G
 - topological (of topological spaces) 425.M
- direct summand (of a direct sum of sets) 381.E
- direct system (of sets) 210.B
- direct transcendental singularity 198.P
- Dirichlet, P. G. L. 119

Dirichlet algebra 164.B
 weak* 164.G
 Dirichlet character 295.D
 Dirichlet discontinuous factor App. A, Table 9.V
 Dirichlet distribution 341.D, App. A, Table 22
 Dirichlet divisor problem 242.A
 Dirichlet domain 120.A
 Dirichlet drawer principle 182.F
 Dirichlet form 261.C
 regular 261.C
 Dirichlet function 84.D 221.A
 Dirichlet functional 334.C
 Dirichlet integral
 (in the Dirichlet problem) 120.F
 (in Fourier's single integral theorem) 160.B
 Dirichlet kernel 159.B
 Dirichlet L -function 450.C
 Dirichlet principle 120.A 323.C
 Dirichlet problem 120 293.F 323.C
 Dirichlet problem with obstacle 440.B
 Dirichlet region 234.C
 Dirichlet series 121.A
 ordinary 121.A
 of the type $\{\lambda_n\}$ 121.A
 Dirichlet space 338.Q
 Dirichlet test (on Abel partial summation) 379.D
 Dirichlet test (on the convergence of Fourier series) 159.B
 Dirichlet theorem
 (of absolute convergence) 379.C
 (on the distribution of primes in arithmetical progression) 123.D
 Dirichlet unit theorem 14.D
 discharge of double negation 411.I
 discharging 157.D
 disconnected, extremely 37.M
 disconnected metric space, totally 79.D
 discontinuity
 of the first kind 84.B
 fixed point of (of a stochastic process) 407.A
 at most of the first kind 84.B
 region of 234.A
 discontinuity formula 146.C 386.C
 discontinuity point 84.B
 of the first kind 84.B
 of the second kind 84.B
 discontinuous distribution, purely 341.D
 discontinuous factor, Dirichlet App. A, Table 9.V
 discontinuous groups 122
 of the first kind 122.B
 discontinuous transformation group 122.A
 properly 122.A
 discontinuum, Cantor 79.D
 discrete covering 425.R
 σ - 425.R
 discrete C^r -flow 126.B
 discrete dynamical system of class C^r 126.B
 discrete flow 126.B
 of class C^r 126.B
 discrete Fourier transform 142.D
 discrete mathematics 66.A
 discrete memoryless channels 213.F
 discrete metric space 273.B
 discrete semiflow 126.B
 of class C^r 126.B
 discrete series (of unitary representations of a semi-simple Lie group) 437.X
 discrete series, principal 258.C
 discrete set 425.O

discrete spectrum 136.E 390.E
 quasi- 136.E
 discrete topological space 425.C
 discrete topology 425.C
 discrete uniformity 436.D
 discrete valuation 439.E
 discrete valuation ring 439.E
 discrete variable method 303.A
 discrete von Neumann algebra 308.E
 discretization error 303.B
 discriminant
 (of an algebraic equation) 337.J
 (of an algebraic number field) 14.B
 (of a binary quadratic form) 348.M
 (of a curve of the second order) 78.I
 (of a family of curves) 93.I
 (of a quadratic form) 348.A
 (of a simple ring) 27.B
 fundamental 295.D
 relative 14.J
 discriminant function, linear 280.I
 discriminant theorem, Dedekind 14.J
 disintegration 270.L
 disjoint family, mutually (of sets) 381.D
 disjoint sets 381.B
 disjoint sum 381.B
 disjoint union 381.B
 of a mutually disjoint family of sets 381.D
 disjoint unitary representations 437.C
 disjunction (of propositions) 411.B
 disjunctive cuts 215.C
 disjunctive programming 264.C
 disk 140
 circular 140
 n - 140
 open 140
 open n - 140
 unit 140
 disk algebra 164.B
 disk theorem (on meromorphic functions) 272.J
 dispersion 397.C
 dispersion relations 132.C
 dispersive
 (flow) 126.E
 (linear operator) 286.Y
 dispersive wave 446
 displacement
 electric 130.A
 parallel (of a tangent vector space) 80.H 364.B
 parallel, along a curve 80.C
 dissection, Farey 4.B
 dissipative (operator) 251.J 286.C
 maximal 251.J
 dissipative part (of a state space) 260.B
 distance
 (in Euclidean geometry) 139.E
 (in a metric space) 273.B
 Euclidean 139.E
 extremal 143.B
 Fréchet (between surfaces) 246.I
 Hamming 63.B 136.E
 Lévy 341.F
 Mahalanobis generalized 280.E
 non-Euclidean (in a Klein model) 285.C
 optical 180.A
 perihelion 309.B
 reduced extremal 143.B
 distance function 273.B
 pseudo- 273.B

Distinct differentiable structures

- distinct differentiable structures 114.B
- distinct system of parameters 284.D
- distinguishable, finitely (hypothesis) 400.K
- distinguished basis, strongly 418.F
- distinguished pseudopolynomial 21.E
- distortion function, rate 213.E
- distortion inequalities 438.B
- distortion measure 213.E
- distortion theorem 438.B
- distributed
 - asymptotically 374.D
 - uniformly 182.H
- distribution(s) 125
 - (on a differentiable manifold) 125.R
 - (of random variables) 342.C
 - (of a vector bundle) 428.D
 - a posteriori 398.B
 - a priori 398.B
 - asymptotically normal 399.K
 - beta 341.D, App. A, Table 22
 - Beurling generalized 125.U
 - binomial 341.D 397.F, App. A, Table 22
 - bivariate 397.H
 - capacity mass 338.K
 - Cauchy 341.D, App. A, Table 22
 - chi-square 374.A, App. A, Table 22
 - conditional probability 342.E
 - continuous 341.D
 - converge in (a sequence of random variables) 342.D
 - covariance 395.C
 - cumulative 397.B
 - Dirac 125.C
 - Dirichlet 341.D, App. A, Table 22
 - double, potential of 338.A
 - entropy of a 403.B
 - equilibrium, Gibbs 136.C
 - equilibrium mass 338.K
 - exponential 341.D, App. A, Table 22
 - exponential family of 396.G
 - F - 341.D 374.B, App. A, Table 22
 - fiducial 401.F
 - finite-dimensional 407.A
 - of finite order 125.J
 - function, empirical 374.D
 - gamma 341.D, App. A, Table 22
 - Gaussian 341.D
 - geometric 341.D, App. A, Table 22
 - hypergeometric 341.D 397.F, App. A, Table 22
 - infinitely divisible 341.G
 - initial 261.A
 - initial law 406.D
 - integrable 125.N
 - invariant (of a Markov chain) 260.A
 - invariant (second quantization) 377.C
 - involutive (on a differentiable manifold) 428.D
 - joint 342.C
 - k -dimensional normal 341.D
 - k -Erlang 260.H
 - k th-order asymptotic 399.O
 - L - 341.G
 - lattice 341.D
 - law, Maxwell-Boltzmann 402.B
 - least favorable 400.B
 - least favorable a priori 398.H
 - limit 250.A
 - logarithmic App. A, Table 22
 - logarithmic normal App. A, Table 22
 - marginal 342.C 397.H
 - multidimensional hypergeometric App. A, Table 22
 - multidimensional normal App. A, Table 22
 - multinomial 341.D
 - multiple hypergeometric 341.D
 - multivariate normal 397.J
 - n -dimensional 342.C
 - n -dimensional probability 342.C
 - negative binomial 341.D 397.F, App. A, Table 22
 - negative multinomial 341.D
 - negative polynomial App. A, Table 22
 - noncentral chi-square 374.B
 - noncentral F - 374.B
 - noncentral t - 374.B
 - noncentral Wishart 374.C
 - normal 341.D 397.C, App. A, Table 22
 - one-dimensional probability, of a random variable 342.C
 - one-side stable for exponent $1/2$ App. A, Table 22
 - operator-valued 150.D
 - p -dimensional noncentral Wishart 374.C
 - Pearson 397.D
 - pluriharmonic 21.C
 - Poisson 341.D 397.F, App. A, Table 22
 - polynomial App. A, Table 22
 - population 396.B 401.F
 - positive 125.C
 - posterior 401.B 403.G
 - predictive 403.G
 - of prime numbers 123
 - prior 401.B 403.G
 - probability 342.B, App. A, Table 22
 - probability, of a random variable 342.C
 - purely discontinuous 341.D
 - quasistable 341.G
 - random 395.H 407.C
 - random, with independent values at every point 407.C
 - random, in the wider sense 395.C 407.C
 - rapidly decreasing 125.O
 - rectangular App. A, Table 22
 - sampling 374.A
 - semistable 341.G
 - simple, potential of 338.A
 - simultaneous 342.C
 - slowly increasing 125.N
 - stable 341.G
 - standard Gaussian 176.A
 - standard normal 341.D
 - strictly stationary random 395.H
 - strongly stationary random 395.H
 - substituted 125.Q
 - t - 341.D 374.B, App. A, Table 22
 - tempered 125.N
 - two-sided exponential App. A, Table 22
 - of typical random variables App. A, Table 22
 - ultra-, of class $\{M_p\}$ or (M_p) 125.U, BB
 - uniform 341.D, App. A, Table 22
 - unit 341.D
 - value 124.A
 - of values of functions of a complex variable 124
 - waiting time 307.C
 - weakly stationary random 395.C
 - Wishart 374.C
 - z - 341.D 374.B, App. A, Table 22
 - distribution curve, cumulative 397.B

distribution derivative 125.E
 distribution-free (test) 371.A
 distribution-free method 371.A
 distribution function 168.B 341.B 342.C
 cumulative 341.B 342.C
 empirical 250.F 396.C
 n -dimensional 342.C
 symmetric 341.H
 unimodal 341.H
 distribution kernel 338.P
 distribution law, Maxwell-Boltzmann 402.B
 distribution polygon, cumulative 397.B
 distribution semigroup 378.F
 distributive algebra 231.A
 distributive lattice 243.E
 distributive law
 (in algebra of sets) 381.B
 (on cardinal numbers) 49.C
 (in a lattice) 243.E
 (on natural numbers) 294.B
 (in a ring) 368.A
 complete (in a lattice-ordered group) 243.G
 disturbance 128.C
 diurnal aberration 392
 div (divergence) 442.D
 diverge 87.B,E 379.A
 to ∞ 87.D
 divergence
 (of a differentiable vector field) 442.D
 (of a vector field with respect to a Riemannian metric) 105.W
 (of a vector field with respect to a volume element) 105.W
 infrared 146.B
 ultraviolet 146.B
 divergence form 323.D
 divergence theorem 94.D
 divergent
 (double series) 379.E
 (infinite product) 379.G
 (integral) 216.E
 (sequence of real numbers) 87.B
 (series) 379.A
 properly 379.A
 divide (a bounded domain) 384.F
 divided difference 223.D
 dividing cycle (on an open Riemann surface) 367.I
 divisibility relation (in a ring) 67.H
 divisible
 (Abelian p -group) 2.D
 (additive group) 2.E
 (element of ring) 67.H 277.D
 (fractional ideal) 14.E
 (general Siegel domain) 384.F
 (number) 297.A
 divisible A -module 277.D
 divisible subgroup (of a discrete Abelian group) 422.G
 division (of a pseudomanifold) 65.A
 simplicial 65.A
 division algebra 29.A
 division algorithm
 of natural numbers 297.A
 of polynomials 337.C
 division ring 368.B
 division theorem
 Späth type (for microdifferential operators) 274.E
 Weierstrass type (for microdifferential operators) 274.E

divisor
 (in an algebraic curve) 9.C
 (of an algebraic function field of dimension 1) 9.D
 (of an algebraic number field) 14.F
 (in an algebraic variety) 16.M
 (in a closed Riemann surface) 11.D
 (in a complex manifold) 72.F
 (of an element of a ring) 67.H
 (of a fractional ideal) 14.E
 (of a number) 297.A
 ample 16.N
 branch (in a covering) 9.I
 canonical (of an algebraic curve) 9.C
 canonical (of an algebraic variety) 16.O
 canonical (of a Jacobian variety) 9.E
 canonical (of a Riemann surface) 11.D
 Cartier 16.M
 common (of elements of a ring) 67.H
 complete linear system defined by 16.N
 complex line bundle determined by 72.F
 differential (of an algebraic curve) 9.C
 of a differential form (on an algebraic variety) 16.O
 effective (on an algebraic curve) 9.C
 effective (on a variety) 16.M
 elementary (of a matrix) 269.E
 embedded prime (of an ideal) 67.F
 finite prime 439.H
 of a function (on an algebraic curve) 9.C
 of a function (on an algebraic variety) 16.M
 greatest common 297.A
 greatest common (of an element of a ring) 67.H
 imaginary infinite prime 439.H
 infinite prime 439.H
 integral (of an algebraic curve) 9.C
 integral (of an algebraic number field) 14.F
 integral (on a Riemann surface) 11.D
 isolated prime (of an ideal) 67.F
 k -rational (on an algebraic curve) 9.C
 linearly equivalent (of a complex manifold) 72.F
 maximal prime (of an ideal) 67.F
 minimal prime (of an ideal) 67.F
 nondegenerate 16.N
 nondegenerate (on an Abelian variety) 3.D
 numerically connected 232.D
 \mathbb{Q} -linearly equivalent (on an algebraic curve) 9.F
 pole (of a function on an algebraic variety) 16.M
 positive (of an algebraic curve) 9.C
 positive (on a Riemann surface) 11.D
 prime (of an algebraic function field of dimension 1) 9.D
 prime (of an algebraic number field or an algebraic function field of one variable) 439.H
 prime (of an ideal) 67.F
 prime (on a Riemann surface) 11.D
 prime rational, over a field (on an algebraic curve) 9.C
 principal (on an algebraic curve) 9.C
 principal (on a Riemann surface) 11.D
 real infinite prime 439.H
 real prime 439.H
 sheaf of ideals of (of a complex manifold) 72.F
 special 9.C
 very ample 16.N

Divisor class (on a Riemann surface)

- zero (of a function on an algebraic variety) 16.M
- zero (of a ring) 368.B
- zero, with respect to M/P 284.A
- divisor class (on a Riemann surface) 11.D
 - canonical 11.D
 - differential 11.D
- divisor class group (of a Riemann surface) 11.D
- divisor function 295.C
 - generalized 295.C
- divisor group (of a compact complex manifold) 72.F
- divisor problem, Dirichlet 242.A
- Dixmier theorem, Rellich- 351.C
- Dixon-Ferrar formula App. A, Table 19.IV
- DK method 301.F
- DKA method 301.F
- DLR equation 402.G
- dn App. A, Table 16.III
- dodecahedron 357.B
- Doetsch three-line theorem 43.E
- Dolbeault cohomology group 72.D
- Dolbeault complex 72.D
- Dolbeault lemma 72.D
- Dolbeault theorem 72.D
- domain(s)
 - (of a correspondence) 358.B
 - (of a mapping) 37.C 381.C
 - (in a topological space) 79.A
 - (of a variable) 165.C
 - angular 333.A
 - annular 333.A
 - of attraction 374.G
 - Brouwer theorem on the invariance of 117.D
 - Cartan pseudoconvex 21.I
 - circular 333.A
 - of class $C^{1,\lambda}$ 323.F
 - closed plane 333.A
 - complete Reinhardt 21.B
 - convergence (of a power series) 21.B
 - Courant-Cheng, theorem 391.H
 - of dependence 325.B
 - Dirichlet 120.A
 - divisible bounded 284.F
 - d -pseudoconvex 21.G
 - effective 88.D
 - fundamental 234.C
 - generated Siegel 384.F
 - holomorphically complete 21.F
 - holomorphically convex 21.H
 - of holomorphy 21.F
 - homogeneous bounded 384.A 412.F
 - individual 411.H
 - of influence 325.B
 - integral 368.B
 - of integration 216.F
 - irreducible symmetric bounded 412.F
 - Jordan 333.A
 - Levi pseudoconvex 21.I
 - of a local homomorphism 423.O
 - locally Cartan pseudoconvex 21.I
 - locally Levi pseudoconvex 21.I
 - nodal 391.H
 - Noetherian 284.A
 - Noetherian integral 284.A
 - object 411.G
 - of operator 409.A
 - operator (of a group) 190.E
 - plane 333
 - principal ideal 67.K
 - pseudoconvex 21.G
 - with regular boundary (in a C^∞ -manifold) 105.U
 - Reinhardt 21.B
 - Siegel 384.A
 - Siegel, generalized 384.F
 - Siegel, irreducible 384.E
 - Siegel, of the first kind 384.A
 - Siegel, of the second kind 384.A
 - Siegel, of the third kind 384.A
 - slit 333.A
 - with smooth boundary (in a C^∞ -manifold) 105.U
 - spectrum of 391.A
 - strongly pseudoconvex 21.G
 - sweepable bounded 284.F
 - symmetric bounded 412.F
 - unique factorization 40.H
 - universal 16.A
 - Weil 21.G
- domain kernel (of a sequence of domains) 333.C
- dominant (of a sequence of functions) 435.A
- dominant integral form (on a Cartan subalgebra) 248.W
- dominate (an imputation of a game) 173.D
- dominated
 - (by a family of topological spaces) 425.M
 - (statistical structures) 396.F
 - weakly (statistical structure) 396.F
- dominated ergodic theorem 136.B
- dominating set 186.I
- domination, number of 186.I
- domination principle 338.L
 - inverse 338.L
- Donsker invariance principle 250.E
- Doob-Meyer decomposition theorem 262.D
- Doolittle method 302.B
- dotted indices 258.B
- dotted spinor of rank k 258.B
- Douady space 23.G
- double chain complex 200.E
- double complex 200.H
- double coset (of two subgroups of a group) 190.C
- double distribution, potential of 338.A
- double exponential formula 299.B
- double integral 216.F
- double integral theorem, Fourier 160.B
- double layer, potential of 338.A
- double mathematical induction 294.B
- double negation, discharge of 411.I
- double point, rational 418.C
- double ratio 343.E
- double sampling inspection 404.C
- double sequence 379.E
- double series 379.E
 - absolutely convergent 379.E
 - conditionally convergent 379.E
 - convergent 379.E
 - divergent 379.E
 - Weierstrass theorem of 379.H
- double suspension theorem 65.C
- double-valued representation 258.B
- doubly invariant 164.H
- doubly periodic function 134.E
- Douglas algebra 164.I
- Douglas functional 334.C
- Douglas-Rad6 solution (to Plateau's problem) 275.C
- downhill method 301.L
- down-ladder 206.B

drawer principle, Dirichlet 182.F
 drift
 transformation by 261.F
 transformation of 406.B
 drift part 406.B
 dual
 (cell) 65.B
 (graded module) 203.B
 (graph) 186.H
 (matroid) 66.H
 (proposition in a projective space) 343.B
 (regular polyhedron) 357.B
 (symmetric Riemannian space) 412.D
 (topological group) 422.C 437.J
 dual algebra 203.F
 dual basis (of a linear space) 256.G
 dual bundle 147.F
 dual category 52.F
 dual cell, $(n - q)$ - 65.B
 dual coalgebra 203.F
 dual complex 65.B
 dual cone 125.BB
 dual convex cone 89.F
 dual curve (of a plane algebraic curve) 9.B
 dual direct product decomposition 422.H
 dual frame 417.B
 dual homomorphism
 (of a homomorphism of algebraic tori) 13.D
 (of lattices) 243.C
 dual Hopf algebra 203.C
 dual isomorphism
 (of lattices) 243.C
 (between ordered sets) 311.E
 duality
 (in field theory) 150.E
 (for symmetric Riemannian space) 412.D
 Martineau-Harvey 125.Y
 Poincaré (in manifolds) 201.O
 Poincaré (in Weil cohomology) 450.Q
 principle of (in projective geometry) 343.B
 duality mapping 251.J
 duality principle
 (for closed convex cone) 89.F
 (for ordering) 311.A
 duality property (of linear space) 256.G
 duality theorem
 (on Abelian varieties) 3.D
 (of linear programming) 255.B
 (in mathematical programming) 292.D
 Alexander 201.O
 for Ω -module 422.L
 Poincaré-Lefschetz 201.O
 Pontryagin (on topological Abelian groups)
 192.K 422.C
 Serre (on complex manifolds) 72.E
 Serre (on projective varieties) 16.E
 of Takesaki 308.I
 Tannaka (on compact groups) 69.D
 Tannaka (on compact Lie group) 249.U
 dual lattice 243.C 310.E 450.K
 dual linear space 256.G
 self- 256.H
 dually isomorphic (lattices) 243.C
 dual mapping (of a linear mapping) 256.G
 dual Martin boundary 260.I
 dual module 277.K
 dual operator
 in Banach space 37.D
 of a differential operator 125.F
 of a linear operator 251.D

Subject Index

 ε -flat

dual ordering 311.A
 dual passive boundary point 260.I
 dual problem 255.B 349.B
 dual process 261.F
 dual representation 362.E
 dual resonance model 132.C
 dual semigroup 378.F
 dual space
 (of a C^* -algebra) 36.G
 (of a linear space) 256.G
 (of a linear topological space) 424.D
 (of a locally compact group) 437.J
 (of a normed linear space) 37.D
 (of a projective space) 343.B
 quasi- (of a locally compact group) 437.I
 strong 424.K
 dual subdivision 65.B
 dual vector bundle 147.F
 du Bois-Reymond and Dedekind, test of 379.D
 du Bois Reymond problem 159.H
 Duffing differential equation 290.C
 Duhamel method 322.D
 Duhem relation, Gibbs- 419.B
 dummy index (of a tensor) 256.J
 Dunford integrable 443.F
 Dunford integral 251.G 443.F
 Dunford-Pettis theorem 68.M
 Donford-Schwartz integral, Bartle- 443.G
 duo-trio test 346.D
 Dupin, cyclide of 111.H
 Dupin indicatrix 111.H
 duplication of a cube 179.A
 Durand-Kerner-Aberth (DKA) method 301.F
 Durand-Kerner (DK) method 301.F
 Dvoretzky-Rogers theorem 443.D
 dyadic compactum 79.D
 dynamical system(s) 126
 classical 126.L 136.G
 continuous 126.B
 differential 126.B
 discrete 126.B
 linear 86.B
 dynamic programming 127 264.C
 dynamic programming model 307.C
 dynamics
 analytical 271.F
 fluid 205.A
 magnetofluid 259
 quantum flavor 132.D
 dynamo theory, hydromagnetic 259
 Dynkin class 270.B
 Dynkin class theorem 270.B
 Dynkin diagram (of a complex semisimple Lie algebra) 248.S
 extended App. A, Table 5.I
 Dynkin formula 261.C
 Dynkin representation of generator 261.C

E

 ε (topology) 424.R
 ε , Eddington's App. A, Table 4.II
 $\mathcal{E}(\Omega)$ ($= C^\infty(\Omega)$) 125.I 168.B
 $\mathcal{E}'(\Omega)$ 125.I
 $\mathcal{E}_{\{M_p\}}, \mathcal{E}_{(M_p)}$ 168.B
 ε -covering 273.B
 ε -entropy 213.E
 ε -expansion 111.C
 ε -factor 450.N
 ε -flat 178.D

- ε -Hermitian form 60.O
- ε -independent partitions 136.E
- ε -induction, axiom of 33.B
- ε -neighborhood (of a point) 273.C
- ε -number 312.C
- ε -operator, Hilbert 411.J
- ε -quantifier, Hilbert 411.J
- ε -sphere (of a point) 273.C
- ε -symbol, Hilbert 411.J
- ε -tensor product 424.R
- ε -theorem (in predicate logic) 411.J
- ε -trace form 60.O
- \mathcal{E} -function 46.C
- \mathcal{E} -space 193.N
- E -function 430.D
- E -optimality 102.E
- E waves 130.B
- Eberlein-Shmul'yan theorem 37.G
- Eberlein theorem 424.V
- eccentric angle
 - of a point on a hyperbola 78.E
 - of a point on an ellipse 78.E
- eccentric anomaly 309.B
- eccentricity (of a conic section) 78.B
- echelon space 168.B
- ecliptic 392
- econometrics 128
- Eddington's ε App. A, Table 4.II
- edge
 - (of a convex cell in an affine space) 7.D
 - (in a graph) 186.B
 - (of a linear graph) 282.A
 - reference 281.C
- edge homomorphism 200.J
- edge of the wedge theorem 125.W
- Edgeworth expansion 374.F
- effect 403.D
 - block 102.B
 - factorial 102.H
 - fixed 102.A
 - fixed, model 102.A
 - main 102.H
 - random 102.A
 - random, model 102.A
 - treatment 102.B
- effective descriptive set theory 356.H
- effective divisor
 - (on an algebraic curve) 9.C
 - (on a variety) 16.M
- effective domain 88.D
- effective genus (of an algebraic curve) 9.C
- effectively (act on a G -space) 431.A
 - almost 431.A
- effectively calculable function 356.C
- effectively given (object) 22.A
- effectively parametrized (at o) 72.G
- effect vector 102.A
- efficiency 399.D
 - asymptotic 399.N
 - Bahadur 400.K
 - second-order 399.O
 - second-order asymptotic 399.O
- efficiency-balanced block design 102.E
- efficient
 - k th order asymptotic 399.O
- efficient estimator 399.D
 - asymptotically 399.N
 - first-order 399.O
 - first-order asymptotic 399.O
- Egerváry theorem, König- 281.E
- Egorov theorem 270.J
- Ehrenfest model of diffusion 260.A
- Ehrenpreis-Malgrange theorem 112.B
- Eichler approximation theorem 27.D
- eigenchain 390.H
- eigenelement (of a linear operator) 390.A
- eigenfunction
 - (of a boundary value problem) 315.B
 - (for an integral equation) 217.F
 - (of a linear operator) 390.A
 - generalized 375.C
- eigenspace
 - (of a linear mapping) 269.L
 - (of a linear operator) 390.A
 - generalized 390.B
 - in a weaker sense 269.L
- eigenvalue(s)
 - (of a boundary value problem) 315.B
 - (of an integral equation) 217.F
 - (of a linear mapping) 269.L
 - (of a linear operator) 390.A
 - (of the Mathieu equation) 268.B
 - (of a matrix) 269.F
 - degenerate 390.A
 - generalized 375.C
 - geometrically simple 390.A
 - index of 217.F
 - multiplicity of 217.F
 - numerical computation of 298
- eigenvalue problem 390.A
 - generalized 298.G
- eigenvector
 - (of a linear mapping) 269.L
 - (of a linear operator) 390.A
 - (of a matrix) 269.F
 - generalized 390.B
- eightfold way 132.D
- eikonal 82.D 180.C
- eikonal equation 324.E 325.L
- Eilenberg-MacLane complexes 70.F
- Eilenberg-MacLane space 70.F
- Eilenberg-MacLane spectrum 202.T
- Eilenberg-Postnikov invariants (of a CW complex) 70.G
- Eilenberg-Steenrod axioms 201.Q
- Eilenberg-Zilber theorem 201.J
- Einstein, A. 129
- Einstein convention (on tensors) 256.J
- Einstein-Kähler metric 232.C
- Einstein metric 364.I
- Einstein relation (in diffusion) 1:8.A
- Einstein space 364.D, App. A, Table 4.II
- Einstein summation convention 417.B
- Eisenstein-Poincaré series 32.F
- Eisenstein series 32.C
 - generalized 450.T
- Eisenstein theorem 337.F
- elastic, total, cross section 386.B
- elasticity
 - modulus of, in shear 271.G
 - modulus of, in tension 271.G
 - small-displacement theory of 271.G
 - theory of 271.G
- elastic limit 271.G
- elastic scattering 375.A
- elation 110.D
- electric displacement 130.A
- electric field 130.A
- electric flux density 130.A
- electric network 282.B

- electric polarization 130.A
 electric susceptibility 130.B
 electric waves 130.B
 transverse 130.B
 electrodynamics, quantum 132.C
 electromagnetic wave 446
 theory of 130.B
 transverse 130.B
 electromagnetism 130
 electron 377.B
 electronic analog computer 19.E
 electronic computer 75.A
 electrostatics 130.B
 element(s) 381.A
 affine arc 110.C
 algebraic (of a field) 149.E
 areal (in a Cartan space) 152.C
 atomic (in a complemented modular lattice) 243.F
 boundary (in a simply connected domain) 333.B
 canonical (in the representation of a functor) 52.L
 Casimir (of a Lie algebra) 248.J
 central (in a lattice) 243.E
 compact (of a topological Abelian group) 422.F
 conformal arc 110.D
 conjugate (in a field) 149.J
 conjugate (in a group) 190.C
 contact 428.E
 contact (in a space with a Lie transformation group) 110.A
 cyclic 251.J
 even (of a Clifford algebra) 61.B
 finite, method 304.C
 function 198.I 339.A
 function, in the wider sense 198.O
 generalized nilpotent (in a commutative Banach algebra) 36.E
 generating 390.G
 greatest (in an ordered set) 311.B
 homogeneous (of a graded ring) 369.B
 homogeneous (of a homogeneous ring) 369.B
 hypersurface 324.B
 idempotent (of a ring) 368.B 450.O
 identity (of an algebraic system) 409.C
 identity (of a field) 149.A
 identity (of a group) 190.A
 identity (of a ring) 368.A
 inseparable (of a field) 149.H
 integral (of a system of total differential equations) 428.E
 inverse (in a group) 190.A
 inverse (in a ring) 368.B
 inverse function 198.L
 invertible (of a ring) 368.B
 irreducible (of a ring) 67.H
 isotropic (with respect to a quadratic form) 348.E
 k -dimensional integral 191.I
 Kepler orbital 309.B
 least (in an ordered set) 311.B
 left inverse (of an element of a ring) 368.B
 line 111.C
 linearly dependent 2.E
 linearly independent 2.E
 matrix 351.B
 maximal (in an ordered set) 311.B
 maximum (in an ordered set) 311.B
 minimal (in an ordered set) 311.B
 minimum (in an ordered set) 311.B
 negative (of an ordered field) 149.N
 neutral (in a lattice) 243.F
 nilpotent (of a ring) 368.B
 odd (of a Clifford algebra) 61.B
 ordinary 191.I
 ordinary integral 428.E
 oriented (in a covering manifold of a homogeneous space) 110.A
 orthogonal (of a ring) 368.B
 osculating 309.D
 polar (of an analytic function in the wider sense) 198.O
 polar (of an integral element) 428.E
 positive (of an ordered field) 149.N
 prime (of a ring) 67.H
 prime (for a valuation) 439.E
 primitive (of an extension of a field) 149.D
 projective line 110.B
 purely inseparable (of a field) 149.H
 quasi-inverse (in a ring) 368.B
 quasi-invertible (of a ring) 368.B
 quasiregular (of a ring) 368.B
 ramified 198.O
 rational 198.O
 regular (of a connected Lie group) 249.P
 regular (of a ring) 368.B
 regular integral 428.E
 right inverse (in a ring) 368.B
 separable (of a field) 149.H
 singular (of a connected Lie group) 249.P
 singular (with respect to a quadratic form) 348.E
 surface 324.B
 surface, union of 324.B
 torsion (of an A -module) 277.D
 transcendental (of a field) 149.E
 transgressive (in the spectral sequence of a fiber space) 148.E
 triangular 304.C
 unit (of a field) 149.A
 unit (of a group) 190.A
 unit (of a ring) 368.A
 unity (of a field) 149.A
 unity (of a ring) 368.A
 volume (of an oriented C^∞ -manifold) 105.W
 volume, associated with a Riemannian metric 105.W
 zero (of an additive group) 2.E 190.A
 zero (of a field) 149.A
 zero (of a linear space) 256.A
 zero (of a ring) 368.A
 elementarily equivalent structures 276.D
 elementary (Kleinian group) 234.A
 elementary (path) 186.F
 elementary Abelian functions 3.M
 elementary Abelian group 2.B
 elementary catastrophe 51.E
 elementary collapsing 65.C
 elementary contract 102.C
 elementary divisor
 (of a matrix) 269.E
 simple (of a matrix) 269.E
 elementary event(s) 342.B
 space of 342.B
 elementary extension 276.D
 elementary function(s) 131
 of class n 131.A
 elementary Hopf algebra 203.D

- ul style="list-style-type: none; padding-left: 0;">
- elementary ideal 235.C
- elementary kernel (of a linear partial differential operator) 320.H
- elementary number theory 297
 - fundamental theorem of, 297.C
- elementary particle(s) 132
- elementary solution App. A, Table 15.V
 - (of a differential operator) 112.B
 - (of a linear partial differential operator) 320.H
 - (of partial differential equations of elliptic type) 323.B
- elementary symmetric function 337.I
- elementary symmetric polynomial 337.I
- elementary topological Abelian group 422.E
- eliminate (variables from a family of polynomials) 369.E
- elimination
 - design for two-way, of heterogeneity 102.K
 - forward 302.B
 - Gaussian 302.B
 - Gauss-Jordan 302.B
- elimination method, Sylvester 369.E
- ellipse 78.A
- ellipsoid 350.B
 - of inertia 271.E
 - of revolution 350.B
- ellipsoidal coordinates 90.C 133.A, App. A, Table 3.V
- ellipsoidal harmonics 133.B
 - four species of 133.C
- ellipsoidal type, special function of 389.A
- elliptic
 - (differential operator) 112.A 323.A 237.H
 - (pseudodifferential operator) 323.K
 - (Riemann surface) 77.B 367.D
 - (solution) 323.D
 - analytically hypo- 323.I
 - analytic hypo- 112.D
 - hypo- 112.D 189.C 323.I
 - microlocally 345.A
 - strongly 112.G 323.H
- elliptic complex (on a compact C^∞ -manifold) 237.H
- elliptic coordinates 90.C 350.E, App. A, Table 3.V
- elliptic curve 9.C
 - L -functions of 450.S
- elliptic cylinder 350.B
- elliptic cylinder function 268.B
- elliptic cylindrical coordinates App. A, Table 3.V
- elliptic cylindrical surface 350.B
- elliptic domain 77.B
- elliptic function(s) 134
 - of the first kind 134.G
 - Jacobi App. A, Table 16.III
 - of the second kind 134.G
 - of the third kind 134.H
 - Weierstrass App. A, Table 16.IV
- elliptic function field 9.D
- elliptic geometry 285.A
- elliptic integral 11.C 134.A, App. A, Table 16.I
 - complete App. A, Table 16.I
 - of the first kind 134.A, App. A, Table 16.I
 - of the first kind, complete 134.B
 - of the first kind, incomplete 134.B
 - of the second kind 134.A, App. A, Table 16.I
 - of the second kind, complete 134.C
 - of the third kind 134.A, App. A, Table 16.I
- elliptic integrals and elliptic functions App. A, Table 16
- elliptic irrational function 134.A
- elliptic modular group 122.D
- elliptic motions 55.A
- elliptic operator 323.H
 - microlocally 345.A
 - strongly 323.H
- elliptic paraboloid 350.B
 - of revolution 350.B
- elliptic point
 - (of a Fuchsian group) 122.C
 - (on a surface) 111.H
- elliptic quadric hypersurface 350.G
- elliptic singularity 418.C
 - minimally 418.C
- elliptic space 285.C
- elliptic surface 72.K
- elliptic theta function 134.I, App. A, Table 16.II
- elliptic transformation 74.F
- elliptic type 323.A.D
 - (Lie algebra) 191.D
 - partial differential equation of 323, App. A, Table 15.VI
- elongation strain 271.G
- embedded
 - (into a topological space) 425.J
 - hyperbolically 21.O
- embedded Markov chain 260.H
- embedded primary component (of an ideal) 67.F
- embedded prime divisor (of an ideal) 67.F
- embedding 105.K
 - (of categories) 52.H
 - (of a C^∞ -manifold) 105.K
 - (of a topological space) 425.J
 - formula of, form 303.D
 - generalized Borel 384.D
 - PL 65.D
 - regular 105.K
 - Tanaka 384.D
 - toroidal 16.Z
 - torus 16.Z
- embedding principle (in dynamic programming) 127.B
- embedding theorem
 - full (of an Abelian category) 52.N
 - Irwin 65.D
 - Menger-Nöbeling 117.D
 - Sobolev-Besov 168.B
 - Tikhonov 425.T
- Emden differential equation 291.F
- Emden function, Lane- 291.F
- emission 325.A
 - backward 325.A
 - forward 325.A
- empirical characteristic function 396.C
- empirical constant 19.F
- empirical distribution function 250.F 374.E 396.C
- empirical formula 19.F
- empiricism, French 156.C
- empty set 33.B 381.A
 - axiom of 33.B
- enantiomorphic pair 92.A
- enantiomorphous 92.A
- encoder 213.D
- encoding 63.A
- end
 - (of an arc) 93.B
 - (of a noncompact manifold) 178.F
 - (of a segment) 155.B
 - (of a segment in an affine space) 7.D
 - Heins 367.E

- lower (of a curvilinear integral) 94.D
 upper (of a curvilinear integral) 94.D
 endogenous variable 128.C
 endomorphism
 (of an algebraic system) 409.C
 (of a group) 190.D
 (of a polarized Abelian variety) 3.G
 (of a probability space) 136.E
 (of a ring) 368.D
 anti- (of a group) 190.D
 anti- (of a ring) 368.D
 aperiodic 136.E
 entropy of 136.E
 exact (of a measure space) 136.E
 periodic (at a point) 136.E
 ring of (of an Abelian variety) 3.C
 endomorphism ring
 (of an Abelian variety) 3.C
 (of a module) 277.B 368.C
 endpoint
 (of an ordinary curve) 93.C
 left (of an interval) 355.C
 right (of an interval) 355.C
 end vertex 186.B
 energy 195.B 338.B
 binding 351.D
 free 340.B 402.G
 Gibbs free 419.C
 Helmholtz free 419.C
 internal 419.A
 kinetic 271.C 351.D
 mean 402.G
 mean free 340.B 402.G
 mutual 338.B
 potential 271.C
 rest 359.C
 total 271.C
 energy density 195.B
 energy equation (for a fluid) 205.A
 energy function 126.L 279.F
 energy inequality 325.C
 energy integral 420.A
 energy minimum principle 419.C
 Gibbs 419.C
 energy-momentum 4-vector 258.C
 energy-momentum operator 258.D
 energy-momentum tensor 150.B 359.D
 energy principle 338.D
 energy spectrum function 433.C
 energy spectrum tensor 433.C
 energy surface 126.L 402.C,G
 Eneström theorem, Kakeya- (on an algebraic equation) 10.E
 Engel theorem 248.F
 engulfing lemma 65.C
 enlargement 293.B
 Enneper formula, Weierstrass- 275.A
 Enneper surface 275.B
 Enriques surface 72.K
 ensemble 402.D
 canonical 402.D
 grand canonical 402.D,G
 microcanonical 402.D
 Enskog method 217.N
 enthalpy 419.C
 enthalpy minimum principle 419.C
 entire algebroidal function 17.B
 entire function 429.A
 rational 429.A
 transcendental 429.A
 entire linear transformation 74.E
 entrance 281.C
 entrance boundary (of a diffusion process) 115.B
 entrance boundary point 260.I
 entropy 212.B
 (of an endomorphism) 136.E
 (in information theory) 213.B
 (in statistical mechanics) 402.B
 (in thermodynamics) 419.A
 closed system 402.G
 completely positive 136.E
 conditional 213.B
 of a distribution 403.B
 of the endomorphism φ 136.E
 ε - (of source coding theorem) 213.E
 maximal 136.C,H
 mean 402.G
 open system 402.G
 of a partition 136.E
 relative 212.B
 topological 136.H
 topological, of f with respect to α 126.K
 entropy condition 204.G
 entropy conjecture 126.K
 entropy maximum principle 419.A
 entropy production, equation of 205.A
 entry (of a matrix) 269.A
 enumerable predicate, recursively 356.D
 enumerable set, recursively 356.D
 enumerating predicate 356.H
 enumeration method, implicit 215.D
 enumeration theorem 356.H
 Pólya's 66.E
 envelope
 (of a family of curves) 93.I
 of holomorphy 21.F
 injective 200.I
 lower 338.M
 upper (of a family of subharmonic functions) 193.R
 envelope power function 400.F
 enveloping algebra 231.A
 special universal (of a Jordan algebra) 231.C
 universal (of a Lie algebra) 248.J
 enveloping surface 111.I
 enveloping von Neumann algebra 36.G
 epicycloid 93.H
 epidemiology 40.E
 epimorphism (in a category) 52.D 200.Q
 epitrochoid 93.H
 Epstein zeta function 450.K
 equality
 Chapman-Kolmogorov 261.A
 Jacobe-Beihler 328
 Parseval 18.B 159.A 160.C 192.K 197.C
 200.B,C,E
 equal weight, principle of 402.D
 equation(s)
 Abelian 172.G
 adjoint differential 252.K
 algebraic 10, App. A, Table 1
 algebraic differential 113 288.A
 approximate functional (for zeta function) 450.B
 basic 320.E
 Bellman 405.B
 bifurcation 286.V
 binomial 10.C
 biquadratic App. A, Table 1
 Boltzmann 41.A 402.B

Equation(s)

- Boussinesq 387.F
 Briot-Bouquet differential 288.B
 Callan-Symanzik 361.B
 canonical 324.E
 canonical forms of the (of surfaces) 350.B
 Cauchy-Riemann (differential) 21.C 198.A
 274.G 320.F
 characteristic (for a homogeneous system of
 linear ordinary differential equations) 252.J
 characteristic (of a linear difference equation)
 104.E
 characteristic (of a linear ordinary differential
 equation) 252.E
 characteristic (of a matrix) 269.F
 characteristic (of an autonomous linear system)
 163.F
 characteristic (of a partial differential equation)
 320.D
 characteristic (of a partial differential equation
 of hyperbolic type) 325.A,F
 Chaplygin differential 326.B
 Chapman-Kolmogorov 260.A 261.A
 Charpit subsidiary 82.C 320.D
 of Codazzi 365.C
 Codazzi-Mainardi 111.H, App. A, Table 4.I
 of a conic section 78.C
 of a conic section, canonical form of 78.C
 of continuity 130.A 204.B 205.A
 cubic 10.D, App. A, Table 1
 cyclic 172.G
 delay-differential 163.A
 de Rham 274.G
 difference \rightarrow difference equation
 difference-differential 163.A
 differential \rightarrow differential equation(s)
 diffusion-convection 304.B
 Diophantine 118.A
 Dirac 377.C 415.G
 DLR 402.G
 eikonal 324.E 325.L
 energy (for a fluid) 205.A
 of entropy production 205.A
 Ernst 359.D
 estimating 399.P
 Euler (calculus of variations) 46.B
 Euler (of a perfect fluid) 204.E
 Euler, of motion (of a perfect fluid) 205.A
 Euler differential (dynamics of rigid bodies)
 271.E
 Euler-Lagrange 46.B
 evolution 378.G
 exterior field 359.D
 field 150.B
 Fokker-Planck 402.I
 functional \rightarrow functional equation
 functional-differential 163.A
 Galois 172.G
 of Gauss (on an isometric immersion) 365.C
 Gauss (on surfaces) 111.H
 Gelfand-Levitan-Marchenko 287.C 387.D
 general 172.G
 general Navier-Stokes 204.F
 Hamilton differential 324.E
 Hamilton-Jacobi 108.B
 Hamilton-Jacobi differential 271.F 324.E
 heat 327.A, App. A, Table 15.VI
 of heat conduction 327.A
 Hill determinantal 268.B
 indicial 254.C
 induction 259
 integral \rightarrow integral equation(s)
 integrodifferential 163.A 222
 integrodifferential, of Fredholm type 222.A
 integrodifferential, of Volterra type 222.A
 interior field 359.D
 Kadomtsev-Petvyashvili 387.F
 KdV 387.B
 Kepler 309.B
 Klein-Gordon 377.C 415.G
 Kolmogorov backward 115.A 260.F
 Kolmogorov forward 115.A 260.F
 Königs-Schröder 44.B
 Korteweg-de Vries 387.A
 Lagrange, of motion 271.F
 Landau 146.C
 Landau-Nakanishi 146.C
 Langevin 45.I 402.K
 Lewy-Mizohata 274.G
 likelihood 399.M
 linear 10.D
 linear, system of 269.M
 linear integral 217.A
 linear ordinary differential 252.A
 linear structure, system 128.C
 Lippman-Schwinger 375.C
 local (of a divisor on an open set) 16.M
 logistic 263.A
 master 402.I
 matrix Riccati differential 86.E
 Maxwell 130.A
 microdifferential 274.G
 minimal surface 275.A
 of motion (of a fluid) 205.A
 of motion (of a model) 264
 of motion (in Newtonian mechanics) 271.A
 of motion (of a particle in a gravitational field)
 359.D
 of motion, Euler (of a perfect fluid) 205.B
 of motion, Heisenberg 351.D
 of motion, Lagrange 271.F
 natural (of a curve) 111.D
 natural (of a surface) 110.A
 Navier-Stokes 204.B 205.C
 normal (in the method of least squares) 403.E
 302.E 397.J
 ordinary differential \rightarrow ordinary differential
 equation(s)
 of oscillation App. A, Table 15.VI
 Painlevé 288.C
 parabolic cylindrical App. A, Table 14.II
 partial differential \rightarrow partial differential
 equation(s)
 Pell 118.A
 Pfaffian 428.A
 Pfaffian, system of 428.A
 Poisson 338.A
 Prandtl boundary layer 205.C
 Prandtl integrodifferential 222.C
 pressure 205.B
 primitive 172.G
 quadratic 10.D, App. A, Table 1
 quartic 10.D, App. A, Table 1
 radial 315.E
 random Schrödinger 340.E
 of a rarefied gas 41.A
 rational differential 288.A
 reciprocal 10.C
 regular local (at an integral point) 428.E
 renewal 260.C
 renormalization 111.B

- resolvent 251.F
- retarded differential 163.A
- Ricci 365.C
- Schlesinger 253.E
- Schrödinger 351.D
- secular 55.B 269.F
- self-adjoint differential 252.K
- self-adjoint system of differential 252.K
- Sine-Gordon 387.A
- singular integral 217.J
- of sound propagation 325.A
- special functional 388
- of state 419.A
- structure (of an affine connection) 417.B
- structure (for a curvature form) 80.G
- structure (for a torsion form) 80.H
- of structure (of a Euclidean space) 111.B
- of structure (for relative components) 110.A
- system of 10.A
- system of linear ordinary differential 252.G
- telegraph 325.A, App. A, Table 15.III
- time-dependant Schrödinger 351.D
- time-independant Schrödinger 351.D
- transport 325.L
- two-dimension KdV 387.F
- variational 316.F 394.C
- of a vibrating membrane 325.A
- of a vibrating string 325.A
- wave 325.A 446.A, App. A, Table 15.III
- Wiener-Hopf integrodifferential 222.C
- Yang-Mills 80.Q
- Yule-Walker 421.D
- equator (of a sphere) 140
- equiangular spiral 93.H
- equianharmonic range of points 343.D
- equicontinuous (family of mappings) 435.D
- equicontinuous group of class (C^0) 378.C
- equicontinuous semigroup of class (C^0) 378.B
- equidistant hypersurface (in hyperbolic geometry) 285.C
- equilateral hyperbola 78.E
- equilateral hyperbolic coordinates 90.C, App. A, Table 3.V
- equilateral triangle solution 420.B
- equilibrium, Nash 173.C
- equilibrium figures 55.D
- equilibrium Gibbs distribution 136.C
- equilibrium mass distribution 338.K
- equilibrium point
 - (of a flow) 126.D
 - (in an N -person differential game) 108.C
 - (in a two-person zero-sum game) 173.C
- equilibrium potential 260.D
- equilibrium principle 338.K
- equilibrium state 136.H 340.B 419.A
- equilibrium statistical mechanics 402.A
- equilibrium system, transformation to 82.D
- equilong transformation 76.B
- equipollent sets 49.A
- equipotential surface 193.J
- equipotent sets 49.A
- equivalence
 - (of categories) 52.H
 - (in a category) 52.D
 - (of complexes) 200.H
 - (of coverings) 91.A
 - anti- (of categories) 52.H
 - chain 200.H
 - cochain 200.F
 - combinatorial 65.A
 - C^* - 126.B
 - CR- 344.A
 - homotopy 202.F
 - Kakutani 136.F
 - Lax, theorem 304.F
 - natural 52.J
 - principle of (in insurance mathematics) 214.A
 - principle of (in physics) 359.D
 - simple homotopy 65.C
 - topological 126.B
- equivalence class 135.B
 - linear (of divisors) 16.M
- equivalence properties 135.A
- equivalence relations 135 358.A
 - proper (in an analytic space) 23.E
- equivalent
 - (additive functionals) 261.E
 - (arc) 246.A
 - (coordinate bundle) 147.B
 - (covering) 91.A
 - (extension by a C^* -algebra) 36.J
 - (fiber bundle) 147.B
 - (formula) 411.E
 - (functions with respect to a subset of C^n) 21.E
 - (G -structures) 191.A
 - (knot) 235.A
 - (linear representation) 362.C
 - (measure) 225.J
 - (methods of summation) 379.L
 - (proposition) 411.B
 - (quadratic form) 348.A
 - (quadratic irrational numbers) 182.G
 - (relation) 135.B
 - (space group) 92.A
 - (stochastic process) 407.A
 - (surfaces in the sense of Fréchet) 246.I
 - (system of neighborhoods) 425.E
 - (unfolding) 51.E
 - (unitary representation) 437.A
 - (valuation) 439.B
 - (word) 31.B
- algebraically (cycles) 16.R
- algebraically, to 0 16.P
- arithmetically 92.B 276.D
- C - 114.H
- certainty 408.B
- chain 200.C
- χ - 431.F
- combinatorially 65.A
- conformally 77.A 191.B 367.A
- countably (under a nonsingular bimeasurable transformation) 136.C
- C^* - 126.B
- elementarily 276.D
- fiber homotopy (vector bundles) 237.I
- finitely (under a nonsingular bimeasurable transformation) 136.C
- flow 126.B
- Γ - (points) 122.A
- geometrically 92.B
- homotopy (systems of topological spaces) 202.F
- k - (C^∞ -manifolds) 114.F
- linearly (divisors) 16.M 72.F
- locally (G -structure) 191.H
- numerically (cycles) 16.Q
- \mathfrak{S} -linearly, divisors 9.F
- Ω - 126.H
- (PL) 65.D
- properly (binary quadratic forms) 348.M

- pseudoconformally 344.A
- quasi- (unitary representations) 437.C
- rationally (cycles) 16.R
- right 51.C
- simple homotopy 65.C
- stably (vector bundles) 237.B
- stably fiber homotopy (vector bundles) 237.I
- topologically 126.B,H
- uniformly (uniform spaces) 436.E
- unitarily (self-adjoint operators) 390.G
- equivalent affinity 7.E
- equivariant Atiyah-Singer index theorem 237.H
- equivariant cohomology 431.D
- equivariant J -group 431.F
- equivariant J -homomorphism 431.F
- equivariant K -group 237.H
- equivariant mapping (map) 431.A
- equivariant point (of a mapping) 153.B
- equivariant point index (of a mapping) 153.B
- Eratosthenes' sieve 297.B
- Erdős theorem, Chung- 342.B
- ergodic
 - (Markov chain) 260.J
 - (transformation) 136.B
- ergodic capacity 213.F
- ergodic class 260.B
 - positive recurrent 260.B
- ergodic decomposition (of a Lebesgue measure space) 136.H
- ergodic homeomorphism
 - strictly 136.H
 - uniquely 136.H
- ergodic hypothesis 136.A 402.C
- ergodic information source 213.C
- ergodic lemma, maximal 136.B
- ergodic Szemerédi theorem 136.C
- ergodic theorem 136.A,B
 - Abelian 136.B
 - dominated 136.B
 - individual 136.B
 - local 136.B
 - mean 136.B
 - multiplicative 136.B
 - pointwise 136.B
 - ratio 136.B
 - subadditive 136.B
- ergodic theory 136 342.A
- Erlang distribution, k - 260.I
- Erlangen program 137
- Ernst equation 359.D
- error(s) 138.A
 - accumulated 138.C
 - burst 63.E
 - discretization 303.B
 - of the first kind 400.A
 - of input data 138.B
 - local truncation 303.E
 - mean square 399.E 403.E
 - propagation of 138.C
 - roundoff 138.B 303.B
 - of the second kind 400.A
 - theory of 138.A
 - truncation 138.B 303.B
- error analysis 138
 - backward 302.B
- error constant 303.E
- error-correcting 63.A
- error-correcting capability 63.B
- error-detecting 63.A
- error estimate, one-step-two-half-steps 303.D
- error function 167.D, App. A, Table 19.II
- error matrix 405.G
- error probability 213.D
- error space 403.E
- error sum of squares (with $n-s$ degree of freedom) 403.E
- error term 403.D
- error vector 102.A
- essential (conformal transformation group) 364.F
- essentially bounded (measurable function) 168.B
- essentially complete class 398.B
- essentially normal 390.I
- essentially self-adjoint 251.E 390.I
- essentially singular point (with respect to an analytic set) 21.M
- essentially unitary 390.I
- essential part 260.I
- essential singularity
 - (of an analytic function in the wider sense) 198.P
 - (of a complex function) 198.D
- essential spectrum 390.E,H
- essential support (of a distribution) 274.D
- essential supremum (of a measurable function) 168.B
- Estes stimulus sampling model 346.G
- estimable (parametric function) 399.C
- estimable parameter 403.E
 - linearly 403.E
- estimate 399.B
 - a priori 323.C
 - nonrandomized 399.B
 - one-step-two-half-steps error 303.D
 - Schauder 323.C
- estimating equation 399.P
- estimating function 399.P
 - likelihood 399.M
- estimating parameters, design for 102.M
- estimation
 - Hadamard App. A, Table 8
 - interval 399.Q 401.C
 - point 371.H 399.B 401.C
 - region 399.O
 - statistical 399.A, App. A, Table 23
- estimation space 403.E
- estimator 399.B
 - asymptotically efficient 399.N
 - BAN 399.K,N
 - based on an estimating function 399.P
 - Bayes 399.F
 - best asymptotically normal 399.K
 - best invariant 399.I
 - best linear unbiased 403.E
 - CAN 399.K
 - consistent 399.K
 - consistent and asymptotically normal 399.K
 - efficient 399.D
 - first-order asymptotic efficient 399.O
 - first-order efficient 399.O
 - generalized least squares 403.E
 - invariant 399.I
 - k th-order AMU 399.O
 - k th-order asymptotically median unbiased 399.O
 - L - 371.H
 - least squares 403.E
 - M - 371.H
 - maximum likelihood 399.M
 - mean unbiased 399.C
 - median unbiased 399.C

- ML 399.M
- modal unbiased 399.C
- moment method 399.L
- Pitman 399.G
- R- 371.H
- randomized 399.B
- ratio 373.C
- state 86.E
- Stein shrinkage 399.G
- superefficient 399.N
- UMV unbiased 399.C
- unbiased 399.C
- uniformly minimum variance unbiased 399.C
- eta function
 - (of a Riemann manifold) 391.L
 - Dedekind 328.A
- étale morphism 16.F
- étale neighborhood 16.AA
- étale site 16.AA
- étale topology 16.AA
- Euclid axiom 139.A
- Euclidean algorithm 297.A
 - of polynomials 337.D
- Euclidean cell complex 70.B
- Euclidean complex 70.B
- Euclidean connection 364.B
 - manifold with 109
- Euclidean distance 139.E
- Euclidean field 150.F
- Euclidean field theory 150.F
- Euclidean geometry 139
 - n -dimensional 139.B 181
 - non- \rightarrow non-Euclidean geometry 285
 - in the wider sense 139.B
- Euclidean group, locally 423.M
- Euclidean Markov field theory 150.F
- Euclidean method 150.F
- Euclidean polyhedron 70.B
- Euclidean simplicial complex 70.B
- Euclidean space(s) 140
 - locally 259.B 425.V
 - non- 285.A
 - theorem on invariance of dimension of 117.D
- Euclidean space form 412.H
- Euclidean type (building) 13.R
- Euclid ring 67.L
- Euler, L. 141
- Euler angles 90.C, App. A, Table 3.V
- Euler characteristic (of a finite Euclidean cellular complex) 201.B
- Euler class (of M) 201.N
- Euler constant 174.A
- Euler criterion 297.H
- Euler differential equation (dynamics of rigid bodies) 271.E
- Euler equation
 - (calculus of variations) 46.B
 - (of a perfect fluid) 204.E
- Euler equation of motion (of a perfect fluid) 205.A
- Euler formula 131.G
- Euler function 295.C
- Euler graph 186.F
- Euler infinite product expansion 450.B
- Euler integral
 - of the first kind 174.C
 - of the second kind 174.A
- Euler-Lagrange differential equation 46.B
- Euler linear ordinary differential equation App. A, Table 14.I
- Euler-Maclaurin formula 379.J
- Euler method
 - of describing the motion of a fluid 205.A
 - of numerical solution of ordinary differential equations 303.E
 - summable by 379.P
 - of summation 379.P
- Euler number 177.C 201.B, App. B, Table 4
- Euler path 186.F
- Euler-Poincaré characteristic 16.E 201.B
- Euler-Poincaré class 56.B
 - (of a manifold) 56.F
 - universal 56.B
- Euler-Poincaré formula 201.B,F
- Euler polynomial 177.C
- Euler product 450.B
- Euler relation 419.B
- Euler square 241.B
- Euler summation formula 295.E
- Euler theorem on polyhedra 201.F
- Euler transformation (of infinite series) 379.I
- evaluabe (locally convex space) 424.I
- evaluation and review technique program 376
- Evans-Selberg theorem 48.E 338.H
- Evans theorem 48.E
- even element (of a Clifford algebra) 61.B
- even function 165.B
- even half-spinor 61.E
- even half-spin representation 61.E
- even-oddness conservation laws 150.D
- even permutation 151.G
- even state 415.H
- event(s) 281.D 342.B
 - complementary 342.B
 - delayed recurrent 260.C
 - elementary 342.B
 - exclusive 342.B
 - impossible 342.B
 - independent 342.B
 - inferior limit 342.B
 - intersection of 342.B
 - measurable 342.B
 - probability of an 342.B
 - with probability 1 342.B
 - product 342.B
 - random 342.B
 - recurrent 250.D 260.C
 - space of elementary 342.B
 - sum 342.B
 - superior limit 342.B
 - sure 342.B
 - symmetric 342.G
 - tail 342.B
- event commutativity 346.G
- event horizon 359.D
- Everett formula (for functions of two variables) App. A, Table 21
- Everett interpolation formula App. A, Table 21
- everywhere
 - almost 270.D 342.D
 - nearly 338.F
 - quasi- 338.F
- evolute (of a curve) 111.E
- evolution equation 378
- evolution operator 378.G
 - holomorphic 378.I
- exact
 - (additive covariant functor) 200.I
 - (differential on a Riemann surface) 367.H
 - (endomorphism) 136.E

Exact differential equation

- (in Galois cohomology) 172.J
- (in sheaf theory) 383.C
- half- 200.I
- left- 200.I
- right- 200.I
- exact differential equation App. A, Table 14.I
- exact differential form 105.Q
- exact functor 52.N
- exact sampling theory 401.F
- exact sequence
 - (of A -homomorphisms of A -modules) 277.E
 - of cohomology 200.F
 - cohomology 201.L
 - of Ext 200.G
 - fundamental (on cohomology groups) 200.M
 - Gysin (of a fiber space) 148.E
 - of homology 200.C
 - homology (of a fiber space) 148.E
 - homology (for simplicial complexes) 201.L
 - homotopy 202.L
 - homotopy (of a fiber space) 148.D
 - homotopy (of a triad) 202.M
 - homotopy (of a triple) 202.L
 - Mayer-Vietoris (for a proper triple) 201.C
 - Puppe 202.G
 - reduced homology 201.F
 - relative Mayer-Vietoris 201.L
 - (R, S) - (of modules) 200.K
 - short 200.I
 - of Tor 200.D
 - Wang (of a fiber space) 148.E
- exceptional
 - (Jordan algebra) 231.A
 - (leaf) 154.D
- exceptional compact real simple Lie algebra 248.T
- exceptional compact simple Lie group 249.L
- exceptional complex simple Lie algebra 248.S
- exceptional complex simple Lie group 249.M
- exceptional curve 15.G
 - of the first kind 15.G
 - of the second kind 15.G
- exceptional function, Julia 272.F
- exceptional orbit 431.C
- exceptional sets 192.R
- exceptional value
 - (of a transcendental entire function) 429.B
 - Borel 272.E
 - Nevanlinna 272.E
 - Picard 272.E
- excess 178.H, App. A, Table 6.III
 - coefficient of 341.H 396.C
 - spherical 432.B
 - total 178.H
- excessive (function) 260.D 261.D
 - α - 261.D
- excessive measure 261.F
- exchange 420.D
- exchange of stability 286.T
- excision isomorphism 201.F, L
- excluded middle, law of 156.C 411.L
- exclusive events 342.B
- exhaustion 178.F
- existence theorem
 - (in class field theory) 59.C
 - (for ordinary differential equations) 316.C
 - Cartan-Kähler 191.I 428.E
 - Cauchy 320.B
 - Cauchy-Kovalevskaya 321.A
- existential proposition 411.B
- existential quantifier 411.C
- exit 281.C
- exit boundary (of a diffusion process) 115.B
- exit boundary point 260.I
- exit time 261.B
- exogenous variable 128.C
- exotic sphere 114.B
- $\exp A$ (exponential function of matrix A) 269.H
- expansion 65.C
 - asymptotic 30.A, App. A, Table 17.I
 - asymptotic (of a pseudodifferential operator) 345.A
 - Cornish-Fisher 374.F
 - Edgeworth 374.F
 - ε - 361.C
 - Laurent 198.D
 - method of matched asymptotic 112.B
 - Minakshisundaram-Pleijel asymptotic 391.B
 - orthogonal 317.A
 - partial wave 375.E 386.B
 - Taylor (of an analytic function of several variables) 21.B
 - Taylor (of a holomorphic function) 339.A
 - Taylor, and remainder App. A, Table 9.IV
 - Taylor, formal 58.C
- expansion coefficient 317.A
- expansion formula, q - 134.I
- expansion method 205.B
- expansion theorem 306.B
 - Hilbert-Schmidt 217.H
 - Laplace (on determinants) 103.D
- expansive 126.J
- expectation 115.B 342.C
 - conditional 342.E
 - mathematical 341.B
- expectation value (of an operator) 351.B
- expected amount of inspection 404.C
- expected value (of a random variable) 342.C
- experiment(s)
 - design of 102
 - factorial 102.H
 - S^k factorial 102.H
 - statistical 398.G
- experimental analysis 385.A
- experimentation model 385.A
- explanatory variable 403.D
- explicit
 - (difference equation in a multistep method) 303.E
 - (Runge-Kutta method) 303.D
- explicit function 165.C
- explicit method 303.E
- explicit reciprocity laws 14.R
- explicit scheme 304.F
- exploratory procedures 397.Q
- exploring a response surface, designs for 102.M
- explosion, Ω - 126.J
- explosion time 406.D
- exponent
 - (of an Abelian extension) 172.F
 - (of an algebra class) 29.E
 - (of a finite group) 362.G
 - (of a Kummer extension) 172.F
 - (of a power) 131.B
 - (of a regular singular point) 254.C
 - (of a stable distribution) 341.G
 - characteristic (of an autonomous linear system) 163.F
 - characteristic (of the Hill differential equation) 268.B
 - characteristic (of a variational equation) 394.C

- conjugate 168.C
- of convergence 429.B
- critical 111.C
- integral 167.D
- one-sided stable process of 5.F
- of the stable process 5.F
- subordinator of 5.F
- exponential curve 93.H
- exponential dichotomy 290.B
- exponential distribution 341.D, App. A, Table 22
- two-sided App. A, Table 22
- exponential family of distributions 396.G
- exponential formula 286.X
- double 299.B
- exponential function 131.D
- with the base a 131.B
- of an operator 306.C
- exponential generating function 177.A
- exponential group 437.U
- exponential Hilbert space 377.D
- exponential integral 167.D, App. A, Table 19.II
- exponential lattice 287.A
- exponential law (on cardinal numbers) 49.C
- exponentially decreasing Fourier hyperfunction 125.BB
- exponentially decreasing real analytic function 125.BB
- exponentially stable 163.G 394.B
- exponential mapping
 - (of a Lie algebra into a Lie group) 249.Q
 - (of a Riemannian manifold) 178.A 364.C
- exponential method, Borel 379.O
- exponential series 131.D
- exponential valuation 439.B
- p -adic 439.F
- exposed, strongly 443.H
- expression
 - field of rational 337.H
 - rational 337.H
- $\exp x$ 131.D
- Ext 200.G
- exact sequence of 200.G
- Ext groups 200.G
- $\text{Ext}_R^n(A, B)$ 200.K
- $\text{Ext}_R^n(M, N)$ 200.G
- extended Dynkin diagram App. A, Table 5.I
- extended hypergeometric function, Barnes 206.C
- extended real number 87.E
- extension
 - (of a connection) 80.F
 - (of a field) 149.B
 - (of a fractional ideal) 14.I
 - (of a group) 190.N
 - (of an ideal of compact operators) 36.J 390.J
 - (of an isomorphism of subfields) 149.D
 - (of a mapping) 381.C
 - (of modules) 200.K
 - (of an operator) 251.B
 - (of a solution of an ordinary differential equation) 316.C
 - (of a valuation) 439.B
 - Abelian (of a field) 172.B
 - algebra 200.L
 - algebraic (of a field) 149.E
 - Artin-Schreier (of a field) 172.F
 - basic \mathbb{Z}_l - 14.L
 - central (of a group) 190.N
 - of the coefficient ring 29.A
 - cone 65.D
 - cyclic (of a field) 172.B
 - cyclotomic \mathbb{Z}_l - 14.L
 - elementary 276.D
 - finite 149.F
 - Friedrichs 112.I 251.E
 - Galois 172.B
 - Γ - 14.L
 - group 200.M
 - inseparable (of a field) 149.H
 - Kummer (of a field) 172.F
 - Lebesgue 270.D
 - linear (of a rational mapping to an Abelian variety) 9.E
 - maximal Abelian 257.F
 - maximal separable (of a field) 149.H
 - natural (of an endomorphism) 136.E
 - normal 149.G 251.K
 - p - (of a field) 59.F
 - p -adic 439.F
 - purely inseparable (of a field) 149.H
 - Pythagorean (of a field) 155.C
 - regular (of a field) 149.K
 - scalar (of an algebra) 29.A
 - scalar (of an A -module) 277.L
 - scalar (of a linear representation) 362.F
 - separable (of a field) 149.H, K
 - separably generated (of a field) 149.K
 - simple (of a field) 149.D
 - split (of a group) 190.N
 - strong (of a differential operator) 112.E, F
 - transcendental 149.E
 - transitive (of a permutation group) 151.H
 - unramified 14.I 257.D
 - weak (of a differential operator) 112.E, F
 - \mathbb{Z}_l - 14.L
- extensionality, axiom of (in axiomatic set theory) 33.B
- extension field 149.B
- Picard-Vessiot 113
- strongly normal 113
- extension theorem
 - first (in the theory of obstructions) 305.B
 - Hahn-Banach 37.F
 - Hopf (in measure) 270.E
 - Kolmogorov 341.I
 - second (in the theory of obstructions) 305.C
 - third (in the theory of obstructions) 305.C
 - Tietze 425.Q
 - Whitney 168.B
- extensive thermodynamical quantity 419.A
- exterior
 - (of an angle) 155.B
 - (of a polygon) 155.F
 - (of a segment) 155.B
 - (of a subset) 425.N
- exterior algebra (of a linear space) 256.O
- exterior capacity, Newtonian 48.H
- exterior derivative (of a differential form) 105.Q
- exterior differential form of degree r 105.Q
- exterior field equation 359
- exterior point (of a subset) 425.N
- exterior power, p -fold
 - (of a linear space) 256.O
 - (of a vector bundle) 147.F
- exterior problem (for the Dirichlet problem) 120.A
- exterior product
 - (of differential forms) 105.Q
 - (of elements of a linear space) 256.O
 - (of a p -vector and a q -vector) 256.O
 - (of two vectors) 442.C
- external (in nonstandard analysis) 293.B

External irregular point

external irregular point 338.L
 external language 75.C
 external law of composition 409.A
 externally stable set 186.I
 external product 200.K
 external space (in the static model in catastrophe theory) 51.B
 external variable 264
 extinction probability 44.B
 extrapolation 176.K 223.A
 extrapolation method
 polynomial 303.F
 rational 303.F
 extremal (Jordan curve) 275.C
 extremal distance 143.B
 reduced 143.B
 extremal function, Koebe 438.C
 extremal horizontal slit mapping 367.G
 extremal length 143
 defined by Hersch and Pfluger 143.A
 with weight 143.B
 extremal quasiconformal mapping 352.C
 extremal vertical slit mapping 367.G
 extremely disconnected 37.M
 extreme point
 of a convex set 89.A
 of a subset of a linear space 424.V
 extremum
 conditional relative (of a function) 106.L
 relative (of a function) 106.L

F

$\bar{\imath}$ (cardinal number of all real-valued functions on $[0, 1]$) 49.A
 F_q (finite field with q elements) 450.Q
 f -metric 136.F
 f_N -metric 136.F
 F -compactification 207.C
 F -distribution 341.D 374.B, App. A, Table 22
 noncentral 374.B
 F -free (compact oriented G -manifold) 431.E
 F -test 400.G
 (F, F') -free (compact oriented G -manifold) 431.G
 F_σ set 270.C
 (F) -space 424.I
 face
 (of a complex) 13.R
 (of a convex cell) 7.D
 i th (of a singular q -simplex) 201.E
 q - (of an n -simplex) 70.B
 face operator (in a semisimplicial complex) 70.E
 factor
 (of an element of a ring) 67.H
 (of a factorial experiment) 102.H
 (of a von Neumann algebra) 308.F
 of automorphy 32.A
 composition (in a composition factor series in a group) 190.G
 determinant (of a matrix) 269.E
 direct (of a direct product of sets) 381.E
 direct (of a group) 190.L
 Dirichlet discontinuous App. A, Table 9.V
 first (of a class number) 14.L
 integrating App. A, Table 14.I
 Krieger 308.I
 p - (of an element of a group) 362.I
 Powers 308.I
 proper (of an element of a ring) 67.H
 second (of a class number) 14.L

Ulm (of an Abelian p -group) 2.D
 factor A -module 277.C
 factor analysis 280.G
 factor analysis model 403.C
 factor group 190.C
 factorial 330, App. A, Table 17.II
 Jordan 330
 factorial cumulant 397.G
 factorial cumulant generating function 397.G
 factorial design 102.H
 balanced fractional 102.I
 fractional 102.I
 orthogonal fractional 102.I
 factorial effect 102.H
 factorial experiment 102.H
 S^k 102.H
 factorial function 174.A
 factorial moment 397.G
 factorial moment generating functions 397.G
 factorial series 104.F 121.E
 factorization
 incomplete 302.C
 regular 251.N
 triangular 302.B
 factorization method 206.B
 factorization theorem
 (of an H^p -function) 43.F
 Neyman 396.F
 unique (in an integral domain) 67.H
 factor loading 280.G 346.F
 factor representation
 of a topological group 437.E
 of type I, II, or III 437.E
 factor ring 368.E, F
 factor score 280.G 346.F
 factor set(s)
 (in cohomology of groups) 200.M
 (of a crossed product) 29.D
 (in extensions of groups) 190.N
 (of a projective representation) 362.J
 associated (in extensions of groups) 190.N
 factor transformation (of a measure-preserving transformation) 136.D
 Faddeev-Popov ghost 132.C 150.G
 fading memory 163.I
 faithful
 (funcator) 52.H
 (linear representation) 362.C
 (permutation representation) 362.B
 (weight on a von Neumann algebra) 308.D
 fully (funcator) 52.H
 faithfully flat A -module 277.K
 faithfully flat morphism 16.D
 false 411.E
 regular 301.C
 false position, method of 301.C
 family 165.D
 algebraic (of cycles on an algebraic variety) 16.R
 of compact complex manifolds 72.G
 of confocal central conics 78.H
 of confocal parabolas 78.H
 of confocal quadrics 350.E
 covering 16.AA
 crystal 92.B
 directed 165.D
 equicontinuous (of mappings) 435.D
 exponential, of distributions 396.G
 of frames (on a homogeneous space) 110.A
 of frames of order 1 110.B

- of functions 165.B,D
 - of functions indexed by a set 165.B
 - indexed by a set 165.D
 - of mappings 165.D
 - normal (of functions) 435.E
 - parameter space 72.G
 - of points 165.D
 - of quasi-analytic functions 58.A
 - quasinormal (of analytic functions) 435.E
 - separating 207.C
 - of sets 165.D 381.B,D
 - of sets indexed by a set 381.D
 - tight (of probability measures) 341.F
 - uniform, of neighborhood system 436.D
- fan 16.Z
- Fannes-Verbeure inequality, Roebstorff- 402.G
- FANR 382.C
- FAR 382.C
- Farcy arc 4.B
- Farey dissection 4.B
- Farey sequence 4.B
- Farkas theorem, Minkowski- 255.B
- fast Fourier transform 142.D
- fast wave 259
- Fatou theorem
 - on bounded functions in a disk 43.D
 - on the Lebesgue integral 221.C
- favorable a priori distribution, least 398.H
- F.D. generator 136.E
- F.D. process 136.E
- feasible (flow) 281.B
- feasible directions, method of (in nonlinear programming) 292.E
- feasible region 264.B 292.A
- feasible solution 255.A 264.B
 - basic 255.A
- feedback control 405.C
- Fefferman-Stein decomposition 168.B
- Feit-Thompson theorem (on finite groups) 151.D
- Fejér kernel 159.C
- Fejér mean 159.C
- Fejér theorem 159.C
- Feller process 261.B
- Feller transition function 261.B
- Fermat, P. de 144
 - last theorem of 145
- Fermat number 297.F
- Fermat principle 180.A 441.C
- Fermat problem 145
- Fermat theorem 297.G
- fermions 132.A 351.H
- Fermi particle 132.A
- Fermi statistics 377.B 402.E
- Ferrari formula, Dixon- App. A, Table 19.IV
- Ferrari formula App. A, Table 1
- Feynman amplitude 146.B
- Feynman diagram 146.B
- Feynman graphs 146.A,B 386.C
- Feynman integrals 146
- Feynman-Kac formula 351.F
- Feynman-Kac-Nelson formula 150.F
- Feynman rule 146.A,B
- F.F. 136.F
- FFT 142.D
- fiber
 - (of a fiber bundle) 147.B
 - (of a fiber space) 148.B
 - (of a morphism) 16.D
 - geometric (of a morphism) 16.D
 - integration along (of a hyperfunction) 274.E
- fiber bundle(s) 147
 - associated 147.D
 - of class C^r 147.O
 - complex analytic 147.O
 - equivalent 147.B
 - isomorphic 147.B
 - orientable 147.L
 - principal 147.C
 - real analytic 147.O
- fibered manifold 428.F
- fiber homotopy equivalent 237.I
 - stably 237.I
- fiber homotopy type, spherical G - 431.F
- fibering, Hopf 147.E
- fiber mapping (map), linear 114.D
- fiber product 52.G
- fiber space(s) 72.I 148
 - algebraic 72.I
 - locally trivial 148.B
 - n -connective 148.D
 - spectral sequence of (of singular cohomology) 148.E
 - Spivak normal 114.J
- fiber sum 52.G
- fiber term (of a spectral sequence) 200.J
- Fibonacci sequence 295.A
- fibration (of a topological space) 148.B
- fibration, Milnor 418.D
- fictitious state 260.F
- fiducial distribution 401.F
- fiducial interval 401.F
- field(s) 149
 - (of sets) 270.B
 - (of stationary curves) 46.C
 - absolute class 59.A
 - algebraically closed 149.I
 - algebraic function, in n variables 149.I
 - algebraic number 14.B
 - alternative 231.A
 - Anosov vector 126.J
 - Archimedean ordered 149.I
 - asymptotic 150.D
 - Axiom A vector 126.J
 - basic (of a linear space) 256.A
 - Borel 270.B,C
 - C_1 - 118.F
 - $C_i(d)$ - 118.F
 - canonical 377.C
 - class 59.B
 - coefficient (of an affine space) 7.A
 - coefficient (of an algebra) 29.A
 - coefficient (of a projective space) 343.C
 - coefficient (of a semilocal ring) 284.D
 - commutative 368.B
 - composite 149.D
 - conjugate 149.J 377.C
 - cyclotomic 14.L
 - decomposition (of a prime ideal) 14.K
 - of definition (for an algebraic variety) 16.A
 - differential 113
 - electric 130.B
 - Euclidean 150.F
 - extension 149.B
 - finite 149.C
 - formal power series, in one variable 370.A
 - of formal power series in one variable 370.A
 - formally real 149.N
 - free 150.A
 - free Dirac 377.C
 - free scalar 377.C

- function 16.A
- Galois 149.M
- Galois theory of differential 113
- ground (of an algebra) 29.A
- ground (of a linear space) 256.A
- Hamiltonian vector 126.L 219.C
- holomorphic vector 72.A
- imaginary quadratic 347.A
- imperfect 149.H
- inertia (of a prime ideal) 14.K
- intermediate 149.D
- invariant 172.B
- Jacobi 178.A
- Lagrangian vector 126.L
- local 257.A
- local class 257.A
- local class, theory 59.G
- linearly disjoint 149.K
- magnetic 130.B
- of moduli 73.B
- Morse-Smale vector 126.J
- noncommutative 149.A
- number 149.C
- ordered 149.N
- p-adic number 257.A 439.F
- perfect 149.H
- Picard-Vessiot extension 113
- power series, in one variable 370.A
- prime 149.B
- Pythagorean 139.B 155.C
- Pythagorean ordered 60.O
- quadratic 347.A
- quasi-algebraically closed 118.F
- of quotients 67.G
- ramification (of a prime ideal) 14.K
- random 407.B
- of rational expressions 337.H
- rational function, in n variables 149.K
- of rational functions 337.H
- real 149.N
- real closed 149.N
- real quadratic 347.A
- relative algebraic number 14.I
- residue class 149.C 368.F
- residue class (of a valuation) 439.B
- scalar 108.O
- scalar (in a 3-dimensional Euclidean space) 442.D
- of scalars (of a linear space) 256.A
- skew 149.A 368.B
- splitting (for an algebra) 362.F
- splitting (for an algebraic torus) 13.D
- splitting (of a polynomial) 149.G
- strongly normal extension 113
- tension 195.B
- tensor \rightarrow tensor field
- topological 423.P
- totally imaginary 14.F
- totally real 14.F
- transversal 136.G
- vector (in a differentiable manifold) 108.M
- vector (in a 3-dimensional Euclidean space) 442.D
- Wightman 150.D
- Yang-Mills 150.G
- field equation 150.B
 - exterior 339.D
 - interior 339.D
- field theory 150
 - constructive 150.F
 - Euclidean 150.F
 - Markov 150.F
 - nonsymmetric unified 434.C
 - quantum 150.C
 - unified 434
 - unitary 434.C
- fifth postulate (in Euclidean geometry) 139.A
- fifth problem of Hilbert 423.N
- figure(s) 137
 - absolute (in the Erlangen program) 137
 - central 420.B
 - equilibrium 55.D
 - fundamental (in a projective space) 343.B
 - linear fundamental 343.B
 - P^r - 343.B
- file 96.B
- filing, inverted, scheme 96.F
- fill-in 302.E
- filter 87.I
 - Cauchy (on a uniform space) 436.G
 - Kalman 86.E
 - Kalman-Bucy 86.E 405.G
 - linear 405.F
 - maximal 87.I
 - nonlinear 405.F,H
 - Wiener 86.E
- filter base 87.I
- filtering 395.E
 - stochastic 342.A 405.F
- filtration 200.J
 - discrete 200.J
 - exhaustive 200.J
- filtration bounded from below 200.J
- filtration degree 200.J
- final object 52.D
- final set
 - (of a correspondence) 358.B
 - (of a linear operator) 251.E
- final state 31.B
- finely continuous 261.C
- finely open (set) 261.D
- fine moduli scheme 16.W
- finer relation 135.C
- finer topology 425.H
- fine topology (on a class of measures) 261.D 338.E
- finitary standpoints 156.D
- finite
 - (cell complex) 70.D
 - (measure) 270.D
 - (morphism) 16.D
 - (potency) 49.A
 - (simplicial complex) 70.C
 - (triangulation) 70.C
 - (von Neumann algebra) 308.E
 - approximately 36.H 308.I
 - geometrically 234.C
 - hyper- 308.I
 - locally \rightarrow locally finite
 - point- (covering) 425.R
 - pro-, group 210.C
 - semi- 308.I
 - σ - \rightarrow σ -finite
 - star- (covering) 425.R
- finite automaton 31.D
- finite-band potentials 387.E
- finite basis (for an ideal) 67.B
- finite branch (of a curve of class C^k) 93.G
- finite character, condition of 34.C

- finite cochain (of a locally finite simplicial complex) 201.P
- finite continued fraction 83.A
- finite covering (of a set) 425.R
- finite differences 223.C
- finite-dimensional distribution 407.A
- finite-dimensional linear space 256.C
- finite-dimensional projective geometry 343.B
- finite-displacement theory 271.G
- finite element method 233.G 290.E 304.C
- finite extension 149.F
- finite field 149.C
- finite field F_q 149.M
- finite-gap potentials 387.E
- finite groups 151.A 190.C
- finite intersection property 425.S
- finite interval (in \mathbf{R}) 355.C
- finite length 277.I
- finitely additive (vector measure) 443.G
- finitely additive class 270.B
- finitely additive measure 270.D
- finitely additive set function 380.B
- finitely determined process 136.E
- finitely distinguishable (hypothesis) 400.K
- finitely equivalent sets (under a nonsingular bi-measurable transformation) 136.C
- finitely fixed 136.F
- finitely generated
(A -module) 277.D
(group) 190.C
- finitely presented (group) 161.A
- finite memory channel 213.F
- finiteness condition for integral extensions 284.F
- finiteness theorem 16.AA
Ahlfors 234.D
- finite order (distribution) 125.J
- finite ordinal number 312.B
- finite part (of an integral) 125.C
- finite population 373.A
- finite presentation 16.E
- finite prime divisor 439.H
- finite projective plane 241.B
- finite rank (bounded linear operator) 68.C
- finite sequence 165.D
- finite series 379.A, App. A, Table 10.I
- finite set 49.A 381.A
hereditary 33.B
- finite subset property 396.F
- finite sum, orthogonality for a 19.G 317.D, App. A, Table 20.VII
- finite type
(graded module) 203.B
(module) 277.D
(morphism of schemes) 16.D
(\mathcal{O} -module) 16.E
algebraic space of 16.W
locally of 16.D
subshift of 126.J
- finite-type power series space 168.B
- finite-valued function 443.B
- finitistic (topological space) 431.B
- Finsler manifold 286.L
- Finsler metric 152.A
- Finsler space 152
- firmware 75.C
- first axiom, Tietze 425.Q
- first boundary value problem 193.F 323.C
- first category, set of 425.N
- first classification theorem (in theory of obstructions) 305.B
- first complementary law of the Legendre symbol 297.I
- first countability axiom 425.P
- first definition (of algebraic K -group) 237.J
- first extension theorem (in the theory of obstructions) 305.B
- first factor (of a class number) 14.L
- first fundamental form (of a hypersurface) 111.G
- first fundamental quantities (of a surface) 111.H
- first fundamental theorem (Morse theory) 279.D
- first homotopy theorem (in the theory of obstructions) 305.B
- first incompleteness theorem 185.C
- first-in first-out memory 96.E
- first-in last-out memory 96.E
- first integral (of a completely integrable system) 428.D
- first isomorphism theorem (on topological groups) 423.J
- first kind
(integral equations of Fredholm type of the) 217.A
Abelian differential of 11.C
Abelian integral of 11.C
differential form of 16.O
- first law of cosines 432.A, App. A, Table 2.II
- first law of thermodynamics 419.A
- first maximum principle (in potential theory) 338.C
- first mean value theorem (for the Riemann integral) 216.B
- first negative prolongational limit set 126.D
- first-order asymptotic efficient estimator 399.O
- first-order derivatives 106.A
- first-order designs 102.M
- first-order efficient estimator 399.O
- first-order predicate 411.K
- first-order predicate logic 411.K
- first positive prolongational limit set 126.D
- first problem, Cousin 21.K
- first prolongation (of P) 191.E
- first quadrant (of a spectral sequence) 200.J
- first quartile 396.C
- first regular integral 126.H
- first-return mapping (map) 126.B
- first separation axiom 425.Q
- first variation 46.B
- first variation formula 178.A
- Fisher consistent 399.K
- Fisher expansions, Cornish- 374.F
- Fisher inequality 102.E
- Fisher information 399.D
- Fisher information matrix 399.D
- Fisher problem, Behrens- 400.G
- Fisher theorem 43.G
- Fisher theorem, Riesz- 168.B 317.A
- Fisher three principles 102.A
- Fisher-Yates-Terry normal score test 371.C
- Fisher z -transformation 374.D
- fish-eye, Maxwell 180.A
- fit, chi-square test of goodness of 400.K
- fit, goodness of 397.Q
- fitting, curve 19.F
- five-disk theorem, Ahlfors 272.J
- fixed branch points (of an algebraic differential equation) 288.A
- fixed component (of a linear system) 16.N
- fixed effect 102.A
- fixed-effect model 102.A
- fixed point
(of a discontinuous group) 122.A

Fixed-point index (of a continuous mapping)

- (of a flow) 126.D
- (of a mapping) 153.A,D
- (of a transformation group) 431.A
- of discontinuity (of an additive process) 5.B
- of discontinuity (of a stochastic process) 407.A
- hyperbolic 126.G
- isolated 126.G
- fixed-point index (of a continuous mapping) 153.B
- fixed point method 138.B
- fixed-point theorem(s) 153
 - Atiyah-Bott 153.C
 - Atiyah-Singer 153.C
 - Brouwer 153.B
 - Kakutani 153.D
 - Lefschetz 153.B
 - Leray-Schauder 286.D
 - Poincaré-Birkoff 153.B
 - Schauder 153.D 286.D
 - Tikhonov 153.D
- fixed singularity (of an algebraic differential equation) 288.A
- fixed variates 403.D
- fixed vector 442.A
- FKG 212.A
- Fl (flèche) 52.A
- flabby resolution 125.W
- flabby sheaf 383.E
- flag (in an affine space) 139.B
- flag manifold 199.B
 - proper 199.B
- flat
 - (connection) 80.E
 - (morphism of schemes) 16.D
 - (Riemannian manifold) 364.E
 - (sphere pair) 235.G
 - conformally 191.B
 - ε - 178.D
 - faithfully (A -module) 277.K
 - faithfully (morphism of schemes) 16.D
 - locally (connection) 80.E
 - locally (PL embedding) 65.D
 - locally (Riemannian manifold) 364.E
 - normally (along a subscheme) 16.L
- flat A -module 277.K
- flat deformation 16.W
- flat F -bundle 154.B
- flat function 58.C
- flat point (of a surface) 111.H
- flat site 16.AA
- flat space
 - concurrently App. A, Table 4.II
 - conformally App. A, Table 4.II
 - projectively App. A, Table 4.II
- flavor dynamics, quantum 132.D
- flex 9.B
- flip model, spin 340.C
- floating point method 138.B
- Floquet theorem 252.J 268.B
- flow
 - (in ergodic theory) 136.D
 - (on a network) 281.B
 - (on a topological space) 126.B
 - Anosov 126.B 136.G
 - associated 136.F
 - Axiom A 126.J
 - built under a function 136.D
 - C^* - 126.B
 - of class C^* 126.B
 - continuous 126.B
 - discrete 126.B
 - equivalent 126.B
 - geodesic 126.L 136.G
 - harmonic 193.K
 - homotropic 205.B
 - horocycle 136.G
 - hypersonic 205.C
 - K - 136.E
 - Kolmogorov 136.E
 - Kronecker 136.G
 - laminar 205.E 433.A
 - maximum, minimum cut theorem 281
 - maximum, problem 281
 - measurable 136.D
 - minimal 126.N
 - minimum-cost, problem 281.C
 - Morse-Smale 126.J
 - multicommodity, problem 281.F
 - network, model 307.C
 - network, problem 281 282.B
 - S - 136.D
 - single-commodity, problem 281.F
 - special 136.D
 - translational 126.L 136.G
 - transonic 205.B
 - transversal 136.G
 - turbulent 205.E 433.A
 - Y -, 136.G
- flow-shop scheduling problem 376
- fluctuation-dissipation theorem 402.K
- fluid 205.A
 - compressible 205.B
 - incompressible 205.B
 - Newtonian 205.C
 - non-Newtonian 205.C
 - perfect 205.B
- fluid dynamics 205.A
- flux
 - (of a regular tube) 193.K
 - vector (through a surface) 442.D
- flux density
 - electric 130.A
 - magnetic 130.A
- focal conic (of a quadric) 350.E
- focal length 180.B
- focal point (of a submanifold of a Riemannian manifold) 364.C
- Fock representation 150.C
- Fock space 377.A
 - antisymmetric 377.A
 - symmetric 377.A
- focus
 - (of a conic section ellipse) 78.B
 - (of an optimal system) 180.B
 - (of a quadric) 350.E
- Fokker-Planck partial differential equation 115.A 402.I
- Foiaş model, Sz. Nagy- 251.N
- folding (of a chamber complex) 13.R
- foliated bundle 154.B,H
- foliated cobordant (C^∞ -foliations) 154.H
- foliated cobordism 154.H
- foliated structure 105.Y
- foliation 154
 - Anosov 126.J
 - compact 154.H
 - C^* - 154.B,G
 - Γ_ϕ - 154.H
 - holomorphic 154.H
 - real analytic 154.H
 - Reeb 154.B

- Riemannian 154.H
- transverse to 154.H
- foliation cycles 154.H
- folium cartesii 93.H
- folium of Descartes 93.H
- foot of the perpendicular 139.E
- force
 - apparent 271.D
 - body 271.G
 - centrifugal 271.D
 - Corioli 271.D
 - line of 193.J
 - Lorentz 130.A
 - restitutive 318.B
- forced oscillation 318.B
- force polygon 19.C
- Ford fundamental region 234.C
- forgetful functor 52.I
- form(s) 337.B
 - anti-Hermitian 256.Q
 - associated (of a projective variety) 16.S
 - automorphic 437.DD 450.O
 - automorphic, of weight k (or of dimension $-k$) 32.B
 - automorphic, of weight m 32.A
 - basic (in linear programming) 255.A
 - bilinear 256.H 277.J 424.G
 - bilinear, associated with a quadratic form 256.H
 - canonical (of $F(M)$) 191.A
 - canonical (of a linear hypothesis) 400.H
 - canonical, of the equation (of a quadric) 350.B
 - canonical 1- (of the bundle of tangent n -frames) 80.H
 - Cantor normal 312.C
 - compact (of a complex semisimple Lie algebra) 248.P
 - complex (of a Fourier series) 159.A
 - complex (of a real Lie algebra) 248.P
 - complex space 365.L
 - connection 80.E 417.B
 - contact 110.E
 - covariant of n -ary, of degree d 226.D
 - covariant with ground 226.D
 - curvature 80.G 364.D
 - cusp (in the case of one variable) 32.B
 - cusp (in Siegel upper half-space) 32.F
 - differential \rightarrow differential form
 - Dirichlet 261.C
 - divergence 323.D
 - dominant integral (on a Cartan subalgebra) 248.W
 - ε -Hermitian 60.O
 - ε -trace 60.O
 - Euclidean space 412.H
 - first fundamental 111.G, App. A, Table 4.I
 - formula of embedding 303.D
 - Fuchsian, of weight k (or of dimension $-k$) 32.B
 - fundamental (associated with a Hermitian metric) 232.A
 - fundamental (of a Finsler space) 152.A
 - games in partition-function 173.D
 - generalized Levi 274.G
 - ground 226.D
 - Hermitian \rightarrow Hermitian form 256.Q
 - Hesse normal (of a hyperplane) 139.H
 - Hilbert modular, of dimension $-k$ 32.G
 - Hilbert modular, of weight k 32.G
 - holomorphic k - 72.A
 - hyperbolic space 412.H
 - integral (on a Cartan subalgebra) 248.W
 - invariants of n -ary, of degree d 226.D
 - Jordan normal 269.G
 - k - (of an algebraic group) 13.M
 - kernel 348.F
 - Khinchin canonical 341.G
 - Killing 248.B
 - Kolmogorov canonical 341.G
 - Legendre-Jacobi standard 134.A, App. A, Table 16.I
 - Levi 344.A
 - Lévy canonical 341.G
 - limit of an indeterminate 106.E
 - linear (on an A -module) 277.E
 - linear (on a linear space) 256.B
 - modular, of level N 32.C
 - multilinear 256.H
 - n -person 173.B-D
 - norm 118.D
 - normal (of an ordinal number) 312.C
 - normal (of an ordinary differential equation) 313.B
 - normal (of partial differential equations) 321.B
 - normal (of a partial differential equation of the first order) 324.E
 - normal (of a surface) 410.B
 - normal real (of a complex semisimple Lie algebra) 248.Q
 - normic (in a field) 118.F
 - Pfaffian 105.Q 428.A
 - polar (of a complex number) 74.C
 - primitive 232.C
 - pseudotensorial 80.G
 - quadratic \rightarrow quadratic form
 - real (of a complex algebraic group) 60.O
 - real (of a complex Lie algebra) 248.P
 - reduced (of a linear structural equation system) 128.C
 - regular Dirichlet 261.C
 - second fundamental 111.G 365.C, App. A, Table 4.I
 - sesquilinear \rightarrow sesquilinear form
 - Siegel modular, of dimension $-k$ 32.F
 - Siegel modular, of weight k 32.F
 - skew-Hermitian 256.Q
 - skew-symmetric multilinear 256.H
 - space 285.E 412.H
 - spherical space 412.H
 - standard (of a difference equation) 104.C
 - standard (of a latin square) 241.A
 - symmetric multilinear 256.H
 - symplectic 126.L
 - tensorial 80.G
 - third fundamental App. A, Table 4.I
 - torsion 80.H
 - Weierstrass canonical (for an elliptic curve) 9.D
 - Weierstrass canonical (of the gamma function) 174.A
 - Weyl 351.C
 - formal adjoint operator 322.E
 - formal degree (of a unitary representation) 437.M
 - formal dimension n , Poincaré pair of 114.J
 - formal geometry 16.X
 - formal group 13.C
 - formalism 156.A,D
 - Gupta-Bleuler 150.G
 - formally real field 149.N
 - formally self-adjoint (differential operator) 112.I

- formally undecidable proposition 185.C
- formal power series 370.A
 - field of, in one variable 370.A
 - rings of 370.A
- formal power series field in one variable 370.A
- formal power series ring 370.A
- formal scheme 16.X
 - separated 16.X
- formal solution (for a system of ordinary differential equations) 289.C
- formal spectrum (of a Noetherian ring) 16.X
- formal system 156.D 411.I
- formal Taylor expansion 58.C
- formal vector fields 105.AA
- formation
 - class 59.H
 - pattern 263.D
- formation, class 59.H
- formation rule 411.D
- form ring 284.D
- formula(s) 411.D
 - Abramov 136.E
 - addition (for e^z) 131.G
 - addition (for sine and cosine) 432.A
 - Adem App.A, Table 6.II
 - algebraic addition 3.M
 - atomic 411.D
 - atomic (of a language) 276.A
 - Bayes 342.F 405.I
 - Bessel interpolation App. A, Table 21
 - Binet 174.A 295.A
 - Bouquet (on space curves) 111.F
 - Campbell-Hausdorff 249.R
 - Cardano App. A, Table 1
 - Cartan (for Steenrod p th power operations) 64.B
 - Cartan (for Steenrod square operations) 64.B
 - Cauchy-Hadamard (on the radius of convergence) 339.A
 - Cauchy integral 198.B
 - Chebyshev (in numerical integration) 299.A
 - Chern (in integral geometry) 218.D
 - Christoffel-Darboux 317.D
 - Clenshaw-Curtis 299.A
 - closed 411.J
 - closed (of a language) 276.A
 - closed type (in numerical integration) 299.A
 - connection 253.A
 - constant variational 163.E
 - Crofton (in integral geometry) 218.B
 - decomposition, of Radon 125.CC
 - De Moivre 74.C
 - dimensional 116
 - discontinuity 146.C 386.C
 - Dixon-Ferrar App. A, Table 19.IV
 - double exponential 299.B
 - Dynkin 261.C
 - of embedding form 303.D
 - empirical 19.F
 - Euler (for $\cos z, \sin z, \cosh z$) 131.G
 - Euler (for e^{by}) 131.G
 - Euler-Maclaurin 379.J
 - Euler-Poincaré (for a finite Euclidean cellular complex) 201.B,F
 - Euler summation 295.E
 - Everett App. A, Table 21
 - Everett interpolation 224.B, App. A, Table 21
 - exponential 286.X
 - Ferrari App. A, Table 1
 - Feynman-Kac 315.F,G
 - Feynman-Kac-Nelson 150.G
 - first variation 178.A
 - Fourier inversion 160.C
 - Fredholm 68.L
 - Frenet (on curves) 111.D
 - Frenet-Serret (on curves) 111.D
 - Gauss (on Gauss sum) 295.D
 - Gauss (on harmonic functions) 193.D
 - Gauss (for integration of a vector field) App. A, Table 3.III
 - Gauss (for isometric immersion) 365.C
 - Gauss (in numerical integration) 299.A
 - Gauss (for the surface integral) 94.F
 - Gauss (in theory of surfaces) 111.H, App. A, Table 4.I
 - Gauss backward interpolation 223.C
 - Gauss-Bonnet 111.H 364.D, App. A, Table 4.I
 - Gauss-Bonnet-Sasaki-Nitsche 275.C
 - Gauss-Chebyshev (in numerical integration) 299.A
 - Gauss forward interpolation 223.C
 - Gauss-Hermite (in numerical integration) 299.A
 - Gauss integration (in the narrow sense) 299.A
 - Gauss interpolation App. A, Table 21
 - Gauss-Laguerre (in numerical integration) 299.A
 - Green (for differential operators) App. A, Table 15.VI
 - Green (for harmonic functions) 193.D
 - Green (for Laplace operator) App. A, Tables 3.III 4.II
 - Green (for ordinary differential equations) 252.K
 - Green (for partial differential equations of parabolic type) 327.D
 - Green (on the plane) 94.F
 - Green-Stokes 94.F
 - Hansen-Bessel App. A, Table 19.III
 - Heron (for plane triangles) App. A, Table 2.II
 - Heron (for spherical triangles) App. A, Table 2.III
 - identically true 411.G
 - IMT 299.B
 - interpolation 223.A
 - interpolatory 299.A
 - inversion (for a characteristic function) 341.C
 - inversion (of cosine transform) 160.C
 - inversion (of Fourier transform) 160.C
 - inversion (of Fourier transform of distributions) 160.H
 - inversion (of Fourier transform on a locally compact Abelian group) 192.K
 - inversion (of generalized Fourier transform) 220.B
 - inversion (of Hilbert transform) 220.E
 - inversion (of integral transform) 220.A
 - inversion (of Laplace-Stieltjes transform) 240.D
 - inversion (on a locally compact group) 437.L
 - inversion (of Mellin transform) 220.C
 - inversion (for a semigroup of operators) 240.I
 - inversion (of Stieltjes transform) 220.D
 - Itô 45.G 406.B
 - Jensen 198.F
 - Klein-Nishina 415.G
 - Kneser-Sommerfeld App. A, Table 19.III
 - Kostant (on representations of compact Lie groups) 248.Z
 - Kronecker limit 450.B

- Kubo 402.K
 Künneth (in an Abelian category) 200.H
 Künneth (in Weil cohomology) 450.Q
 Lagrange (for the vector triple product) 442.C
 of a language 276.A
 lattice-point 222.B
 Lefschetz fixed-point 450.Q
 Leibniz (in differentiation) 106.D, App. A, Table 9.III
 Leibniz (in infinite series) App. A, Table 10.III
 Liouville 252.C
 Machin 332
 Mehler App. A, Table 19.III
 Milne-Simpson 303.E
 Möbius inversion (in combinatorics) 66.C
 Möbius inversion (in number theory) 295.C
 Nakano-Nishijima-Gell-Mann 132.A
 Newton (on interpolation) App. A, Table 21
 Newton (on symmetric functions) 337.I
 Newton backward interpolation 223.C
 Newton-Cotes (in numerical integration) 299.A
 Newton forward interpolation 223.C
 Newton interpolation App. A, Table 21
 Nicholson App. A, Table 19.IV
 open (in numerical integration) 299.A
 Ostrogradskii 94.F
 overall approximation 303.C
 Picard-Lefschetz 418.F
 Plancherel (on a unimodular locally compact group) 437.L
 Plücker (on plane algebraic curves) 9.B
 Poincaré (in integral geometry) 218.C
 Poisson (on Bessel functions) App. A, Table 19.III
 Poisson (for a flat torus) 391.J
 Poisson integral 198.B
 Poisson summation 192.C, L
 prime 411.D
 prime (of a language) 276.A
 principal, of integral geometry 218.C
 product (for the Hilbert norm-residue symbol) 14.R
 product (on invariant Haar measures) 225.F
 product (for the norm-residue symbol) 14.Q
 product (on valuations) 439.H
 q -expansion (on theta functions) 134.I
 recurrence, for indefinite integrals App. A, Table 9.II
 reduction (of a surface) 110.A
 Ricci 417.B, App. A, Table 4.II
 Riemann-Hurwitz (on coverings of a non-singular curve) 9.I
 Rodrigues 393.B
 Schläfli App. A, Table 19.III
 Schwarz-Christoffel transformation 77.D
 second variation 178.A
 set-theoretic 33.B
 Sommerfeld App. A, Table 19.III
 Sonine-Schafheitlin App. A, Table 19.III
 Steinberg (on representations of compact Lie groups) 248.Z
 Stirling 174.A 212.C, App. A, Table 17.I
 Stirling interpolation App. A, Table 21
 Stokes 94.F
 Stokes (on a C^∞ -manifold) 108.U
 Stokes (for integration of a vector field) App. A, Table 3.III
 Taylor (for a function of many variables) 106.J
 Taylor (for a function of one variable) 106.E, App. A, Table 9.IV
 theoretical 19.E
 Θ - (on ideles) 6.F
 trace (on unitary representations) 437.DD
 transformation (for the generating function of the number of partitions) 328.A
 transformation (of a theta function) 3.I
 transformation (for theta series) 348.L
 Trotter product 351.F
 valid 411.G
 Villat integration App. A, Table 15.VI
 Wallis App. A, Table 10.VI
 Watson 39.D, App. A, Table 19.IV
 Watson-Nicholson App. A, Table 19.III
 Weber App. A, Table 19.IV
 Weber-Sonine App. A, Table 19.III
 Weierstrass-Enneper 275.A
 Weingarten (for isometric immersion) 365.C
 Weingarten (in theory of surfaces) 111.H, App. A, Table 4.I
 Weyl 323.M
 Weyl character (on representations of compact Lie groups) 248.Z
 Weyl integral 225.I
 Weyrich App. A, Table 19.III
 Wiener 160.B
 Wu's App. A, Table 6.V
 Forti paradox, Burali- 319.B
 forward analysis 138.C
 forward difference 304.E, App. A, Table 21
 forward emission 325.A
 forward equation, Kolmogorov 115.A 260.F
 forward interpolation formula
 Gauss 223.C
 Newton 223.C
 forward type 304.D, F
 foundation, axiom of 33.B
 foundations of geometry 155
 foundations of mathematics 156
 four arithmetic operations 294.A
 four color conjecture 186.I
 four-color problem 157
 4-current density 150.B
 four-group 151.G
 Fourier, J. B. J. 158
 Fourier analysis App. A, Table 11
 Fourier analysis on the adèle group 6.F
 Fourier-Bessel series 39.D
 Fourier-Bessel transform 39.D
 Fourier coefficient 159.A, App. A, Table 11.I
 (of an almost periodic function) 18.B
 (in a Hilbert space) 197.C
 (in an orthogonal system) 317.A
 Fourier cosine series App. A, Table 11.I
 Fourier cosine transform 160.C, App. A, Table 11.II
 Fourier double integral theorem 160.B
 Fourier-Hermite polynomial 176.I
 Fourier hyperfunction 125.BB
 exponentially decreasing 125.BB
 modified 125.BB
 Fourier integral 160.A
 conjugate 160.D
 Fourier integral operator 274.C 345.B
 Fourier inversion formula 160.C
 Fourier kernel 220.B
 Fourier-Laplace transform 192.F
 Fourier reciprocity 160.C
 Fourier series 159, App. A, Table 11.I

- (of an almost periodic function) 18.B
- (of a distribution) 125.P
- (in a Hilbert space) 197.C
- Fourier sine series App. A, Table 11.I
- Fourier sine transform 160.C, App. A, Table 11.II
- Fourier single integral theorem 160.C
- Fourier-Stieltjes transform 192.B,O
- Fourier theorem (on real roots of an algebraic equation) 10.E
- Fourier transform 160, App. A, Table 11.II
 - (of a distribution) 125.O
 - (in topological Abelian groups) 36.L 192.I
 - discrete 142.D
 - fast 142.D
 - generalized 220.B
 - inverse 125.O
- Fourier ultrahyperfunction 125.BB
- 4-momentum operators 258.A,D
- fourth separation axiom 425.Q
- four-vector 258.C 359.C
- four-vector, energy-momentum 258.C
- four-vertex theorem 111.E
- fractal 246.K
- fraction(s)
 - continued 83
 - partial App. A, Table 10.V
- fractional cutting algorithm 215.B
- fractional factorial design 102.I
 - balanced 102.I
 - orthogonal 102.I
- fractional group, linear 60.B
- fractional ideal 67.J
 - of an algebraic number field 14.E
 - principal 67.K
- fractional power 378.D
- fractional programming 264.D
- fractional step 304.F
- Fraenkel set theory, Zermelo- 33.A,B
- fractional transformation, linear 74.E
- frame(s) 90.B
 - (of an affine space) 7.C
 - (of a C^∞ -manifold) 191.A
 - (in E^n) 111.B
 - (of a group manifold) 110.A
 - (in projective geometry) 343.C
 - (of a real line) 355.E
 - affine 7.C
 - bundle of tangent r - 105.H
 - Darboux 110.B
 - dual 417.B
 - family of (on a homogeneous space) 110.A
 - family of, of order 1 110.B
 - Frenet 110.A 111.D
 - Gaussian (of a surface) 111.H
 - k - (in \mathbf{R}^n) 199.B
 - method of moving 110.A
 - moving 90.B 111.C 417.B
 - natural moving 417.B
 - normal 110.B
 - of order 0 110.C
 - of order 1 110.B,C
 - of order 2 110.B,C
 - of order 3 110.B,C
 - of order 4 110.B
 - orthogonal (in a Euclidean space) 111.B 139.E
 - orthogonal k - (in \mathbf{R}^n) 199.B
 - orthogonal moving 417.D
 - projective 343.C
 - Stiefel manifold of k - 199.B
 - stochastic moving 406.G
 - tangent r - 105.H
- frame bundle
 - orthogonal 364.A
 - tangent orthogonal n - 364.A
 - tangent r - 105.H 147.F
- framed link 114.L
- framing 114.L
- Fréchet axiom 425.Q
- Fréchet curve 246.A
- Fréchet derivative 286.E
- Fréchet differentiable function 286.E
- Fréchet differential 286.E
- Fréchet distance (between surfaces) 246.I
- Fréchet L -space 87.K
- Fréchet manifold 286.K
- Fréchet space
 - (quasinormed space) 37.O
 - (topological linear space) 424.I
 - (topological space) 425.CC
 - locally convex 424.I
 - in the sense of Banach 37.O
 - in the sense of Bourbaki 424.I
- Fréchet surface 246.I
- Fréchet-Uryson space 425.CC
- Fredholm alternative theorem 68.E 217.F
- Fredholm determinant 217.E
- Fredholm first minor 217.E
- Fredholm formula 68.L
- Fredholm integral equation 217.A
 - of the first kind 217.A
 - of the second kind 217.A
 - of the third kind 217.A
- Fredholm mapping 286.E
- Fredholm operator 68.F 251.D
 - (in the sense of Grothendieck) 68.K
- Fredholm r th minor 217.E
- Fredholm type
 - integral equation of 217.A
 - integrodifferential equation of 222.A
- free
 - (discontinuous group) 122.A
 - distribution- 371.A
 - F - (oriented G -manifold) 431.E
 - (F, F') - (oriented G -manifold) 431.E
- free Abelian group 2.C
- free additive group 2.E
- free derivative 235.C
- free Dirac field 377.C
- freedom
 - asymptotic 361.B
 - degrees of (of a dynamical system) 271.F
 - n degrees of (sampling distribution) 374.B,C
- free energy 340.B 402.G
 - Gibbs 419.C
 - Helmholtz 419.C
 - mean 340.B 402.G
- free fields 150.A
- free grammar, context- 31.D
- free groups 161
- free Hamiltonian 351.D
- free homotopy 202.B
- free Lagrangian density 150.B
- freely act 122.A 431.A
- free module 277.G
- free product (of groups) 190.M
- free scalar field 377.C
- free semigroup 161.A
- free special Jordan algebra 231.A
- free vacuum vector 150.C
- free variable 411.C

- free vector 442.A
- French empiricism 156.C
- Frenet formula 111.D
- Frenet frame 110.A 111.D
- Frenet-Serret formulas (on curves) 111.D,
App. A, Table 4.I
- frequency
 - (of an oscillation) 318.A
 - (of samples) 396.C 397.B
 - (of a translational flow) 126.L 136.G
 - (of a wave) 446
 - angular (of a wave) 446
 - circular (of a simple harmonic motion) 318.B
 - relative (of samples) 396.C
- frequency distribution 397.B
- frequency function 397.D
- frequency response function 421.E
- Fresnel integral 167.D, App. A, Tables 9.V 19.II
- Freudenthal theorem 202.U
- Friedrichs extension 112.I 251.I
- Friedrichs scheme 304.F
- Friedrichs theorem 323.H 326.D
- frieze group 92.F
- Frobenius algebra 29.H
 - quasi- 29.H
- Frobenius automorphism (of a prime ideal) 14.K
- Frobenius group 151.H
- Frobenius integrability condition 154.B
- Frobenius method App. A, Table 14.I
- Frobenius morphism 450.P
- Frobenius substitution (of a prime ideal) 14.K
- Frobenius theorem
 - (on Abelian varieties) 3.D
 - (on matrices with nonnegative entries) 269.N
 - (on polynomials of a matrix) 390.B
 - (on representations of finite groups) 362.G
 - (on total differential equations and on foliations)
154.B 286.H 428.D
- Frobenius theorem, Perron- 310.H
- Froissart bound 386.B
- Froissart-Martin bound 386.B
- frontier point (of a subset) 425.N
- front set, analytic wave 274.D
- front set, wave 274.B 345.A
- Frostman maximum principle 338.C
- Froude number 116.B
- Fubini theorem 221.E
- Fuchsian form of weight k (or of dimension $-k$)
32.B
- Fuchsian function 32.B
- Fuchsian group 122.C
 - of the first kind 122.C
 - of the second kind 122.C
- Fuchsian relation 253.A, App. A, Table 18.I
- Fuchsian type (visibility manifold) 178.F
- Fuchsian type, equation of 253.A
- Fuchsoid group 122.C
- Fuks cohomology, Gelfand- 105.AA
- full discrete approximation 304.B
- full embedding theorem (of an Abelian category)
52.N
- full group 136.F 258.A
- full homogeneous Lorentz group 258.A
- full inhomogeneous Lorentz group 258.A
- full international notation 92.E
- full linear group 60.B
- full matrix algebra 269.B
- full Poincaré group 258.A
- full subcategory 52.A
- fully complete (locally convex space) 424.Y
- fully faithful functor 52.H
- fully normal space 425.X
- fully transitive 92.C
- Fulton and Hansen, general connectedness theorem
of 16.I
- function(s) 165 381.C
 - Abelian 3.J
 - absolutely integrable 214.E
 - additive interval 380.B
 - additive set 380.C
 - admissible 46.A 304.B
 - Ahlfors 43.G 77.E
 - algebraic 11.A
 - almost periodic 18
 - almost periodic, on a group 18.C
 - almost periodic, with respect to ρ 18.C
 - almost periodic, in the sense of Bohr 18.B
 - α -excessive 261.D
 - alternating 337.I
 - amplitude (of a Fourier integral operator)
274.C 345.B
 - analytic \rightarrow analytic function(s)
 - analytic almost periodic 18.D
 - analytic operator 37.K
 - Anger 39.G, App. A, Table 19.IV
 - Appell hypergeometric, of two variables
206.D, App. A, Table 18.I
 - argument 46.A
 - arithmetic 295.A
 - Artin-Hasse 257.H
 - associated Legendre 383.C, App. A,
Table 18.III
 - asymptotically developable 30.A
 - automorphic 32
 - automorphic, with respect to Γ 32.A
 - b - 125.EE 418.H
 - \mathfrak{B} -measurable 270.J
 - Baire 84.D
 - Barnes extended hypergeometric 206.C,
App. A, Table 18.I
 - base 304.B
 - Bellman 127.G
 - Bergman kernel 188.G
 - Bessel 39, App. A, Table 19.III
 - beta 174.C, App. A, Table 17.I
 - bispectral density 421.C
 - Borel measurable 270.J
 - boundary 160.E
 - bounded 43.A
 - of bounded variation 166
 - Busemann 178.F
 - C^∞ - (of many variables) 58.B
 - C^∞ -, slowly increasing 125.O
 - C^r -, in a C^∞ -manifold 105.G
 - canonical (on a nonsingular curve) 9.E
 - characteristic (of a density function) 397.G
 - characteristic (of a graded R -module) 369.F
 - characteristic (of a meromorphic function)
272.B
 - characteristic (of an n -person cooperative game)
173.D
 - characteristic (for an optical system) 180.C
 - characteristic (of a probability measure) 341.C
 - characteristic (of a subset) 381.C
 - characteristic operator 251.N
 - Chebyshev App. A, Table 20.II
 - Chebyshev q - 19.G, App. A, Table 20.VII
 - choice 33.B 34.A
 - circular 131.F 432.A
 - class (on a compact group) 69.B

Function(s)

- of class C^n , C^0 , C^1 , C^∞ , or C^ω 106.K
- of class C^r at a point 105.G
- of class C^r in a C^∞ -manifold 105.G
- of class C^∞ (of many variables) 58.B
- of class n , 0, 1, ξ , or ω 84.D
- cn App. A, Table 16.III
- completely additive set 380.C
- completely monotonic 240.E,K
- completely multiplicative number-theoretic 295.B
- complex 165.B
- complex-valued 165.B
- composite 106.I
- concave 88.A
- of confluent type 167
- of confluent type and Bessel functions App. A, Table 19
- conical App. A, Table 18.II
- conjugate 159.E 160.D
- conjugate harmonic 193.C
- constant 381.C
- continuous (on a metric space) 84
- continuous additive interval 380.B
- convex 88.A
- coordinate (of a fiber bundle) 147.B
- coordinate (in the Ritz method) 304.B
- cosigma 134.H, App. A, Table 16.IV
- counting (of a meromorphic function) 272.B
- covariance 386.A 395.A
- criterion 127.A
- cross spectral density 421.E
- cumulative distribution 341.B 342.C
- cylindrical 39.B, App. A, Table 19.III
- Daniell-Stone integrable 310.E
- decision 398.A
- decision, space of 398.A
- Dedekind eta 328.A
- Dedekind zeta 14.C 450.D
- defining (of a hyperfunction) 125.V
- density 397.D
- derived 106.A
- digamma 174.B
- dimension (on a continuous geometry) 85.A
- D-integrable 100.D
- Dirac delta App. A, Table 12.II
- Dirichlet 84.D 221.A
- distance 273.B
- distribution 168.B 341.B 342.C
- divisor 295.C
- divisor of (on an algebraic curve) 9.C
- divisor of (on an algebraic variety) 16.M
- dn App. A, Table 16.III
- doubly periodic 134.E
- E- 430.D
- \mathcal{E} - 46.C
- effectively calculable 356.C
- eigen- (of a boundary value problem) 315.B
- eigen- (for an integral equation) 271.F
- eigen- (of a linear operator) 390.A
- elementary 131
- elementary, of class n 131.A
- elementary Abelian 3.M
- elliptic 134 323.A,D
- elliptic, of the first kind 134.G
- elliptic, of the second kind 134.G
- elliptic, of the third kind 134.H
- elliptic cylinder 268.B
- elliptic irrational 134.A
- elliptic theta 134.I, App. A, Table 16.II
- empirical characteristic 396.C
- empirical distribution 250.F 374.E 396.C
- energy 126.L 279.F
- energy spectrum 433.C
- entire 429.A
- entire algebroidal 17.B
- envelope power 400.F
- error 167.D, App. A, Table 19.II
- estimating 399.D
- η - 391.L
- Euler 295.C
- even 165.B
- explicit 165.C
- exponential 131.D
- exponential, with the base a 131.B
- exponential, of an operator 306.C
- exponential generating 177.A
- factorial 174.A
- family of quasi-analytic 58.A
- Feller transition 261.B
- finitely additive set 380.B
- finite-valued 443.B
- flat 58.C
- Fréchet differentiable 286.E
- frequency 397.D
- frequency response 421.E
- Fuchsian 32.B
- gamma 150.D 174, App. A, Table 17.I
- Gelfand-Shilov generalized 125.S
- generalized 125.S
- generalized divisor 295.C
- generalized rational 142.B
- general Mathieu 268.B
- general recursive 356.C,F
- generating (of an arithmetic function) 295.E
- generating (of a contact transformation) 82.A 271.F
- generating (of a sequence of functions) 177.A
- of Gevrey class 168.B
- global implicit, theorem 208.D
- grand partition 402.D
- Green 188.A 189.B
- Green (α -order) 45.D
- Green (of a boundary value problem) 315.B
- Green, method 402.J
- Gudermann 131.F, App. A, Table 16.III
- half-Bessel 39.B
- Hamiltonian 219.C 271.F
- Hankel 39.B, App. A, Table 19.III
- harmonic 193
- harmonic kernel 188.H
- hazard 397.O
- Heaviside 125.E 306.B, App. A, Table 12.II
- Hey zeta 27.F
- higher transcendental 289.A
- Hilbert characteristic (of a coherent sheaf on a projective variety) 16.E
- Hilbert modular 32.G
- Hill 268.E
- holomorphic 198
- holomorphic (of many variables) 21.C
- holomorphic (on an open set in a complex manifold) 72.A
- holomorphic, germ of 21.E
- hyper- 125.V
- hyperarithmetical 356.H
- hyperbolic 131.F
- hypergeometric 206.A, App. A, Table 18.I
- hypergeometric (of the hyperspherical differential equation) 393.E

- hypergeometric, of confluent type 167.A,
 App. A, Table 19.I
 hypergeometric, with matrix argument 206.E
 identity 381.C
 implicit 165.C 208
 implicit, theorem 208.A 286.G
 implicit, theorem (in Banach algebra) 36.M
 implicit, theorem (in locally convex spaces)
 286.J
 impulse 306.B, App. A, Table 12.II
 incomplete beta 174.C, App. A, Table 17.I
 incomplete gamma 174.A, App. A, Table 17.I
 increment 380.B
 indicator (of a subset) 342.E 376.C
 inner 43.F
 integral 429.A
 integral of, with respect to a volume element
 (on a C^∞ -manifold) 105.W
 interpolation 223.A
 interval 380.A
 invariant decision 398.E
 inverse 198.L 381.C
 inverse analytic 198.L
 inverse trigonometric 131.E
 Jacobi elliptic App. A, Table 16.III
 joint density 397.J
 Julia exceptional 272.F
 jump 306.C
 K -pseudoanalytic 352.B
 K -quasiregular 352.B
 k -valued algebroidal 17.A
 Kelvin 39.G, App. A, Table 19.IV
 kernel 188.G
 Koebe extremal 438.C
 Kummer 167.A, App. A, Table 19.I
 L - \rightarrow L -function
 Lagrangian 271.F 292.A
 Laguerre App. A, Table 20.VI
 λ - 32.C
 Lamé, of the first kind 133.B
 Lamé, of the first species 133.C
 Lamé, of the fourth species 133.C
 Lamé, of the second kind 133.C
 Lamé, of the second species 133.C
 Lamé, of the third species 133.C
 Lane-Emden 291.F
 Laplace spherical 393.A
 Lebesgue measurable 270.J
 Legendre 393.B, App. A, Table 18.II
 likelihood 374.J 399.M
 likelihood estimating 399.M
 linear 74.E
 linear discriminant 280.I
 linear fractional 74.E
 linear regression 397.H,J 403.D
 locally integrable 168.B
 logarithmic, to the base a 131.B
 loss 398.A
 lower limit 84.C
 lower semicontinuous (in a set) 84.C
 Lyapunov 126.F 163.G
 major 100.F
 Mangoldt 123.B
 many-valued 165.B
 of many variables 165.C
 Mathieu 268
 Mathieu, of the first kind 268.B
 Mathieu, of the second kind 268.D
 maximal concentration 341.E
 mean concentration 341.E
 measurable 270.J
 measurable vector 308.G
 meromorphic 21.J 272
 meromorphic (on an analytic space) 23.D
 meromorphic (on a complex manifold) 72.A
 minimax decision 398.B
 minor 100.F
 Möbius 66.C 295.C
 modified Bessel 39.G, App. A, Table 19.IV
 modified indicator 341.C
 modified Mathieu 268.A
 modified Mathieu, of the first kind 268.D
 modified Mathieu, of the second kind 268.D
 modified Mathieu, of the third kind 268.D
 modular (of a locally compact group) 225.D
 modular, of level N 32.C
 moment-generating 177.A 341.C
 monotone 166.A
 monotone decreasing 166.A
 monotone increasing 166.A
 monotonic 166.A
 Morse 279.B
 of at most class 1 84.D
 μ -conformal 352.B
 multidimensional gamma 374.C
 multiplicative 32.A
 multiplicative automorphic 32.A
 multiplicity (of a mapping) 246.G
 multivalent 438
 multivalued 165.B
 Nash-Moser implicit, theorem 286.J
 n -dimensional distribution 342.C
 n th derived 106.D
 n -times continuously differentiable 106.K
 n -times differentiable 106.D
 of n variables 165.C
 nice (on a C^∞ -manifold) 114.F
 nondecreasing 166.A
 nondegenerate theta 3.I
 nonincreasing 166.A
 nontangential maximal 168.B
 normal (of ordinal numbers) 312.C
 normal density 397.D
 null 310.I
 number-theoretic 295.A 356.A
 objective 264.B 307.C
 odd 165.B
 operating 192.N
 order (of a meromorphic function) 272.B
 orthogonal 317, App. A, Table 20
 orthogonal, Haar system of 317.C
 orthogonal, Rademacher system of 317.C
 orthogonal, Walsh system of 317.C
 outer 43.F
 P -, of Riemann 253.B
 \wp -, of Weierstrass 134.F, App. A, Table 16.IV
 Painlevé transcendental 288.C
 parabolic cylinder 167.C
 parametric 102.A 399.A
 partial 356.E
 partition 402.D
 payoff 173.B
 pentagramma 174.B
 periodic 134.E
 phase (of a Fourier integral operator) 274.C
 345.B
 piecewise continuous 84.B
 plurisubharmonic 21.G
 point 380.A
 polygamma 174.B, App. A, Table 17.I

- positive real 282.C
- of positive type 192.B,J
- power 400.A
- primitive 216.C
- primitive, derivatives and App. A, Table 9.I
- primitive recursive 356.A,B,F
- probability generating 341.F
- proper (of a boundary value problem) 315.B
- proper convex 88.D
- propositional 411.C
- proximity (of a meromorphic function) 272.B
- pseudo- 125.C
- psi 174.B
- quadratic loss 398.A 399.E
- quasi-analytic 58.F
- quasi-analytic, family of 58.A
- quasi-analytic, set of 58.F
- quasicontinuous 338.I
- radial maximal 168.B
- rank 66.F
- rapidly decreasing C^∞ - 168.B
- rate distortion 213.E
- rational, field of 337.H
- rational, on a variety 16.A
- rational entire 429.A
- real 165.B
- real analytic 106.K 198.H
- real-valued 165.B
- recursive \rightarrow recursive function(s)
- regression 397.I
- regular 198
- regular, on an open set (of a variety) 16.B
- regular, at a subvariety 16.B
- representative (of a compact Lie group) 249.U
- representing (of a predicate) 356.B
- representing (of a subset) 381.C
- reproduction 263.A
- Riemann (of a Cauchy problem) 325.D
- Riemann integrable 216.A
- Riemann P App. A, Tables 14.I 18.I
- Riemann theta 3.L
- Riemann ζ - 450.B
- right continuous 84.B
- right majorizing 316.E
- right superior 316.E
- risk 398.A
- sample 407.A
- sample covariance 395.G
- with scattered zeros 208.C
- schlicht 438.A
- Schwinger 150.F
- selection 354.E
- self-reciprocal 220.B
- semicontinuous (at a point) 84.C
- sequential decision 398.F
- set 380
- of several variables 106.I,J
- shape 223.G
- Siegel modular, of degree n 32.F
- σ -, of Weierstrass 134.F, App. A, Table 16.IV
- simple 221.B 443.B
- simple loss 398.A
- simplest Chebyshev q - 19.G
- simply periodic 134.E
- single-valued 165.B
- singular inner function 43.F
- slope 46.C
- sn App. A, Table 16.III
- special App. A, Table 14.II
- special, of confluent type 389.A
- special, of ellipsoidal type 389.A
- special, of hypergeometric type 389.A
- spherical (on a homogeneous space) 437.Y
- spherical Bessel 39.B
- spherical harmonic 193.C
- spheroidal wave 133.E
- standard defining 125.Z
- stationary 46.B
- statistical decision 398.A
- stream 205.B
- strictly concave 88.A
- strictly convex 88.A
- strictly decreasing 166.A
- strictly increasing 166.A
- strictly monotone 166.A
- strictly monotone (of ordinal numbers) 312.C
- strictly monotone decreasing 166.A
- strictly monotone increasing 166.A
- strictly monotonic 166.A
- structure 191.C
- Struve 39.G, App. A, Table 19.IV
- subharmonic 193.A
- superharmonic 193.P
- supporting 125.O
- symmetric 337.I
- Szegő kernel 188.H
- τ - 150.D
- test 130.DD 400.A
- tetragamma 174.B
- Theodorsen 39.E
- theory of 198.Q
- theory of, of a complex variable 198.Q
- theta 134.I
- theta (on a complex torus) 3.I
- time ordered 150.D
- torus App. A, Table 18.III
- transcendental, of Painlevé 288.C
- transcendental entire 429.A
- transcendental meromorphic 272.A
- transfer 86.D
- transfinite logical choice 411.J
- transition (of a fiber bundle) 147.B
- transition (of a Markov chain) 250.A 261.B
- trigamma 174.B
- trigonometric 432.A, App. A, Table 2
- truncated Wightman 150.D
- truth 341.A 411.E
- ultradifferentiable 168.B
- uniformly almost periodic 18.B
- unit 306.B, App. A, Table 12.II
- universally measurable 270.L
- upper limit 84.C
- upper semicontinuous 84.C
- value 108.B
- on a variety 16.A
- von Neumann 39.B 188.H, App. A, Table 19.III
- Wagner 39.E
- wave 351.D
- Weber 39.G 167.C, App. A, Tables 19.IV, 20.IV
- Weierstrass elliptic App. A, Table 16.IV
- Weierstrass \wp - 134.F, App. A, Table 16.IV
- Weierstrass sigma 134.F, App. A, Table 16.IV
- weight (interpolatory) 299.A
- weight (for the mean of a function) 211.C
- weight (in orthogonality) 317.A
- Whittaker 167.B, App. A, Table 19.II
- Wightman 150.D
- \mathfrak{X} -valued holomorphic 251.G
- zeta \rightarrow zeta functions

- zonal spherical (on a homogeneous space) 437.Y
- functional 46.A 162 165.B
 - additive (of a Markov process) 261.E
 - algebraic linear 424.B
 - analytic 168.C
 - areal 334.B
 - bilinear 424.G
 - Brownian 176.I
 - characteristic (of a probability distribution) 407.C
 - Dirichlet 334.C
 - Douglas 334.C
 - linear 37.C 197.F 424.B
 - martingale additive 261.E
 - multiplicative (of a Markov process) 261.E
 - multiplicative, transformation by (in Markov process) 261.F
 - perfect additive 261.E
 - subadditive 88.B
 - supporting (of a convex set) 89.G
 - Yang-Mills 80.Q
- functional analysis 162
- functional analysis, nonlinear 286
- functional cohomology operation 202.S
- functional-differential equation 163
 - system of 163.E
- functional equation
 - Abel 388.D
 - approximate (of zeta function) 450.B
 - Schröder 388.D
 - special 388.A
 - of zeta function 450.B
- function algebra 164.A
- functionally dependent (components of mapping) 208.C
 - of class C^* 208.C
- functional model 251.N
- functional paper 19.D
- functional Φ -operation 202.S
- functional relation 208.C
 - of class C^* 208.C
 - of gamma function 174.A
- function element 198.I 339.A
 - inverse 198.L
 - in the wider sense 198.O
- function field
 - (of an algebraic curve over a field) 9.C
 - (of an algebraic variety) 16.A
 - Abelian 3.J
 - algebraic, over k of dimension 1 9.D
 - algebraic, over k of transcendence degree 1 9.D
 - algebraic, in n variables 149.K
 - elliptic 9.D
 - rational, in n variables 149.K
- function group 234.A
- function matrix
 - rational 86.D
 - transfer 86.B
- functions on a variety 16.A
- function space(s) 168 435.D
 - test 125.S
- function symbol 411.H
- function-theoretic null sets 169
- function variable 411.H
- functor 52.H
 - ∂ - 200.I
 - ∂^* - 200.I
 - additive 52.N
 - adjoint 52.K
 - cohomological 200.I
 - connected sequences of 200.I
 - contravariant 52.H
 - covariant 52.H
 - derived 200.I
 - exact 52.N 200.I
 - faithful 52.H
 - forgetful 52.I
 - fully faithful 52.H
 - half-exact 200.I
 - homological 200.I
 - left adjoint 52.K
 - left balanced 200.I
 - left derived 200.I.Q
 - left exact 200.I
 - partial derived 200.I
 - relative derived 200.K
 - representable 53.L
 - right adjoint 52.K
 - right balanced 200.I
 - right derived 200.I.Q
 - right exact 200.I
 - spectral 200.J
 - universal ∂ - 200.I
- functorial isomorphism 53.J
- functorial morphism 53.J
- fundamental absolute neighborhood retract (FANR) 382.C
- fundamental absolute retract (FAR) 382.C
- fundamental cell (of a symmetric Riemann space) 413.F
- fundamental class
 - (of an Eilenberg-MacLane space) 70.F
 - (of a Poincaré pair) 114.J
 - (of a Thom complex) 114.G
 - with coefficient \mathbb{Z}_2 65.B
- fundamental conjecture (in topology) 70.C
- fundamental curve (with respect to a birational mapping) 16.I
- fundamental cutset matrix 186.G
- fundamental cycle
 - (of an oriented pseudomanifold) 65.B
 - (in a resolution of a singularity) 418.C
- fundamental differential invariants (of a surface) 110.B
- fundamental discriminant 295.D
- fundamental domain 234.C
- fundamental exact sequence 200.M
- fundamental figure(s) 343.B
 - linear 343.B
- fundamental form
 - (associated with a Hermitian metric) 232.A
 - (of a Finsler space) 152.A
 - first 111.G, App. A, Table 4.I
 - second 111.G 360.G 365.C
- fundamental group 170
 - algebraic 16.U
- fundamental homology class 201.N
 - around K 201.N
- fundamental invariants (of a space with a Lie transformation group) 110.A
- fundamental kernel 320.H
- fundamental lemma
 - in the calculus of variations 46.B
 - Neyman-Pearson 400.B
- fundamental open set 122.B
- fundamental operator 163.H
- fundamental period (of a periodic function) 134.E
- fundamental period parallelogram 134.E

- ul>
- fundamental point
 - (with respect to a birational mapping) 16.I
 - (of a projective space) 343.C
- fundamental quantities
 - first 111.H
 - second 111.H
- fundamental region (of a discrete transformation group) 122.B
 - Ford 234.C
- fundamental relations
 - (of gamma functions) 174.A
 - (among the generators of a group) 161.A
 - (in thermodynamics) 419.A
- fundamental retract 382.C
- fundamental root system (of a semisimple Lie algebra) 248.N
- fundamental sequence
 - of cross cuts (in a simply connected domain) 333.B
 - of rational numbers 294.E
 - of real numbers 355.B
 - in a uniform space 436.G
- fundamental set (of a transformation group) 122.B
- fundamental solution(s)
 - (of a differential operator) 112.B 189.C
 - (of an elliptic equation) 323.B
 - (of an evolution equation) 189.C
 - (of a hyperbolic equation) 325.D
 - (of a parabolic equation) 327.D
 - (of a partial differential equation) 320.H
 - system of (of a system of linear equations) 269.M
- fundamental space 125.S
- fundamental subvariety (with respect to a birational mapping) 16.I
- fundamental system
 - (of eigenfunctions to an eigenvalue for an integral equation) 217.F
 - (for a linear difference equation) 104.D
 - (of a root system) 13.J
 - of irreducible representations (of a complex semisimple Lie algebra) 248.W
 - of neighborhoods 425.E
 - of solutions (of a homogeneous linear ordinary differential equation) 252.B
 - of solutions (of a homogeneous system of linear differential equations of the first order) 252.H
- fundamental tensor(s)
 - (of a Finsler space) 152.A
 - (of a Riemannian manifold) 364.A
 - Lie 413.G
 - second 417.F
- fundamental theorem(s)
 - of algebra 10.E
 - Bonnet (on surfaces) 111.H
 - of calculus 216.C
 - of elementary number theory 297.C
 - the first (of Morse theory) 279.D
 - Gentzen 411.J
 - Nevanlinna first 272.B
 - Nevanlinna second 272.E
 - of the principal order \mathfrak{o} 14.C
 - of projective algebraic varieties 72.F
 - of projective geometry 343.D
 - of proper mapping 16.X
 - the second (of Morse theory) 279.D
 - of Stein manifolds 21.L 72.E
 - on symmetric polynomials 337.I
 - of the theory of curves 111.D
 - of the theory of surfaces 111.G
 - Thom 114.H
 - of the topology of surfaces 410.B
 - of ultraproducts 276.E
- fundamental tieset matrix 186.G
- fundamental unit 414.A 116
- fundamental units (of an algebraic number field) 14.D
- fundamental vector field 191.A
- fundamental vectors (in a vector space) 442.A
- future cone 258.A
- G**
- $\gamma \rightarrow$ gamma
 - $GL(n, k)$ (general linear group) 60.B
 - γ -matrices, Dirac 415.G
 - γ -perfect 186.J
 - γ -perfectness 186.J
 - γ -point of the k th order (of a holomorphic function) 198.C
 - Γ -equivalent (points) 122.A
 - Γ -extension 14.L
 - $\Gamma_{\mathcal{F}}$ -foliation 154.H
 - Γ -structure 90.D 105.Y
 - $\Gamma_{\mathcal{F}}$ -structure 154.H
 - Γ_q^* -structure 154.E
 - g -lattice (of a separable algebra) 27.A
 - integral 27.A
 - normal 27.A
 - G -bundle 147.B
 - G -connections, Yang-Mills 80.Q
 - G -fiber homotopy type, spherical 431.F
 - G -group 172.J
 - G -invariant
 - (element) 226.A
 - (statistics) 396.I
 - almost 396.I
 - G -invariant measure 225.B
 - G -isomorphism 191.A
 - G -manifold 431.C
 - oriented 431.E
 - G -mapping (G -map) 362.B 431.A
 - G -set
 - k -ply transitive 362.B
 - left 362.B
 - quotient 362.B
 - right 362.B
 - simply transitive 362.B
 - sub- 362.B
 - G -space 178.H 431.A
 - with nonpositive curvature 178.H
 - G -stationary
 - strictly 395.I
 - weakly 395.I
 - G -structure 191
 - G -subset 362.B
 - G -surface 178.H
 - G -vector bundle 237.H
 - G_{δ} -set 270.C
 - gain, heat 419.A
 - Galerkin method 290.E 303.I 304.B
 - Galilei transformation 359.C
 - Galois, E. 171
 - Galois cohomology 172.J 200.N
 - Galois equation 172.G
 - Galois extension (of a field) 172.B
 - Galois field 149.M
 - Galois group
 - of an algebraic equation 172.G
 - of a Galois extension 172.B

- of a polynomial 172.G
- Galois theory 172
 - of differential fields 113
- Galton-Watson process 44.B
 - multi (k)-type 44.C
- game
 - bimatrix 173.C
 - constant-sum 173.A
 - cooperative 173.A
 - differential 108
 - general-sum 173.A
 - with infinitely many players 173.D
 - matrix 173.C
 - multistage 173.C
 - noncooperative 173.A
 - n -person, in extensive form 173.B
 - n -person, in normal form 173.C
 - n -person cooperative, in characteristic-function form 173.D
 - in partition-function form 173.D
 - without side payments 173.D
 - zero-sum 173.A
 - zero-sum two-person 108.B
- game-theoretic model 307.C
- game theory 173
- gamma density 397.D
- gamma distribution 341.D, App. A, Table 22
- gamma function 150.D 174, App. A, Table 17.I
 - incomplete 174.A, App. A, Table 17.I
 - multidimensional 374.C
- gamma function and related functions App. A, Table 17
- gap (at a point) 84.B
- gap theorem 339.D
 - Hadamard 339.D
- gap value (of a point on a Riemann surface) 11.D
- Gårding, hyperbolic in the sense of 325.F
- Gårding inequality 112.G 323.H
- Garnier system 253.E
- Garside-Jarratt-Mack method 301.N
- gases, kinetic theory of 402.B
- Gâteaux derivative 286.E
- Gâteaux differentiable 286.E
- gauge theory 105.G
 - lattice 150.G
- gauge transformation
 - (in electromagnetism) 130.A
 - (in a lattice spin system) 402.G
 - (of a principal fiber bundle) 80.Q
 - (in unified field theory) 434.B
 - of the first kind 150.B
- Gauss, C. F. 175
- Gauss-Argand plane 74.C
- Gauss backward interpolation formula 223.C
- Gauss-Bonnet formula 111.H 364.D, App. A, Table 4.I
- Gauss-Bonnet-Sasaki-Nitsche formula 275.C
- Gauss-Chebyshev formula (in numerical integration) 299.A
- Gauss circle problem 242.A
- Gauss criterion App. A, Table 10.II
- Gauss equation
 - (on an isometric immersion) 365.C
 - (on surfaces) 111.H
- Gauss formula
 - (on Gauss sum) 295.D
 - (on harmonic functions) 193.D
 - (for integration of a vector field) App. A, Table 3.III
 - (for isometric immersion) 365.C
 - (in numerical integration) 299.A
 - (for the surface integral) 94.F
 - (in theory of surfaces) 111.H, App. A, Table 4.I
- Gauss forward interpolation formula 223.C
- Gauss-Hermite formula (in numerical integration) 299.A
- Gauss hypergeometric differential equation App. A, Table 14.II
- Gaussian
 - (system of random variables) 176.A
 - complex 176.B
- Gaussian curvature
 - (of a surface) 111.H, App. A, Table 4.I
 - total (of a surface) 111.H
- Gaussian differential equation 206.A
- Gaussian distribution 341.D
 - standard 176.A
- Gaussian elimination 302.B
- Gaussian frame (of a surface) 111.H
- Gaussian integer 14.U
- Gaussian plane 74.C
- Gaussian process 176 342.A
 - complex 176.C
 - N -ple Markov 176.F
 - N -ple Markov, in the restricted sense 176.F
 - stationary 176.C
- Gaussian random field
 - Markov, in the McKean sense 176.F
 - Markov, in the Nelson sense 176.F
- Gaussian random measure 407.D
- Gaussian random variable, complex 176.B
- Gaussian source, autoregressive 213.E
- Gaussian sum 295.D 450.C
 - local 450.F
- Gaussian system
 - (of random variables) 176.A
 - complex 176.B
- Gaussian white noise 407.C
- Gauss integral 338.J
- Gauss integration formula (in the narrow sense) 299.A
- Gauss interpolation formula App. A, Table 21
- Gauss-Jordan elimination 302.B
- Gauss kernel 327.D
- Gauss-Laguerre formula (in numerical integration) 299.A
- Gauss-Manin connection (of a variety) 16.V
- Gauss mapping (in geometric optics) 180.B
- Gauss-Markov theorem 403.E
- Gauss-Seidel method 302.C
- Gauss series 206.A
- Gauss symbol 83.A
- Gauss theorem
 - (on algebraic closedness of \mathbb{C}) 10.E
 - (on primitive polynomials) 337.D
- Gauss theorem egregium (on surfaces) 111.H
- Gauss transformation App. A, Table 16.III
- Gauss variational problem 338.J
- G.C.D. (greatest common divisor) 67.H 297.A
- GCR algebra 36.H
- Gegenbauer polynomials 317.D 393.E, App. A, Table 20.I
- Gel'fand-Fuks cohomology 105.AA
- Gel'fand integrable 443.F
- Gel'fand-Levitan-Marchenko equation
 - (for KdV equations) 387.D
 - (for nonlinear lattice) 287.C
- Gel'fand-Mazur theorem 36.E
- Gel'fand-Naimark theorem 36.G
- Gel'fand-Pettis integrable 443.F

Gel'fand-Pettis integral 443.F
 Gel'fand-Pyatetskii-Shapiro reciprocity law 437.DD
 Gel'fand representation (of a commutative Banach algebra) 36.E
 Gel'fand-Shilov generalized function 125.S
 Gel'fand theorem, Stone- 168.B
 Gel'fand topology 36.E
 Gel'fand transform 36.E
 Gel'fand triplet 424.T
 Gell-Mann formula, Nakano-Nishijima- 132.A
 general addition theorem 388.C
 general analytic space 23.G
 general angle 139.D
 general associative law (for group composition) 190.C
 general boundary value problem 323.H
 general Cantor set 79.D
 general Cayley algebra 54
 general connectedness theorem due to Fulton and Hansen 16.I
 general curve 93.D
 general derivative (of a set function) 380.D
 general geometry of paths 152.C
 generalization (in étale topology) 16.AA
 generalized absolute continuity (*) 100.C
 generalized absolute continuity in the restricted sense 100.C
 generalized absolutely continuous function 100.C
 generalized Bayes solution 398.B
 generalized Bernoulli shift 136.D
 generalized Bernshtein problem 275.F
 generalized Boolean algebra 42.B
 generalized Boolean ring 42.C
 generalized Borel embedding 384.D
 generalized Clifford torus 275.F
 generalized cohomology theories 201.A
 generalized cohomology theory with E-coefficient 202.T
 generalized conformal mapping 246.I
 generalized continuum hypothesis 49.D
 generalized convergence 331.C
 generalized convolution (of distributions) 125.M
 generalized coordinates (in analytical dynamics) 271.F
 generalized cylindrical coordinates App. A, Table 3.V
 generalized decomposition number (of a finite group) 362.I
 generalized derivative 125.E
 generalized distance, Mahalanobis 280.E
 generalized distribution, Beurling 125.U
 generalized divisor function 295.C
 generalized eigenfunction 375.C
 generalized eigenspace (of a linear operator) 390.B
 generalized eigenvalue 375.C
 generalized eigenvalue problem 298.G
 generalized eigenvector 390.B
 generalized Eisenstein series 450.T
 generalized Fourier transform 220.B
 generalized function 125.S
 generalized Hardy class 164.G
 generalized helix 111.F
 generalized homology theory 201.A
 generalized homology theory with E-coefficient 202.T
 generalized Hopf homomorphism 202.U
 generalized Hopf invariant 202.Q
 generalized Hurewicz theorem 202.N

generalized isoperimetric problem 46.A 228.A
 generalized Jacobian (of a set function) 246.H
 generalized Jacobian variety 9.F 11.C
 generalized Lamé differential equation 167.E
 generalized least squares estimator 403.E
 generalized Lebesgue measure 270.E
 generalized Levi form 274.G
 generalized limit 37.F
 generalized minimal immersion 275.B
 generalized module 143.B
 generalized momentum 271.F
 generalized nilpotent (operator) 251.F
 generalized nilpotent element 36.E
 generalized nilpotent group 190.K
 generalized peak point 164.D
 generalized peak set 164.D
 generalized Pfaff problem 428.B
 generalized Poincaré conjecture 65.C
 generalized quaternion group 151.B
 generalized rational function 142.B
 generalized Riccati differential equation App. A, Table 14.I
 generalized Riemann-Roch theorem (on algebraic curves) 9.F
 generalized Schlömilch series 39.C
 generalized solvable group 190.K
 generalized stochastic process 407.C
 generalized suspension theorem 202.T
 generalized Tauberian theorem 36.L 160.G
 of Wiener 192.D
 generalized topological space 425.D
 generalized trigonometric polynomial 18.B
 generalized trigonometric series 18.B
 generalized uniserial algebra 29.I
 generalized valuation 439.B
 generalized variance 280.E 397.J
 generalized variance, sample 280.E
 generalized wave operator 375.B
 generalized Whitehead theorem 202.N
 general knot conjecture 235.B
 general law of reciprocity 14.O
 Artin 59.C
 general linear group 60.B 226.B 256.D
 of degree n over K 60.B 226.B 256.D
 over a noncommutative field 60.O
 projective 60.B
 general linear hypothesis 400.C
 general linear Lie algebra 248.A
 general lower derivative (of a set function) 380.D
 general Markov chain 260.J
 general Mathieu function 268.B
 general Navier-Stokes equations 204.F
 general position
 (complexes) 70.B
 (PL mappings) 65.D
 (in a projective space) 343.B
 theorem 65.D
 general principle of relativity 358
 general projective geometry 343.B
 general random walk 260.A
 general recursive function 356.C,F
 general recursive predicate 356.C
 general recursive set 97
 general Runge-Kutta method 303.D
 general sense, derivable in the 380.D
 general set theory 33.B
 general solution
 (of a difference equation) 104.D
 (of an ordinary differential equation) 313.A
 (of a partial differential equation) 320.C

- (of a system of partial differential equations) 428.B
- general sum 173.A
- general theory
 - of perturbations 420.E
 - of relativity 358
- general topology 426
- general type 72.H
 - surface of 72.K
- general uniformization theorem 367.G
- general upper derivative (of a set function) 380.D
- generate
 - (an A -module) 277.D
 - (a completely additive class) 270.B
 - (a field over k) 149.D
 - (a filter) 87.I
 - (an ideal) 67.B
 - (a linear subspace) 256.F
 - (a subgroup) 190.C
 - (a subring) 368.E
 - (a topology) 425.F
- generated, finitely 277.D
- generating curve 111.I
- generating element (with respect to a self-adjoint operator) 390.G
- generating function(s)
 - (of an arithmetic function) 295.E
 - (of a canonical transformation) 82.B
 - (of a contact transformation) 82.A
 - (of an infinitesimal transformation) 271.F
 - (of a sequence of functions) 177.A
 - exponential 177.A
 - factorial cumulant- 397.G
 - factorial moment 397.G
 - joint moment 397.I,J
 - moment- 177.A 341.C 397.G,J
 - probability- 341.F 397.G
- generating line
 - (of a circular cone) 78.A
 - (of a quadric hypersurface) 343.E
 - (of a quadric surface) 350.B
 - (of a ruled surface in differential geometry) 111.I
- generating representation (of a compact Lie group) 249.U
- generating space (of a quadric hypersurface) 343.E
- generator
 - (of an Abelian category) 200.I
 - (of a cyclic code) 63.D
 - (of an endomorphism) 136.E
 - (of a group) 190.C
 - (of a Markov process) 261.C
 - (of a semigroup) 378.D
 - Bott 237.D
 - Dynkin representation of 261.B
 - F.D. 136.E
 - infinitesimal 378.B
 - system of (of an A -module) 277.D
 - topological (of a compact Abelian group) 136.D
 - two-sided 136.E
- generic (property) 126.H
- generic point 16.A
- Gentzen fundamental theorem 411.J
- genuine solution 323.G
- genus
 - (of an algebraic curve) 9.C
 - (of a differential ideal) 428.E
 - (of an ideal group) 59.E
 - (in integral representation theory) 362.K
 - (of a knot) 235.A
 - (of a lattice group) 13.P
 - (of a quadratic field) 347.F
 - (of a quadratic form) 348.H
 - (of a surface) 410.B
 - (of a transcendental integral function) 429.B
 - arithmetic (of an algebraic curve) 9.F
 - arithmetic (of an algebraic surface) 15.C
 - arithmetic (of a complete variety) 16.E
 - arithmetic, of a divisor (on an algebraic surface) 15.C
 - boundary 410.B
 - effective (of an algebraic curve) 9.C
 - of the function field K/k 9.D
 - geometric (of an algebraic surface) 15.E
 - geometric (of a complete variety) 16.O
 - geometric (of a singular point) 418.C
 - i - 15.E
 - linear 15.G
 - measure of (of a positive definite symmetric matrix) 348.K
 - \odot (of an algebraic curve) 9.F
 - principal (of an ideal group) 59.E
 - principal (of a quadratic field) 347.F
 - virtual arithmetic (of a divisor) 16.E
- geocentric parallax 392
- Geöcze area (of a surface) 246.E
- Geöcze problem 246.D
- geodesic 80.L,I 111.H 178 364.C, App. A, Table 4.I
 - null 359.D
 - totally, submanifold 365.D
- geodesic arc 178.H 364.B
- geodesic coordinates 80.J
- geodesic coordinate system in the weak sense 232.A
- geodesic correspondence (between surfaces) 111.I
- geodesic curvature 111.I, App. A, Table 4.I
- geodesic flow 126.L 136.G
- geodesic line 178.H
- geodesic point 111.H 365.D
- geodesic polar coordinates 90.C
- geodesic triangle 178.A
- geodesic variation 178.A
- geometrically finite 234.C
- geometrically reductive 226.B
- geometrically simple eigenvalue 390.A
- geometrical mean 397.C
- geometric complex 70.B
- geometric construction problem 179.A
- geometric crystal class 92.B
- geometric difference equation 104.G
- geometric dimension (of a vector bundle) 114.D
- geometric distribution 341.D, App. A, Table 22
- geometric fiber (of a morphism) 16.D
- geometric genus
 - of an algebraic surface 15.E
 - of a complete variety 16.O
 - of a singular point 418.C
- geometric mean
 - (of a function) 211.C
 - (of numbers) 211.C
- geometric multiplicity (of an eigenvalue) 390.A
- geometric number theory 296.B
- geometric optics 180
- geometric point (of a scheme) 16.D
- geometric probability 218.A
- geometric programming 264.D
- geometric progression 379.I, App. A, Table 10.VI
- geometric quotient 16.W
- geometric realization (of the s.s. complex) 70.E

- geometric series 379.B, App. A, Table 10.I
- geometry 181
 - affine 7
 - affine, in the narrower sense 7.E
 - affine differential 110.C
 - algebraic 12.A
 - analytic 181
 - circle 76.C
 - conformal 76.A
 - conformal differential 110.D
 - continuous 85.A
 - differential 109
 - differential, of curves and surfaces 111
 - differential, in specific spaces 110
 - elliptic 285.A
 - Euclidean, in the wider sense 139.B
 - finite-dimensional projective 343.B
 - formal 16.X
 - general, of paths 152.C
 - general projective 343.B
 - hyperbolic 285.A
 - hypersphere 76.C
 - integral 218.A
 - Laguerre 76.B
 - Lobachevskii non-Euclidean 285.A
 - Möbius 76.A
 - natural 110.A
 - n -dimensional Euclidean 139.B 181
 - non-Archimedean 155.D
 - non-Desarguesian 155.D 343.C
 - non-Euclidean 285.A
 - parabolic 285.A
 - plane 181
 - projective 343.B
 - projective, of paths 109
 - projective differential 110.B
 - pseudoconformal 344.A
 - pure 181
 - Riemannian 137, App. A, Table 4.II
 - Riemann non-Euclidean 285.A
 - solid 181
 - space 181
 - of a space subordinate to a group 137
 - spectral 391
 - sphere 76.C
 - spherical 285.D
 - on a surface 111.G
 - synthetic 181
 - wave 434.C
- geometry of numbers 182
- germ(s) 383.B
 - of an analytic set 23.B
 - of a C^∞ -function at the origin 58.C
 - of a holomorphic function 21.E
 - irreducible 23.B
 - reducible 23.B
 - sheaf of, of holomorphic functions 23.C
 - sheaf of, of regular functions 16.B
- Germain curvature (of a surface) 111.H, App. A, Table 4.I
- Gevrey class 58.G 125.U 325.I
 - function of 168.B
- ghost component (of an infinite-dimensional vector) 449.A
- ghost, Faddeev-Popov 132.C 150.G
- GHS inequality 212.A
- Gibbs distribution, equilibrium 136.C
- Gibbs-Duhem relation 419.B
- Gibbs free energy 419.C
 - minimum principle 419.C
- Gibbs measure 136.C
- Gibbs phenomenon 159.D
- Gibbs state 340.B
- Gilbert-Sacks bound, Varsharmov 63.B
- Gill method, Runge-Kutta- 303.D
- Gini coefficient of concentration 397.E
- Giraud theorem 323.C
- Girsanov theorem 406.B
- Girsanov transformation 406.B
- Girshick-Savage theorem 399.F
- Givens method 298.D
- Givens transformation 302.E
- GKS first inequality 212.A
- GKS second inequality 212.A
- Glashow-Weinberg-Salam model 132.D
- Glauert approximation, Prandtl- 205.B
- Glauert law of similarity, Prandtl- 205.D
- g.l.b. (greatest lower bound) 311.B
- Gleason part (for a function algebra) 164.F
- Gleason theorem 351.L
- glide operation 92.E
- glide reflection 92.E
- Glivenko-Cantelli theorem 374.E
- global analysis 183
- global dimension
 - (of an analytic set) 23.B
 - (of a ring) 200.K
 - left (of a ring) 200.K
 - right (of a ring) 200.K
 - weak (of a ring) 200.K
- global discretization error 303.B
- global Hecke algebra 450.O
- global implicit function theorem 208.D
- globally asymptotically stable 126.F
- globally symmetric Riemannian space 412.A
- global property (in differential geometry) 109
- global roundoff error 303.B
- global truncation error 303.B
- gluing theorem 21.I
- gluons 132.D
- GNS construction 308.D
- Godbillion-Vey classes 154.G
- Gödel, Kurt 184
- Gödel completeness theorem 411.J
- Gödel incompleteness theorem 156.E
- Gödel number(s) 185 356.C,E
- Gödel numbering 185.A
- Gödel set theory 33.C
 - Bernays- 33.A
- Goldbach problem 4.C
- Golden-Thompson inequality 212.B
- Goldstein method, Ince- 268.C
- Goldstone boson, Nambu- 132.C
- Goldstone theorem 132.C
- Gomory cut 215.B
- Goodner-Kelley theorem, Nachbin- 37.M
- goodness of fit 397.Q
 - test 401.E
- good reduction (of an Abelian variety) 3.N
 - potential (of an Abelian variety) 3.N
- good resolution 418.C
- Goppa code 63.E
- Gordan coefficients, Clebsch- 258.B 353.B
- Gordan equation, Klein- 351.G 377.C
- Gordon equation, Sine- 387.A
- Gorenstein ring 200.K
- Goursat kernel, Pincherle- 217.F
- Goursat theorem 198.B
- grad (gradient) 442.D
- graded algebra 203.B

graded A -module 200.B
 graded coalgebra 203.B
 graded Hopf algebra 203.C
 graded ideal 369.B
 graded module
 connected 203.B
 dual 203.B
 graded object 200.B
 graded ring 369.B
 associated 284.D
 gradient 442.D, App. A, Table 3.II
 gradient method 292.E
 Arrow-Hurwicz-Uzawa 292.E
 conjugate (CG) 302.D
 gradient projection method, Rosen 292.E
 Graeffe method 301.N
 Gramian (determinant) 103.G 208.E
 grammar
 Chomsky 31.D
 context-free 31.D
 context-sensitive 31.D
 regular 31.D
 Gram-Schmidt orthogonalization 317.A
 Gram theorem 226.E
 grand canonical ensemble 402.D
 grand partition function 402.D
 graph 186.B
 (of a knot) App. A, Table 7
 (= linear graph) 282.A
 (of a mapping) 381.C
 (of a meromorphic mapping) 23.D
 (of an operator) 251.B
 (of a relation) 358.A
 bipartite 186.C
 complete 186.C
 complete bipartite 186.C
 direct 186.B
 Euler 186.F
 Feynman 146.A,B 386.C
 labeled 186.B
 linear 282.A
 oriented 186.B
 partial 186.C
 planar 186.H
 regular 186.C
 reoriented 186.B
 section 186.C
 sub- 186.D
 undirected 186.B
 unicursal, theorem (Euler's) 186.F
 unlabeled 186.B
 unoriented 186.B
 graphic 66.H
 graphical calculation 19.B
 graphical differentiation 19.B
 graphical integration 19.B
 graphical mechanics 19.D
 graphical method of statistical inference 19.B
 graph norm 251.D
 graph theorem, closed 37.I 251.D 424.X
 graph theory 186
 Grashoff number 116.B
 Grassmann algebra (of a linear space) 256.O
 Grassmann coordinates (in a Grassmann manifold) 90.B
 Grassmann manifold 119.B 427.D
 complex 199.B
 formed by oriented subspaces 199.B
 infinite 147.I

Subject Index

Group(s)

real 199.B
 Grauert theorem (on proper holomorphic mappings) 23.E 72.E
 gravitation, law of universal 271.B
 gravitational units, system of 414.B
 gravity, center of 271.E
 gravity wave 205.F
 long 205.F
 grazing ray 325.L
 great circle (of a sphere) 140
 greater than (another compactification) 207.B
 greatest common divisor 67.H 297.A
 greatest element (in an ordered set) 311.B
 greatest lower bound (of an ordered set) 310.C 311.B
 greedy algorithm 66.G
 Greek mathematics 187
 Greek quadratrix 187
 Greek quadrivium 187
 Greek three big problems 187
 Green formula
 (for differential operators) App. A, Table 15.VI
 (for harmonic functions) 193.D
 (for Laplace operator) App. A, Tables 3.III, 4.II
 (for ordinary differential equations) 252.K
 (for partial differential equations of parabolic type) 327.D
 (on the plane) 94.F
 Green function method 402.J
 Green functions 188 189.B
 (α -order) 45.D
 (of a boundary value problem) 315.B
 Green line 193.J
 Green measure 193.J
 Green operator 189.A,B 194.C
 Green space 193.N
 Green-Stokes formula 94.F
 Green tensor 188.E
 Green theorem 105.W
 Griffith first inequality 212.A
 Griffith second inequality 212.A
 Gross area (of a Borel set) 246.G
 Grössencharakter 6.D
 Hecke L -function with 450.F
 Gross theorem 272.I
 Grothendieck category 200.I
 Grothendieck construction 237.B
 Grothendieck criterion of completeness 424.L
 Grothendieck group
 (of a compact Hausdorff space) 237.B
 (of a ring) 237.J
 Grothendieck theorem of Riemann-Roch type 366.D
 Grothendieck topology 16.AA
 ground field (of a linear space) 256.A
 ground form 226.D
 covariant with 226.E
 ground ring (of a module) 277.D
 group(s) 190.A
 Abelian 2 190.A
 Abelian linear 60.L
 absolute homology 201.L
 additive 2.E 190.A
 adele (of an algebraic group) 13.P
 adele (of a linear algebraic group) 6.C
 adjoint (isogenous to an algebraic group) 13.N
 adjoint (of a Lie algebra) 248.H
 adjoint (of a Lie group) 249.P
 of affine transformations 7.E

affine Weyl (of a symmetric Riemann space) 413.G
 algebra 192.H
 algebra class 29.E
 algebraic 13
 algebraic fundamental 16.U
 algebraic homotopy 16.U
 alternating, of degree n 151.G
 automorphism (of a Lie algebra) 248.A
 of automorphisms (of a group) 190.D
 *-automorphism 36.K
 Betti (of a complex) 201.B
 black and white 92.D
 black and white point 92.D
 boundary 234.B
 braid 235.F
 Brauer (of algebra classes) 29.E
 Brauer (of a commutative ring) 29.K
 Bravais 92.B
 bundle (of a fiber bundle) 147.B
 \mathcal{C} 52.M
 of canonical transformations 271.F
 category of 52.B
 cellular homology 201.F,G
 character (of an Abelian group) 2.G
 character (of a topological Abelian group) 422.B
 Chevalley 151.I
 of classes of algebraic correspondences 9.H
 classical 60.A
 Clifford 61.D
 closed 362.J
 coefficient (of a homology group) 201.Q
 cohomology (of a complex) 200.O
 cohomology (of a group) 200.M
 cohomology (of a Lie algebra) 200.O
 cohomology, with coefficient sheaf \mathcal{F} 283.E
 cohomotopy 202.I
 of collineations 343.D
 color point 92.D
 color symmetry (colored symmetry) 92.D
 commutative 2.A 190.A
 commutator (of two elements) 190.H
 commutator (of two subsets of a group) 190.H
 compact 69.A
 completely reducible 190.L
 complex 60.L
 complex cobordism 114.H
 complex orthogonal 60.I
 complex special orthogonal 60.I
 complex symplectic 60.L
 of congruence classes modulo m^* 14.H
 of congruent transformations 285.C
 covering 91.A 423.O
 covering transformation 91.A
 Coxeter 13.R
 crystallographic 92.A
 crystallographic space 92.A
 cyclic 190.C
 decomposition (of a prime ideal) 14.K
 defect (of a block of representations) 362.I
 defect (of a conjugate class in a group) 362.I
 derived (of a group) 190.H
 difference (of an additive group) 190.C
 of differentiable structures on combinatorial spheres App. A, Table 6.I
 differentiable transformation 431.C
 dihedral 151.G
 direct product 190.L
 discontinuous, of the first kind 122.B

discontinuous transformation 122.A
 divisor (of a compact complex manifold) 72.F
 divisor class (of a Riemann surface) 11.D
 elementary topological Abelian 422.E
 elliptic modular 122.D
 equicontinuous, of class (C^0) 378.C
 equivariant J - 431.C
 exponential 437.U
 extension (cohomology of groups) 200.M
 factor 190.C
 finite 151.A 190.C
 finitely generated 190.C
 finitely presented 161.A
 of the first kind 122.C
 formal 13.C
 four- 151.G
 free 161.A
 free product (of the system of groups) 190.M
 frieze 92.F
 Frobenius 151.H
 Fuchsian 122.C
 Fuchsoid 122.C
 full 136.F
 full linear 60.B
 full Poincaré 258.A
 function 234.A
 fundamental (of a topological space) 170
 Galois (of an algebraic equation) 172.G
 Galois (of a Galois extension) 172.B
 Galois (of a polynomial) 172.G
 generalized nilpotent 190.K
 generalized quaternion 151.B
 generalized solvable 190.K
 general linear 60.B 226.B
 general linear (over a noncommutative field) 60.O
 Grothendieck (of a compact Hausdorff space) 237.B
 Grothendieck (of a ring) 237.J
 h -cobordism (of homotopy n -spheres) 114.I,
 App. A, Table 6.I
 Hamilton 151.B
 Hausdorff topological 423.B
 Hilbert modular 32.G
 holonomy 80.D 364.E
 homogeneous holonomy 364.E
 homogeneous Lorentz 359
 homology (of a chain complex) 201.B
 homology (of a group) 200.M
 homology (of a Lie algebra) 200.O
 homology (of a polyhedron) 201.D
 homotopy 202.J
 hyper- 190.P
 icosahedral 151.G
 ideal, modulo m^* 14.H
 ideal class 14.E 67.K
 idele 6.C
 idele class 6.D
 indecomposable 190.L
 inductive limit 210.C
 inductive system of 210.C
 inertia (of a finite Galois extension) 257.D
 inertia (of a prime ideal) 14.K
 infinite 190.C
 infinite classical 147.I 202.V
 infinite orthogonal 202.V
 infinite symplectic 202.V
 infinite unitary 202.V
 inhomogeneous Lorentz 359
 of inner automorphisms (of a group) 190.D

- of inner automorphisms (of a Lie algebra) 248.H
- integral homology (of a polyhedron) 201.D
- integral homology (of a simplicial complex) 201.C
- integral singular homology 201.E
- isotropy 362.B
- J - 237.I
- of Janko-Ree type 151.J
- k - 13.A
- K - (of a compact Hausdorff space) 237.B
- Klein four- 151.G
- Kleinian 122.C 243.A
- knot 235.B
- L - 450.N
- lattice 182.B
- lattice (of a crystallographic group) 92.A
- lattice-ordered Archimedean 243.G
- Lie 249.A 423.M
- Lie transformation 431.C
- linear fractional 60.B
- linear isotropy (at a point) 199.A
- linear simple 151.I
- link 235.D
- little 258.C
- local Lie 423.L
- local Lie, of local transformations 431.G
- locally Euclidean 423.M
- local one-parameter, of local transformations 105.N
- Lorentz 60.J 258 359.B
- magnetic 92.D
- Mathieu 151.H
- matric 226.B
- matrix 226.B
- maximally almost periodic 18.I
- minimally almost periodic 18.I
- mixed 190.P
- Möbius transformation 76.A
- modular 122.D
- monodromy (of an n -fold covering) 91.A
- monodromy (of a system of linear ordinary differential equations) 253.B
- monothetic 136.D
- of motions 139.B
- of motions in the wider sense 139.B
- multiplicative 190.A
- multiplicative (of a field) 149.A 190.B
- Néron-Severi (of a variety) 15.D 16.P
- nilpotent 151.C 190.J
- octahedral 151.G
- Ω 190.E
- one-parameter, of transformations (of a C^∞ -manifold) 105.N
- one-parameter, of transformations of class C^r 126.B
- one-parameter semi-, of class C^0 378.B
- one-parameter sub- 249.Q
- ordered 243.G
- ordered additive 439.B
- of orientation-preserving diffeomorphisms 114.I
- oriented cobordism 114.H
- of oriented differentiable structures on a combinatorial sphere 114.I
- orthogonal 60.I 139.B 151.I
- orthogonal (over a field with respect to a quadratic form) 60.K
- orthogonal (over a noncommutative field) 60.O
- orthogonal transformation 60.I
- of outer automorphisms (of a group) 190.D
- of outer automorphisms (of a Lie algebra) 248.H
- p - 151.B
- periodic 2.A
- permutation 190.B
- permutation, of degree n 151.G
- π - 151.F
- π -solvable 151.F
- Picard (of a commutative ring) 237.J
- Poincaré 170 258.A
- point (of a crystallographic group) 92.A
- polychromatic 92.D
- principal isotropy 431.C
- profinite 210.C
- projective class 200.K
- projective general linear 60.B
- projective limit 210.C
- projective special linear 60.B,O
- projective special unitary 60.H
- projective symplectic 60.L
- projective system of 210.C
- of projective transformations 343.D
- projective unitary 60.F
- proper Lorentz 60.J 258.A 359.B
- proper orthogonal 60.I 258.A
- pseudo- (of topological transformations) 105.Y
- p -torsion, of exceptional groups App. A, Table 6.IV
- q th homology 201.B
- quasi- 190.P
- quasi-Fuchsian 234.B
- quaternion 151.B
- quaternion unimodular 412.G
- quotient 190.C
- quotient (of a topological group) 423.E
- of quotients (of a commutative semigroup) 190.P
- ramification (of a finite Galois extension) 257.D
- ramification (of a prime ideal) 14.K
- rational cohomology 200.O
- reductive 13.Q
- Ree 151.I
- of Ree type 151.J
- regular polyhedral 151.G
- relative homotopy 202.K
- relative singular homology 201.L
- renormalization 111.A
- restricted holonomy 364.E
- restricted homogeneous holonomy 364.E
- Riemann-Roch 366.D
- Riesz 36.H
- rotation 60.I 258.A
- Schottky 234.B
- semi- 190.P 396.A
- separated topological 423.B
- sequence of factor (of a normal chain) 190.G
- shape 382.C
- Siegel modular (of degree n) 32.F
- simple 190.C
- simply connected (isogenous to an algebraic group) 13.N
- singular homology 201.G,L
- solvable 151.D 190.I
- space 92.A
- special Clifford 61.D
- special linear 60.B

special linear (over a noncommutative field)
60.O
special orthogonal 60.I,K
special unitary 60.F,H,O
spinor 60.I 61.D
stability 362.B
stable homotopy 202.T
stable homotopy (of classical group) 202.V
stable homotopy (of the Thom spectrum)
114.G
Steinberg (of a ring) 237.J
structure (of a fiber bundle) 147.B
supersolvable 151.D
Suzuki 151.I
symmetric 190.B
symmetric, of degree n 151.G
symplectic 60.L 151.I
symplectic (over a noncommutative field)
60.O
symplectic transformation 60.L
Tate-Shafarevich 118.D
tetrahedral 151.G
theoretic approach 215.C
Tits simple 151.I
 T_2 -topological 423.B
topological 423
topological Abelian 422.A
topological transformation 431.A
torsion 2.A
torsion (of a finite simplicial complex) 201.B
torus 422.E
totally ordered 243.G
totally ordered additive 439.B
total monodromy 418.F
transformation 431, App. A, Table 14.III
transitive permutation 151.H
of translations 7.E 258.A
of twisted type 151.I
type I 308.L 437.E
underlying (of topological group) 423.A
unimodular 60.B
unimodular locally compact 225.C
unit (of an algebraic number field) 14.D
unitary 60.F 151.I
unitary (over a field) 60.H
unitary (relative to an ε -Hermitian form) 60.O
unitary symplectic 60.L
unitary transformation 60.F
universal covering 91.B 423.O
unoriented cobordism 114.H
value (of a valuation) 439.B,C
vector 422.E
Wall 114.J
WC (Weil-Châtelet) 118.D
weakly wandering under 136.F
web 234.B
weight 92.C
Weil 6.E 450.H
Weil-Châtelet 118.D
Weyl (of an algebraic group) 13.H
Weyl (of a BN pair) 13.R
Weyl (of a Coxeter complex) 13.R
Weyl (of a root system) 13.J
Weyl (of a semisimple Lie algebra) 248.R
Weyl (of a symmetric Riemannian space)
413.F
Weyl, affine 413.F
Weyl, k - 13.Q
White 92.D
Whitehead (of a ring) 237.J

Witt (of nondegenerate quadratic forms)
348.E
Zassenhaus 151.H
group algebra 29.C 36.L
 C^* - 36.L
group code 63.C
group extension 200.M
grouplike 203.F
group manifold (of a Lie transformation group)
110.A
group measure space construction 136.F
group minimization problem 215.C
group object (in a category) 52.M
groupoid 190.P
hyper- 190
group pair (of topological Abelian groups) 422.I
orthogonal 422.I
group ring (of a compact group) 69.A
group scheme 16.H
group system 235.B
group theorem (on fractional ideals) 67.J
group-theoretic approach 215.C
group variety 13.B 16.H
algebraic 13.B
group velocity 446
growth, infra-exponential 125.AA
Grunsky inequality 438.B
Gudermann function (Gudermannian) 131.F,
App. A, Table 16.III
guide, wave 130.B
Guignard constraint qualification 292.B
Gupta-Bleuler formalism 150.G
Gysin exact sequence (of a fiber space) 148.E
Gysin homomorphism 201.O
Gysin isomorphism, Thom- 114.G
of a fiber space 148.E

H

H_p (Hardy spaces) 168.B
 $H^1(\Omega)$ (Sobolev spaces) 168.B
 $H_0^1(\Omega)$ (Sobolev spaces) 168.B
 h -cobordant oriented manifolds 114.I
 h -cobordism group of n -dimensional homotopy
spheres 114.I, App. A, Table 6.I
 h -cobordism theorem 114.F
 H -function 402.B
 H -series, principal 437.X
 H -space 203.D
 H -theorem 402.B
(H, p)-summable 379.M
 H -closed space 425.U
Haag-Araki axioms 150.E
Haag-Kastler axioms 150.E
Haag-Ruelle scattering theory 150.D
Haag theorem 150.C
Haar condition (on best approximation) 336.B
Haar measure
left-invariant 225.C
right-invariant 225.C
Haar space 142.B
Haar system of orthogonal functions 317.C
Haar theorem 225.C
Hadamard estimation App. A, Table 8
Hadamard formula, Cauchy- 339.A
Hadamard gap theorem 339.D
Hadamard multiplication theorem 339.D
Hadamard theorem
(on meromorphic functions) 272.E
(on singularities of power series) 339.D

Hadamard three-circle theorem 43.E
 hadrons 132.B
 Haefliger structure 154.E
 C^* - 154.E
 Hahn-Banach (extension) theorem
 (in a normed space) 37.F
 (in a topological linear space) 424.C
 half Bessel function 39.B
 half-exact (functor) 200.I
 half-life 132.A
 half-line 155.B
 closed (in affine geometry) 7.D
 half-periodic solution (of Hill equation) 268.E
 half-plane 155.B 333.A
 half-space
 of an affine space 7.D
 closed (of an affine space) 7.D
 principal (of a flag) 139.B
 Siegel upper (of degree n) 32.F
 supporting (of a convex set) 89.A
 half-spinor (even, odd) 61.E
 half-spin representation (even, odd) 61.E
 half-trajectory
 negative 126.D
 positive 126.D
 half-width 132.A
 Hall subgroup 151.E
 Hällström-Kametani theorem 124.C
 Halmos theorem, von Neumann- 136.E
 Hamburger moment problem 240.K
 Hamilton canonical equation 271.F
 Hamilton-Cayley theorem 269.F
 Hamilton differential equation 324.E
 Hamilton group 151.B
 Hamiltonian 271.F 351.D 442.D
 bilinear 377.A
 cluster decomposition 375.F
 free 351.D
 Hamiltonian function 219.C 271.F
 Hamiltonian operator 351.D
 Hamiltonian system 126.L
 Hamiltonian vector field 126.L 219.C
 Hamilton-Jacobi differential equation 271.F 324.E
 Hamilton-Jacobi equation 108.B
 Hamilton path 186.F
 Hamilton principle 441.B
 Hamilton quaternion algebra 29.B
 Hammerstein integral equation 217.M
 Hamming bound (of a code) 63.B
 Hamming code 63.C
 Hamming distance 63.B 136.E
 handle 410.B
 attaching 114.F
 Casson 114.K
 manifold with 114.F
 handlebody 114.F
 Hankel asymptotic representation App. A,
 Table 19.III
 Hankel determinant 142.E
 Hankel functions 39.B, App. A, Table 19.III
 Hankel transform 220.B
 Hansen, general connectedness theorem due to
 Fulton and 16.I
 Hansen-Bessel formula App. A, Table 19.III
 hard, NP- 71.E
 hard Lefschetz theorem 450.Q
 hardware 75.C
 Hardy class 43.F 159.G
 generalized 164.E
 Hardy inequality App. A, Table 8

Subject Index

Hasse norm-residue symbol, Hilbert-

Hardy-Littlewood-Sobolev inequality 224.E
 Hardy-Littlewood supremum theorem App. A,
 Table 8
 Hardy-Littlewood theorem
 on bounded functions 43.E
 on trigonometric systems 317.B
 Hardy space 168.B
 Hardy theorem
 on bounded functions 43.E
 on the Cauchy product of two series 379.F
 harmonic
 (form) 194.B
 (function) 193.A
 (function on a state space) 260.D
 (mapping) 195.B
 harmonically separated points (in a projective space)
 343.D
 harmonic analyzer 19.E
 harmonic analysis 192
 harmonic boundary 207.B
 harmonic conjugates 343.D
 harmonic continuation 193.M 198.G
 harmonic differential (on a Riemann surface) 367.H
 harmonic dimension (of a Heins end) 367.E
 harmonic flow 193.K
 harmonic functions 193
 conjugate 193.C
 spherical 193.C
 harmonic integrals 194
 harmonic kernel function 188.H
 harmonic majorant (of subharmonic function)
 193.S
 harmonic mapping 195
 harmonic mean
 (of a distribution) 397.C
 (of a function) 211.C
 (of numbers) 211.C
 harmonic measure
 inner 169.B
 outer 169.B
 harmonic motion, simple 318.B
 harmonic oscillation 318.B
 harmonic range of points 343.D
 harmonics
 ellipsoidal 133.B
 ellipsoidal, of four species 133.C
 solid 393.A
 spherical 193.C 393.A
 surface 393.A
 tesseral 393.D
 zonal 393.D
 Harnack condition (on the D-integral) 100.E
 Harnack first theorem 193.I
 Harnack lemma 193.I
 Harnack second theorem 193.I
 Hartogs continuation theorem 21.F
 Hartogs-Osgood theorem 21.H
 Hartogs theorem
 of continuity 21.H
 of holomorphy 21.C
 Hartshorne conjecture 16.R
 Harvey duality, Martineau- 125.Y
 hashing 96.B
 Hasse character, Minkowski- (of a nondegenerate
 quadratic form) 348.D
 Hasse conjecture 450.S
 Hasse function, Artin- 257.H
 Hasse invariant (of a central simple algebra over
 a p -adic field) 29.G
 Hasse norm-residue symbol, Hilbert- 14.R

- Hasse principle 348.G
- Hasse theorem, Minkowski- (on quadratic forms over algebraic number fields) 348.G
- Hasse-Witt matrix 9.E
- Hasse zeta function 450.S
- Haupt theorem 268.E
- Hauptvermutung (in topology) 65.C 70.C
- Hausdorff axiom 425.Q
- Hausdorff dimension 117.G 234.E 246.K
- Hausdorff formula, Campbell- 249.R
- Hausdorff measure 169.D
- Hausdorff moment problem 240.K
- Hausdorff space 425.Q
 - collectionwise 425.AA
- Hausdorff theorem, Baire- 273.J 425.N
- Hausdorff topological group 423.B
- Hausdorff uniform space 436.C
- Hausdorff-Young inequality 224.E
- Hausdorff-Young theorem 317.B
- Hawaiian earring 79.C
- hazard function 397.O
- hazard rate 397.O
- heat
 - Joule 130.B
 - specific, at constant pressure 419.B
 - specific, at constant volume 419.B
- heat bath 419.B
- heat conduction, equation of 327.A
- heat equation 327.A, App. A, Table 15.VI
- heat gain 419.A
- heat loss 419.A
- Heaviside calculus 306.A
- Heaviside function 125.E 306.B, App. A, Table 12.II
- Hecke algebra 29.C 32.D
 - global 450.O
- Hecke character 6.D
- Hecke L -function 450.E
 - with Grössencharakter 450.F
- Hecke operator 32.D
 - zeta function defined by 450.M
- Hecke ring 32.D
- Heegaard decomposition 65.C
- height
 - (of an algebraic number) 430.B
 - (of an element in a lattice) 243.F
 - (of an ideal) 67.E
 - (of an isogeny) 3.F
 - (of a lattice) 243.F
 - (of a prime ideal) 67.E
 - infinite (element of an Abelian p -group) 2.D
- Heilbronn phenomenon, Deuring- 123.D
- Heine-Borel theorem 273.F
- Heine series 206.C
- Heins end 367.E
- Heisenberg equation of motion 351.D
- Heisenberg picture 351.D
- Heisenberg uncertainty relation 351.C
- helicity 258.C
- helicoid
 - ordinary 111.I
 - right 111.I
- helicoidal surface 111.I
- Helinger-Hahn theorem 390.G
- helix
 - generalized 111.F
 - ordinary 111.F
- Helly theorem 94.B
- Helmholtz differential equation 188.D, App. A, Table 15.VI
- Helmholtz free energy 419.C
- minimum principle 419.C
- Helmholtz theorem (on vector fields) 442.D, App. A, Table 3.III
- Helmholtz vorticity theorem 205.B
- Helson set 192.R
- hemisphere
 - northern 140
 - southern 140
- Henselian ring 370.C
- Henselization 370.C
- Hensel lemma 118.C
- Hensel ring 370.C
- Herbrand lemma 200.N
- Herbrand quotient 200.N
- hereditarily normal space 425.Q
- hereditarily quotient mapping 425.G
- hereditarily weak topology 425.M
- hereditary finite set 33.B
- hereditary ring 200.K
 - left 200.K
 - right 200.K
- Herglotz integral representation 43.I
- Herglotz theorem 192.B
- Hermite differential equation App. A, Table 14.II
- Hermite differential equation, Weber- 167.C
- Hermite formula, Gauss- (in numerical integration) 299.A
- Hermite interpolation polynomial 223.E
- Hermite polynomials 317.D, App. A, Table 20.IV
- Hermite polynomials, Fourier- 176.I
- Hermitian (integral operator) 251.O
- Hermitian form 256.Q 348.F
 - anti- 256.Q
 - ε - 60.O
 - indefinite 348.F
 - negative definite 348.F
 - positive definite 348.F
 - semidefinite 398.F
 - skew- 256.Q
 - v - 384.A
- Hermitian homogeneous space, complex 199.A
- Hermitian hyperbolic space 412.G
- Hermitian inner product 256.Q
- Hermitian kernel 217.H
- Hermitian linear space 256.Q
- Hermitian matrix 269.I
 - anti- 269.I
 - skew 269.I
- Hermitian metric 232.A
- Hermitian operator 251.E
- Hermitian space
 - irreducible symmetric 412.E
 - symmetric 412.E
- Heron formula
 - (for plane triangles) App. A, Table 2.II
 - (for spherical triangles) App. A, Table 2.III
- Hersch and Pfluger, extremal length defined by 143.A
- Hersch problem 391.E
- Hessenberg method 298.D
- Hesse normal form (of a hyperplane) 139.H
- Hessian
 - (on a differential manifold) 279.B,F
 - (form) 226.D
 - (of a plane algebraic curve) 9.B
- heterogeneity, design for two-way elimination of 102.K
- heuristic algorithm 215.E
- Hewitt-Savage zero-one law 342.G
- hexagon 155.F

- hexagonal (system) 92.E
- hexahedron 357.B
- Hey zeta function 27.F
- hidden variable theories 351.L
- hierarchy 356.H
 - analytic 356.H
 - arithmetical 356.H
 - arithmetical, of degrees of recursive unsolvability 356.H
 - C-analytic 356.H
 - C-arithmetical 356.H
 - Chomsky 31.D
 - hyperarithmetical, of degrees of recursive unsolvability 356.H
- hierarchy theorem 356.H
- Higgs mechanism 132.D
- higher algebraic K -theory 237.J
- higher differentiation (in a commutative ring) 113
- higher order, of (for infinitesimals) 87.G
- higher-order derivative (of a differentiable function) 106.D
- higher-order ordinary differential equation
 - App. A, Table 14.I
- higher-order partial derivative 106.G
- higher-order predicate logic 411.K
- higher-transcendental function 389.A
- highest weight (of a representation of a Lie algebra) 248.W
- high-precision computation 138.B
- Hilbert, D. 196
- Hilbert basis theorem 284.A
- Hilbert characteristic function
 - (of a coherent sheaf) 16.E
 - (of a graded module) 369.F
- Hilbert cube 382.B
- Hilbert ε -operator 411.J
- Hilbert ε -quantifier 411.J
- Hilbert ε -symbol 411.J
- Hilbert fifth problem 423.N
- Hilbert-Hasse norm-residue symbol 14.R
- Hilbertian space, countably 424.W
- Hilbert inequality App. A, Table 8
- Hilbert invariant integral 46.C
- Hilbert irreducibility theorem (on polynomials) 337.F
- Hilbert manifold 105.Z 286.K
- Hilbert modular form
 - of dimension $-k$ 32.G
 - of weight k 32.G
- Hilbert modular function 32.G
- Hilbert modular group 32.G
- Hilbert modular surface 15.H
- Hilbert norm-residue symbol 14.R
- Hilbert Nullstellensatz 369.D
- Hilbert polynomial
 - (of an algebraic curve) 9.F
 - (of a coherent sheaf) 16.E
 - (of a graded module) 369.F
- Hilbert problem (in calculus of variations) 46.A
- Hilbert problem, Riemann-
 - (for integral equations) 217.J
 - (for ordinary differential equations) 253.D
- Hilbert scheme 16.S
- Hilbert-Schmidt class 68.I
- Hilbert-Schmidt expansion theorem 217.H
- Hilbert-Schmidt norm 68.I
- Hilbert-Schmidt type
 - integral operator of 68.C
 - kernel of 217.I
- Hilbert spaces 197
- adjoint 251.E
- channel 375.F
- complex 197.B
- exponential 377.D
- physical 150.G
- pre- 197.B
- real 197.B
- rigged 424.T
- Hilbert-Speiser theorem 172.J
- Hilbert system of axioms (foundations of geometry) 155.B
- Hilbert syzygy theorem 369.F
- Hilbert theorem 90 172.J
- Hilbert transform 160.D 220.E
- Hilbert zero point theorem 369.D
- Hilferty approximations, Wilson- 374.F
- Hill determinant 268.B
- Hill determinantal equation 268.B
- Hill differential equation 268.B
- Hille-Yosida theorem 378.B
- Hill function 268.E
- Hill method of solution 268.B
- Hirsch theorem, Leray- 201.J
- Hirzebruch index theorem (for differentiable manifolds) 56.G
- Hirzebruch signature theorem (on algebraic surface) 72.K
- Hirzebruch surface 15.G
- Hirzebruch theorem of Riemann-Roch type 366.B
- histogram 397.B
- Hitchcock method 301.E
- hitting probability 5.G
- hitting time 260.B 261.B 407.B
- Hlawka theorem, Minkowski- 182.D
- Hochschild cohomology group 200.L
- Hodge conjecture 450.S
- Hodge index theorem 15.D
- Hodge manifold 232.D
- Hodge metric 232.D
- Hodges-Lehmann theorem 399.E,H
- Hodge spectral sequence 16.U
- Hodge structure (of a vector space) 16.V
 - mixed 16.V
 - polarized 16.V
- Hodgkin-Huxley differential equation 291.F
- hodograph method 205.B
- hodograph plane 205.B
- hold almost everywhere (in a measure space) 270.D
- hold at almost all points
 - in a measure space 270.D
- Hölder condition of order α 84.A
- Hölder inequality 211.C, App. A, Table 8
- Hölder integral inequality 211.C
- Hölder method of order p , summable by 379.M
- Hölder sequence, Jordan- (in a group) 190.G
- Hölder space 168.B
- Hölder theorem 104.F
- Hölder theorem, Jordan- (in group theory) 190.G
- Hölder theorem, Jordan- (on representations of algebras) 362.D
- hole theory, Dirac 415.G
- Holmgren type theorem (of Kashiwara-Kawai) 125.DD
- Holmgren uniqueness theorem 321.F
- holohedral 92.B
- holohedry 92.B
- holomorphic
 - (family of linear operators) 331.C
 - (function) 198.A
 - (in the sense of Riemann) 21.C

Holomorphically complete domain

- (vector-valued function) 37.K
- holomorphically complete domain 21.F
- holomorphically complete space 23.F
- holomorphically convex domain 21.H
- holomorphic automorphism 21.J
- holomorphic differential (on a Riemann surface) 367.H
- holomorphic differential form of degree k 72.A
- holomorphic distribution (with respect to a parameter) 125.H
- holomorphic evolution operator 378.I
- holomorphic foliation 154.H
- holomorphic function(s) 198
 - (on a complex manifold) 72.A
 - (of many variables) 21.A,C
 - germ of a 21.E
 - sheaf of germs of 23.C 383.D
- holomorphic functional calculus 36.M
- holomorphic hull 21.H
- holomorphic k -form 72.A
- holomorphic local coordinate system 72.A
- holomorphic mapping 21.J
 - (of a complex manifold) 72.A
 - nondegenerate (between analytic spaces) 23.C
- holomorphic microfunction 274.F
- holomorphic modification (of an analytic space) 23.D
- holomorphic part (in a Laurent expansion) 198.D
- holomorphic sectional curvature 364.D
- holomorphic semigroup 378.D
- holomorphic tangent vector 72.A
- holomorphic vector field 72.A
- holomorphy
 - domain of 21.F
 - envelop of 21.F
 - Hartogs theorem of 21.C
- holonomic (coherent \mathcal{E} -module) 274.H
- holonomic systems
 - with regular singularities 274.H
 - simple 274.H
- holonomy 154.C
- holonomy group 80.D 154.C 364.D
 - homogeneous 364.E
 - restricted 80.D 364.E
 - restricted homogeneous 364.E
- holonomy homomorphism 154.C
 - linear 154.C
- holosymmetric class 92.B
- homentropic flow 205.B
- homeomorphic 425.G
- homeomorphism 425.G
 - minimal 136.H
 - PL 65.A
 - strictly ergodic 136.H
 - uniquely ergodic 136.H
- homeomorphism problem 425.G
- homoclinic point 126.J
 - transversal 126.J
- homogeneous
 - (A -submodule) 200.B
 - (boundary value problem) 315.B
 - (difference equation) 104.C
 - (lattice) 182.B
 - (linear ordinary differential equation) 252.A
 - (system of linear differential equations of the first order) 252.G
 - spatially (process) 261.A
 - temporally (additive process) 5.B
 - temporally (process) 261.A
 - weighted (analytic function) 418.D
- homogeneous bounded domain 384.A 412.F
- homogeneous coordinate ring 16.A
- homogeneous coordinates 343.C
- homogeneous difference equation 104.C
- homogeneous element
 - of a graded ring 369.B
 - of a homogeneous ring 369.B
- homogeneous equations, system of linear 269.M
- homogeneous holonomy group 364.E
 - restricted 364.E
- homogeneous hypersurface 344.A
- homogeneous ideal
 - of a graded ring 369.B
 - of a polynomial ring 369.B
- homogeneous integral equation 217.F
- homogeneous Lorentz group 258.A 359
- homogeneously regular 275.C
- homogeneous Markov process 5.H 261.A
- homogeneous n -chain (for a group) 200.M
- homogeneous ordinary differential equation
 - App. A, Table 14.I
 - of higher order App. A, Table 14.I
- homogeneous part (of a formal power series) 370.A
- homogeneous polynomial 337.B
- homogeneous ring 369.B
- homogeneous Siegel domain, irreducible 384.E
 - topology of Lie groups and 427
- homogeneous space(s) 199 249.F 362.B
 - complex Hermitian 199.A
 - Kähler 199.A
 - linearly connected 199.A
 - reductive 199.A
 - Riemannian 199.A
 - symmetric 412.B
 - symmetric Riemannian 412.B
- homogeneous turbulence 433.C
- homological algebra 200
 - relative 200.K
- homological dimension
 - of a module 200.K
 - of a topological space 117.F
- homological functor 200.I
- homological mapping 200.C
- homologous 201.B
- homologous to zero 198.B
- homology 200.H
 - intrinsic 114.H
- homology basis, canonical 11.C
- homology class 200.H
 - fundamental 201.N
 - fundamental, around K 201.N
 - q -dimensional 201.B
- homology exact sequence 201.L
 - (of fiber space) 148.E
 - reduced 201.F
- homology group(s)
 - (of a chain complex) 201.B
 - (of a group) 200.M
 - (of a Lie algebra) 200.O
 - (of a polyhedron) 201.D
 - (of a simplicial complex) 201.G
 - absolute 201.L
 - Čech 201.M
 - cellular 201.F,G
 - with coefficients in G 201.G
 - integral 201.C,D
 - integral singular 201.E
 - local 201.N
 - reduced 201.E
 - relative Čech 201.M

- relative singular 201.L
- simplicial 201.D
- singular 201.G,L,R
- homology manifold 65.B
- homology module 200.C
- homology theory 201
 - generalized 201.Q
 - generalized, with E-coefficient 202.T
 - uniqueness theorem of 201.Q
- homomorphic
 - (algebraic system) 409.C
 - (groups) 190.D
 - (topological groups) 423.J
 - order (ordered sets) 311.E
- homomorphic image (of a measure-preserving transformation) 136.D
- homomorphism
 - (of Abelian varieties) 3.C
 - (of algebraic systems) 409.C
 - (of fields) 149.B
 - (of groups) 190.D
 - (of lattices) 243.C
 - (of Lie algebras) 248.A
 - (of linear representations) 362.C
 - (of presheaves) 383.A
 - (of rings) 368.D
 - (of sheaves) 363.B
 - A -(of A -modules) 277.E
 - A -, of degree p (of graded A -modules) 200.B
 - admissible (of Ω -groups) 190.E
 - algebra 29.A
 - allowed (of A -modules) 277.E
 - analytic (of Lie groups) 249.N
 - anti- (of groups) 190.D
 - anti- (of rings) 368.D
 - bialgebra 203.G
 - Bokstein 64.B
 - boundary (on homology groups) 201.L
 - boundary (in homotopy exact sequences) 202.L
 - C^∞ -(between Lie groups) 249.N
 - canonical (on direct products of rings) 368.E
 - coalgebra 203.F
 - coboundary (on cohomology groups) 201.L
 - connecting (in homology) 200.C 201.L
 - connecting (on homology groups) 201.C
 - continuous (of topological groups) 423.J
 - crossed (of an associative algebra) 200.L
 - dual (of a homomorphism of algebraic tori) 13.D
 - dual (of lattices) 243.C
 - edge 200.J
 - equivariant J - 431.F
 - generalized Hopf 202.V
 - Gysin 201.O
 - holonomy 154.C
 - Hopf algebra 203.H
 - Hurewicz 202.N
 - induced by a continuous mapping (between homotopy groups) 202.K
 - J -(in homotopy theory) 202.V
 - J -(in K -theory) 237.I
 - Jordan (of Jordan algebras) 231.A
 - lattice- 243.C
 - local (of a topological group) 423.O
 - module of (of modules) 277.B
 - module of A -(of A -modules) 277.E
 - Ω -(of Ω -groups) 190.E
 - open continuous (of topological groups) 423.J
 - operator (of A -modules) 277.E
 - operator (of Ω -groups) 190.E
 - order 311.E
 - rational (of Abelian varieties) 3.C
 - rational (of algebraic groups) 13.A
 - ring 368.D
 - $*$ - 36.F
 - Umkehr 201.O
 - unitary (of rings) 368.D
 - zero (of two A -modules) 277.H
 - homomorphism theorem
 - on groups 190.D
 - on Lie algebras 248.A
 - on topological groups 423.J
 - on topological linear spaces 424.X
 - homothetic correspondence (between surfaces) 111.I
 - homothety
 - in conformal differential geometry 110.D
 - in Euclidean geometry 139.B
 - homotopic 154.E,F 202.B
 - chain (chain mappings) 200.C
 - integrably 154.F
 - null- (continuous mapping) 202.B
 - regularly (immersions) 114.D
 - relative to a subspace 202.B
 - to zero 202.B
 - homotopy 202.B
 - composite 202.B
 - free 202.B
 - linear 114.D
 - restricted 202.B
 - homotopy-associative (multiplication) 203.D
 - homotopy category of topological spaces 52.B
 - homotopy chain 200.C
 - homotopy class 202.B
 - compact 286.D
 - homotopy cochain 200.F
 - homotopy commutative (multiplication) 203.D
 - homotopy equivalence 202.F
 - simple 65.C
 - weak 202.F
 - homotopy equivalent systems (of topological spaces) 202.F
 - homotopy exact sequence 202.L
 - of a fiber space 148.D
 - of a triad 202.M
 - of a triple 202.L
 - homotopy extension property 202.E
 - homotopy group(s) 202.J
 - algebraic 16.U
 - of a compact connected Lie group App. A, Table 6.VII
 - realization theorem of 202.N
 - of a real Stiefel manifold App. A, Table 6.VII
 - relative 202.K
 - of a sphere App. A, Table 6.VI
 - stable 202.T, App. A, Table 6.VII
 - stable (of classical groups) 202.V
 - stable (of the k -stem) 202.U
 - stable (of Thom spectrum) 114.G
 - of a triad 202.M
 - homotopy identity (of an H -space) 203.D
 - homotopy invariance (of a homology group) 201.D
 - homotopy invariant 202.B
 - homotopy inverse (for an H -space) 203.D
 - homotopy n -spheres
 - group of 114.I
 - h -cobordism group of 114.I
 - homotopy operations 202.O
 - homotopy set 202.B

- homotopy sphere 65.C
- homotopy theorem
 - first (in obstruction theory) 305.B
 - second (in obstruction theory) 305.C
 - simple 65.C
 - third (in obstruction theory) 305.C
- homotopy theory 202
 - de Rham 114.L
- homotopy type 202.F
 - (of a link) 235.D
 - spherical G -fiber 431.F
- homotopy type invariant 202.F
- Hooke law 271.G
- Hopf algebra(s) 203
 - dual 203.C
 - elementary 203.D
 - graded 203.C
- Hopf algebra homomorphism 203.H
- Hopf bifurcation 126.M
- Hopf bundle 147.E
- Hopf classification theorem 202.I
- Hopf comultiplication 203.D
- Hopf coproduct 203.D
- Hopf extension theorem 270.E
- Hopf fibering 147.E
- Hopf homomorphism, generalized (of homotopy groups of spheres) 202.U
- Hopf integrodifferential equation, Wiener- 222.C
- Hopf invariant 202.U
 - generalized 202.Q
 - modulo p 202.S
- Hopf mapping (Hopf map) 147.E
- Hopf surface 72.K
- Hopf theorem (continuous vector field) 153.B
- Hopf weak solution 204.C
- horizon, event 359.F
- horizontal components
 - of a homogeneous space 110.A
 - of a vector field 80.C
- horizontal slit mapping, extremal 367.G
- horizontal subspace 191.C
- horizontal vector (in a differentiable principal fiber bundle) 80.C
- Hörmander theorem 112.C,D
- horned sphere, Alexander 65.G
- Horner method 301.C
- horocycle flow 136.G
- horosphere 218.G
- horseshoe diffeomorphism 126.J
- Hosokawa polynomial 235.D
- Hotelling T^2 statistic 280.B
 - noncentral 374.C
- Householder method 298.D
- Householder transformation 302.E
- Hugoniot relation, Rankine- 204.G 205.B
- Hukuhara theorem, Dini- 314.D
- Hukuhara problem 315.C
- hull
 - closed convex 424.H
 - convex 89.A
 - convex (in an affine space) 7.D
 - convex (of a boundary curve) 275.A
 - convex (in linear programming) 255.D
 - holomorphic 21.H
- hull-kernel topology 36.D
- human death and survival, model of 214.A
- Hunt process 261.B
- Hunt-Stein lemma 400.F
- Hurewicz homomorphism 202.N
- Hurewicz isomorphism theorem 202.N
- Hurewicz-Steenrod isomorphism theorem 148.D
- Hurewicz theorem, generalized 202.N
- Hurewicz-Uzawa gradient method, Arrow- 292.E
- Hurwitz formula, Riemann- (on coverings of a nonsingular curve) 9.I
- Hurwitz relation
 - (on homomorphisms of Abelian varieties) 3.K
 - Riemann- 367.B
- Hurwitz theorem 10.E
- Hurwitz zeta function 450.B
- Huxley differential equation, Hodgkin- 291.F
- Huygens principle 325.B 446
 - in the wider sense 325.D
- hybrid computer 19.E
- hydrodynamics 205
- hydromagnetic dynamo theory 259
- hydromagnetics 259
- hydrostatics 205.A
- hyperalgebra 203.I
- hyperarithmetical function 356.H
- hyperarithmetical hierarchy of degrees of recursive unsolvability 356.H
- hyperarithmetical predicate 356.H
- hyperbola 78.A
 - conjugate 78.E
 - equilateral 78.E
 - rectangular 78.E
- hyperbolic
 - (closed invariant set of a dynamical system) 126.J
 - (differential operator) 112.A 325.H
 - (linear mapping) 126.G
 - (partial differential equation) 325.A,E
 - (Riemann surface) 367.D,E
 - (simply connected domain) 77.B
 - (space form) 412.H
 - complete 21.O
 - regularly 325.A,F
 - in the sense of Gårding 325.F
 - in the sense of Petrovskii 325.F
 - in the strict sense 325.F
 - strongly 325.H
 - symmetric (in the sense of Friedrichs) 325.G
 - weakly 325.H
- hyperbolically embedded 21.O
- hyperbolic closed orbit 126.G
- hyperbolic coordinates
 - equilateral 90.C, App. A, Table 3.V
 - rectangular 90.C
- hyperbolic cosecant 131.F
- hyperbolic cosine 131.F
- hyperbolic cotangent 131.F
- hyperbolic cylinder 350.B
- hyperbolic cylindrical coordinates App. A, Table 3.V
- hyperbolic cylindrical surface 350.B
- hyperbolic differential equations, system of (in the sense of Petrovskii) 325.G
- hyperbolic-elliptic motion 420.D
- hyperbolic fixed point 126.G
- hyperbolic function 131.F
- hyperbolic geometry 285.A
- hyperbolic knot 235.E
- hyperbolic manifold 21.O 235.E
- hyperbolic motion 420.D
- hyperbolic-parabolic motion 420.D

hyperbolic paraboloid 350.B
 hyperbolic plane 122.C
 hyperbolic point (on a surface) 111.H
 hyperbolic quadric hypersurface 350.I
 hyperbolic secant 131.F
 hyperbolic sine 131.F
 hyperbolic singular point 126.G
 hyperbolic space 285.C
 Hermitian 412.G
 quaternion 412.G
 real 412.G
 hyperbolic spiral 93.H
 hyperbolic tangent 131.F
 hyperbolic transformation 76.F
 hyperbolic type, partial differential equation of 321.E 325
 hyperbolic type, primitive 92.C
 hyperboloidic position 350.B
 hyperboloid of one sheet 350.B
 hyperboloid of revolution of one or two sheets 350.B
 hyperboloid of two sheets 350.B
 hypercohomology 200.J
 hyperconstructive ordinal 81.E
 hypercubic type, primitive 92.C
 hyperelliptic curve 9.D
 hyperelliptic integral 11.C
 hyperelliptic Riemann surface 11.C
 hyperelliptic surface 72.K
 hyperfinite 293.B 308.I
 hyperfunction 125
 in the Dirichlet problem 120.C
 exponentially decreasing Fourier 125.BB
 Fourier 125.BB
 Fourier ultra- 125.BB
 modified Fourier 125.BB
 Sato 125.V
 hypergeometric differential equation 260.A,
 App. A, Table 18.I
 confluent 167.A, App. A, Tables 14.II, 19.I
 Gauss App. A, Table 14.II
 hypergeometric distribution 341.D 397.F,
 App. A, Table 22
 multidimensional App. A, Table 22
 multiple 341.D
 hypergeometric function(s) 209, App. A, Table 18.I
 Appell, of two variables 206.D, App. A,
 Table 18.I
 Barnes extended 206.G, App. A, Table 18.I
 of confluent type 167.A, App. A, Table 19.I
 of the hyperspherical differential equation 393.E
 with matrix argument 206.E
 and spherical functions App. A, Table 18
 hypergeometric integral 253.B
 hypergeometric series 206.A
 hypergeometric type, special function of 389.A
 hypergroup 190.P
 hypergroupoid 190.P
 hyperinvariant (under an operator) 251.L
 hyperplanar symmetry (of an affine space) 139.B
 hyperplane(s)
 in an affine space 7.A
 characteristic (of a partial differential equation
 of hyperbolic type) 325.A
 at infinity (in affine geometry) 7.B
 pencil of (in a projective space) 343.B
 in a projective space 343.B

Subject Index

Hypotrochoid

regression 403.D
 tangent (of a quadric hypersurface) 343.E
 hyperplane coordinates
 of an affine frame 7.C
 in projective geometry 343.C
 hyperplane section 418.I
 hyperquadric
 in an affine space 350.G
 in a projective space 343.D 350.I
 hypersonic flow 205.C
 hypersphere 76.A
 imaginary 76.A
 limiting (in hyperbolic geometry) 285.C
 non-Euclidean 285.C
 oriented real 76.A
 point 76.A
 proper (in hyperbolic geometry) 285.C
 real 76.A
 hypersphere geometry 76.A
 hyperspherical coordinates, $(n+2)$ - 76.A 90.B
 hyperspherical differential equation 393.E
 hypersurface(s)
 (of an algebraic variety) 16.A
 (in a Euclidean space) 111.A
 central quadric 350.G
 characteristic (of a partial differential equation
 of hyperbolic type) 325.A
 coordinate (in a Euclidean space) 90.C
 elliptic quadric 350.G
 homogeneous 344.A
 hyperbolic quadric 350.G
 integral (partial differential equations) 320.A
 noncentral quadric 350.G
 nondegenerate 344.A
 parabolic quadric 350.G
 pencil of quadric 343.E
 properly $(n-1)$ -dimensional quadric 350.G
 quadric 343.D 350.G, I
 quadric conical 350.Q
 quadric cylindrical 350.Q
 regular quadric 343.E
 singular quadric (of the h th species) 343.E
 spherical real 344.C
 hypersurface element(s) 82.A 324.B
 union of 82.A
 hypocontinuous (bilinear mapping) 424.Q
 hypocycloid 93.H
 hypo-Dirichlet 164.B
 hypoelliptic 112.D 189.C 323.I
 analytically 112.D 323.I
 hypofunction (in the Dirichlet problem) 120.C
 hyponormal 251.K
 hypothesis
 alternative 400.A
 composite 400.A
 continuum \rightarrow continuum hypothesis
 ergodic 136.A 402.C
 general linear 400.H
 Lindelöf 123.C
 null 400.A
 Riemann 450.B, P
 simple 400.A
 statistical 400.A
 Suslin 33.F
 hypothesis testing 401.C, App. A, Table 23
 statistical 400, App. A, Table 23
 hypothetical infinite population 397.P
 hypotrochoid 93.H

i-genus

I

i-genus 15.E*I*-adic topology (of a ring) 16.X*i*th component (of an *n*-tuple) 256.A,C*i*th coordinate 256.C*i*th coordinate axis (of a Euclidean space) 140

icosahedral group 151.G

icosahedron 357.B

ideal(s)

(of an algebra) 29.A

(of an algebraic number field) 14.B

(of a lattice) 42.C

(of a Lie algebra) 248.A

(of a ring) 368.F

Abelian (of a Lie algebra) 248.C

Alexander (of a knot) 235.C

ambig (of a quadratic field) 347.F

conjugate (of a fractional ideal) 14.I

defining (of a formal spectrum) 16.X

differential (of a differential ring) 113

differential (on a real analytic manifold) 428.E

elementary 235.C

fractional (of an algebraic number field) 14.E

graded 369.B

homogeneous (of a graded ring) 369.B

homogeneous (of a polynomial ring) 369.B

integral (of an algebraic number field) 14.C

integral left 27.A

integral right 27.A

integral two-sided α - 27.A

involutive differential 428.E

largest nilpotent (of a Lie algebra) 248.D

left (of a ring) 368.F

left α - 27.A

maximal 67.C

maximal (left or right) 368.F

maximal (with respect to *S*) 67.C

maximal, space (of a Banach algebra) 36.E

minimal (left or right) 368.F

mixed 284.D

nilpotent (of a Lie algebra) 248.C

order (of a vector lattice) 310.B

 \mathfrak{p} -primary 67.F

primary 67.F

prime 67.C

prime (of a maximal order) 27.A

prime differential (of a differential ring) 113

primitive (of a Banach algebra) 36.E

principal 67.K

principal (of an algebraic number field) 14.E

principal, theorem (in class field theory) 59.D

principal fractional 67.K

pure 284.D

right (of a ring) 368.F

right α , 27.A

semiprime (of a commutative ring) 113

semiprime differential (of a differential ring)

113

sheaf of (of a divisor of a complex manifold)

72.F

solvable (of a Lie algebra) 248.C

two-sided (of a ring) 368.P

two-sided α - 27.A

unmixed 284.D

valuation (of a valuation) 439.B

ideal boundary 207.A

in the narrow sense 14.G

ideal class 14.E

ideal class group 14.E 67.K

ideal group modulo m^* 14.H

ideal point (in hyperbolic geometry) 285.C

idele (of an algebraic number field) 6.C

principal 6.C

idele class 6.D

idele class group 6.D

idele group 6.C

idempotent element (of a ring) 368.B

elementary 450.O

primitive 368.B

idempotent law (in a lattice) 243.A

idempotent measure 192.P

idempotent set (of a ring) 368.B

idempotent theorem 36.M

identically true formula 411.G

identification (in factor analysis) 280.G

identification space (by a partition) 425.L

identified equation 128.C

just 128.C

over- 128.C

identity (identities) 231.A

Bianchi 80.J 417.B

Bianchi first App. A, Table 4.II

Bianchi second App. A, Table 4.II

homotopy (of an *H*-space) 203.D

Jacobi (on the bracket of two vector fields)

105.M

Jacobi (in a Lie algebra) 248.A

Jacobi (with respect to Whitehead product)

202.P

in Jordan algebras 231.A

Lagrange 252.K

Parseval 18.B 159.A 160.C 192.K 197.C

220.B,C,E

resolution of 390.D

theorem of (of one variable) 198.C

theorem of (of several variables) 21.C

identity character (of an Abelian group) 2.G

identity component (of a topological group) 423.F

identity element

of an algebraic system 409.C

of a field 149.A

of a group 190.A

of a local Lie group 423.L

of a ring 368.A

identity function 381.C

identity mapping 381.C

identity matrix 269.A

identity operator (in a linear space) 37.C

identity relation 102.I

Ihara zeta function 450.U

Ikehara-Landau theorem, Wiener- 123.B

ill-conditioned (coefficient matrix in numerical

solution of linear equations) 302.D

image

(of a group homomorphism) 190.D

(of a linear mapping) 256.F

(of a line in P^3) 343.E

(of a mapping) 381.C

(of a morphism) 52.N

(of an operator homomorphism) 277.E

(of a sheaf homomorphism) 383.D

closed (of a variety) 18.I

continuous 425.G

direct (of a sheaf) 383.G

homomorphic (of a measure preserving

transformation) 136.D

inverse (of a set) 381.C

inverse (of a sheaf) 383.G

inverse (of a uniformity) 436.E

perfect 425.CC
 perfect inverse 425.CC
 image measure 270.K
 imaginary axis 74.C
 imaginary field, totally 14.F
 imaginary hypersphere 76.A
 imaginary infinite prime divisor 439.H
 imaginary number 74.A
 purely 74.A
 imaginary part 74.A
 imaginary prime divisor 439.H
 imaginary quadratic field 347.A
 imaginary root (of an algebraic equation) 10.E
 imaginary transformation, Jacobi's 134.I, App. A, Table 16.III
 imaginary unit 74.A 294.F
 imbedded Markov chain 260.H
 imbedding 105.K
 imbedding principle 127.B
 immersed submanifold (of a Euclidean space) 111.A
 immersion
 (of a C^∞ -manifold) 105.K
 (of a Riemann surface) 367.G
 branched minimal 275.B
 generalized minimal 275.B
 isometric 365.A
 Kähler 365.L
 minimal 275.A
 minimum 365.O
 tight 365.O
 totally real 365.T
 imperfect field 149.H
 implication 411.B
 strict 411.L
 implicit
 (difference equation in a multistep method) 303.E
 (Runge-Kutta method) 303.D
 implicit enumeration method 215.D
 implicit functions 165.C 208
 implicit function theorem 208.A 286.G
 (in Banach algebras) 36.M
 (in locally convex spaces) 286.J
 global 208.D
 Nash-Moser 286.J
 implicit method 303.E
 implicit scheme 304.F
 impossible construction problem 179.A
 impossible event 342.B
 impredicative (object) 156.B
 imprimitive (permutation group) 151.H
 improper integral 216.D,E
 convergent 216.E
 divergent 216.E
 improper Riemann integral 216.E
 improvement, iterative 302.C
 impulse control 405.E
 impulse function 306.B, App. A, Table 12.II
 imputation 173.D
 imputed costs 292.B
 imputed prices 292.C
 IMT formula 299.B
 inaccessible, strongly 33.F
 inaccessible cardinal number
 strongly 33.E
 weakly 33.E
 inaccessible ordinal number
 strongly 312.E
 weakly 312.E

Subject Index

Independent

Ince definition 268.D
 Ince-Goldstein method 268.C
 incidence matrix
 of a block design 102.B
 of a graph 186.G
 incidence number 146.B 201.B
 incidence relation 282.A
 inclination, curve of constant 111.F
 inclination problem, critical 55.C
 inclusion 381.C
 inclusion mapping 381.C
 incoming wave operator 375.B
 incompatible system (of partial differential equations) 428.B
 incomplete beta function App. A, Table 17.I
 incomplete blocks 102.B
 incomplete elliptic integral of the first kind 134.B
 incomplete factorization 302.C
 incomplete gamma function 174.A, App. A, Table 17.I
 incompleteness theorem, Gödel 156.E
 first 185.C
 second 185.C
 incompressible (measurable transformation) 136.C
 incompressible fluid 205.B
 inconsistent problem (of geometric construction) 179.A
 inconsistent system (of algebraic equations) 10.A
 increasing (sequence function or distribution)
 monotone 166.A 380.B
 monotonically 87.B
 non- 166.A
 slowly (C^∞ -function) 125.O
 slowly (distribution) 125.N
 slowly (in the sense of Deny) 338.O
 slowly, sequence 168.B
 strictly 166.A
 strictly monotone 166.A
 increasing directed set 308.A
 increasing process 406.B
 integrable 406.B
 increment
 of a function 106.B
 process with independent 5.B
 increment function 380.B
 Ind (large inductive dimension) 117.B
 ind (small inductive dimension) 117.B
 indecomposable A -module 277.I
 indecomposable continuum 79.D
 indecomposable group 190.L
 indecomposable linear representation 362.C
 indecomposable vector bundle 16.Y
 indefinite D -integral 100.D
 indefinite Hermitian form 348.F
 indefinite integral 198.B
 in Lebesgue integral 221.D
 in Riemann integral 216.C
 indefinite quadratic form 348.C
 indefinite sum (of a function) 104.B
 in degree 186.B
 independence, number of 186.I
 independence theorem (on valuations) 439.G
 independent
 (axioms) 35.B
 (complexes) 70.B
 (differential operators) 324.C
 (events) 342.B
 (frequency) 126.L
 (partitions) 136.E
 (points) 7.A

Independent increments, process with

- (random variables) 342.C
- algebraically 149.K 369.A
- analytically (in a complete ring) 370.A
- ε - (partitions) 136.E
- linearly 2.E 256.C,E 277.G
- path 346.G
- independent increments, process with 5.B
- independent of the past history 406.D
- independent process 136.E
- independent set 66.G 186.I
- independent system, maximal (of an additive group) 2.E
- independent variable 165.C
- independent vector 66.F
- indeterminacy, set of points of (of a proper meromorphic mapping) 23.D
- indeterminate 369.A
 - in the algebraic sense 337.C
- indeterminate coefficients, Lagrange's method of 106.L
- indeterminate form, limit of 106.E
- indeterminate system (of algebraic equations) 10.A
- $\text{Ind}_g a$ 297.G
- index
 - (of a central simple algebra) 29.G
 - (of a critical point) 279.B,E 286.N
 - (of a divisor) 3.D
 - (of an ε -Hermitian form) 60.O
 - (of an eigenvalue) 217.F
 - (of a Fredholm mapping) 286.E
 - (of a Fredholm operator) 68.F 251.D
 - (of a manifold) 56.G
 - (of a number) 297.G
 - (of an orthogonal array) 102.L
 - (of a quadratic form) 348.E
 - (of a recursive function) 356.F
 - (of the Riemann-Hilbert problem) 217.J
 - (of a stable distribution) 341.G
 - (of a stable process) 5.F
 - (of a subgroup) 190.C
 - analytic (of an elliptic complex) 237.H
 - analytic (of an elliptic differential operator) 237.H
 - contravariant (of a component of a tensor) 256.J
 - covariant (of a component of a tensor) 256.J
 - cycle 66.E
 - deficiency (of a closed symmetric operator) 251.I
 - deficiency (of a differential operator) 112.I
 - degeneracy 17.C
 - differential (in a covering of a nonsingular curve) 9.I
 - dummy (of a tensor) 256.J
 - fixed-point (of a continuous mapping) 153.B
 - of inertia (of a quadratic form) 348.E
 - Keller-Maslov 274.C
 - Kronecker 201.H
 - multi- 112.A
 - α -speciality (of a divisor) 9.F
 - p - (on a central simple algebra) 29.G
 - ramification (of an algebroidal function) 17.C
 - ramification (of a finite extension) 257.D
 - ramification (of a prime ideal) 14.I
 - ramification (of a valuation) 439.I
 - ramification, relative (of a prime ideal) 14.I
 - of relative nullity 365.D
 - Schur (of a central simple algebra) 29.E
 - Schur (of an irreducible representation) 362.F
 - of a singular point (of a continuous vector field) 153.B
 - speciality (of a divisor) 9.C 15.D
 - topological (of an elliptic complex) 237.H
 - of total isotropy (of a quadratic form) 348.E
 - indexing set (of a family of elements) 381.D
 - index set
 - (of a balanced array) 102.L
 - (of a family) 165.D 381.D
 - index theorem
 - Atiyah-Singer 237.H
 - for differentiable manifolds 56.G
 - Hirzebruch (for differentiable manifolds) 56.G
 - of Hodge 15.D
 - Morse 279.F
 - Indian mathematics 209
 - indicator function
 - modified 341.C
 - of a subset 342.E
 - indicatrix
 - Dupin 111.H
 - spherical (of a space curve) 111.F
 - indicial equation 254.C
 - indirect least squares method 128.C
 - indirect transcendental singularity (of an analytic function) 198.P
 - indiscrete pseudometric space 273.B
 - indiscrete topology 425.C
 - individual 411.H
 - individual domain 411.H
 - individual ergodic theorem 136.B
 - individual risk theory 214.C
 - individual symbol 411.H
 - individual variables 411.H
 - indivisibilis 265
 - induced
 - (Cartan connection) 80.O
 - (unfolding) 51.D
 - induced bundle 147.G
 - induced homomorphism 202.K
 - induced module 277.L
 - induced representation
 - (of a finite group) 362.G
 - (of a unitary representation) 437.O
 - induced topology 425.I
 - induced von Neumann algebra 308.C
 - induction
 - (of a von Neumann algebra) 308.C
 - axiom of mathematical 294.B
 - complete 294.B
 - double mathematical 294.B
 - magnetic 130.A
 - mathematical 294.B
 - multiple mathematical 294.B
 - transfinite (in a well-ordered set) 311.C
 - induction equation 259
 - inductive dimension
 - small 117.B
 - large 117.B
 - inductive limit
 - (in a category) 210.D
 - (of an inductive system of sets) 210.B
 - (of a sequence of topological spaces) 425.M
 - (of sheaves) 383.I
 - strictly (of a sequence of locally convex spaces) 424.W
 - inductive limit and projective limit 210
 - inductive limit group 210.C
 - inductive limit space 210.C
 - inductively ordered set 34.C

- inductive system
 - (in a category) 210.D
 - (of groups) 210.C
 - (of sets) 210.B
 - of topological spaces 210.C
- inelastic (scattering) 375.A
- inequality (inequalities) 211, App. A, Table 8
 - absolute 211.A
 - Bell's 351.L
 - Bernshtein (for trigonometric polynomials) 336.C
 - Bessel 197.C
 - Bhattacharyya 399.D
 - Bunyakovskii 211.C, App. A, Table 8
 - Carleman App. A, Table 8
 - Cauchy 211.C, App. A, Table 8
 - Cauchy-Schwarz 211.C, App. A, Table 8
 - Chapman-Robbins-Kiefer 399.D
 - Chebyshev 342.C
 - conditional 211.A
 - correlation 212.A
 - Cramér-Rao 399.D
 - distortion 438.B
 - energy 325.C
 - Fisher 102.E
 - FKG 212.A
 - Gårding 112.G
 - GHS 212.A
 - GKS first 212.A
 - GKS second 212.A
 - Golden-Thompson 212.B
 - Griffiths's first 212.A
 - Griffiths's second 212.A
 - Grunsky 438.B
 - Hardy App. A, Table 8
 - Hardy-Littlewood-Sobalen 224.E
 - Hausdorff-Young 224.E
 - Hilbert App. A, Table 8
 - Hölder 211.C, App. A, Table 8
 - Hölder integral 211.C
 - isoperimetric 228.B
 - Jordan App. A, Table 8
 - Klein 212.B
 - Markov (for polynomials) 336.C
 - maximal (maximal ergodic lemma) 136.B
 - Minkowski 211.C, App. A, Table 8
 - Morse 279.D,E
 - Peierls-Bogolyubov 212.B
 - Powers-Størmer 212.B
 - quasivariational 440.D
 - Riemann's period 3.L
 - Riemann-Roch (on algebraic surfaces) 15.D
 - Roepstorff-Araki-Sewell 402.G
 - Roepstorff-Fannes-Verbeure 402.G
 - Schwarz 211.C
 - stationary variational 440.B
 - triangle 273.A
 - variational, of evolution 440.C
 - von Neumann 251.M
 - Wirtinger App. A, Table 8
 - Wolfowitz 399.J
 - Young 224.E, App. A, Table 8
- inertia
 - ellipsoid of 271.E
 - index of (of an a quadratic form) 348.B
 - law of 271.A
 - law of, Sylvester (on a quadratic form) 348.B
 - moment of 271.E
 - principal axis of 271.E
 - principal moment of 271.E
 - product of 271.E
- inertia field (of a prime ideal) 14.K
- inertia group 14.K 257.D
- inertial system 271.D 359
- inf (infimum) 311.B
- inference
 - rule of 411.I
 - statistical 401
 - statistical, graphical method of 19.B
- inferior limit
 - (of a sequence of real numbers) 87.C
 - (of a sequence of subsets of a set) 270.C
- inferior limit event 342.B
- infimum
 - (of an ordered set) 311.B
 - (of a subset of a vector lattice) 310.C
- infinite, purely (von Neumann algebra) 308.E
- infinite branch (of a curve of class C^k) 93.G
- infinite cardinal number 49.A
- infinite classical group 147.I 202.V
- infinite continued fraction 83.A
- infinite determinant (in Hill's method of solution) 268.B
- infinite-dimensional complex projective space 56.C
- infinite-dimensional linear space 256.C
- infinite-dimensional normal space 117.B
- infinite-dimensional real projective space 56.B
- infinite Grassmann manifold 147.I
- infinite group 190.C
- infinite height (element of an Abelian p -group) 2.D
- infinite interval 355.C
- infinite lens space 91.C
- infinitely differentiable (function) 106.K
- infinitely divisible distribution 341.G
- infinitely recurrent (measurable transformation) 136.C
- infinite matrix 269.K
- infinite order (of an element in a group) 190.C
- infinite orthogonal group 202.V
- infinite population 401.E
- infinite prime divisor 439.H
 - imaginary 439.H
 - real 439.H
- infinite product 379.G, App. A, Table 10.VI
 - divergent 379.G
- infinite product expansion, Euler's 436.B 450.B
- infinite sequence 165.D
- infinite series 379.A, App. A, Table 10.III
- infinite set 49.F 381.A
 - countably 49.A
- infinitesimal
 - (for a function) 87.G
 - (for a hyperreal number) 293.D
 - (for a sequence of random variables) 250.B
 - order of (of a function) 87.G
- infinitesimal birth rate 260.G
- infinitesimal calculus (in nonstandard analysis) 293.D
- infinitesimal death rate 260.G
- infinitesimal deformation to the direction $\partial/\partial s$ 72.G
- infinitesimal element (in nonstandard Hilbert space) 276.E
- infinitesimal generator (of a semigroup) 378.B
- infinitesimal motion (of a Riemannian manifold) 364.E
- infinitesimal real number 276.E
- infinitesimal transformation
 - (of a differentiable transformation group) 431.G

(of a one-parameter transformation group) 105.N
 infinitesimal wedge 125.V
 infinite Stiefel manifold 147.I
 infinite symplectic group 202.V
 infinite type (Lie algebra) 191.D
 infinite type power series space 168.B
 infinite unitary group 202.V
 infinity 87.D,G
 axiom of 33.B 381.G
 axiom of strong 33.E
 hyperplane at (in affine geometry) 7.B
 minus 87.D
 negative 87.D 355.C
 order of (of a function) 87.G
 plus 87.D
 point at 7.B 74.D 178.F 285.C
 positive 87.D 355.C
 regular at the point at 193.B
 space at (in affine geometry) 7.B
 inflation 200.M
 inflection, point of 9.B 93.G
 influence, domain of 325.B
 influence curve 371.I
 information
 Fisher 399.D
 limited (in maximum likelihood method) 128.C
 loss of 138.B
 mutual 213.E
 self- 213.B
 information bit 63.C
 information compression 96.B
 information matrix 102.I 399.D
 information number, Kullback-Leibler (K-L) 398.G
 information retrieval 96.F
 information retrieval system 96.F
 information sciences 75.F
 information set 173.B
 information source 213.A
 ergodic 213.C
 information theory 213
 informatiques 75.F
 informative, more (statistical experiment) 398.G
 infra-exponential growth 125.AA,BB
 infrared divergence 132.C 146.B
 ingoing subspaces 375.H
 inhomogeneous
 (boundary value problem) 315.B
 (difference equation) 104.C
 (linear ordinary differential equation) 252.A
 (system of linear differential equations) 252.G
 inhomogeneous coordinates 343.C
 inhomogeneous lattice (in \mathbf{R}^n) 182.B
 inhomogeneous Lorentz group 359.B
 inhomogeneous polarization 3.G
 initial blocks 102.E
 initial-boundary value problem 325.K
 initial condition
 (for ordinary differential equations) 316.A
 (for partial differential equations) 321.A
 initial data 321.A
 initial distribution
 (of a Markov process) 261.A
 (for a stochastic differential equation) 406.D
 initial function (of a functional-differential equation) 163.C
 initial law (for a stochastic differential equation) 406.D

initial number 312.D
 initial object 52.D
 initial ordinal number 49.E
 initial phase (of a simple harmonic motion) 318.B
 initial point
 (of a curvilinear integral) 94.D
 (of a path) 170
 (of a position vector) 7.A
 (of a vector) 442.A
 initial set
 (of a correspondence) 358.B
 (of a linear operator) 251.E
 initial state 31.B
 initial surface 321.A
 initial term (of an infinite continued fraction) 83.A
 initial value
 (for an ordinary differential equation) 316.A
 (for a partial differential equation) 321.A
 (for a stochastic differential equation) 406.D
 initial value problem
 (for functional-differential equations) 163.D
 (for hyperbolic partial differential equations) App. A, Table 15.III
 (for integrodifferential equations) 222.B
 (for ordinary differential equations) 313.C 316.A
 (for partial differential equations) 321.A
 Navier-Stokes 204.B
 singular (for partial differential equations of mixed type) 326.C
 initial vertex 186.B
 injection 381.C
 (in a category) 52.D
 (homomorphism of cohomology groups) 200.M
 canonical 381.C,E
 canonical (on direct sums of modules) 277.F
 canonical (on free products of group) 190.M
 canonical (from a subgroup) 190.D
 natural (from a subgroup) 190.D
 injective
 (Banach space) 37.M
 (C^* -algebra) 36.H
 (mapping) 381.C
 (object in an Abelian category) 200.I
 injective A -module 277.K
 injective class 200.Q
 injective dimension 200.K
 injective envelope 200.I
 injective module, (R, S) - 200.K
 injective resolution (in an Abelian category) 200.I
 j - 200.Q
 right (of an A -module) 200.F
 injectivity, rational 200.O
 injectivity radius 178.C
 inner area 216.F 270.G
 inner automorphism(s)
 (of a group) 190.D
 (of a ring) 368.D
 group of (of a group) 190.D
 group of (of a Lie algebra) 248.H
 inner capacity, Newtonian 48.F
 inner derivation
 (of an associative algebra) 200.L
 (of a Lie algebra) 248.H
 inner function 43.F
 inner harmonic measure 169.B
 inner measure 270.E
 Lebesgue 270.F

- inner product
 - (in a Hermitian linear space) 256.Q
 - (in a Hilbert space) 197.B
 - (of hyperspheres) 76.A
 - (between a linear space and its dual space) 256.G
 - (in a metric vector space) 256.H
 - (of a pair of linear spaces) 424.G
 - (of vectors) 256.A 442.B
 - Hermitian 256.Q
- inner product space 442.B
- inner solution 112.B
- inner topology (of a Lie subgroup) 249.E
- inner transformation (in the sense of Stoilow) 367.B
- inner variable 112.B
- inner volume 270.G
- innovation 405.H
- input, Poisson 260.H
- input data, error of 138.B
- inrevolvable oval 89.E
- inscribe (in a sphere) 139.I
- inscribed circle (of a regular polygon) 357.A
- inseparable, purely (rational mapping) 16.I
- inseparable element (of a field) 149.H
 - purely 149.H
- inseparable extension (of a field) 149.H
 - purely 149.H
- inseparable polynomial 337.G
- inspection
 - expected amount of 404.C
 - sampling (\Rightarrow sampling inspection) 404.C
- instantaneous state 260.F 261.B
- in-state 150.D 386.A
- instruction 31.B
 - single-address 75.C
- insurance
 - amount of 214.A
 - cost of 214.B
 - death 214.B
 - mixed 214.B
 - survival 214.B
- insured, amount 214.A
- integer(s) 294.C
 - algebraic 14.A
 - Cartan (of a semisimple Lie algebra) 248.N
 - Gaussian 14.U
 - p -adic 439.F
 - p -adic, ring of 439.F
 - rational 294.B
- integer polyhedron 215.C
- integer programming 215 264.C
- integer programming problem
 - all- 215.A
 - mixed 215.A
 - pure 215.A
 - 0-1 215.A
- integrability
 - (of multivalued vector functions) 443.I
 - strong 443.I
- integrability condition, complete 428.C
- integrable
 - (function) 221.B
 - (G -structure) 191.A
 - (increasing process) 262.D 406.B
 - (representation) 437.X
 - (in the sense of Riemann) 216.A
 - absolutely 216.E,F
 - Birkhoff (function) 443.E
 - Bochner (function) 443.C
 - completely (system of independent 1-forms) 154.B 428.D
 - D - (function) 100.D
 - Daniell-Stone (function) 310.I
 - Denjoy (in the wider sense) 100.D
 - Dunford 443.F
 - Gel'fand-Pettis (function) 443.F
 - Lebesgue (function) 221.B
 - locally, function 168.B
 - μ - (function) 221.B
 - Perron (function) 100.F
 - Pettis 443.F
 - Riemann (function) 216.A
 - scalarly 443.F,I
 - square (function) 168.B
 - square (unitary representation) 437.M
 - termwise (series) 216.B
 - uniformly (family of random variables) 262.A
 - weakly (function) 443.E
- integrable distribution 125.N
- integrable increasing process 262.D 406.B
- integrable process of bounded variation 406.B
- integrable system 287.A
- integrably homotopic 154.F
- integral(s)
 - (of differential forms) 105.T
 - (of a distribution with respect to λ) 125.H
 - (of a function) 221.B
 - (= integrally dependent) 67.I
 - (of a Monge-Ampère equation) 278.B
 - (of multivalued vector functions) 443.I
 - (scheme) 16.D
 - Abelian 11.C
 - action 80.Q
 - Airy App. A, Table 19.IV
 - almost (element of a ring) 67.I
 - of angular momentum 420.A
 - Banach 310.I
 - Bartle-Dunford-Schwartz 443.G
 - Bessel 39.B
 - Birkhoff 443.E
 - Bochner 443.C
 - Bromwich 240.D 322.D, App. A, Table 12.I
 - Carson App. A, Table 12.II
 - of Cauchy type 198.B
 - of the center of mass 420.A
 - complete additivity of the (in Lebesgue integral) 221.C
 - complete elliptic App. A, Table 16.I
 - complete elliptic, of the first kind 134.B
 - complete elliptic, of the second kind 134.C
 - conjugate Fourier 160.D
 - constant 216.C
 - cosine 167.D, App. A, Table 19.II
 - curvilinear 94.A
 - curvilinear (with respect to a line element) 94.D
 - curvilinear (with respect to a variable) 94.D
 - D (*)- 100.D
 - Daniell-Stone 310.I
 - definite App. A, Table 9.V
 - definite (of a hyperfunction) 125.X
 - definite (in a Riemann integral) 216.C
 - definite D - 100.D
 - Denjoy 100
 - Denjoy (in the restricted sense) 100.D
 - Denjoy (in the wide sense) 100.D
 - direct 308.G
 - direct (of unitary representations) 437.H
 - Dirichlet (in Dirichlet problem) 120.F

Integral bilinear functional

- Dirichlet (in Fourier's single integral theorem) 160.B
 double (in Riemann integral) 216.F
 Dunford 251.G 443.F
 elliptic 11.C 134.A, App. A, Table 16.I
 elliptic, of the first kind 134.A
 elliptic, of the second kind 134.A
 elliptic, of the third kind 134.A
 energy 420.A
 Euler, of the first kind 174.C
 Euler, of the second kind 174.C
 exponential 167.D, App. A, Table 19.II
 Feynman 146
 first (of a completely integrable system) 428.D
 Fourier 160.A
 Fresnel 167.D, App. A, Tables 9.V 19.II
 of a function with respect to a volume element 105.W
 Gauss 338.J
 Gel'fand 443.F
 Gel'fand-Pettis 443.F
 harmonic 194.A
 Hilbert's invariant 46.C
 hyperelliptic 11.C
 hypergeometric 253.B
 improper (in Riemann integral) 216.D,E
 incomplete elliptic (of the first kind) 134.B
 indefinite (in Lebesgue integral) 221.D
 indefinite (in Riemann integral) 198.B 216.C
 indefinite D - 100.D
 intermediate App. A, Table 15.III
 intermediate (of a Monge-Ampère equation) 278.B
 iterated (in Lebesgue integral) 221.E
 iterated (in Riemann integral) 216.G
 Jacobi 420.F
 L - 221.B
 with respect to λ (of a distribution) 125.H
 Lebesgue 221.B
 Lebesgue-Radon 94.C
 Lebesgue-Stieltjes 94.C 166.C
 logarithmic 167.D, App. A, Table 19.II
 Lommel 39.C
 multiple (in Lebesgue integral) 221.E
 multiple (in Riemann integral) 216.F
 n -tuple (in Riemann integral) 216.F
 over an oriented manifold 105.T
 Pettis 443.F
 Poisson 168.B 193.G
 probability App. A, Table 19.II
 regular first 126.H
 repeated (in Lebesgue integral) 221.E
 repeated (in Riemann integral) 216.G
 Riemann 37.K 216.A
 Riemann lower 216.A
 Riemann-Stieltjes 94.B 166.C
 Riemann upper 216.A
 scalar 443.F,I
 sine 167.D, App. A, Table 19.II
 singular 217.J
 over a singular chain 105.T
 spectral 390.D
 Stieltjes 94.B
 stochastic 261.E 406.B
 stochastic, of Stratonovich type 406.C
 surface 94.A,E
 surface (with respect to a surface element) 94.E
 trigonometric 160.A
 vector 443.A
 of a vector field App. A, Table 3.III
 vector-valued 443.A
 integral bilinear functional 424.R
 integral calculus 216
 integral character (of the homology group of a Riemann surface) 11.E
 integral closure (of a ring) 67.I
 integral cohomology group 201.H
 integral constant 216.C
 integral cosine 167.D
 integral current 275.G
 integral curvature (of a surface) 111.H
 integral curve
 (of a Monge equation) 324.F
 (of ordinary differential equations) 316.A
 integral direct sum 308.G
 integral divisor
 (of an algebraic curve) 9.C
 (of an algebraic number field) 14.F
 (on a Riemann surface) 11.D
 integral domain 368.B
 Noetherian 284.A
 integral element 428.E
 k -dimensional 191.I
 ordinary 428.E
 regular 428.E
 integral equation(s) 217
 Abel 217.L
 associated 217.F
 Fredholm 217.A
 of Fredholm type 217.A
 Hammerstein 217.M
 homogeneous 217.F
 linear 217.A
 nonlinear 217.M
 numerical solution of 217.N
 singular 217.J
 transposed 217.F
 Volterra 217.A
 of Volterra type 217.A
 integral exponent 167.D
 integral form 248.W
 integral formula
 Cauchy 198.B
 Poisson 198.B
 Villat App. A, Table 15.VI
 Weyl 225.I
 integral function 429.A
 integral g -lattice 27.A
 integral geometry 218
 principal formula of 218.C
 integral homology group
 of a polyhedron 201.D
 of a simplicial complex 201.C
 integral hypersurface (of a partial differential equation) 320.A
 integral ideal (of an algebraic number field) 14.C
 integral inequality, Hölder 211.C
 integral invariant(s) 219
 absolute 219.A
 Cartan's relative 219.B
 relative 219.A
 integral kernel 217.A 251.O
 integral left ideal 27.A
 integral logarithm 167.D
 integrally closed
 (in a ring) 67.I
 completely (ring) 67.I

integrally closed ring 67.I
 integrally dependent element (of a ring) 67.I
 integral manifold 428.A,B,D
 k-dimensional 191.I
 ordinary (of a differential ideal) 428.E
 regular (of a differential ideal) 428.E
 singular (of a differential ideal) 428.E
 integral method, summable by Borel's 379.O
 integral operator 68.N 100.E 250.O
 Calderón-Zygmund singular 217.J 251.O
 Fourier 274.C 345.B
 of Hilbert-Schmidt type 68.C
 integral point 428.E,F
 integral quotient (in the division algorithm of
 polynomials) 337.C
 integral representation 362.G,K
 Cauchy 21.C
 Herglotz 43.I
 Laplace-Mehler App. A, Table 18.II
 Schläfli 393.B
 integral right ideal 27.A
 integral sine 167.D
 integral singular homology group 201.E
 integral test, Cauchy (for convergence) 379.B
 integral theorem
 Cauchy 198.A,B
 Fourier double 160.B
 Fourier single 160.B
 stronger form of Cauchy 198.B
 integral transforms 220.A
 integral two-sided \mathfrak{o} -ideal 27.A
 integral vector 428.E
 integrand 216.A
 integrate 216.A
 (an ordinary differential equation) 313.A
 integrating factor App. A, Table 14.I
 integration
 along a fiber (of a hyperfunction) 274.E
 automatic, scheme 299.C
 contour of (of curvilinear integral) 94.D
 domain of 216.F
 graphical 19.B
 Jacobi's second method of 324.D
 numerical 299
 path of (of curvilinear integral) 94.D
 Romberg 299.C
 integration by parts 216.C
 (on *D*-integral) 100.G
 (in the Stieltjes integral) 94.C
 integration constant (in a general solution of
 a differential equation) 313.A
 integration formula
 based on variable transformation 299.B
 Gauss (in the narrow sense) 299.A
 Poisson App. A, Table 15.VI
 Villat App. A, Table 15.VI
 integrodifferential equation(s) 163.A 222
 of Fredholm type 222.A
 Prandtl's 222.C
 of Volterra type 222.A
 Wiener-Hopf 222.C
 intensity, traffic 260.H
 intensive (thermodynamical quantity) 419.A
 interaction 102.H
 interest, assumed rate of 214.A
 interference (of waves) 446
 interior
 (of an angle) 139.D 155.B
 (of a manifold) 105.B
 (of a polygon) 155.F

Subject Index

Intersect

 (of a segment) 155.B
 (of a set) 425.B
 (of a simplex) 70.C
 interior capacity, Newtonian 48.F
 interior cluster set 62.A
 interior field equation 359.D
 interior operator 425.B
 interior point 425.B
 interior problem (in Dirichlet problems) 120.A
 interior product (of a differential form with a vector
 field) 105.Q
 intermediate convergent (of an irrational number)
 83.B
 intermediate field 149.D
 intermediate integrals App. A, Table 15.III
 of Monge-Ampère equation 278.B
 intermediate-value theorem 84.C
 intermittent structure 433.C
 internal (in nonstandard analysis) 293.B
 internal energy 419.A
 internal irregular point 338.L
 internal law of composition (of a set) 409.A
 internally stable set 186.I
 internally thin set 338.G
 internal product 200.K
 internal space in catastrophe theory (in static
 model) 51.B
 internal state 31.B
 internal symmetry 150.B
 international notation (for crystal classes) 92.B
 international system of units 414.A
 interpolating (for a function algebra) 164.D
 interpolating sequence 43.F
 interpolation
 (of a function) 223, App. A, Table 21
 (of a stationary process) 176.K 395.E
 Chebyshev 223.A 336.J
 inverse 223.A
 Lagrange 223.A
 of operators 224
 spline 223.F
 interpolation coefficient, Lagrange's 223.A
 interpolation formula 223.A
 Bessel App. A, Table 21
 Everett App. A, Table 21
 Gauss App. A, Table 21
 Gauss's backward 223.C
 Gauss's forward 223.C
 Newton App. A, Table 21
 Newton's backward 223.C
 Newton's forward 223.C
 Stirling App. A, Table 21
 interpolation function 223.A
 interpolation method 224.A
 interpolation polynomial 223.A
 Hermite 223.E
 Lagrange 336.G, App. A, Table 21
 Newton 336.G
 trigonometric 336.E
 interpolation problem 43.F
 interpolation scheme, Aitken 223.B
 interpolation space 224.A
 complex 224.B
 real 224.C
 interpolation theorem 224.B,C
 interpolatory formula 299.A
 interquartile range 397.C
 intersect 155.B
 properly (on a variety) 16.G
 transversally 105.L

Intersection

- intersection
 - (of events) 342.B
 - (of projective subspaces) 343.B
 - (of sets) 381.B
 - (of subspaces of an affine space) 7.A
 - complete 16.A
- intersection chart 19.D
- intersection multiplicity (of two subvarieties) 16.Q
- intersection number
 - (of divisors) 15.C
 - (of homology classes) 65.B 201.O
 - (of sheaves) 16.E
 - self- 15.C
- intersection product
 - (in algebraic varieties) 16.Q
 - (in homology theory) 201.O
- intersection property, finite 425.S
- intersection theorem
 - (of affine geometry) 7.A
 - (of projective geometry) 343.B
 - Cantor's 273.F
 - Krull 284.A
- interval
 - (in a Boolean algebra) 42.B
 - (in a lattice) 243.C
 - (in a vector lattice) 310.B
 - (in an ordered set) 311.B
 - (in real number space) 355.C
 - of absolute stability 303.G
 - basic 4.B
 - closed 140 355.C
 - confidence 399.Q
 - of continuity (for a probability distribution) 341.C
 - fiducial 401.F
 - finite 355.C
 - infinite 355.C
 - open 140 355.C
 - principle of nested 87.C
 - of relative stability 303.G
 - supplementary 4.B
 - tolerance 399.R
- interval estimation 399.Q 401.C
- interval function 380.A
 - additive 380.B
 - continuous additive 380.B
- in the large (in differential geometry) 109
- in the small (in differential geometry) 109
- intra-block analysis 102.D
- intransitive (permutation group) 151.H
- intrinsic angular momentum 415.G
- intrinsic homology 114.H
- intuitionism 156.A
 - semi- 156.C
- intuitionistic logic 411.L
- invariance
 - of a confidence region 399.Q
 - of dimension, theorem on (of Euclidean spaces) 117.D
 - of domain, Brouwer theorem on 117.D
 - homotopy 201.D
 - isospin 351.J
 - Lorentz 150.B
 - of speed of light, principle of 359.B
 - topological (homology groups) 201.A
- invariance principle
 - (of hypothesis testing) 400.E
 - (of wave operators) 375.B
 - Donsker's 250.E
 - Strassen's 250.E
- invariance theorem of analytic relations 198.K
- invariant(s) App. A, Table 14.III
 - (of an Abelian group) 2.B
 - (of a cohomology class of a Galois group) 59.H 257.E
 - (decision problem) 398.E
 - (element under a group action) 226.A
 - (of an elliptic curve) 73.A
 - (in the Erlangen program) 137
 - (under flow) 126.D
 - (function algebra) 164.H
 - (hypothesis) 400.E
 - (measure) 136.B 225 270.L
 - (S -matrices) 386.B
 - (of a normal simple algebra) 257.G
 - (subspace of a Banach space) 251.L
- absolute 12.A 226.A
- absolute integral 219.A
- almost G - 396.I
- Arf-Kervaire 114.J
- basic 226.B
- birational 12.A
- Browder-Livesay 114.L
- Cartan (of a finite group) 362.I
- Cartan relative integral 219.B
- conformal 77.E
- covering linkage 235.E
- differential (on an m -dimensional surface) 110.A
- Eilenberg-Postnikov (of a CW-complex) 70.G
- fundamental (of a space with a Lie transformation group) 110.A
- fundamental differential (of a surface) 110.B
- G - (element) 226.A
- G - (measure) 225.A
- G - (statistics) 396.I
- generalized Hopf 202.Q
- Hasse (of a central simple algebra) 29.G
- homotopy 202.B
- homotopy type 202.F
- Hopf 202.S,U
- Hopf, modulo p 202.S
- integral 219
- isomorphism (on a measure space) 136.E
- Iwasawa 14.L
- k - (of a CW-complex) 70.G
- left (metric in a topological group) 423.I
- left, Haar measure 225.C
- left, tensor field 249.A
- metric (on a measure space) 136.E
- Milnor 235.D
- of n -ary form of degree d 226.D
- negatively 126.D
- normal 114.J
- of order p 110.A
- p - (of a central simple algebra) 29.G
- PCT 386.B
- Poincaré's differential 74.G
- positively 126.D
- rcarrangement 168.B
- relative 12.A 226.A
- relative integral 219.A
- right, Haar measure 225.C
- right, tensor field 249.A
- sampling procedure 373.C
- semi- 226.A
- semi- (of a probability distribution) 341.C
- shape 382.C
- spectral 136.E
- TCP- 386.B

- topological 425.G
- U - (subspace of a representation space of a unitary representation) 437.C
- uniformly most powerful 399.Q
- vector 226.C
- of weight w 226.D
- invariant decision function 398.E
- invariant derivation (on an Abelian variety) 3.F
- invariant differential form (on an Abelian variety) 3.F
- invariant distribution(s)
 - (of a Markov chain) 260.A
 - (of second quantization) 377.C
- invariant estimator 399.I
 - best 399.I
- invariant field 172.B
- invariant integral, Hilbert's 46.C
- invariant level α test, uniformly most powerful (UMP) 400.E
- invariant Markov process 5.H
- invariant measure(s) 225
 - (of a Markov chain) 260.A
 - (of a Markov process) 261.F
 - (under a transformation) 136.B
 - G - 225.B
 - quasi- 225.J
 - relatively 225.H
 - smooth 126.J
 - sub- 261.F
 - transverse 154.H
- invariant measure problem 136.C
- invariants and covariants 226
- invariant statistic 396.I
 - maximal 396.I
- invariant subgroup (of a group) 190.C
- invariant subspace (of a linear operator) 164.H
 - doubly 164.H
- invariant tensor field
 - left 249.A
 - right 249.A
- invariant test 400.E
 - almost 400.E
- invariant torus 126.L
- inventory control 227
- inventory model 307.C
- inverse
 - (in a group) 190.A
 - (of a mapping) 381.C
 - homotopy (for an H -space) 203.D
 - quasi- (on a Banach algebra) 36.C
 - right (in nonlinear functional analysis) 286.G
- inverse analytic function 198.L
- inverse assumption 304.D
- inverse correspondence 358.B
- inverse domination principle 338.L
- inverse element
 - (in a group) 190.A
 - (in a ring) 368.B
 - left (in a ring) 368.B
 - quasi- (in a ring) 368.B
 - right (in a ring) 368.B
- inverse Fourier transform (of a distribution) 125.O
- inverse function 198.L 381.C
- inverse function element 198.L
- inverse image
 - (of a set) 381.C
 - (of a sheaf) 383.G
 - (of a uniformity) 436.E
 - perfect 425.CC
- inverse interpolation 223.A
- inverse iteration 298.C
- inverse limit (of an inverse system of sets) 210.B
- inverse mapping 381.C
- inverse mapping theorem 208.B
- inverse matrix 269.B
- inverse morphism 52.D
- inverse operator 37.C 251.B
- inverse path 170
- inverse problem
 - (in potential scattering) 375.G
 - Jacobi 3.L
- inverse relation 358.A
- inverse system (of sets) 210.B
- inverse transform (of an integral transform) 220.A
- inverse trigonometric function 131.E
- inversion
 - (with respect to a circle) 74.E
 - (of a domain in \mathbf{R}^n) 193.B
 - (with respect to a hypersphere) 76.A
 - Laguerre 76.B
 - space 258.A
 - space-time 258.A
- inversion formula
 - (for a characteristic function) 341.C
 - (of a cosine transform) 160.C
 - (of a Fourier transform) 160.C
 - (of a Fourier transform of distributions) 160.H
 - (of a Fourier transform on a locally compact Abelian group) 192.K
 - (of a generalized Fourier transform) 220.B
 - (of a Hilbert transform) 220.E
 - (of an integral transform) 220.A
 - (of a Laplace-Stieltjes transform) 240.D
 - (on a locally compact group) 437.L
 - (of a Mellin transform) 220.C
 - (for a semigroup of operators) 240.I
 - (of a Stieltjes transform) 220.D
 - Fourier 160.C
 - Möbius (in combinatorics) 66.C
 - Möbius (in number theory) 295.C
- inverted filing scheme 96.F
- invertible element
 - quasi- 368.B
 - of a ring 368.B
- invertible jet 105.X
- invertible knot 235.A
- invertible matrix 269.B
- invertible sheaf 16.E
- involute (of a curve) 111.E
- involution
 - (of an algebraic correspondence) 9.H
 - (in a Banach algebra) 36.F
 - (of a division ring) 348.F
 - (of a homotopy sphere) 114.L
 - Cartan 427.X
- involutive
 - (cross section) 286.H
 - (differential ideal) 428.E
 - (differential system) 191.I
 - (distribution) 154.B 428.D
 - (Lie group) 191.H
- involutive automorphism (of a Lie group) 412.B
- involutive correlation 343.D
- involutive distribution (on a differentiable manifold) 428.D
- involutive subspace 428.F
- involutory (involutive) system
 - (of differential forms) 428.F
 - (of nonlinear equations) 428.C

Irrational function, elliptic

- (of partial differential equations) 428.F
- (of partial differential equations of first order) 324.D
- irrational function, elliptic 134.E
- irrational number(s) 294.E 355.A
 - space of 22.A
- irrational real number 294.E
- irreducibility theorem, Hilbert's (on polynomials) 337.F
- irreducible
 - (algebraic curve) 9.B
 - (algebraic equation) 10.B
 - (algebraic variety) 16.A
 - (coalgebra) 203.F
 - (complemented modular lattice) 243.F
 - (continuous geometry) 85.A
 - (continuum) 79.D
 - (Coxeter complex) 13.R
 - (discrete subgroup of a semisimple Lie group) 122.F
 - (germ of an analytic set) 23.B
 - (linear representation) 362.C
 - (linear system) 16.N
 - (linear system in control theory) 86.C
 - (3-manifold) 65.E
 - (Markov chain) 260.B
 - (polynomial) 337.F
 - (positive matrix) 269.N 310.H
 - (projective representation) 362.J
 - (representation of a compact group) 69.B
 - (Riemannian manifold) 364.E
 - (root system) 13.L
 - (scheme) 16.D
 - (Siegel domain) 384.E
 - (transition matrix) 126.J
 - (unitary representation) 437.A
 - absolutely (representation) 362.F
 - at 0 (for an algebraic set) 23.B
- irreducible character
 - (of an irreducible representation) 362.E
 - absolutely 362.E
- irreducible component
 - (of an algebraic variety) 16.A
 - (of an analytic space) 23.C
 - (of a linear representation) 362.D
- irreducible element (of a ring) 67.H
- irreducible representation(s)
 - (of a Banach algebra) 36.D
 - fundamental system of (of a complex semisimple Lie algebra) 248.W
- irreducible symmetric bounded domain 412.F
- irreducible symmetric Hermitian space 412.E
- irreducible symmetric Riemannian space 412.C, App. A, Table 5.III
- irreducible tensor of rank k 353.C
- irredundant (intersection of primary ideals) 67.F
- irregular
 - (boundary point) 120.D
 - (prime number) 14.L
- irregularity
 - (of an algebraic surface) 15.E
 - number of (of an algebraic variety) 16.O
- irregular point
 - (of Brownian motion) 45.D
 - (of a Markov process) 261.D
 - (in potential theory) 338.L
 - external 338.L
 - internal 338.L
- irregular singular point
 - (of a solution) 254.B
- (of a system of linear ordinary differential equations) 254.B
- irreversible processes, statistical mechanics of 402.A
- irrotational
 - (fluid) 205.B
 - (vector field) 442.D
- Irwin's embedding theorem 65.D
- Ising model 340.B 402.G
 - stochastic 340.C
- island (in a Riemann surface) 272.J
- isobaric polynomial 32.C
- isogenous
 - (Abelian varieties) 3.C
 - (algebraic groups) 13.A
- isogeny 13.A
- isolated fixed point 126.G
- isolated ordinal number 312.B
- isolated point
 - (of a curve) 93.G
 - (in a topological space) 425.O
- isolated primary component (of an ideal) 67.F
- isolated prime divisor (of an ideal) 67.F
- isolated singularity (of an analytic function) 198.D, M
- isolated singular point 198.D 418.D
- isolated vertex 186.B
- isometrically isomorphic (normed spaces) 37.C
- isometric immersion 365.A
- isometric mapping 111.H 273.B
- isometric operator 251.E
 - partially 251.E
- isometric Riemannian manifolds 364.A
- isometric spaces 273.B
- isometry
 - (= isometric operator) 251.E
 - (between Riemannian manifold) 364.A
- isomonodromic deformation 253.E
- isomorphic
 - (algebraic systems) 409.C
 - (block bundles) 147.Q
 - (cohomology theories) 201.Q
 - (complex manifolds) 72.A
 - (fiber bundles) 147.B
 - (groups) 190.D
 - (Lie algebras) 248.A
 - (Lie groups) 249.N
 - (measure spaces) 398.D
 - (normed spaces) 37.C
 - (objects) 52.D
 - (PL-embeddings) 65.D
 - (representations) 362.C
 - (simplicial complexes) 70.C
 - (s.s. complexes) 70.E
 - (structures) 276.E
 - (topological groups) 423.A
 - (unitary representations) 437.A
- anti- (lattices) 243.C
- Borel 270.C
- dually (lattices) 243.C
- isometrically (normed spaces) 37.C
- locally 423.O
- metrically (automorphisms on a measure space) 136.E
- order 311.E
- similarly (ordered fields) 149.N
- spatially (automorphisms on a measure space) 136.E
- spectrally 136.E
- weakly 136.E

- isomorphic mapping, Borel 270.C
- isomorphic relations among classical Lie algebras
 App. A, Table 5.IV
- isomorphism
 - (of Abelian varieties) 3.C
 - (of algebraic systems) 409.C
 - (of block bundles) 147.Q
 - (of fields) 149.B
 - (of functors) 52.J
 - (of groups) 190.D
 - (of lattices) 243.C
 - (of Lie algebras) 248.A
 - (of linear spaces) 256.B
 - (of objects) 52.D
 - (of prealgebraic varieties) 16.C
 - (of rings) 368.D
 - (of topological groups) 423.A
 - (of unfoldings) 51.D
 - admissible (of Ω -groups) 190.E
 - algebra 29.A
 - analytic 21.J
 - analytic (of Lie groups) 249.N
 - anti- (of groups) 190.D
 - anti- (of lattices) 243.C
 - anti- (of ordered sets) 311.E
 - anti- (of rings) 368.D
 - birational (of Abelian varieties) 3.C
 - birational (of algebraic groups) 13.A
 - Bott 237.D
 - C^ω - (of Lie groups) 249.N
 - dual (of lattices) 243.C
 - dual (of ordered sets) 311.E
 - excision (on homology groups) 201.F,L
 - functorial 52.J
 - G - 191.A
 - k - (of algebraic groups) 13.A
 - k - (of extension fields of k) 149.D
 - lattice- 243.C
 - local (of topological groups) 423.O
 - mod p (in a class of Abelian groups) 202.N
 - Ω - (of Ω -groups) 190.E
 - operator (of Ω -groups) 190.E
 - order 311.E
 - ring 368.D
 - suspension (for homology) 201.E
 - Thom-Gysin 114.G
 - Thom-Gysin (of a fiber space) 148.E
 - uniform 436.E
- isomorphism invariant (on a measure space) 136.E
- isomorphism problem
 - (in ergodic theory) 136.E
 - (for graphs) 186.J
 - (for integral group algebras) 362.K
- isomorphism theorem
 - (in class field theory) 59.C
 - (on groups) 190.D
 - (on rings) 368.F
 - (on topological groups) 423.J
 - Ax-Kochen (on ultraproduct) 276.E
 - first (on topological groups) 423.J
 - Hurewicz 202.N
 - Hurewicz-Steenrod (on homotopy groups of fiber spaces) 148.D
 - Keisler-Shelah (in model theory) 276.E
 - second (on topological groups) 423.J
 - third (on topological groups) 423.J
- isoparametric (hypersurface) 365.I
- isoparametric method 304.C
- isoperimetric (curves) 228.A
- isoperimetric constant 391.D
- isoperimetric inequality 228.B
- isoperimetric problem(s) 111.E 228.A
 - generalized 46.A 228.A
 - special 228.A
- isospectral 391.B
- isospectral deformation 387.C
- isospin 351.J
- isospin invariance 351.J
- isothermal compressibility 419.B
- isothermal coordinates 90.C
- isothermal curvilinear coordinate system App. A,
 Table 3.V
- isothermal parameter 334.B
 - (for an analytic surface) 111.I
- isothermal process 419.B
- isotopic 65.D 202.B
 - (braids) 235.F
 - (embeddings) 114.D
 - (latin square) 241.A
 - ambient 65.D
- isotopy 65.D 202.B
 - ambient 65.D
- isotopy lemma, Thom's first 418.G
- isotopy type (of knots) 235.A
- isotropic
 - (with respect to a quadratic form) 348.E
 - k - (algebraic group) 13.G
 - totally (subspace) 60.O 348.E
- isotropic point 365.D
- isotropic submanifold 365.D
- isotropic turbulence 433.C
- isotropy, index of total (of a quadratic form) 348.E
- isotropy group 362.B
 - linear 199.A
 - principal 431.C
- isotropy representation 431.C
- isotropy subgroup (of a topological group) 431.A
- isotropy type (of a transformation group) 431.A
- iterated integral
 - (in Lebesgue integral) 221.E
 - (in Riemann integral) 216.G
- iterated kernel (for a Fredholm integral equation)
 217.D
- iterated logarithm
 - Khinchin's law of 250.C
 - law of 45.F
- iterated series
 - by columns (of a double series) 379.E
 - by rows (of a double series) 379.E
- iteration
 - (in a Banach space) 286.B
 - inverse 298.C
 - method of successive (for Fredholm integral equations) 217.D
 - two-body 271.C
- iteration matrix 302.C
- iterative improvement 302.C
- iterative method 302.C
- iterative process, linear stationary 302.C
- Itô circle operation 406.C
- Itô decomposition, Wiener- 176.I
- Itô formula 45.G 406.B
- Itô process 406.B
- Itô theorem, Lévy- (on Lévy processes) 5.E
- Itô type, stochastic integral of 406.C
- ITPFI 308.I
- Iversen-Beurling-Kunugi theorem 62.B
- Iversen theorem 272.I
- Iwahori subgroup 13.R
- Iwasawa decomposition

(of a Lie group) 249.T
(of a real semisimple Lie algebra) 248.V
Iwasawa group 384.C
Iwasawa invariant 14.L
Iwasawa main conjecture 450.J
Iwasawa theorem, Cartan-Mal'tsev- (on maximal compact subgroups) 249.S

J

J-group 237.I
 equivariant 431.F
J-homomorphism 202.V 237.I
 equivariant 431.F
J-method 224.C
Jackson's theorem (on the degree of approximation) 336.C
Jacobi, C. G. J. 229
Jacobian 208.B
Jacobian, generalized (of a set function) 246.H
Jacobian criterion (on regularity of local rings) 370.B
Jacobian determinant 208.B
Jacobian matrix 208.B
Jacobian variety 9.E 11.C 16.P
 canonically polarized 3.G 9.E
 generalized 9.F 11.C
Jacobi-Biehler equality 328
Jacobi condition 46.C
Jacobi differential equation App. A, Tables 14.II 20.V
 Hamilton- 271.F 324.E
Jacobi elliptic functions App. A, Table 16.III
Jacobi equation, Hamilton- 108.B
Jacobi field 178.A
Jacobi identity
 (on the bracket of two vector fields) 105.M
 (in a Lie algebra) 248.A
 (with respect to Whitehead product) 202.P
Jacobi imaginary transformation 134.I
Jacobi integral 420.F
Jacobi inverse problem 3.L
Jacobi last multiplier App. A, Table 14.I
Jacobi matrix 390.G
Jacobi method
 (in numerical computation of eigenvalues) 298.B
 (in numerical solution of linear equations) 302.C
 cyclic 298.B
 threshold 298.B
Jacobi polynomial 317.D, App. A, Table 20.V
Jacobi second method of integration 324.D
Jacobi symbol 297.I
 complementary law of 297.I
 law of quadratic reciprocity of 297.I
Jacobi standard form, Legendre- 134.A, App. A, Table 16.I
Jacobi transformation App. A, Table 16.III
Jacobson radical (of a ring) 67.D
Jacobson topology 36.D
James theorem 37.G
Janko-Ree type, group of 151.J
Janzen area (of a Borel set) 246.G
Japanese mathematics (wasan) 230
Japanese ring, universally 284.F
Jarrat-Mack method, Garside- 301.N
Jeffreys method 112.B
Jensen formula 198.F
Jensen measure 164.K

jet

invertible 105.X
 of order r 105.X
job 281.D
job-shop scheduling 307.C
job-shop scheduling problem 376
John-Nirenberg space (= *BMO*) 168.B
join
 (in a Boolean algebra) 42.A
 (in a lattice) 243.A
 (of points) 155.B
 (of projective spaces) 343.B
 (of sets) 381.B
 (of simplicial complexes) 70.C
 (of subgroups of a group) 190.G
 reduced (of homotopy classes) 202.Q
 reduced (of mappings) 202.F
 reduced (of topological spaces) 202.F
joined by an arc 79.B
joint cumulant 397.I
joint density 397.I
joint distribution 342.C
joint moment generating function 397.I.J
joint random variable 342.C
joint sensity function 397.J
joint spectrum 36.M
Jordan algebras 231
 exceptional 231.A
 free special 231.A
 semisimple 231.B
 special 231.A
Jordan arc 93.B
Jordan canonical form (of a matrix) 269.G
Jordan content 270.G
Jordan curve 93.B
Jordan curve theorem 93.K
Jordan decomposition
 (of an additive set function) 380.C
 (of a function of bounded variation) 166.B
 (of a linear mapping) 269.L
 (in an ordered linear space) 310.B
 multiplicative 269.L
Jordan domain 333.A
Jordan elimination, Gauss- 302.B
Jordan factorial 330
Jordan-Hölder sequence (in a group) 190.G
Jordan-Hölder theorem (in group theory) 190.G
Jordan-Hölder theorem (on representations of algebras) 362.D
Jordan homomorphism (between Jordan algebras) 231.A
Jordan inequality App. A, Table 8
Jordan measurable set (of \mathbf{R}^n) 270.G
Jordan measure 270.D,G
Jordan module 231.C
Jordan normal form 269.G
Jordan test (on the convergence of Fourier series) 159.B
Jordan-Zassenhaus theorem (on integral representation of a group) 362.K
Joule heat 130.B
Joule's law 130.B
Julia's direction (of a transcendental entire function) 272.F 429.C
Julia's exceptional function 272.F
jump (at a point) 84.B
jump function 306.C
jumping of the structures 72.G
just identified (equation) 128.C

K

- κ -recursiveness 356.G
- k -almost simple algebraic group 13.O
- k -anisotropic algebraic group 13.G
- k -array 330
- k -Borel subgroup (of an algebraic group) 13.G
- k -closed algebraic set 13.A
- k -combination 330
- k -compact algebraic group 13.G
- k -connect (graph) 186.F
- k -dimensional integral element 191.I
- k -dimensional integral manifold 191.I
- k -dimensional normal distribution 341.D
- k -equivalent C^∞ -manifolds 114.F
- k -Erlang distribution 260.H
- k -fold mixing automorphism 136.E
- k -fold screw glide with pitch 92.E
- k -form
 - (of an algebraic group) 13.M
 - holomorphic 72.A
- k frame 199.B
 - orthogonal 199.B
- k -group 13.A
- k -invariants (of a CW-complex) 70.G
- k -isomorphism (between algebraic groups) 13.A
- k -isotropic algebraic group 13.G
- k -morphism (between algebraic groups) 13.A
- k -movable 382.C
- k -permutation 330
- k -ply transitive G -set 362.B
- k -ply transitive permutation group 151.H
- k -quasisplit algebraic group 13.O
- k -rank (of a connected reductive algebraic group) 13.Q
- k -rational divisor (on an algebraic curve) 9.C
- k -rational point (of an algebraic variety) 16.A
- k -root 13.Q
- k -sample problem 371.D
- k -simple algebraic group 13.O
- k -solvable algebraic group 13.F
- k -space 425.CC
- k -split algebraic group 13.N
- k -split torus 13.D
 - maximal 13.Q
- k -step method, linear 303.E
- k -subgroup
 - minimal parabolic 13.Q
 - standard parabolic 13.Q
- k -subset 330
- k -transitive permutation group 151.H
- k -trivial torus 13.D
- k -valued algebroidal function 17.A
- k -vector bundle, normal 114.J
- k -way contingency table 397.K
- k -Weyl group 13.Q
- k' -space 425.CC
- k th prolongation
 - (of G structure) 191.E
 - (of a linear Lie algebra) 191.D
 - (of a linear Lie group) 191.D
- k th saturated model 293.B
- k th transform 160.F
- K -complete analytic space 23.F
- K -complete scheme 16.D
- K -flow 136.E
- K -group 237.B
 - equivariant 237.H
- K -method 224.C
- K -pseudoanalytic function 352.B
- K -quasiregular function 352.B
- K -rational (= algebraic over K) 369.C
- K -regular measure 270.F
- K -theory 237
 - algebraic 237.J
 - higher algebraic 237.J
- $K3$ surface 15.H 72.K
 - marked 72.K
- Kac formula, Feynman- 351.F
- Kac- Nelson formula, Feynman- 150.F
- Kadomtsev-Petvyashvili equation 387.F
- Kähler existence theorem, Cartan- 191.I 428.E
- Kähler homogeneous space 199.A
- Kähler immersion 365.L
- Kähler manifold 232
- Kähler metric 232.A
 - standard (of a complex projective space) 232.D
- Kähler metric, Einstein- 232.C
- Kähler submanifold 365.L
- Takeya-Eneström theorem (on an algebraic equation) 10.E
- Katutani equivalence 136.F
- Kakutani fixed point theorem 153.D
- Kakutani theorem
 - (on complemented subspace problems of Banach spaces) 37.N
 - (on statistical decision problems) 398.G
- Kakutani unit 310.G
- Källén-Lehmann representation 150.D
- Källén-Lehmann weight 150.D
- Kalman-Bucy filter 86.E 405.G
- Kalman filter 86.E
- Kaluza's 5-dimensional theory 434.C
- Kametani theorem, Hällström- 124.C
- Kan complex 70.F
- Kaplansky's density theorem 308.C
- Kapteyn series 39.D, App. A, Table 19.III
- Karlin's theorem 399.G
- Kastler axioms, Haag- 150.E
- Kato perturbation 351.D
- Kato theorem, Rellich- 331.B
- Kawaguchi space 152.C
- KdV equation 387.B
 - two-dimensional 387.F
- Keisler-Shelah isomorphism theorem 276.E
- Keller-Maslov index 274.C
- Kelly theorem, Nachbin-Goodner- 37.M
- Kelvin function 39.G, App. A, Table 19.IV
- Kelvin transformation 193.B
- Kendall notation 260.H
- Kendall's rank correlation 371.K
- Kepler's equation 309.B
- Kepler's first law 271.B
- Kepler's orbital elements 309.B
- Kepler's second law 271.B
- Kepler's third law 271.B
- Kerékjártó-Stoilow compactification 207.C
- kernel
 - (of a bargaining set) 173.D
 - (distribution) 125.L
 - (of a group homomorphism) 190.D
 - (of an integral equation) 217.A
 - (of an integral operator) 251.O
 - (of an integral transform) 220.A
 - (of a linear mapping) 256.F
 - (of a morphism) 52.N
 - (of an operator homomorphism) 277.E
 - (of a potential) 338.B

- (of a set) 425.O
- (of a sheaf homomorphism) 383.C
- (of a system of Suslin) 22.B
- adjoint 338.B
- of Calderón-Zygmund type 217.J
- of Carleman type 217.J
- consistent 338.E
- degenerate 217.F
- diffusion 338.N
- Dirichlet 159.B
- distribution 338.P
- domain (of a sequence of domains) 333.C
- elementary 320.H
- Fejér 159.C
- Fourier 220.B
- fundamental 320.H
- Hermitian 217.H
- of Hilbert-Schmidt type 217.I
- Hunt 338.O
- integral 217.A 251.O
- iterated 217.D
- Kuramochi 207.C
- Martin 207.C
- perfect (in potential theory) 338.E
- Pincherle-Goursat 217.F
- Poisson 159.C
- positive 217.H
- positive definite 217.H 338.D
- positive semidefinite 217.H
- reproducing 188.G
- semidefinite 217.H
- separated 217.F
- singular 217.J
- symmetric 217.G
- symmetric, of positive type 338.D
- weak potential 260.D
- Wiener 95
- kernel differential 188.G
- kernel form 348.E
- kernel function 188.G
 - Bergman 188.G
 - harmonic 188.H
 - Szegő 188.H
- kernel representation (of a Green's operator) 189.B
- kernel theorem (of Schwartz) 125.L 424.S
- Kerner-Aberth (DKA) method, Durand- 301.F
- Kerner (DK) method, Durand- 301.F
- Kerr metric 359.E
- Kervaire invariant, Arf- 114.J
- Khachyan's method 255.C
- Khinchin canonical form (of an infinitely divisible probability distribution) 341.G
- Khinchin decomposition (of a covariance function) 395.B
- Khinchin's law of the iterated logarithm 250.C
- Kiefer inequality, Chapman-Robbins- 399.D
- killing 261.F
- Killing curvature, Lipschitz- 279.C
- Killing differential equation 364.F
- Killing form 248.B
- killing measure 115.B
- killing method (of obtaining a homotopy group) 202.N
- killing time 260.A
- Killing vector field 364.F
- kind
 - Abelian differential of the first, second, third 11.C
 - Abelian integral of the first, second, third 11.C
 - associated Legendre function of the first, second 393.C
 - Beltrami differential operator of the first, second App. A, Table 4.II
 - Bessel function of the first, second, third 39.B
 - complete elliptic integral of the first, second 134.B,C
 - differential form of the first (on an algebraic variety) 16.O
 - canonical coordinates of the first, second 249.Q
 - differential form of the first, second, third (on a nonsingular curve) 9.E
 - discontinuity of the first, second 84.B
 - discontinuous group of the first 122.B
 - elliptic function of the first, second, third 134.G,H
 - elliptic integral of the first, second, third 134.A
 - error of the first, second 400.A
 - Euler integral of the first, second 174.A,C
 - exceptional curve of the first, second 15.G
 - Fredholm integral equation of the first, second, third 217.A
 - Fuchsian group of the first, second 122.C
 - Hankel function of the first, second 39.B
 - incomplete elliptic integral of the first 134.B
 - Lamé function of the first, second 133.C
 - Legendre function of the first, second 393.B
 - Mathieu function of the first, second 268.B
 - modified Mathieu function of the first, second, third 268.D
 - perfect members of the second 297.D
 - Stirling number of the second 66.D
 - kinetic density 218.A
 - kinetic energy 271.C 351.D
 - kinetic measure 218.A
 - kinetic theory of gases 402.B
 - Kirby calculus 114.L
 - Kirchhoff laws 282.B
 - Kirchhoff solution 325.D
 - Kleene's normal form theorem 356.C
 - Klein, F. 233
 - Klein bottle 410.B
 - Klein combination theorem 234.D
 - Klein four-group 151.G
 - Klein-Gordon equation 351.G 377.C
 - Kleinian group 122.C 234.A
 - Klein inequality 212.B
 - Klein line coordinates 90.B
 - Klein model (of non-Euclidean geometry) 285.C
 - Klein-Nishina formula 351.G
 - Klein transform 150.D
 - K-L (= Kullback-Leibler) information number 398.G
 - Kloosterman sum 32.C
 - KMS condition 308.H 402.G
 - Kneser-Nagumo theorem 316.E
 - Kneser-Sommerfeld formula App. A, Table 19.III
 - Knopp-Schmidt theorem 208.C
 - Knopp-Schnee theorem (on method of summation) 379.M
 - knot 235.A
 - alternating 235.A, App. A, Table 7
 - amphicheiral 235.A
 - clover leaf 235.C
 - equivalent 235.A
 - hyperbolic 235.E
 - invertible 235.A
 - (p, q)- 235.G
 - (p, q)-ball 235.G

slice 235.G
 trefoil 235.C
 knot cobordism 235.G
 knot complement conjecture 235.B
 knot conjecture, general 235.B
 knot group (of a knot) 235.B
 knot projection, regular 235.A
 knotted 235.A
 knot theory 235, App. A, Table 7
 knot types 235.A
 Kobayashi pseudodistance 21.0
 Kochen isomorphism theorem, Ax- 276.E
 Kodaira dimension (of a compact complex manifold) 72.I
 Kodaira-Spencer mapping (map) 72.G
 Kodaira theorem
 (on Hodge manifolds) 232.D
 Chow- 72.F
 Kodaira theory, Weyl-Stone-Titchmarsh- 112.O
 Kodaira vanishing theorem 232.D
 Koebe extremal function 438.C
 Koebe theorem 193.E
 Kojima-Schur theorem (on linear transformations of sequences) 379.L
 Kolchin theorem, Lie- (on solvable algebraic groups) 13.F
 Kollektiv 342.A
 Kolmogorov automorphism 136.E
 Kolmogorov axiom T_0 425.Q
 Kolmogorov backward equation 115.A 260.F
 Kolmogorov canonical form 341.G
 Kolmogorov-Chaitin complexity 354.D
 Kolmogorov equality, Chapman- 261.A
 Kolmogorov equation, Chapman- 260.A
 Kolmogorov extension theorem 341.I 407.D
 Kolmogorov flow 136.E
 Kolmogorov forward equation 115.A 260.F
 Kolmogorov-Smirnov test 371.F
 Kolmogorov-Smirnov test statistic 374.E
 Kolmogorov space 425.Q
 Kolmogorov-Spanier cohomology theory, Alexander- 201.M
 Kolmogorov spectrum 433.C
 Kolmogorov test 45.F
 Kolmogorov theorem 250.F
 Kolmogorov zero-one law 342.G
 Kondô uniformization theorem 22.F
 König-Egerváry theorem 281.E
 Königs-Schröder equation 44.B
 Korteweg-de Vries (KdV) equation 387.A
 Kostant's formula (on representations of compact Lie groups) 248.Z
 Kovalevskaya existence theorem, Cauchy- 321.A
 Kovalevskaya theorem, abstract Cauchy- 286.Z
 Kreĭn-Mil'man property 443.H
 Kreĭn-Mil'man theorem 424.U
 Kreĭn-Schmul'yan theorem 37.E 424.O
 Kreĭn theorem 424.V
 Krieger's factor 308.I
 Kronecker, L. 236
 Kronecker approximation theorem 422.K
 Kronecker delta 269.A, App. A, Table 4.II
 Kronecker flow 136.G
 Kronecker index
 (in homology theory) 201.H
 (of divisors on a surface) 15.C
 Kronecker limit formula 450.B
 Kronecker product (of matrices) 269.C
 Kronecker set 192.R
 Kronecker symbol (for a quadratic field) 347.D

Kronecker theorem
 (on an Abelian extension of \mathbb{Q}) 14.L
 (on an algebraic equation) 10.B
 Krull-Akizuki theorem 284.F
 Krull altitude theorem 28.A
 Krull-Azumaya lemma 67.D
 Krull dimension
 (of a commutative ring) 67.E
 (of an ideal) 67.E
 Krull intersection theorem 284.A
 Krull-Remak-Schmidt theorem (in group theory) 190.L
 Krull ring 67.J
 Krull topology (for an infinite Galois group) 172.I
 Kruskal coordinates 359.F
 Kruskal-Wallis test 371.D
 Kubo formula 402.K
 Kuhn-Tucker theorem 292.B
 Kullback discrimination information 213.D
 Kullback-Leibler (K-L) information number 398.G
 Kummer criterion 145, App. A, Table 10.II
 Kummer differential equation App. A, Table 19.I
 Kummer extension 172.F
 Kummer function 167.A, App. A, Table 19.I
 Kummer surface 15.H
 Künneth formula
 (in Abelian category) 200.H
 (in Weil cohomology) 450.Q
 Künneth theorem 200.E 201.J
 Kunugi theorem, Iversen-Beurling- 62.B
 Kuo (PLK) method, Poincaré-Lighthill- 25.B
 Kuramochi compactification 207.C
 Kuramochi kernel 207.C
 Kuranishi prolongation theorem 428.G
 Kuranishi space 72.G
 Kuratowski space 425.Q
 kurtosis 396.C 397.C
 population 396.C
 Kutta-Gill method, Runge- 303.D
 Kutta methods, Runge- 303.D

L

Λ^s (Lipschitz spaces) 168.B
 λ -function 32.C
 (l_p) or l_p (a sequence space) 168.B
 $L_p(\Omega)$ (the space of measurable functions $f(x)$ on Ω such that $|f(x)|^p$, $1 \leq p \leq \infty$, is integrable) 168.B
 $L_\infty(\Omega)$ 168.B
 $L_{(p,q)}(\Omega)$ (the Lorentz spaces) 168.B
 l -adic cohomology 450.Q
 l -adic coordinate system 3.E
 l -adic representation 3.E
 L -distribution 341.G
 L -estimator 371.H
 L -function
 Artin 450.G,R
 of automorphic representation 450.N
 Dirichlet 450.C
 of elliptic curves 450.S
 Hecke 450.E
 Hecke (with Größencharakter) 450.F
 p -adic 450.J
 Weil 450.H
 L -group 450.N
 L -integral 221.B
 L -space 87.K
 abstract 310.G
 Fréchet 87.K

L_1 -algebra

- L_1 -algebra (of a locally compact Hausdorff group) 36.L
- L_p -space, abstract 310.G
- L^* -space 87.K
- (LF)-space 424.W
- labeled graph 186.B
- lacunas for hyperbolic operators 325.J
- lacunary structure (of a power series) 339.E
- ladder
 - down- 206.B
 - up- 206.B
 - ladder method 206.B
- lag 163.A
- lagged variables 128.C 274.C
- Lagrange, J. L. 238
- Lagrange bracket 82.A 324.D
- Lagrange-Charpit method 322.B, App. A, Table 15.II
- Lagrange differential equation 320.A, App. A, Table 14.I
- Lagrange differential equation, Euler- 46.B
- Lagrange equations of motion 271.F
- Lagrange formula (for vector triple product) 442.C
- Lagrange identity 252.K
- Lagrange interpolation 223.A
- Lagrange interpolation coefficients 223.A
- Lagrange interpolation polynomial 223.A 336.G, App. A, Table 21
- Lagrange manifold, conic 345.B
- Lagrange method
 - (of describing the motion of a fluid) 205.A
 - of indeterminate coefficients 106.L
 - of variation of constants 252.D
- Lagrange multiplier 46.B
 - method of 106.L
- Lagrange partial differential equation App. A, Table 15.II
- Lagrange problem (in calculus of variations) 46.A
- Lagrange remainder App. A, Table 9.IV
- Lagrange resolvent 172.F
- Lagrange stable 126.E
 - negatively 126.E
 - positively 126.E
- Lagrange-stable motion 420.D
- Lagrangian density 150.B
 - free 150.B
- Lagrangian derivative 205.A
- Lagrangian function 271.F 292.A
- Lagrangian manifold, conic 274.C
- Lagrangian vector field 126.L
- Laguerre differential equation App. A, Tables 14.II 20.VI
- Laguerre formula, Gauss- (in numerical integration) 299.A
- Laguerre function App. A, Table 20.VI
- Laguerre geometry 76.B
- Laguerre inversion 76.B
- Laguerre polynomials 317.D, App. A, Table 20.VI
 - associated 317.D
- Laguerre transformation 76.B
- Lambert series 339.C
- Lamé differential equation 133.B
 - generalized 167.E
- Lamé function
 - of the first kind 133.B
 - of the first species 133.C
 - of the fourth species 133.C
 - of the second kind 133.C
 - of the second species 133.C
 - of the third species 133.C
- lamellar vector field 442.D
- laminar flow 205.E 433.A
- Lanczos method 298.D, E 301.N
- Landau constant 77.F
- Landau equation 146.C
- Landau-Nakanishi equation 146.C
- Landau-Nakanishi variety 146.C 386.C
- Landau symbols (O, o) 87.G
- Landau theorem 43.J
 - Wiener-Ikehara 123.B
- Landau variety 146.C
- Landen transformation 134.B, App. A, Table 16.III
- Lane-Emden function 291.F
- Langevin equation 45.I 402.K
- language 31.D 276.A
 - accepted by 31.D
 - external 75.C
 - machine 75.C
- Laplace, P. S. 239
- Laplace-Beltrami operator 194.B
- Laplace differential equation 323.A, App. A, Table 15.III
 - 2-dimensional case App. A, Table 15.VI
 - 3-dimensional case App. A, Table 15.VI
- Laplace expansion theorem (on determinants) 103.D
- Laplace-Mehler integral representation App. A, Table 18.II
- Laplace method 30.B
- Laplace operator 323.A 442.D
- Laplace spherical functions 393.A
- Laplace-Stieltjes transform 240.A
- Laplace theorem, de Moivre- 250.B
- Laplace transform 240, App. A, Table 12.I
 - Fourier- 192.F
- Laplace transform and operational calculus App. A, Table 12
- Laplacian 323.A 442.D, App. A, Table 3.II
 - in the large 109
 - large deviation 250.B
- large inductive dimension (Ind) 117.B
- large numbers
 - law of 250.B
 - strong law of 250.C
 - weak law of 395.B
- larger, stochastically 371.C
- larger topology 425.H
- large sample theory 401.E
- large semigroup algebra 29.C
- large sieve method 123.E
- largest nilpotent ideal (of a Lie algebra) 248.D
- Lashnev space 425.CC
- last multiplier, Jacobi App. A, Table 14.I
- last-out memory, first-in 96.E
- last theorem of Fermat 145
- last theorem of Poincaré 153.B
- latin rectangle 241.E
- Latin square(s) 102.K 241
- lattice(s)
 - (of a crystallographic group) 92.A
 - (= lattice ordered set) 243
 - (of a Lie group) 122.G
 - (in \mathbf{R}^n) 182.B
 - A - 92.E
 - anti-isomorphic 243.C
 - Archimedean vector 310.C
 - B - 92.E
 - Banach 310.F

- black and white 92.D
- Boolean 42.A 243.E
- Boolean, of sets 243.E
- Bravais 92.B
- C- 92.E
- color 92.D
- complemented 243.E
- complete 243.D
- complete vector 310.C
- conditionally complete 243.D
- conditionally σ -complete 243.D
- critical, in M with respect to S 182.B
- distributive 243.E
- dual 243.C 310.E 450.K
- dually isomorphic 243.C
- F- 92.E
- g- (of a separable algebra) 27.A
- homogeneous (in \mathbf{R}^n) 182.B
- integral g- 27.A
- I- 92.E
- inhomogeneous (in \mathbf{R}^n) 182.B
- modular 243.F
- normal g- 27.A
- normal vector 310.F
- one-dimensional 287.A
- P- 92.E
- primitive 92.E
- quotient 243.C
- of sets 243.E
- σ -complete 243.D
- σ -complete vector 310.C
- Toda 287.A 387.A
- vector 310.B
- weight 92.C
- lattice constants (of a lattice group) 92.C
- lattice distribution 341.D
- lattice equivalent 92.D
- lattice gauge theory 150.G
- lattice group(s) 182.B
 - (of a crystallographic group) 92.A
- lattice-homomorphism 243.C
- lattice-isomorphism 243.C
- lattice-ordered group 243.G
 - Archimedean 243.G
- lattice-ordered linear space 310.B
- lattice-ordered set 243.A
- lattice-point formula 220.B
- lattice-point problems 242
- lattice-spin systems 402.G
- latus rectum 78.D,E
- Laurent expansion 198.D
- Laurent series 339.A
- law(s)
 - (of a solution of a stochastic differential equation) 406.D
 - absorption (in algebra of sets) 381.B
 - absorption (in a lattice) 243.A
 - of action and reaction 271.A
 - adiabatic 205.B
 - alternating (in a Lie algebra) 248.A
 - antisymmetric (for ordering) 311.A
 - arcsine (for Brownian motion) 45.E
 - arcsine (for distribution function) 250.D
 - arcsine (for random walk) 260.E
 - associative (for addition) 294.B
 - associative (in algebra of sets) 381.B
 - associative (of correspondences) 358.B
 - associative (of group composition) 190.A
 - associative (in a lattice) 243.A
 - associative (for multiplication) 294.B
 - associative (in a ring) 368.A
 - Blumenthal zero-one 261.B
 - Brandt's 241.C
 - cancellation (for addition) 294.B
 - cancellation (in a commutative semigroup) 190.P
 - cancellation (for multiplication) 294.B
 - commutative (for addition) 294.B 368.A
 - commutative (in algebra of sets) 381.B
 - commutative (of group composition) 190.A
 - commutative (in a lattice) 243.A
 - commutative (for multiplication) 294.B 368.A
 - complementary, of quadratic reciprocity of Jacobi symbol 297.I
 - complementary, of reciprocity 14.O
 - of complementation (in a Boolean algebra) 42.A
 - complete distributive (in a lattice-ordered group) 243.G
 - of composition 409.A
 - of composition, external 409.A
 - of composition, internal 409.A
 - of cosines (on spherical triangles) 432.B, App. A, Table 2.III
 - of cosines, first 432.A, App. A, Table 2.II
 - of cosines, second 432.A, App. A, Table 2.II
 - of cotangents App. A, Table 2.III
 - de Morgan's (in algebra of sets) 381.B
 - de Morgan's (in a Boolean algebra) 42.A
 - differential 107.A
 - distributive (in algebra of sets) 381.B
 - distributive (in a lattice) 243.E
 - distributive (of natural numbers) 294.B
 - distributive (in a ring) 368.A
 - even-oddness conservation 150.D
 - of excluded middle 156.C 411.L
 - explicit reciprocity (of norm-residue symbol) 14.R 257.H
 - first complementary, of quadratic reciprocity of Legendre symbol 297.I
 - Gel'fand-Pyatetskii-Shapiro reciprocity (on unitary representations) 437.DD
 - general associative (for group composition) 190.C
 - Hewitt-Savage zero-one 342.G
 - Hooke's 271.G
 - idempotent (in a lattice) 243.A
 - of inertia 271.A
 - of inertia, Sylvester (for a quadratic form) 348.C
 - initial (for stochastic differential equation) 406.D
 - of iterated logarithm 45.F
 - of iterated logarithm, Khinchin 250.C
 - Joule's 130.B
 - Kepler's first 271.B
 - Kepler's second 271.B
 - Kepler's third 271.B
 - Kirchhoff 282.B
 - Kolmogorov zero-one 342.G
 - of large numbers 250.B 395.B
 - of large numbers, strong 250.C
 - Maxwell-Boltzmann distribution 402.B
 - modular (in a lattice) 243.F
 - of motion 271.A
 - of motion, Newton's three 271.A
 - Newton (on frictional stresses) 205.C
 - Newton's first 271.A
 - Newton's second 271.A
 - Newton's third 271.A

Lax equivalence theorem

- Ohm 130.B 259
- of quadratic reciprocity of Jacobi symbol 297.I
- of quadratic reciprocity of Legendre symbol 297.I
- of reaction 271.A
- of reciprocity 14.O 297.I
- of reciprocity, Artin general 59.C
- of reciprocity, general 14.O
- reciprocity, of Shafarevich 257.H
- reflexive (for an equivalence relation) 135.A
- reflexive (for ordering) 311.A
- second complementary, of quadratic reciprocity of Legendre symbol 297.I
- of similarity, Prandtl-Glauert 205.D
- of similarity, Reynolds 205.C
- of similitude 116
- of sines 432.A, App. A, Table 2.II
- of sines (for spherical triangles) 432.B, App. A, Table 2.III
- of sines and cosines App. A, Table 2.III
- of small numbers 250.B
- symmetric (for an equivalence relation) 135.A
- of symmetry (for the Hilbert norm-residue symbol) 14.R
- of tangents App. A, Table 2.III
- transitive (for an equivalence relation) 135.A
- transitive (for ordering) 311.A
- of universal gravitation 271.B
- zero-one 342.G
- Lax equivalence theorem 304.F
- Lax representation 287.B 387.C
- Lax-Wendroff scheme 304.F
- layout, two-way 155.H
- layer
 - boundary 205.C
 - potential of double 338.A
 - potential of single 338.A
- LBA problem 31.F
- L.B. (loosely Bernoulli) 136.F
- LCL (lower control limit) 404.B
- L.C.M. (least common multiple) 67.H 297.A
- leaf (leaves)
 - (of a foliation) 154.B
 - compact 154.D
 - growth of 154.H
- leaf topology 154.D
- learning model 346.G
- least action, principle of 441.B
- least common multiple 67.H 297.A
- least element (in an ordered set) 311.B
- least favorable distribution 400.B
- least favorable a priori distribution 398.H
- least square approximation 336.D
- least squares, method of
 - (for estimation) 403.E
 - (for higher-dimensional data) 397.J
 - (for numerical solution) 303.I
- least squares estimator 403.E
 - generalized 403.E
- least squares method
 - indirect (in econometrics) 128.C
 - three-stage 128.C
 - two-stage 128.C
- least squares problem, linear 302.E
- least upper bound (ordered sets) 310.C 311.B
- Lebesgue, H. L. 244
- Lebesgue area (of a surface) 246.C
- Lebesgue convergence theorem 221.C
- Lebesgue decomposition theorem 270.L 380.C
- Lebesgue density theorem 100.B
- Lebesgue dimension 117.B
- Lebesgue extension 270.D
- Lebesgue integrable 221.B
- Lebesgue integral 221.B
- Lebesgue measurability and the Baire property 33.F
- Lebesgue measurable (set) 270.G
- Lebesgue measurable function 270.J
- Lebesgue measure 270.G
 - generalized 270.E
- Lebesgue measure space (with a finite σ -finite measure) 136.A
- Lebesgue method of summation 379.S
- Lebesgue number 273.F
- Lebesgue outer measure 270.G
- Lebesgue-Radon integral 94.C
- Lebesgue spectrum, countable 136.E
- Lebesgue-Stieltjes integral 94.C 166.C
- Lebesgue-Stieltjes measure 166.C 270.L
- Lebesgue test (on the convergence of Fourier series) 159.B
- Lebesgue theorem
 - (on the dimension of \mathbf{R}^n) 117.D
 - Borel- 273.H
 - Cantor- 159.J
 - Riemann- 159.A 160.A
- Le Cam theorem 399.K
- Lefschetz duality theorem, Poincaré- 201.O
- Lefschetz fixed-point formula 450.Q
- Lefschetz fixed-point theorem 153.B
- Lefschetz formula, Picard- 418.F
- Lefschetz number
 - (of a continuous mapping) 153.B
 - (of a variety) 16.P
- Lefschetz pencil 16.U
- Lefschetz theorem
 - strong 16.U
 - weak 16.U
- Lefschetz transformation, Picard- 16.U
- left, limit on the 87.F
- left A -module 277.D
- left adjoint functor 52.K
- left adjoint linear mapping 256.Q
- left annihilator 29.H
- left Artinian ring 368.F
- left balanced functor 200.I
- left continuous 84.B
- left coset 190.C
- left coset space 423.E
- left decomposition, Peirce (in a unitary ring) 368.F
- left derivative 106.A
- left derived functor 200.I, Q
- left differentiable 106.A
- left distributive law 312.C
- left endpoint (of an interval) 355.C
- left exact functor 200.I
- left G -set 362.B
- left global dimension (of a ring) 200.K
- left hereditary ring 200.K
- left ideal 368.F
 - integral 27.A
- left invariant Haar measure 225.C
- left invariant metric (of a topological group) 423.I
- left invariant tensor field (on a Lie group) 249.A
- left inverse element (in a ring) 368.B
- left linear space 256.A
- left Noetherian ring 368.F
- left \mathfrak{D}_r -ideal 27.A
- left operation 409.A

left order (of a g -lattice) 27.A
 left parametrix 345.A
 left projective resolution (of an A -module) 200.C
 left projective space 343.F
 left quotient space (of a topological group) 423.E
 left regular representation
 (of an algebra) 362.C
 (of a group) 362.B
 left resolution (of an A -module) 200.C
 left satellite 200.I
 left semihereditary ring 200.K
 left semi-integral 68.N
 left shunt 115.B
 left singular point (of a diffusion process) 115.B
 left translation 249.A 362.B
 left uniformity (of a topological group) 423.G
 Legendre associated differential equation 393.A
 Legendre coefficient 393.B
 Legendre differential equation 393.B, App. A,
 Table 14.II
 Legendre function 393.B, App. A, Table 18.II
 associated App. A, Table 18.III
 associated (of the first kind) 393.C, App. A,
 Table 18.III
 associated (of the second kind) 393.C, App.
 A, Table 18.III
 of the first kind 393.B, App. A, Table 18.II
 of the second kind 393.B, App. A, Table 18.II
 Legendre-Jacobi standard form 134.A, App. A,
 Table 16.I
 Legendre polynomial 393.B, App. A, Table 18.II
 Legendre relation 134.F, App. A, Table 16.I
 Legendre symbol 297.H
 first complementary law of reciprocity of 297.I
 law of quadratic reciprocity of 297.I
 second complementary law of reciprocity of
 297.I
 Legendre transform 419.C
 Legendre transformation (contact transform) 82.A,
 App. A, Table 15.IV
 Lehmann representation, Källén- 150.D
 Lehmann-Scheffé theorem 399.C
 Lehmann-Stein theorem 400.B
 Lehmann theorem 371.C
 Hodges- 399.E,H
 Lehmann weight, Källén- 150.D
 Lehmer method 301.K
 Leibler (K-L) information number, Kullback-
 398.G
 Leibniz, G. W. 245
 Leibniz formula (in differentiation) 106.D, App.
 A, Table 9.III
 Leibniz test (for convergence) 379.C
 lemniscate 93.H
 Bernoulli 93.H
 length 246
 (of a broken line) 139.F
 (of a curve) 93.F 246.A
 (of a descending chain in a lattice) 243.F
 (of a module) 277.I
 (of a multi-index) 112.A
 (of a normal chain in a group) 190.G
 (of a path) 186.F
 (of a segment) 139.C
 (of a Witt vector) 449.B
 affine 110.C
 affine arc 110.C
 extremal (of a family of curves) 143.A
 extremal, defined by Hersch-Pfluger 143.A
 extremal, with weight 143.B

Subject Index

Lie algebra(s)

of finite (module) 277.I
 focal 180.B
 π - (of a group) 151.F
 queue 260.H
 wave 446
 lens, Luneburg's 180.A
 lens space 91.C
 infinite 91.C
 Leopoldt's conjecture 450.J
 leptons 132.B
 Leray-Hirsch theorem 201.J
 Leray-Schauder degree 286.D
 Leray-Schauder fixed-point theorem 286.D 323.D
 letter
 (in information theory) 63.A 213.B
 (= variable) 369.A
 level
 (of a factor) 102.H
 (of a modular form) 32.C
 (of a modular function) 32.C
 (of an orthogonal array) 102.L
 (of a principal congruence subgroup) 122.D
 (of a test) 400.A
 (of a tolerance region) 399.R
 average outgoing quality 404.C
 confidence 399.Q
 level α test 400.A
 minimax 400.F
 most stringent 400.F
 unbiased 400.C
 uniformly most powerful (UMP) invariant
 400.E
 uniformly most powerful (UMP) unbiased
 400.C
 level n structure (on an Abelian variety) 3.N
 level set (of a C^∞ -function) 279.D
 level surface 193.J
 Levi-Civita, parallel in the sense of 111.H
 Levi-Civita connection 364.B
 Levi condition 321.G 325.H
 Levi decomposition
 (on algebraic groups) 13.Q
 (on Lie algebras) 248.F
 Levi form 344.A
 generalized 274.G
 Levi problem 21.I,F
 Levi pseudoconvex domain 21.I
 locally 21.I
 Levi subgroup 13.Q
 Levitan-Marchenko equation, Gel'fand-
 (for KdV equations) 387.D
 (for a nonlinear lattice) 287.C
 Lévy canonical form 341.G
 Lévy continuity theorem 341.F
 Lévy distance 341.F
 Lévy-Itô theorem (on Lévy processes) 5.E
 Lévy measure 5.E
 Lévy process 5.B
 Lévy theorem, Wiener- 159.I
 Lewy-Mizohata equation 274.G
 lexicographic linear ordering 248.M
 lexicographic ordering 311.G
 liability reserve 214.B
 Lie, M. S. 247
 Lie algebra(s) 248, App. A, Table 5.I
 (of an algebraic group) 13.C
 (of a Lie group) 249.B
 Abelian 248.C
 adjoint 248.B
 algebraic 13.C

- classical compact real simple 248.T
- classical complex simple 248.S
- compact real 248.P
- compact real simple App. A, Table 5.I
- complex 248.A
- complex (of a complex Lie group) 249.M
- complex simple App. A, Table 5.I
- of derivations 248.H
- exceptional compact real simple 248.T
- exceptional complex simple 248.S
- general linear 248.A
- isomorphic 248.A
- nilpotent 248.C
- noncompact real simple App. A, Table 5.II
- quotient 248.A
- real 248.A
- reductive 248.G
- restricted 248.V
- semisimple 248.E
- simple 248.E
- solvable 248.C
- Lie derivative 105.O.Q
- Lie fundamental theorem (on a local Lie group of local transformations) 431.G
- Lie group(s) 249.423.M
 - Abelian 249.D
 - Banach 286.K
 - classical compact simple 249.L
 - classical complex simple 249.M
 - commutative 249.D
 - complex 249.A
 - direct product of 249.H
 - exceptional compact simple 249.L
 - exceptional complex simple 249.M
 - isomorphic 249.N
 - local 423.L
 - local (of local transformations) 431.G
 - nilpotent 249.D
 - quotient 249.G
 - semisimple 249.D
 - simple 249.D
 - simply connected covering 249.C
 - solvable 249.D
 - topology of, and homogeneous spaces 427
- Lie-Kolchin theorem (on solvable algebraic groups) 13.F
- Lie line-sphere transformation 76.C
- Lie minimal projection 76.B
- Liénard differential equation 290.C
- lies over (of a compactification) 207.B
- Lie subalgebra 248.A
 - associated with a Lie subgroup 249.D
- Lie subgroup
 - (of a Lie group) 249.D
 - connected 249.D
- Lie theorem (on Lie algebras) 248.F
- Lie transformation (in circle geometry) 76.C
- Lie transformation group (of a differentiable manifold) 431.C
- lifetime 260.A 261.B
 - (of a particle by a scattering) 132.A
- lift
 - (along a curve in a covering surface) 367.B
 - (of a differentiable curve) 80.C
 - (of a vector field) 80.C
 - inflation 200.M
- lifting (in nonstandard analysis) 293.D
- lifting theorem 251.M
- light cone 258.A
- Lighthill-Kuo (P.L.K.) method, Poincaré- 25.B
- Lighthill method 25.B
- lightlike 258.A 359.B
- likelihood 374.J
- likelihood equation 399.M
- likelihood estimating function 399.M
- likelihood estimator, maximum 399.M
- likelihood function 374.J 399.M
- likelihood method, maximum 399.M
- likelihood ratio 400.I
 - monotone 374.J
- likelihood ratio test 400.I
- limaçon of Pascal 93.H
- liminal C^* -algebra 36.H
- limit
 - (of a function) 87.F
 - (of an indeterminate form) 106.E
 - (of a mapping) 87.F
 - (of a net) 87.H
 - (of a sequence of lattices) 182.B
 - (of a sequence of points) 87.E 273.D
 - (of a sequence of real numbers) 87.B 355.B
 - (of a sequence of sets) 270.C
 - (of a spectral sequence) 200.J
 - Banach 37.F
 - confidence 399.Q
 - direct (of a direct system) 210.B
 - elastic 271.G
 - generalized 37.F
 - inductive (in a category) 210.D
 - inductive (group) 210.C
 - inductive (of an inductive system) 210.B
 - inductive (of sheaves) 383.I
 - inductive (space) 210.C
 - inductive (of topological spaces) 425.M
 - inferior (event) 342.B
 - inferior (of a sequence of real numbers) 87.C
 - inferior (of a sequence of sets) 270.C
 - inverse (of an inverse system) 210.B
 - on the left (of a real-valued function) 87.F
 - lower (function) 84.C
 - lower (of a Riemann integral) 216.A
 - lower (of a sequence of real numbers) 87.C
 - lower control 404.B
 - in the mean 168.B
 - order (of an order convergent sequence) 310.C
 - projective (in a category) 210.D
 - projective (of a family of continuous homomorphisms) 423.K
 - projective (group) 210.C
 - projective (of a projective system) 210.B
 - projective (space) 210.C
 - on the right (of a real-valued function) 87.F
 - strictly inductive (of a sequence of locally convex spaces) 424.W
 - superior (event) 342.B
 - superior (of a sequence of real numbers) 87.C
 - superior (of a sequence of sets) 270.C
 - thermodynamic 402.G
 - tolerance 399.R
 - upper (function) 84.C
 - upper (of a Riemann integral) 216.A
 - upper (of a sequence of real numbers) 87.C
 - upper control 404.B
- limit circle type (boundary point) 112.I
- limit cycle 126.I
- limit distribution 250.A
- limit formula, Kronecker's 450.B
- limited information maximum likelihood method 128.C
- limit inferior

- (of a sequence of real numbers) 87.C
- (of a sequence of sets) 270.C
- limiting absorption principle 375.C
- limiting hypersphere (in hyperbolic geometry) 285.C
- limit ordinal number 312.B
- limit point
 - (of a discontinuous group) 122.C
 - (of a sequence) 87.B,E
 - α - 126.D
 - negative 126.D
 - ω - 126.D
 - positive 126.D
- limit point type (boundary point) 112.I
- limit set 234.A
 - α - 126.D
 - first negative prolongational 126.D
 - first positive prolongational 126.D
 - ω - 126.D
 - residual 234.E
- limit superior
 - (of a sequence of real numbers) 87.C
 - (of a sequence of sets) 270.C
- limit theorem(s) 250.A
 - basic 260.C
 - central 250.B
 - local 250.B
 - in probability theory 250
- limit value (of a mapping) 87.F
- Lindeberg condition 250.B
- Lindelöf asymptotic value theorem 43.C
- Lindelöf hypothesis 123.C
- Lindelöf space 425.S
- Lindelöf theorem 43.F
 - Phragmén- 43.C
- Lindemann-Weierstrass theorem 430.D
- Lindstedt-Poincaré method 290.E
- line(s) 7.A 93.A 155.B
 - (of a graph) 186.B
 - broken 155.F
 - complexes, linear 343.E
 - complex of 110.B
 - concurrent (in projective geometry) 343.B
 - congruence of 110.B
 - congruences, linear 343.E
 - of curvature (on a surface) 111.H
 - of force 193.J
 - generating (of a circular cone) 78.A
 - generating (of a quadric hypersurface) 343.E
 - generating (of a quadric surface) 350.B
 - generating (of a ruled surface) 111.I
 - geodesic 178.H
 - Green 193.J
 - Green, regular 193.J
 - half- 155.B
 - long 105.B
 - normal (to a curve) 93.G
 - Pascal 78.K
 - pencil of (in a projective plane) 343.B
 - projective 343.B
 - real 355.E
 - regression 403.D
 - of regression 111.F,I
 - straight 93.A 155.B
 - stream 205.B
 - supporting (function) 89.C
 - supporting (of an oval) 89.C
 - of swiftest descent 93.H
 - tangent 93.G 111.C,F
 - tangent, oriented 76.B
 - vector (of a vector field) 442.D
 - vortex 205.B
- linear algebra 8
- linear algebraic group 13.A
- linear boundary operators, 315.B
- linear bounded automation
 - deterministic 31.D
 - nondeterministic 31.D
- linear code 63.C
- linear combination 256.C
 - of ovals 89.D
- linear connection 80.H
- linear difference equation 104.C
- linear differential equations, system of (of the first order) 252.G
- linear discriminant function 280.I
- linear dynamical system 86.B
- linear equation(s) 10.D 16.M 269.M
- linear equivalence class (of divisors) 16.M
- linear extension (of a rational mapping to an Abelian variety) 9.E
- linear fiber mapping (map) 114.D
- linear filter 405.F
- linear form 256.B 277.E
- linear fractional function 74.E
- linear fractional group 60.B
- linear fractional programming 264.D
- linear fractional transformation 74.E
- linear function 74.E
- linear functional 37.C 197.F 424.B
 - algebraic 424.B
- linear fundamental figure (of a projective space) 343.B
- linear genus 15.G
- linear graph 282.A
- linear group
 - Abelian (over K) 60.L
 - full 60.B
 - general 60.B 256.D
 - general (of degree n over K) 60.B 226.B
 - general (over a noncommutative field) 60.O
 - projective general 60.B
 - projective general (of degree n over K) 60.B
 - projective special 60.B
 - projective special (over a noncommutative field) 60.O
 - special 60.B
 - special (of degree n over K) 60.B
 - special (over a noncommutative field) 60.O
- linear holonomy 154.C
- linear homogeneous equations, system of 269.M
- linear homotopy 114.D
- linear hypothesis, general 400.H
- linear integral equation 217.A
- linear isotropy group (at a point) 199.A
- linearized operator 183 286.E
- linearized stability, principle of 286.S
- linear k -step method 303.E
- linear least squares problem 302.E
- linear Lie algebra, general 248.A
- linear line complex 343.E
- linear line congruence 343.E
- linear logistic model 403.C
- linearly compact 422.L
 - locally 422.L
- linearly connected homogeneous space 199.A
- linearly dependent (with respect to a difference equation) 104.D

Linearly dependent elements

- linearly dependent elements
 - (in an additive group) 2.E
 - (in a linear space) 256.C
- linearly disjoint (fields) 149.K
- linearly equivalent (divisors) 16.M 172.F
 - \mathcal{O} - (on an algebraic curve) 9.F
- linearly estimable parameter 403.E
- linearly independent elements
 - (in an A -module) 277.G
 - (in an additive group) 2.E
 - (in a linear space) 256.C
- linearly independent family (of elements in a linear space) 256.E
- linearly ordered set 311.A
- linearly reductive 226.B
- linearly representable 66.H
- linear mapping 70.C 256.B
 - A - (of an A -module) 277.E
 - piecewise 70.C
- linear model 403.D
 - log 403.C
 - multivariate 280.B
 - normal 403.C
- linear multistep method 303.E
- linear network 282.C
- linear operator(s) 37.C 251
 - (between linear spaces) 256.B
 - bounded 37.C
- linear ordering 311.A
 - lexicographic 248.M
 - theorem of 155.B
- linear ordinary differential equation(s) 252 253 254 313.A
 - with constant coefficients App. A, Table 14.I
 - of the first order App. A, Table 14.I
 - of higher order App. A, Table 14.I
- linear parameter 102.A
- linear partial differential equation 320.A
- linear pencil 16.N
- linear prediction theory 395.D
- linear predictor 395.D
 - optimal 395.D
- linear programming 255 264.C
- linear programming problem 255.A
- linear recurrent sequence 295.A
- linear regression 397.J
- linear regression function 397.J 403.D
- linear representation
 - (of an algebra) 362.C
 - (of a group) 362.C
 - (of a Lie algebra) 248.B
 - associated with representation module 362.C
 - completely reducible 362.C
 - direct sum of 362.C
 - equivalent 362.C
 - faithful 362.C
 - homomorphism of 362.C
 - indecomposable 362.C
 - irreducible 362.C
 - isomorphic 362.C
 - reciprocal (of an algebra) 362.C
 - reducible 362.C
 - semisimple 362.C
 - similar 362.C
 - simple 362.C
 - tensor product of 362.C
- linear simple group 151.I
- linear space(s) 256
 - category of (over a ring) 52.B
 - complex 256.A
 - dual 256.G
 - over a field 256.A
 - finite-dimensional 256.C
 - Hermitian 256.Q
 - infinite-dimensional 256.C
 - lattice-ordered 310.B
 - left 256.A
 - normed 37.B
 - ordered 310.B
 - quasinormed 37.O
 - quotient 256.F
 - real 256.A
 - right 256.A
 - self-dual 256.H
- linear stationary iterative process 302.C
- linear structural equation system 128.C
- linear structure 96.C
- linear subspace
 - closed 197.E
 - of a linear space 256.F
- linear system
 - (of divisors) 15.C 16.N
 - (of functional-differential equations) 163.E
 - ample 16.N
 - characteristic (of an algebraic family) 15.F
 - complete 9.C 16.N
 - complete (defined by a divisor) 16.N
 - irreducible 16.N
 - reducible 16.N
 - very ample 16.N
- linear time-invariant (dynamical system) 86.B
- linear time-varying system 86.B
- linear topological space 424.A
- linear topology 422.L
- linear transformation
 - (on a Banach space) 251.A
 - (on a linear space) 256.B
 - (on a Riemann sphere) 74.E
 - (of series) 379.L
 - entire 74.E
 - semisimple 256.P
 - triangular 379.L
- linear unbiased estimator, best 403.E
- linear variety 422.L
 - linearly compact 422.L
- line bundle 147.F
 - complex 72.F
 - complex (determined by a divisor) 72.F
 - tautological 16.P
- line coordinates (of a line) 343.C
- line element 111.C
 - characteristic 82.C
 - of higher order, space of 152.C
 - projective 110.B
- line-sphere transformation, Lie 76.C
- linguistics, mathematical 75.E
- link 235.D
 - framed 114.L
- linkage invariant, covering 235.E
- link group 235.D
- linking number 99.C
- link polynomial 235.D
 - reduced 235.D
- link type 235.D
- Linnik's constant 123.D
- Liouville formula 252.C
- Liouville number 430.B
- Liouville operator, Sturm- 112.I
- Liouville problem, Sturm- 315.B
- Liouville theorem

on bounded entire functions 272.A
 first 134.E
 fourth 134.E
 on integral invariants 219.A
 second 134.E
 third 134.E
 Lip α (Lipschitz) 84.A
 of order α 84.A
 Lippman-Schwinger equation 375.C
 Lipschitz condition 84.A 163.D 286.B 316.D
 of order α 84.A
 Lipschitz-Killing curvature 279.C
 Lipschitz space 168.B
 Lipschitz test, Dini- (on the convergence of Fourier series) 159.B
 list (representation) 96.D 186.D
 little group 258.C
 Littlewood-Paley theory 168.B
 Littlewood-Sobolev inequality, Hardy- 224.E
 Littlewood supremum theorem, Hardy- App. A, Table 8
 Littlewood theorem, Hardy-
 (on bounded functions) 43.E
 (on trigonometric systems) 317.B
 lituus 93.H
 Livesay invariant, Browder- 114.L
 loading 214.A
 factor 280.G 346.F
 Lobachevskii non-Euclidean geometry 285.A
 local base (in a topological space) 425.E
 local canonical parameter (for power series) 339.A
 local class field theory 59.G
 local cohomology group 125.W
 local concept (in differential geometry) 109
 local continuity 45.F
 local control 102.A
 local coordinates
 (on an algebraic variety) 16.O
 (on a differentiable manifold) 105.C
 transformation of 90.D
 local coordinate system 90.D 105.C
 holomorphic 72.A
 local cross section (in a topological group) 147.E
 local degree of a mapping 99.B
 local dimension (of an analytic set at a point) 23.B
 local equation
 (of a divisor) 16.M
 regular (at an integral point) 428.E
 local ergodic theorem 136.B
 local field 257.A
 local Gaussian sum 450.F
 local homology group 201.N
 local homomorphism (of a topological group) 423.O
 local isomorphism (of topological groups) 423.O
 localization
 of a linear representation relative to a prime ideal 362.F
 principle of (on convergence tests of Fourier series) 159.B
 strict 16.AA
 local Lie group 423.L
 (of local transformations) 431.G
 local limit theorem 250.B
 locally (on a topological space) 425.J
 locally absolutely p -valent (function) 438.E
 locally arcwise connected (space) 79.B
 locally Cartan pseudoconvex (domain) 21.I
 locally closed (set) 425.J

Subject Index

Local property

locally compact space 425.V
 uniformly 425.V
 locally connected (space) 79.A
 locally constructible (constant sheaf) 16.AA
 locally contractible
 (at a point) 79.C
 (space) 79.C 202.D
 locally convex (topological linear space) 424.E
 locally convex Fréchet space 424.I
 locally countable cell complex 70.D
 locally countable simplicial complex 70.C
 locally dense 154.D
 locally equicontinuous semigroup 378.F
 locally equivalent (G -structure) 191.H
 locally Euclidean group 423.M
 locally Euclidean space 425.V
 locally finite
 (algebra) 29.J
 (cell complex) 70.D
 (covering) 425.R
 (simplicial complex) 70.C
 σ -(covering) 425.R
 locally flat
 (connection) 80.E
 (injection between topological manifolds) 65.D
 (PL embedding) 65.D
 (Riemannian manifold) 364.E
 locally integrable function 168.B
 locally isomorphic (topological groups) 423.O
 locally Levi pseudoconvex (domain) 21.I
 locally linearly compact (Ω -module) 422.L
 locally Macaulay ring 284.D
 locally n -connected
 (at a point) 79.C
 (space) 79.C
 locally Noetherian (scheme) 16.D
 locally Noetherian formal scheme 16.X
 locally of finite type (for a morphism) 16.D
 locally ω -connected (space) 79.C
 locally p -valent 438.E
 locally quadratic transformation
 (of an algebraic surface) 15.G
 (of an algebraic variety) 16.K
 (of a complex manifold) 72.H
 locally rectifiable (curve) 143.A 246.A
 locally symmetric Riemannian space 412.A, App. A, Table 4.II
 locally symmetric space 364.D
 affine 80.J
 locally symmetrizable (diffusion processes) 115.D
 locally totally bounded (uniform space) 436.H
 locally trivial fiber space 148.B
 locally uniformized 367.C
 local martingale 262.E
 local maximum modulus principle 164.C
 local moduli space (of a compact complex manifold) 72.G
 local one-parameter group of local transformations 105.N
 local operator 125.DD
 local orientation (in an oriented manifold) 201.N
 local parameter
 (around a cusp of a Fuchsian group) 32.B
 (of a nonsingular algebraic curve) 9.C
 (of a Riemann surface) 367.A
 local problem (on the solutions of differential equations) 289.A
 local property
 (in differential geometry) 109

Local regime

- (of a pseudodifferential operator) 345.A
- micro-pseudo- 345.A
- pseudo- (of a pseudodifferential operator) 345.A
- local regime (in static model in catastrophe theory) 51.B
- local ring 284.D
 - (of a prime ideal) 67.G
 - (of a subvariety) 16.B
 - analytically normal 284.D
 - analytically unramified 284.D
 - complete 284.D
 - Macaulay 284.D
 - Noetherian 284.D
 - Noetherian semi- 284.D
 - quasi- 284.D
 - quasisemi- 284.D
 - regular 284.D
 - semi- 284.D
 - structure theorem of complete 284.D
- local-ringed space 383.H
- local section 126.E
- local strategy 173.B
- local system of groups (over a topological space) 201.R
- local time 45.G
- local transformations, local Lie group of 431.G
- local truncation error 303.E
- local uniformizing parameter 367.A
- location parameter 396.I 400.E
- locus
 - cut 178.A
 - singular (of a variety) 16.F
- Loeb measure 293.D
- Loeb space 293.D
- $\log x$ 131.D
- $\log_a x$ 131.B
- $\log z$ (logarithm) 131.G
- logarithm 131.B
 - common 131.C
 - integral 167.D
 - Khinchin law of the iterated 250.C
 - law of iterated 45.F
 - Napierian 131.D
 - natural 131.D
- logarithmically convex 21.B
- logarithmic branch point (of a Riemann surface) 367.B
- logarithmic capacity 48.B
- logarithmic criterion App. A, Table 10.II
- logarithmic curve 93.H
- logarithmic decrement (of a damped oscillation) 318.B
- logarithmic differentiation App. A, Table 9.I
- logarithmic distribution App. A, Table 22
- logarithmic function to the base a 131.B
- logarithmic integral 167.D, App. A, Table 19.II
- logarithmic normal distribution App. A, Table 22
- logarithmic paper 19.F
 - semi- 19.F
- logarithmic potential 338.A
- logarithmic series 131.D
- logarithmic singularity
 - (of an analytic function) 198.M
 - (of an analytic function in the wider sense) 198.P
- logarithmic spiral 93.H
- logic
 - algebra of 411.A
 - classical 411.L
 - intuitionistic 411.L
 - many-valued 411.L
 - mathematical 411.A
 - modal 411.L
 - predicate 411.G
 - predicate, with equality 411.J
 - propositional 411.E
 - quantum 351.L
 - symbolic 411.A
 - three-valued 411.L
 - two-valued 411.L
- logical axiom 411.I
- logical choice function, transfinite 411.J
- logical operator 411.E
- logical product (of propositions) 411.B
- logical sum (of propositions) 411.B
- logical symbol 411.B
- logicism 156.A,B
- logistic equation 263.A
- logistic model, linear 403.C
- log linear model 403.C
- logmodular algebra 164.B
- Lommel integral 39.C
- Lommel polynomials App. A, Table 19.IV
- long gravity wave 205.F
- longitude (of a knot) 235.B
- longitudinal wave 446
- long line 105.B
- long water wave 205.F
- look-up, table 96.C
- Looman-Men'shov theorem 198.A
- loop 170 190.P
 - self- 186.B
- loop space 202.C
- loop theorem (on 3-manifolds) 65.E
- loosely Bernoulli 136.F
- Lopatinskii condition, Shapiro- 323.H
- Lopatinskii determinant 325.K
- Lorentz condition 130.A
- Lorentz force 130.A
- Lorentz group 60.J 258.A 359.B
 - full homogeneous 258.A
 - full inhomogeneous 258.A
 - homogeneous 359.B
 - inhomogeneous 359.B
 - proper 60.J 359.B
 - proper complex 258.A
- Lorentz invariance 150.B
- Lorentz space 168.B
- Lorentz transformation 359.B
- Lorenz curve 397.E
- loss
 - heat 419.A
 - of information 138.B
- loss function 398.A
 - quadratic 398.A 399.E
 - simple 398.A
- lot tolerance percent defective 404.C
- Löwenheim theorem, Skolem- 156.E
- lower bound 311.B
 - greatest 310.C 311.B
- lower central series (of a group) 190.J
- lower class
 - with respect to local continuity 45.F
 - with respect to uniform continuity 45.F
- lower control limit 404.B
- lower derivative
 - general (of a set function) 380.D
 - ordinary (of a set function) 380.D

lower end (of a curvilinear integral) 94.D
 lower envelope principle 338.M
 lowering the superscripts 417.D
 lower integral, Riemann 216.A
 lower limit
 (of a Riemann integral) 216.A
 (of a sequence of real numbers) 87.C
 lower limit function 84.C
 lower order
 (for infinity) 87.G
 (of a meromorphic function) 272.C
 lower semicontinuity (of length) 264.A
 lower semicontinuous (at a point) 84.C
 lower semicontinuous function 84.C
 lower semilattice 243.A
 lower triangular matrix 269.B
 lower variation (of a set function) 380.B
 Löwner differential equation 438.B
 loxodromic transformation 74.F
 LP (linear programming) 255
 LSZ asymptotic condition 150.D
 l.u.b. (least upper bound) 311.B
 Lubanski vector, Pauli- 258.D
 Luenberger observer 86.E
 lumping, mass 304.D
 Luneburg lens 180.A
 Lüroth theorem 16.J
 Lutz-Mattuck theorem 118.E
 Luzin first principle (in analytic set theory) 22.C
 Luzin second principle (in analytic set theory) 22.C
 Luzin space 22.I 425.CC
 Luzin theorem 270.J
 Denjoy- 159.I
 Luzin unicity theorem (in analytic set theory) 22.C
 Lyapunov characteristic number 314.A
 Lyapunov condition 250.B
 Lyapunov convexity theorem 443.G
 Lyapunov function 126.F 163.G 394.C
 Lyapunov-Schmidt procedure 286.V
 Lyapunov stable 126.F
 in both directions 394.A
 in the negative direction 394.A
 in the positive direction 394.A
 uniformly 126.F
 Lyapunov theorem 398.C
 Lyusternik-Shnirel'man theory 286.Q

M

$M(\Omega)$ (the set of all essentially bounded measurable functions on Ω) 168.B
 $\{M_p\}$, ultradistribution of class 125.U
 (M_p) , ultradistribution of class 125.U
 μ -absolutely continuous (additive set function) 380.C
 μ -completion 270.D
 μ -conformal function 352.B
 μ -constant stratum 418.E
 μ -integrable 221.B
 μ -measurable 270.D
 μ -null set 370.D
 μ -operator, bounded 356.B
 μ -singular (additive set function) 380.C
 m -dissipative 251.J
 $m \times n$ matrix 269.A
 m th root 10.C
 M -estimator 371.H
 M -port network 282.C
 M -set 159.J
 M -space 425.Y

Subject Index Manifold(s)

(M)-space (= Montel space) 424.O
 M space, abstract 310.G
 M waves 130.B
 Macaulay local ring 284.D
 Macaulay ring 284.D
 locally 284.D
 Mach cone 205.B
 machine
 Turing 31.B
 universal Turing 31.C
 machine-language program 75.C
 machine scheduling problem 376
 machine sequencing problem 376
 Machin formula 322
 Mach number 116.B 205.B
 Mach wave 205.B
 Mackey-Arens theorem 424.N
 Mackey space 424.N
 Mackey theorem 424.M
 Mackey topology 424.N
 Mack method, Garside-Jarratt- 301.N
 MacLane complexes, Eilenberg- 70.F
 MacLane space, Eilenberg- 70.F
 MacLane spectrum, Eilenberg- 202.I
 Maclaurin formula, Euler- 379.J
 macroeconomic data 128.A
 macroscopic causality (of S -matrix) 386.C
 magnetic field 130.A
 magnetic flux density 130.A
 magnetic group 92.D
 magnetic induction 130.A
 magnetic quantum number, orbital 351.E
 magnetic permeability 130.B
 magnetic polarization 130.A
 magnetic Reynolds number 259
 magnetic susceptibility 130.B
 magnetic viscosity 259
 magnetic wave 130.B
 transverse 130.B
 magnetofluid dynamics 259
 magnetohydrodynamics 259
 magnetostatics 130.B
 magnitude (of a vector) 442.B
 Mahalanobis generalized distance 280.E
 Mainardi equations, Codazzi- 111.H, App. A, Table 4.I
 main classes 241.A
 main effect 102.H
 main theorem
 (in class field theory) 59.C
 Zariski's 16.I
 majorant
 (of a sequence of functions) 435.A
 harmonic (of a subharmonic function) 193.S
 method of 316.G
 majorant series 316.G 435.A
 major arc 4.B
 major axis (of an ellipse) 78.C
 major function 100.F
 majorizing function, right 316.E
 Malfatti problem (in geometric construction) 179.A
 Malgrange theorem, Ehrenpreis- 112.B
 Mal'tsev-Iwasawa theorem, Cartan- (on maximal compact subgroups) 249.S
 Mal'tsev theorem, Wedderburn- (on algebras) 29.F
 Malus theorem 180.A
 Mandelstam representation 132.C
 Mangoldt function 123.B
 manifold(s)
 almost complex 72.B

Manin connection, Gauss- (of a variety)

- almost contact 110.E
- almost parallelizable 114.I
- analytic \rightarrow analytic manifold
- Banach 105.Z 286.K
- Blaschke manifold 178.G
- with boundary 105.B
- without boundary 105.B
- C^r - 105.D
- C^r -, with boundary 105.E
- C^r -, without boundary 105.E
- center, theorem 286.V
- characteristic (of a partial differential equation) 320.B
- characteristic classes of 56.F
- closed 105.B
- coherently oriented pseudo- 65.B
- combinatorial 65.C
- compact C^r - 105.D
- complex \rightarrow complex manifold(s)
- complex analytic 72.A
- conic Lagrange 345.B
- conic Lagrangian 274.C
- contact 110.E
- covering 91.A
- covering differentiable 91.A
- differentiable, with boundary of class C^r 105.E
- differentiable, of class C^r 105.D
- with Euclidean connection 109
- fibred 428.F
- Finsler 286.L
- flag 199.B
- Fréchet 286.K
- G - 431.C
- Grassmann \rightarrow Grassmann manifold
- group (of a Lie transformation) 110.A
- with a handle attached by f 114.F
- h -cobordant oriented 114.I
- Hilbert 105.Z 286.K
- Hodge 232.D
- homology 65.B
- Hopf 232.E
- hyperbolic 21.O 235.E
- integral 428.A,B,D
- irreducible 3- 65.E
- Kähler 232
- k -dimensional integral 191.I
- nontrivial 3- 65.E
- ordinary integral (of a differential ideal) 428.E
- orientable (C^r -manifold) 105.F
- orientation 201.N
- oriented 105.F 201.N
- oriented G - 431.E
- paracompact C^r - 105.D
- parallelizable 114.I
- π - 114.I
- PL- 65.C
- Poincaré 105.A
- at a point 178.G
- prime 3- 65.E
- proper flag 199.B
- pseudo- 65.B
- pseudo-Hermitian 344.F
- Q - 382.D
- real analytic 105.D
- regular integral (of a differential ideal) 428.E
- Riemannian \rightarrow Riemannian manifold
- singular integral (of a differential ideal) 428.E
- SC^r - 178.G
- smooth 105.D 114.B
- space-time 359.D
- s -parallelizable 114.I
- stable 126.G,J
- stably almost complex 114.H
- stably parallelizable 114.I
- Stein 21.L
- Stiefel \rightarrow Stiefel manifold
- symplectic 219.C
- topological 105.B
- triangulated 65.B
- unstable 126.G,J
- visibility 178.F
- weakly almost complex 114.H
- weakly 1-complete 21.L
- Manin connection, Gauss- (of a variety) 16.V
- Mannheim curve 111.F
- Mann-Whitney U -test 371.C
- MANOVA (multivariate analysis of variance) 280.B
- mantissa (of the common logarithm) 131.C
- many body problem 402.F 420.A
- many-valued (analytic function) 198.J
- many-valued function 165.B
- many-valued logic 411.L
- map 381.C (also \rightarrow mapping)
 - bundle 147.B
 - covering 91.A
 - cubic 157.B
 - equivariant 431.A
 - first-return 126.C
 - G - 431.A
 - Gauss 111.G
 - Kodaira-Spencer 72.G
 - linear fiber 114.D
 - normal 114.J
 - PL 65.A
 - Poincaré 126.C
 - time-one 126.C
 - trivalent 157.B
- mapping 381.C
 - A -balanced 277.J
 - affine 7.E
 - alternating multilinear 256.H
 - analytic 21.J
 - antiholomorphic 195.B
 - antisymmetric multilinear 256.H
 - biadditive 277.J
 - biholomorphic 21.J
 - bijective 381.C
 - bilinear 256.H 277.J
 - birational 16.I
 - biregular (between prealgebraic varieties) 16.C
 - Borel isomorphic 270.C
 - of bounded variation 246.H
 - bundle 147.B
 - C^r - 105.J
 - c_1 - 237.G
 - CE 382.D
 - cellular (between cell complexes) 70.D
 - chain 200.C
 - chain (between chain complexes) 201.B
 - characteristic (in the classification theorem of fiber bundles) 147.G
 - class 202.B
 - of class C^r 208.B
 - classifying 147.G
 - closed 425.G
 - cochain 200.F 201.H
 - complete 241.C
 - conformal 198.A

- conjugation (of a Hopf algebra) 203.E
- constant 381.C
- continuous 425.G
- covering 91.A
- degenerate 208.B
- degree 99.A
- degree of 99.A
- diagonal (of a graded coalgebra) 203.B,F
- differentiable, of class C^r 105.J
- dual (of a linear mapping) 256.G
- duality 251.J
- equivariant 431.A
- essential 202.B
- exponential 178.A 249.Q 364.C
- extremal horizontal slit 367.G
- extremal quasiconformal 352.C
- extremal vertical slit 367.G
- first-return 126.C
- Fredholm 286.E
- G- 362.B 431.A
- Gauss (in geometric optics) 180.B
- generalized conformal 246.I
- of group algebra 192.Q
- harmonic 195.B
- hereditarily quotient 425.G
- holomorphic 21.J 72.A
- homological 200.C
- homotopy-associative 203.D
- Hopf 147.E
- identity 381.C
- inclusion 381.C
- inverse 381.C
- inverse, theorem 208.B
- isometric 111.H 273.B
- Kodaira-Spencer 72.G
- linear (between linear spaces) 256.B
- linear (between polyhedrons) 70.C
- linear fiber 114.D
- local degree of 99.B
- meromorphic 23.D
- monotone 311.E
- multilinear 256.H
- nondegenerate holomorphic (between analytic spaces) 23.C
- nonexpansive 286.B
- nonsingular, of class C^1 208.B
- normal 114.J
- normal coordinate 364.C
- one-to-one 381.C
- onto 381.C
- open 425.G
- order-preserving 311.E
- orientation-preserving 99.A
- orientation-reversing 99.A
- partial (of a mapping) 381.C
- perfect 425.W
- perspective (in projective geometry) 343.B
- piecewise affine 192.Q
- piecewise linear (between polyhedra) 70.C
- PL 65.A
- Poincaré 126.C,G
- product 425.K
- projective (in projective geometry) 343.B
- proper 425.W
- purely inseparable rational 16.I
- quasiconformal 352.B
- quasiperfect 425.CC
- quotient 425.G
- rational 16.I
- regular (between prealgebraic varieties) 16.C
- regular, of class C^1 208.B
- semicontinuous (in a topological linear space) 153.D
- semilinear 256.P 277.L
- separable (rational) 16.I
- simplicial 70.C
- simplicial (between polyhedra) 70.C
- simplicial (relative to triangulations) 70.C
- skew-symmetric multilinear 256.H
- space 202.C
- space of continuous 435.D
- spin 237.G
- s.s. (semisimplicial) (between s.s. complexes) 70.E
- s.s., realization of 70.E
- surjective 381.C
- symmetric multilinear 256.H
- Teichmüller 352.C
- time-one 126.C
- topological 425.G
- topology induced by a 425.I
- transposed (of a diffusion kernel) 338.N
- transposed (of a linear mapping) 256.G
- uniformly continuous 273.I 436.E
- unit 203.F
- mapping chain 201.B
- mapping class 202.B
- mapping cone 202.E
- reduced 202.F
- mapping cylinder 202.E
- mapping space 435.D
- mapping theorem
 - Brouwer 99.A
 - open 37.I 424.X
 - Riemann 77.B
 - spectral 251.G
- mapping truck 202.G
- Marchenko equation, Gelfand-Levitan-
 - (for KdV equations) 387.D
 - (for a nonlinear lattice) 287.C
- Marcinkiewicz theorem 224.E
- marginal density functions 397.I
- marginal distribution 342.C 397.H
- marked K3 surface 72.K
- Markov branching process 44.D
 - multitype 44.E
- Markov chains 260.A 342.A
 - embedded 260.H
 - general 260.J
 - imbedded 260.H
 - (non) recurrent 260.B
- Markov field theory, Euclidean 150.F
- Markovian decision process 127.E
- Markovian policy 405.C
- Markovian type (stochastic differential equation) 406.D
- Markov inequality (for polynomials) 336.C
- Markov measure 136.D
- Markov operators 136.B
- Markov partition (for an automorphism) 136.G
- Markov process(es) 261 342.A
 - branching 44.E
 - homogeneous 5.H
 - invariant 5.H
 - strong 261.B
- Markov property 261.B
 - strong 261.B
- Markov shift 136.D
- Markov statistical mechanics 340.C
- Markov subshift 126.J

Markov theorem, Gauss-

- Markov theorem, Gauss- 403.E
- Markov time 261.B 407.B
- Martin axiom (in set theory) 33.F
- Martin bound, Froissart- 386.B
- Martin boundary 207.C 260.I
 - dual 260.I
- Martin compactification 207.C
- Martineau-Harvey duality 125.Y
- martingale 262 342.A
 - $\{F_t\}$ -Wiener 406.B
 - local 262.E
- martingale additive functional 261.E
- martingale part 406.B
- martingale problem 115.C 261.C 406.A
- Martin kernel 207.C
- Maslov bundle 274.C
- Maslov index, Keller- 274.C
- mass 132.A 258.C 271.E
 - (of a current) 275.G
 - center of 271.E
 - integrals of the center of 420.A
- mass distribution
 - capacitary 338.K
 - equilibrium 338.K
- Massey theorem, Blakers- 202.M
- mass lumping 304.D
- mass matrix 304.D
- master equation 402.I
- matched asymptotic expansions, method of 25.B
- mathematical axiom 411.I
- mathematical expectation (of a probability distribution) 341.B
- mathematical induction 294.B
 - axiom of 294.B
 - definition by 294.B
 - doulbe 294.B
 - multiple 294.B
- mathematical linguistics 75.E
- mathematical logic 411.A
- mathematical modeling 40.G 300
- mathematical models in biology 263
- mathematical object 52.A
- mathematical programming 264.A
- mathematical programming problem 264.B
- mathematical structure 409.B
- mathematical system (for a structure) 409.B
- mathematics
 - actuarial 214.A
 - combinatorial 66.A
 - discrete 66.A
- mathematics in the 18th century 266
- mathematics in the 19th century 267
- mathematics in the 17th century 265
- Mathieu differential equation 268.A
 - modified 268.A
- Mathieu functions 268
 - modified 268.A
 - modified, of the first kind 268.D
 - modified, of the second kind 268.D
 - modified, of the third kind 268.D
 - of the second kind 268.D
- Mathieu group 151.H
- Mathieu method 268.C
- matric group 226.B
- matrix (matrices) 269
 - adjacement 186.G
 - adjoint 269.I
 - Alexander (of a knot) 235.C
 - alternating 269.B
 - amplification 304.F
 - anti-Hermitian 269.I
 - antisymmetric 269.B
 - association 102.J
 - asymptotic covariance 399.K
 - of a bilinear form 256.H
 - bounded 269.K
 - circuit 254.B
 - column finite 269.K
 - companion 301.I
 - complex orthogonal 269.J
 - correlation 397.J
 - covariance 341.B 397.J
 - density 351.B
 - design 102.A 403.D
 - diagonal 269.A
 - Dirac 377.C
 - Dirac's γ 351.G
 - error 405.G
 - Fisher information 399.D
 - fundamental cutset 186.G
 - fundamental tieset 186.G
 - group 226.B
 - Hasse-Witt 9.E
 - Hermitian 269.I
 - identity 269.A
 - incidence (of a block design) 102.B
 - incidence (of a graph) 186.G
 - infinite 269.K
 - information 102.I
 - inverse 269.B
 - invertible 269.B
 - iteration 302.C
 - Jacobi 390.G
 - Jacobian 208.B
 - lower triangular 269.B
 - mass 304.D
 - m by n 269.A
 - of (m, n) -type 269.A
 - $m \times n$ 269.A
 - moment 341.B
 - monodromy 254.B
 - nilpotent 269.F
 - noncentrality 374.C
 - nonsingular 269.B
 - normal 269.I
 - orthogonal 269.J
 - parity check 63.C
 - Pauli spin 258.A 351.G
 - period (of a closed Riemann surface) 11.C
 - period (of a complex torus) 3.H
 - port-admittance 282.C
 - port-impedance 282.C
 - positive definite 269.I
 - positive semidefinite 269.I
 - principal 3.I
 - projection 269.I
 - proper orthogonal 269.J
 - Q - 260.F
 - of quadratic form 348.A
 - rational function 86.D
 - rectangular 269.A
 - regular 269.B
 - Riemann 3.I
 - row finite 269.K
 - S - 150.D 386
 - sample correlation 280.E
 - scalar 269.A
 - scale 374.C

- Seifert 235.C
- semisimple 269.G
- of sesquilinear form 256.Q
- similar square 269.G
- skew h - 269.I
- skew-Hermitian 269.I
- skew-symmetric 269.B
- square 269.A
- stiffness 304.C
- stochastic 260.A
- of the sum of squares between classes 280.B
- of the sum of squares within classes 280.B
- symmetric 269.B
- symplectic 60.L
- transfer function 86.B
- transition 126.J 260.A
- transposed 269.B
- triangular 269.B
- tridiagonal 298.D
- unipotent 269.F
- unit 269.A
- unitary 269.I
- upper triangular 269.B
- variance 341.B
- variance-covariance 341.B 397.J
- weighting 86.B
- zero 269.B
- matrix algebra
 - full 269.B
 - total 269.B
- matrix convex of order m 212.C
- matrix element 351.B
- matrix game 173.C
- matrix group 226.B
- matrix monotone decreasing of order m 212.C
- matrix monotone increasing of order m 212.C
- matrix representation 362.D
- matrix Riccati differential equation 86.E
- matrix Riccati equation 405.G
- matrix unit 269.B
- matroid 66.G
 - p -ary 66.H
 - poly- 66.F
 - operations for 66.H
- Mattuck theorem, Lutz- 118.E
- Maupertuis principle 180.A
- Maurer-Cartan
 - differential form of 249.R
 - system of differential equations of 249.R
- maximal
 - (hypersurface in Minkowski space) 275.H
 - (ideal) 368.F
 - (in prediction theory) 395.D
 - (Riemann surface) 367.F
- maximal concentration function 341.E
- maximal condition 311.C
- maximal deficiency (of an algebraic surface) 15.E
- maximal dilatation 352.B
- maximal dissipative operator 251.J
- maximal element (in an ordered set) 311.B
- maximal entropy 136.C,H
- maximal ergodic lemma 136.B
- maximal filter 87.I
- maximal function
 - nontangential 168.B
 - radial 168.B
- maximal ideal 67.C 368.F
 - with respect to S 67.C
- maximal ideal space (of a Banach algebra) 36.E
- maximal independent system (of an additive group)
 - 2.E
- maximal inequality (= maximal ergodic lemma)
 - 136.B
- maximal invariant statistic 396.I
- maximal k -split torus 13.Q
- maximally almost periodic group 18.I
- maximally overdetermined (= holonomic) 274.H
- maximal operator 112.E
- maximal order 27.A
- maximal prime divisor (of an ideal) 67.F
- maximal separable extension (of a field) 149.H
- maximal toroidal subgroup (of a compact Lie group)
 - 248.X
- maximal torsion subgroup (of an Abelian group)
 - 2.A
- maximal torus (of a compact Lie group) 248.X
- maximum, relative (of a function) 106.L
- maximum element (in an ordered set) 311.B
- maximum-flow minimum-cut theorem 281.C
- maximum-flow problem 281.C
- maximum likelihood estimator 399.M
- maximum likelihood method 399.M
 - limited information 128.C
- maximum modulus principle (for a holomorphic function)
 - 43.B
 - local 164.C
 - Cartan 338.L
 - complete 338.M
- maximum principle
 - (for analytic functions) 43.B
 - (in control theory) 86.F
 - (for harmonic functions) 193.E
 - (for minimal surface) 275.B
 - (for parabolic operators) 327.D
 - dilated (in potential theory) 338.C
 - entropy 419.A
 - first (in potential theory) 338.C
 - Frostman's 338.C
 - Hopf (for equations of elliptic type) 323.C
 - strong (for equations of elliptic type) 323.C
 - Ugaheri's 338.C
- maximum return 127.B
- maximum-separation minimum-distance theorem
 - 281.C
- maximum solution (of a scalar equation) 316.E
- maximum spectral measure 390.G
- Maxwell-Boltzmann distribution law 402.B
- Maxwell convention 51.F
- Maxwell equations 130.A
- Maxwell fisheye 180.A
- Maxwell relations 419.B
- Maxwell stress tensor 130.A
- Maxwell theorem (on spherical functions) 393.D
- Mayer-Vietoris exact sequence (for a proper triple)
 - 201.C
 - relative 201.L
- Mazur theorem 37.F
 - Gel'fand- 36.E
- meager set 425.N
 - non- 425.N
- mean
 - (of an almost periodic function) 18.B,E
 - (of numbers or a function) 211.C
 - (of a probability distribution) 341.B
 - (of a random variable) 342.C
 - (of a statistical data) 397.C
 - (of a weakly stationary process) 395.A
 - α -trimmed 371.H

Mean absolute deviation

- arithmetic 211.C 397.C
- arithmetic-geometric 134.B
- bounded, oscillation 168.B
- conditional (of a random variable) 342.E 397.I
- continuous in the 217.M 407.A
- convergence in the, of order p 168.B 342.D 407.A
- convergence in the, of power p 168.B
- of degree r (of a function with respect to a weight function) 211.C
- Fejér 159.C
- geometric 211.C
- geometrical 397.C
- harmonic 211.C 397.C
- limit in the 168.B
- moment about the (k th) 341.B
- population 396.C
- sample 396.C
- mean absolute deviation 397.C
- mean anomaly 309.B
- mean concentration function 341.E
- mean content (of a tolerance region) 399.R
- mean curvature 111.H 364.D 365.D, App. A, Table 4.I
- total 365.O
- mean curvature vector 365.D
- mean energy 402.G
- mean entropy 402.G
- mean ergodic theorem 136.B
- mean free energy 340.B 402.G
- mean motion 309.B
- mean number of sheets (of a covering surface of a Riemann sphere) 272.J
- mean oval (of two ovals) 89.D
- mean p -valent, areally 438.E
- mean p -valued, circumferentially 438.E
- mean recurrence time 260.C
- mean square error 399.E 403.E
- mean unbiased, asymptotically 399.K
- mean unbiased estimator 399.C
- mean value
 - (of a continuous function on a compact group) 69.A
 - (of a weakly stationary process) 395.C
- mean value theorem
 - (in differential calculus) 106.E
 - (on harmonic functions) 193.E
 - first (in the Riemann integral) 216.B
 - second (in the D -integral) 100.G
 - second (in the Riemann integral) 216.B
 - second (for the Stieltjes integral) 94.C
 - Siegel 182.E
 - Vinogradov 4.E
- mean vector 341.B
- measurability 443.I
 - strong 443.I
- measurability theorem, Péttis 443.B
- measurable
 - (flow) 136.D
 - (multivalued vector function) 443.I
 - (operator function) 308.G
 - (set) 270.D,G
 - (in set theory) 33.F
 - (stochastic process) 407.A
 - (transformation) 136.B
 - (vector valued function) 443.B
- absolutely 270.L
- \mathfrak{B} - 270.C
- Baire 270.L
- Jordan 270.D,G
- Lebesgue 270.G
- μ - 270.D
- nearly Borel 261.D
- progressively (stochastic process) 407.A
- real-valued (in set theory) 33.F
- with respect to a family of random variables 342.C
- with respect to μ^* 270.E
- scalarly 443.I
- strongly 443.B,I
- universally 270.L
- weakly 443.B,I
- measurable cardinal number 33.E
- measurable event 342.B
- measurable function 270.J
 - \mathfrak{B} - 270.J
 - Baire 270.L
 - Borel 270.J
 - Lebesgue 270.J
 - universally 270.L
- measurable space(s) 270.C
 - analytic 270.C
 - complete 270.D
 - isomorphic 398.D
 - standard 270.C
- measurable vector function 308.G
- measure 270.D,G
 - of an angle 139.D
 - of association 397.K
 - atomless probability 398.C
 - \mathfrak{B} -regular 270.F
 - Borel 270.G
 - bounded 270.D
 - canonical 115.B 260.G
 - characteristic 407.D
 - Carathéodory 270.E
 - Carathéodory outer 270.E
 - complete 270.D
 - completely additive 270.D
 - complex spectral 390.D
 - convergence in 168.B
 - δ - 270.D
 - distortion 213.E
 - excessive 261.F
 - finitely additive 270.D
 - G -invariant 225.B
 - Gaussian random 407.D
 - generalized Lebesgue 270.E
 - of genus (of a positive definite symmetric matrix) 348.K
 - Gibbs 136.C
 - Green 193.J
 - Haar 225.C
 - harmonic 120.C 169.B 207.B 260.I
 - Hausdorff 169.D
 - idempotent 192.P
 - image 270.K
 - inner harmonic 169.B
 - invariant 136.B 255.B 260.A,I 261.F
 - Jensen 164.K
 - Jordan 270.D,G
 - killing 115.B
 - kinetic 228.A
 - K -regular 270.F
 - Lebesgue 270.G
 - Lebesgue outer 270.G
 - Lebesgue-Stieltjes 166.C 270.L
 - left invariant Haar 225.C
 - of length 139.C
 - Lévy 5.E

- of location 397.C
- Loeb 293.D
- Markov 136.D
- orthogonal 164.C
- outer 270.E,G
- outer harmonic 169.B
- Plancherel (of a locally compact group) 437.L
- Poisson random 407.D
- positive Radon 270.I
- probability 341 342.B
- product 270.H
- quasi-invariant 225.J
- quotient 225.H
- Radon 270.G
- real spectral 390.D
- regular 270.F
- relatively invariant 225.H
- representing 164.C
- right invariant Haar 225.C
- σ -additive 270.D
- σ -finite 270.D
- signed 380.C
- smooth (for a Riemannian metric) 136.G
- smooth invariant 126.J
- spectral 390.D,K 395.B,C
- speed 115.B
- subinvariant 261.F
- superharmonic 260.I
- of variability 397.C
- vector 443.G
- Weil 225.G
- Wiener 45.B 250.E
- measure algebra 192.O
- measure-preserving (transformation) 136.B
- measure problem, invariant 136.C
- measure space 270.D
 - bounded 270.D
 - complete 270.D
 - complete product 270.H
 - Lebesgue, with a finite (or σ -finite) measure 136.A
 - product 270.H
 - σ -finite 270.D
- measure theory 270
- mechanics
 - celestial 55.A
 - classical 271.A
 - classical statistical 402.A
 - equilibrium statistical 402.A
 - graphical 19.C
 - Markov statistical 340.C
 - Newtonian 271.A
 - nonlinear 290.A
 - quantum 351
 - quantum statistical 402.A
 - statistical 342.A 402
- mechanism, Higgs 132.D
- median 341.H 396.C 397.C
 - sample 396.C
- mediant (of two fractions in Farey sequence) 4.B
- median unbiased estimator 399.C
- medieval mathematics 372
- meet
 - (in a Boolean algebra) 42.A
 - (in an ordered set) 243.A
 - (of sets) 381.B
- Mehler formula App. A, Table 19.III
- Mehler integral representation, Laplace- App. A, Table 18.II
- Mellin transform 220.C
- member (of a set) 381.A
- membrane
 - equation of a vibrating 325.A
 - permeable 419.A
- memory
 - fading 163.I
 - first-in first-out 96.E
 - first-in last-out 96.E
- memory channel
 - almost finite 213.F
 - finite 213.F
- memoryless channel 213.C
 - discrete 213.F
- memory unit 75.B
- Menelaus theorem (in affine geometry) 7.A
- Menger-Nöbeling embedding theorem 117.D
- Men'shov theorem, Looman- 198.A
 - Rademacher- 317.B
- Mercer theorem 217.H
- merging 96.C
- meridian
 - (of a knot) 235.B
 - (of a surface of revolution) 111.H
- meromorphic (in a domain) 272.A
- meromorphic curve 272.L
- meromorphic differential (on a Riemann surface) 367.H
- meromorphic function(s) 21.J 272.A
 - (on an analytic set) 23.D
 - (on a complex manifold) 72.A
 - transcendental 272.A
- meromorphic mapping, proper (between analytic spaces) 23.D
- meromorphy
 - circle of (of a power series) 339.D
 - radius of (of a power series) 339.D
- Mersenne number 297.E, App. B, Table 1
- Mersenne prime 297.E
- Mertens theorem (on the Cauchy product of two series) 379.F
- mesh of a covering (in a metric space) 273.B
- mesons 132.B
- meta-Abelian group 190.H
- metabolic model (in catastrophe theory) 51.F
- metamathematics 156.D
- metastable range (of embeddings) 114.D
- method(s)
 - Abel, of summation 379.N
 - Adams-Bashforth 303.E
 - Adams-Moulton 303.E
 - ADI 304.F
 - alternating direction implicit (ADI) 304.F
 - Arrow-Hurwicz-Uzawa gradient 292.E
 - of averaging 290.D
 - Bairstow 301.E
 - Bernoulli 301.J
 - Borel, of summation 379.O
 - branch and bound 215.D
 - Cesàro, of summation of order α 379.M
 - Cholesky 302.B
 - circle 4.B
 - collocation 303.I
 - congruence 354.B
 - conjugate gradient (C.G.) 302.D
 - constructive 156.D
 - continuation 301.M
 - Crout 302.B
 - cyclic Jacobi 298.B
 - d'Alembert, of reduction of order 252.F
 - Danilevskii 298.D

- Davidenko, of differentiation with respect to a parameter 301.M
 Dejon-Nickel 301.G
 difference 303.A
 direct (in the calculus of variations) 46.E
 discrete variable 303.A
 distribution-free 371.A
 Doolittle 302.B
 downhill 301.L
 Duhamel 322.D
 Durand-Kerner (DK) 301.F
 Durand-Kerner-Aberth (DKA) 301.F
 Enskog 217.N
 Euclidean 150.F
 Euler (of describing the motion of a fluid) 205.A
 Euler (of numerical solution of ordinary differential equations) 303.E
 Euler (of summation) 379.P
 expansion 205.B
 extrapolation 303.F
 factorization 206.B
 of false position 301.C
 of feasible directions (in nonlinear programming) 292.E
 finite element 223.G 304.C
 fixed point 138.B
 floating point 138.B
 Frobenius App. A, Table 14.I
 Galerkin 303.I 304.B
 Garside-Jarratt-Mack 301.N
 Gauss-Seidel 302.C
 Givens 298.D
 gradient 292.E
 Graeffe 301.N
 graphical, of statistical inference 19.B
 Green function 402.J
 of harmonic balance 290.D,E
 Hessenberg 298.D
 Hill, of solution 268.B
 Hitchcock 301.E
 hodograph 205.B
 Horner 301.C
 Householder 298.D
 implicit 303.E
 implicit enumeration 215.D
 Ince-Goldstein 268.C
 indirect least squares 128.C
 interpolation 224.A
 isoparametric 304.C
 Jacobi (of numerical computation of eigenvalues) 298.B
 Jacobi (in numerical solution of linear equations) 302.C
 Jacobi second, of integration 324.D
 Jeffreys 25.B
 killing (of obtaining a homotopy group) 202.N
 ladder 206.B
 Lagrange (of describing the motion of a fluid) 205.A
 Lagrange (of indeterminate coefficients) 106.L
 Lagrange, of variation of constants 252.D
 Lagrange, of variation of parameters 252.D
 Lagrange-Charpit 322.B, App. A, Table 15.II
 of Lagrange multipliers 106.L
 Lanczos 298.D 301.N
 Laplace 30.B
 of least squares (for estimation) 403.E
 of least squares (for numerical solution of linear equations) 397.J
 of least squares (for numerical solution of ordinary differential equations) 303.I
 Lebesgue, of summation 379.S
 Lehmer 301.K
 Lighthill 25.B
 limited information maximum likelihood 128.C
 of linearization 290.D
 linear k -step 303.E
 linear multistep 303.E
 of majorants 316.G
 of matched asymptotic expansions 25.B
 Mathieu 268.C
 maximum likelihood 399.M
 middle-square 354.B
 Milne 303.E
 modified minimum chi-square 400.K
 moment 399.L
 Monte Carlo 385.C
 of moving frames 110.A
 of multiple scales 290.E
 multistep 303.E
 multivalued 303.E
 Newton-Raphson 301.D
 nonparametric 371.A
 Nörlund, of summation 379.Q
 of orthogonal projection 323.G
 ($p + 1$)-stage 303.D
 penalty 292.E
 Perron (in Dirichlet problem) 120.C
 perturbation 25.B
 Poincaré 25.B
 Poincaré-Lighthill-Kuo (P.L.K.) 25.B
 polynomial extrapolation 303.F
 power 298.C
 predictor-corrector (PC) 303.E
 projective approximation 304.B
 QR 298.F
 of quadrature 313.D
 QZ 298.G
 rational extrapolation 303.F
 Rayleigh-Ritz 46.F 271.G
 renormalization 361.A
 Richardson 302.C
 Riemann, of summation 379.S
 Riesz, of summation of the k th order 379.R
 Ritz 46.F 303.I 304.B
 robust 371.A
 Rosen gradient projection 292.E
 Runge-Kutta 303.D
 Runge-Kutta-Gill 303.D
 saddle point 25.C
 scaling 346.E
 scoring 397.M
 simplex 255.C
 spectral 304.B
 stationary phase 30.B
 of steepest descent 25.C
 step by step 163.D
 Strum 301.C
 of successive approximation (for an elliptic partial differential equation) 323.D
 of successive approximation (for Fredholm integral equations of the second kind) 217.D
 of successive approximation (for ordinary differential equations) 316.D
 of successive iteration (for Fredholm integral equations of the second kind) 217.D
 summable by Abel's 379.N
 summable by Borel's exponential 379.O

- summable by Borel's integral 379.O
 - summable by Cesàro's, of order α 379.M
 - summable by Euler's 379.P
 - summable by Hölder's, of order p 379.M
 - summable by Nörlund's 379.Q
 - summable by M. Riesz's, of order k 379.R
 - of summation 379.L
 - Sylvester elimination 369.E
 - three-stage least squares 128.C
 - threshold Jacobi 298.B
 - two-phase simplex 255.C
 - two-stage least squares 128.C
 - variational 438.B
 - of variation of constants 55.B 252.I
 - of variation of parameters App. A, Table 14.I
 - WKB 25.B
 - WKBJ 25.B
- metric 273.B
 - Bergman 188.G
 - Einstein 364.I
 - Einstein-Kähler 232.C
 - \bar{f} - 136.F
 - f_N - 136.F
 - Finsler 152.A
 - Hermitian 232.A
 - Hodge 232.D
 - Kähler 232.A
 - Kerr 359.E
 - left invariant (in a topological group) 423.I
 - Petersson 32.B
 - Poincaré 74.G
 - pseudo- 273.B
 - pseudo-Riemannian 105.P
 - Riemannian 105.P
 - Robertson-Walker 359.E
 - standard Kähler (of a complex projective space) 232.P
 - Teichmüller 416
- metrically isomorphic automorphisms (on a measure space) 136.E
- metric comparison theorem 178.A
- metric connection 80.K
- metric invariant (on a measure space) 136.E
- metric multidimensional scaling 346.E
- metric space(s) 273
 - compact 273.F
 - complete 273.J
 - discrete 273.B
 - indiscrete pseudo- 273.B
 - induced by a mapping 273.B
 - precompact 273.B
 - product 273.B
 - pseudo- 273.B
 - separable 273.E
 - totally bounded 273.B
- metric structure
 - almost contact 110.E
 - contact 110.E
- metric subspace 273.B
- metric topology 425.C
- metric vector space 256.H
- metrizable topological group 423.I
- metrizable topological space 273.K
- metrizable uniform space 436.F
 - pseudo- 436.F
- Meusnier theorem (on surfaces) 111.H
- Meyer decomposition theorem, Doob- 262.D
- Michael theorem 425.X
- micro-analytic 125.CC 274.E
- microbundle 147.P
 - normal PL 147.P
 - PL 147.P
 - tangent PL 147.P
- microcanonical ensemble 402.D
- microdifferential equation 274.G
- microdifferential operator 274.F
 - of finite order 274.F
 - of infinite order 274.F
- microfunction 274.E
 - holomorphic 274.F
- microlocal analysis 274 345.A
- microlocally elliptic (operator) 345.A
- microlocal operator 274.F
- micro-pseudolocal property 345.A
- middle point 7.C
- middle-square method 354.B
- midpoint 7.C
- midpoint rule 303.E
- midrange 374.G
- Mikusinski operator 306.B
- Mills equation, Yang- 80.G
- Mills field, Yang- 150.G
- Mills functional, Yang- 80.Q
- Mills G -connection, Yang- 80.Q
- Mil'man property, Krein- 443.H
- Mil'man theorem 37.G
 - Krein- 424.U
- Milne corrector 303.F
- Milne method 303.E
- Milne predictor 303.E
- Milne-Simpson formula 303.E
- Milnor fibering theorem 418.D
- Milnor fibration 418.D
- Milnor invariant 235.D
- Milnor monodromy 418.D
- Milnor number (in Milnor fibering theorem) 418.D
- Minakshisundaram-Pleijel asymptotic expansion 391.B
- minimal
 - (algebraic surface) 15.G
 - (algebraic variety) 16.I
 - (ideal) 368.F
 - (immersion) 275.A
 - (superharmonic function) 260.I
 - (transformation) 136.H
 - relatively 15.G 16.I
- minimal basis (of a principal order or an algebraic number field) 14.B
- minimal chain (for a transition probability) 260.F
- minimal complete class 398.B
- minimal complex 70.E
- minimal condition
 - (in an ordered set) 311.C
 - restricted (in a commutative ring) 284.A
- minimal diffeomorphism 126.N
- minimal element (in an ordered set) 311.B
- minimal flow 126.N
- minimal function, \mathfrak{K} - 367.E
- minimal immersion 275.A
 - branched 275.B
 - generalized 275.B
- minimality 16.I
 - absolute 16.I
- minimally almost periodic group 18.I
- minimally elliptic singularity 418.C
- minimal model 15.G
 - (for the algebra of differential forms) 114.L
 - Néron (of an Abelian variety) 3.N
 - relatively 15.G
- minimal operator 112.E

Minimal parabolic k -subgroup

- minimal parabolic k -subgroup 13.Q
- minimal polynomial
 - (of an algebraic element) 149.E
 - (of a linear transformation) 269.L
 - (of a matrix) 269.F
- minimal prime divisor (of an ideal) 67.F
- minimal projection, Lie 76.B
- minimal realization 86.D
- minimal resolution 418.C
- minimal set 126.E
- minimal splitting field (of a polynomial) 149.G
- minimal submanifold 275.A 365.D
- minimal sufficient σ -field 396.E
- minimal surface 111.I 334.B
 - affine 110.C
 - branched 275.B
- minimal surface equation 275.A
- minimal variety 275.G
- minimax (estimator) 399.H
- minimax decision function 398.B
- minimax level α test 400.F
- minimax principle
 - (for eigenvalues of a compact operator) 68.H
 - (for statistical decision problem) 398.B
 - for λ_k 391.G
- minimax solution 398.B
- minimax theorem 173.C
- minimization problem, group 215.C
- minimizing sequence 46.E
- minimum (minima)
 - of a function 106.L
 - relative (at a point) 106.L
 - successive (in a lattice) 182.C
 - weak 46.C
- minimum chi-square method, modified 400.K
- minimum-cost flow problem 281.C
- minimum curvature property 223.F
- minimum element (in an ordered set) 311.B
- minimum immersion 365.O
- minimum norm property 223.F
- minimum principle
 - energy 419.C
 - enthalpy 419.C
 - Gibbs free energy 419.C
 - Helmholtz free energy 419.C
 - for λ_k 391.G
 - of λ 391.D
- minimum solution (of a scalar equation) 316.E
- minimum variance unbiased estimator, uniformly 399.C
- Minkowski-Farkas theorem 255.B
- Minkowski-Hasse character (of a nondegenerate quadratic form) 348.D
- Minkowski-Hasse theorem (on quadratic forms over algebraic number fields) 348.G
- Minkowski-Hlawka theorem 182.D
- Minkowski inequality 211.C, App. A, Table 8
- Minkowski reduction theory (on fundamental regions) 122.E
- Minkowski space 258.A
- Minkowski space-time 359.B
- Minkowski theorem 182.C
 - on discriminants 14.B
 - on units 14.D
- Minlos theorem 424.T
- minor
 - (of a matrix) 103.D
 - (of a matroid) 66.H
 - Fredholm's first 203.E
 - Fredholm's r th 203.E
 - principal (of a matrix) 103.D
- minor arc 4.B
- minor axis (of an ellipse) 78.C
- minor function 100.F
- minus infinity 87.D
- minute (an angle) 139.D
- Mittag-Leffler theorem 272.A
- mixed Abelian group 2.A
- mixed area (of two ovals) 89.D
- mixed group 190.P
- mixed Hodge structure 16.V
- mixed ideal 284.D
- mixed initial-boundary value problem (for hyperbolic operator) 325.K
- mixed insurance 214.B
- mixed integer programming problem 215.A
- mixed model 102.A
- mixed periodic continued fraction 83.C
- mixed problem 322.D
- mixed spinor rank (k, n) 258.B
- mixed strategy 173.C
- mixed tensor 256.J
- mixed type, partial differential equation of 304.C 326.A
- mixing (automorphism)
 - k -fold 136.E
 - strongly 136.E
 - weakly 136.E
- mixture 351.B
- Mizohata equation, Lewy- 274.G
- ML estimator 399.M
- mobility, axiom of free (in Euclidean geometry) 139.B
- Möbius band 410.B
- Möbius function 66.C 295.C
- Möbius geometry 76.A
- Möbius strip 410.B
- Möbius transformation 74.E 76.A
- Möbius transformation group 76.A
- mod 1, real number 355.D
- mod p (modulo p)
 - Hopf invariant 202.S
 - isomorphism (in a class of Abelian groups) 202.N
- modal logic 411.L
- modal proposition 411.L
- modal unbiased estimator 399.C
- mode 396.C 397.C
 - sample 396.C
- model
 - (of an algebraic function field) 9.D
 - (of a mathematical structure) 409.B
 - (of a symbolic logic) 276.D 411.G
 - Bayesian 403.G
 - Bradley-Terry 346.C
 - Bush-Mosteller 346.G
 - canonical 251.N
 - component 403.F
 - components-of-variance 403.C
 - countable (of axiomatic set theory) 156.E
 - derived normal (of a variety) 16.F
 - dual resonance 132.C
 - Estes stimulus-sampling 346.G
 - factor analysis 403.C
 - fixed effect 102.A
 - functional 251.N
 - game theoretic 307.C
 - Glashow-Weinberg-Salam 132.D

- of human death and survival 214.A
- Ising 340.B 402.G
- Klein (of non-Euclidean geometry) 285.C
- learning 346.G
- linear 403.D
- linear logistic 403.C
- log linear 403.C
- Luce β - 346.G
- mathematical, in biology 263
- metabolic (in catastrophe theory) 51.F
- minimal 15.G
- minimal (for the algebra of differential forms) 114.L
- mixed 102.A
- multiple 403.E
- multivariate linear 280.B
- natural (in axiomatic set theory) 33.C
- normal linear 403.C
- Poincaré (of geometry) 285.D
- queuing 260.H
- random-effects 102.A 403.C
- relatively minimal 15.G
- Sakata 132.D
- Scheffé 346.C
- simple 403.F
- spin-flip 340.C
- static (in catastrophe theory) 51.B
- stochastic Ising 340.C
- stochastic programming 307.C
- string 132.C
- Sz.-Nagy-Foiaş 251.N
- Thurstone-Mosteller 346.C
- Veneziano 132.C 386.C
- Whittaker 450.O
- model experimentation 385.A
- modeling, mathematical 300
- model scheduling 307.C
- model selection 401.D
- model theory 276
- modification
 - (of a stochastic process) 407.A
 - holomorphic (of an analytic space) 23.D
 - proper (of an analytic space) 23.D
 - spherical 114.F
- modified Bessel functions App. A, Table 19.IV
- modified Fourier hyperfunction 125.BB
- modified indicator function 341.C
- modified Mathieu differential equation 268.A
- modified Mathieu function 268.A
 - of the first kind 268.D
 - of the second kind 268.D
 - of the third kind 268.D
- modified minimum chi-square method 400.K
- modified wave operator 375.B
- modular, weakly (in quantum mechanics) 351.L
- modular automorphism 308.H
- modular character (of a modular representation) 362.I
- modular form
 - Hilbert, of dimension $-k$ 32.G
 - Hilbert, of weight k 32.G
 - of level N 32.C
 - Siegel, of dimension $-k$ 32.F
 - Siegel, of weight k 32.F
- modular function
 - (of a locally compact group) 225.D
 - Hilbert 32.G
 - of level N 32.C
 - Siegel, of degree n 32.F
- modular group 122.D
 - elliptic 122.D
 - Hilbert 32.G
 - Siegel, of degree n 32.F
- modular lattice 243.F
- modular law (in a lattice) 243.F
- modular operator 308.H
- modular representation (of a finite group) 362.G
- modular surface, Hilbert 15.H
- module(s) 277
 - (of a family of curves) 143.A
 - A - 277.C
 - over A 277.C
 - of A -homomorphisms (between A -modules) 277.E
 - Artinian 277.I
 - of boundaries 200.C
 - category of left (right) R - 52.B
 - character (of an algebraic group) 13.D
 - of coboundaries 200.F
 - of cocycles 200.F
 - coefficient 200.L
 - cohomology 200.F
 - connected graded 203.B
 - of cycles 200.C
 - defining (of a linear system) 16.N
 - degenerate 118.D
 - divisible A - 277.D
 - dual 277.K
 - dual graded 203.B
 - duality theorem for Ω - 422.L
 - factor A - 277.C
 - faithfully flat A - 277.K
 - of finite length 277.I
 - flat A - 277.K
 - free 277.G
 - generalized 143.B
 - graded A - 200.B
 - homology 200.C
 - of homomorphisms (between two modules) 277.B
 - induced 277.L
 - injective A - 277.K
 - Jordan 231.C
 - left A - 277.D
 - Noetherian 277.I
 - \mathcal{O} -, 383.I
 - with operator domain A 277.C
 - projective A - 277.K
 - of quotients of an R -module with respect to S 67.G
 - (R, S)-injective 200.K
 - (R, S)-projective 200.K
 - representation (of a linear representation) 362.C
 - of representations (of a compact group) 69.D
 - right A - 277.D
 - torsion A - 277.D
- moduli functor 16.W
- moduli scheme 16.W
 - coarse 16.W
 - fine 16.W
- moduli space 16.W 72.G
 - of curves of genus g 9.J
 - local 72.G
- modulus (moduli)
 - (of a complex number) 74.B
 - (of a complex torus of dimension 1) 32.C
 - (= a conformal invariant) 11.B 77.E

- (of a congruence) 297.G
- (of an elliptic integral) 134.A, App. A, Table 16.I
- (in Jacobi elliptic functions) 134.J, App. A, Table 16.III
- (of a locally multivalent function) 438.E
- (of a ring) 77.E
- complementary (of an elliptic integral) App. A, Table 16.I
- complementary (in Jacobi elliptic functions) 134.J, App. A, Table 16.III
- of continuity (of a function) 84.A
- of continuity of k th order (of a continuous function) 336.C
- of elasticity in shear 271.G
- of elasticity in tension 271.G
- field of 73.B
- local maximum, principle 164.C
- maximum, principle (for a holomorphic function) 43.B
- periodicity (of an elliptic integral) 134.A
- of rigidity 271.G
- Young's 271.G
- modulus number 418.E
- modus ponens 411.I
- Moishezon criterion, Nakai- (of ampleness) 16.E
- Moishezon space 16.W
- mole numbers 419.A
- moment 397.C
 - absolute (k th) 341.B
 - bivariate 397.H
 - central 397.C
 - conditional 397.I
 - factorial 397.G
 - of inertia 271.E
 - (k th) 341.B
 - about the mean (k th) 341.B
 - population (of order k) 396.C
 - principal, of inertia 271.E
 - sample (of order k) 396.C
- moment generating function 177.A 341.C 397.G,J
- moment matrix 341.B
- moment method 399.L
- moment method estimator 399.L
- moment problem
 - Hamburger 240.K
 - Hausdorff 240.K
 - Stieltjes 240.K
- momentum 271.A,E
 - angular 271.E
 - generalized 271.F
 - integrals of angular 420.A
 - intrinsic angular 351.G
 - orbital angular 351.E
 - theorem of 271.E
 - theorem of angular 271.E
- momentum density, angular 150.B
- momentum 4-vector, energy- 258.C
- momentum operator
 - angular 258.D
 - energy- 258.D
- momentum phase space 126.L
- momentum representation 351.C
- momentum tensor
 - angular 258.D
 - energy- 150.D 359.D
- monad
 - (in homology theory) 200.Q
 - (in nonstandard analysis) 293.D
- Monge-Ampère equations 278, App. A, Table 15.III
- Monge differential equation 324.F
- monic polynomial 337.A
- monoclinic system 92.E
- monodromy group
 - (of an n -fold covering) 91.A
 - (of a system of linear ordinary differential equations) 253.B
 - Milnor 418.D
 - total 418.F
- monodromy matrix 254.B
- monodromy theorem (on analytic continuation) 198.J
- monogenic function
 - in the sense of E. Borel 198.Q
 - in the sense of Cauchy 198.Q
- monoid, unitary 409.C
- monoidal transformation
 - (of an analytic space) 23.D
 - (of a complex manifold) 172.H
 - (by an ideal sheaf) 16.K
 - with center W 16.K
 - real 274.E
- monomial 337.B
 - (module) 277.D
 - admissible (in Steenrod algebra) 64.B
- monomial representation (of a finite group) 362.G
- monomorphism (in a category) 52.D
- monothetic group 136.D
- monotone
 - (curve) 281.B
 - operator 212.C
- monotone class 270.B
- monotone class theorem 270.B
- monotone decreasing
 - (set function) 380.B
 - matrix, of order m 212.C
- monotone decreasing function 166.A
 - strictly 166.A
- monotone function 166.A
 - strictly 166.A
 - strictly (of ordinal numbers) 312.C
- monotone increasing
 - (set function) 380.B
 - matrix, of order m 212.C
- monotone increasing function 166.A
 - strictly 166.A
- monotone likelihood ratio 374.J
- monotonely very weak Bernoulli 136.F
- monotone mapping 311.E
- monotone operator (in a Hilbert space) 286.C
- monotone sequence (of real numbers) 87.B
- monotonically decreasing (sequence of real numbers) 87.B
- monotonically increasing (sequence of real numbers) 87.B
- monotonic function, completely 240.E,K
- Monte Carlo method 385.C
- Montel space 424.O
- Montel theorem 435.E
- moon argument, behind-the- 351.K
- Moore-Smith convergence 87.H
- Moore space 273.K 425.AA
- Moore space problem, normal 425.AA
- Mordell conjecture 118.E
- Mordell-Weil theorem 118.E
 - weak 118.E
- more informative (experiment) 398.G

- Morera theorem 198.A
 Morgenstern solution, von Neumann- 173.D
 morphism
 (in a category) 52.A
 (of chain complexes) 200.H
 (of complexes) 13.R
 (of filtered modules) 200.J
 (of inductive systems) 210.D
 (of unfoldings) 51.D
 affine 16.D
 connecting 200.H
 diagonal (in a category) 52.E
 étale 16.F
 finite 16.D
 flat 16.D
 faithfully flat 16.D
 Frobenius 450.P
 functorial 52.J
 inverse 52.D
 k - (between algebraic groups) 13.A
 projective 16.E
 proper (between schemes) 16.D
 quasiprojective 16.E
 S - 52.G
 of schemes 16.D
 separated 16.D
 shape 382.A
 smooth 16.F
 strict (between topological groups) 423.J
 structure 52.G
 Morse function 279.B
 Morse index theorem 279.F
 Morse inequalities 279.D
 Morse lemma 279.B
 Morse-Smale diffeomorphism 126.J
 Morse-Smale flow 126.J
 Morse-Smale vector field 126.J
 Morse theory 279
 fundamental theorems of 279.D
 Morsification 418.F
 Moser implicit function theorem, Nash- 286.J
 Mosteller model
 Bush- 346.G
 Thurstone- 346.C
 most powerful (test) 400.A
 most probable cause 401.E
 most probable value 401.E
 most stringent level α test 400.F
 motion(s)
 (in dynamical system) 126.B
 (Euclidean) 139.B
 Brownian 5.D 45 342.A
 Brownian (d -dimensional) 45.C
 Brownian, on Lie groups 406.G
 Brownian, with an N -dimensional time parameter 45.I
 central 126.E
 elliptic 55.A
 equation of (of a fluid) 205.A
 equation of (of a particle in a gravitation field) 359.D
 equations of (in Newtonian mechanics) 271.A
 Euler equation of (of a perfect fluid) 205.B
 $\{\mathcal{F}_t\}$ -Brownian 45.B 406.B
 group of (in Euclidean geometry) 139.B
 group of, in the wider sense 139.B
 Heisenberg equation of 351.D
 hyperbolic 420.D
 hyperbolic-elliptic 420.D
 hyperbolic-parabolic 420.D
 infinitesimal (of a Riemannian manifold) 364.F
 Lagrange equation of 271.F
 Lagrange-stable 420.D
 law of 271.A
 mean 309.B
 Newton three laws of 271.A
 Ornstein-Uhlenbeck Brownian 45.I
 oscillating 420.D
 parabolic 420.D
 parabolic-elliptic 420.D
 perpetual 402.G
 proper (in Euclidean geometry) 139.B
 proper (of a star) 392
 quasiperiodic 136.G
 right-invariant Brownian 406.G
 simple harmonic 318.B
 space-time Brownian 45.F
 Moulton method, Adams- 303.E
 movability 382.C
 movable 382.C
 k - 382.C
 movable branch point (of an algebraic differential equation) 288.A
 movable singularity (of an algebraic differential equation) 288.A
 move 173.B
 chance 173.B
 moving average 397.N
 weighted 397.N
 moving average process 421.D
 autoregressive 421.D
 autoregressive integrated 421.G
 moving average representation
 backward 395.D
 canonical backward 395.D
 moving coordinates App. A, Table 3.IV
 moving coordinate system 90.B
 moving frame(s) 90 111.C 417.B
 method of 110.A
 natural 417.B
 orthonormal 417.D
 stochastic 406.G
 multicollinearity 128.B
 multicommodity flow problem 281.F
 multidagonal type 304.C
 multidimensional diffusion process 115.A,C
 multidimensional gamma function 374.C
 multidimensional hypergeometric distribution App. A, Table 22
 multidimensional normal distribution App. A, Table 22
 multidimensional scaling 346.E
 nonmetric 346.E
 multi-index 112.A 168.B
 multilinear form 256.H
 multilinear mapping 256.H
 alternating 256.H
 antisymmetric 256.H
 skew-symmetric 256.H
 symmetric 256.H
 multinomial distribution 341.D
 negative 341.D
 multinomial theorem 330
 multi-objective model 307.C
 multiobjective programming 264.C
 multiplanar coordinates 90.C
 multiple 297.A
 (of an element of a ring) 67.H

- (of a fractional ideal) 14.E
 - common (of elements of a ring) 67.H
 - least common 297.A
 - least common (of elements of a ring) 67.H
 - scalar (of an element of a module) 277.D
 - scalar (of a linear operator) 37.C
 - scalar (in a linear space) 256.A
 - scalar (of a vector) 442.A
 - multiple complex 200.H
 - multiple correlation coefficient 397.J
 - sample 280.E
 - multiple covariant 226.E
 - absolute 226.E
 - multiple hypergeometric distribution 341.D
 - multiple integral
 - (in Lebesgue integral) 227.E
 - (in Riemann integral) 216.F
 - multiple mathematical inductions 294.B
 - multiple model 403.F
 - multiple point
 - (on an arc) 93.B
 - (of a plane algebraic curve) 9.B
 - (on a variety) 16.F
 - multiple root (of an algebraic equation) 10.B
 - multiple sampling inspection 404.C
 - multiple-valued (analytic function) 198.J
 - multiple Wiener integral 176.I
 - multiplication
 - (of an algebra) 203.F
 - (of a graded algebra) 203.B
 - (in a group) 190.A
 - (of an H -space) 203.D
 - (of local Lie groups) 423.L
 - (by a natural number) 294.B
 - (in a ring) 368.A
 - associative (of a graded algebra) 203.B
 - commutative (of a graded algebra) 203.B
 - commutative law for (in a ring) 368.A
 - complex 73.A
 - homotopy associative 203.D
 - homotopy commutative 203.D
 - Pontryagin 203.D
 - scalar (in a module) 277.D
 - scalar (on vectors) 442.A
 - symmetric 406.C
 - multiplication theorem, Hadamard 339.D
 - multiplicative (arithmetic function) 295.B
 - multiplicative automorphic function 32.A
 - multiplicative class 270.B
 - multiplicative congruence 14.H
 - multiplicative ergodic theorem 136.B
 - multiplicative function 32.A 295.B
 - multiplicative functional
 - (of a Markov process) 261.E
 - transformation by 261.F
 - multiplicative group 149.A 190.A
 - of a field 190.B
 - multiplicative Jordan decomposition (of a linear transformation) 269.L
 - multiplicatively closed subset (of a ring) 67.C
 - multiplicative valuation 439.C
 - multiplicator (of a relative invariant measure) 225.H
 - multiplicity
 - (of a covering surface) 367.B
 - (of an eigenvalue for an integral equation) 217.F
 - (of an eigenvalue of a matrix) 390.B
 - (of a Gaussian process) 176.E
 - (of a local ring) 284.D
 - (of a representation) 362.D
 - (of a root of an algebraic equation) 10.B
 - (of a weight) 248.W
 - algebraic (of an eigenvalue) 390.B
 - geometric (of an eigenvalue) 390.A
 - intersection (of two subvarieties) 16.Q
 - representation without 437.G
 - set of 159.J
 - multiplicity function (of a mapping) 246.G
 - multiplier
 - (of a group) 362.J
 - (of a semi-invariant) 226.A
 - characteristic (of a closed orbit) 126.G
 - characteristic (of a periodic linear system) 163.F
 - Jacobi's last App. A, Table 14.I
 - Lagrange 46.B
 - method of Lagrange 106.L
 - Stokes 254.D
 - multiplier algebra 36.K
 - multiply connected domain 333.A
 - multiply transitive (permutation group) 151.H
 - multipolar coordinates 90.C
 - multiprocessor scheduling problem 376
 - multistage allocation process 127.A
 - multistage choice process 127.A
 - multistage game 173.C
 - multistage programming 264.C
 - multistage sampling 373.E
 - multistep method 303.E
 - linear 303.E
 - multitype Galton-Watson process 44.C
 - multitype Markov branching process 44.E
 - multivalent function 438.E
 - multivalued function 165.B
 - multivalue method 303.E
 - multivariate (data) 397.A
 - multivariate analysis 280
 - of variance 280.B
 - multivariate linear model 280.B
 - multivariate normal distribution 397.J
 - Muntz theorem (on polynomial approximation) 336.A
 - mutual energy 338.B
 - mutual information 213.E
 - mutually associated diagrams (for $O(n)$ diagrams) 60.J
 - mutually disjoint family (of sets) 381.D
 - mutually noncomparable (summations) 379.L
 - mutually orthogonal (latin squares) 241.B
 - M.V.W.B. (= monotonely very weak Bernoulli) 136.F
- N**
- N (natural numbers) 294.A,B
 - NP 71.E
 - n -ary predicate 411.G
 - n -ary relation 411.G
 - n -ball 140
 - open 140
 - n -body problem 420.A
 - n -cell 70.D 140
 - open 70.D
 - topological 140
 - n -classifying space (of a topological group) 147.G
 - n -cochain (for an associative algebra) 200.L
 - n -connected
 - (pair of topological spaces) 202.L

(space) 79.C 202.L
 locally 79.C
n-connective fiber space 148.D
n-cube, unit 140
n-cylinder set 270.H
n-decision problem 398.A
n degrees of freedom (sampling distribution)
 374.B.C
n-dimensional (normal space) 117.B
n-dimensional distribution 342.C
n-dimensional distribution function 342.C
n-dimensional Euclidean geometry 139.B 181
n-dimensional probability distribution 342.C
n-dimensional random variable 342.C
n-dimensional sample space 396.B
n-dimensional statistic 396.B
n-disk 140
 open 140
n-element 140
n-fold covering 91.A
n-fold reduced suspension (of a topological space)
 202.F
n-gon, regular 357.A
n-gonal number 296.A
n-particle subspace 377.A
n-person game 173.B–D
n-ply connected (plane domain) 333.A
n-section (in a cell complex) 70.D
n-sheeted (covering surface) 367.B
n-simple
 (pair of topological spaces) 202.L
 (space) 202.L
n-simplex
 (in a Euclidean simplicial complex) 70.B
 (in a semisimplicial complex) 70.E
 (in a simplicial complex) 70.C
n-sphere 140
 open 140
 solid 140
n-sphere bundle 147.K
n-times continuously differentiable (function)
 106.K
n-times differentiable (function) 106.D
n-torus 422.E
n-tuple 256.A 381.B
n-universal bundle 147.G
n-valued (analytic function) 198.J
 ($n + 2$)-hyperspherical coordinates 79.A 90.B
N-ple Markov Gaussian process 176.E
 in the restricted sense 176.F
nth approximation (of an *n*-times differentiable
 function) 106.E
nth convergent (of an infinite continued fraction)
 83.A
nth derivative (of a differentiable function)
 106.D
nth derived function 106.D
nth differential (of a differentiable function)
 106.D
nth partial quotient 83.A
nth order, differential of 106.D
nth order partial derivatives 106.H
nth term 165.D
 nabla 442.D, App. A, Table 3.II
 Nachbin-Goodner-Kelley theorem 37.M
 Nagumo theorem, Kneser- 316.E
 Naimark theorem, Gelfand- 36.G
 Nakai-Moishezon criterion (of ampleness) 16.E
 Nakanishi equation, Landau- 146.C
 Nakanishi variety, Landau- 146.C 386.C

Nakano-Nishijima-Gell-Mann formula 132.A
 Nakayama lemma 67.D
 Nambu-Goldstone boson 132.C
 Napier analogies App. A, Table 2.III
 Napierian logarithm 131.D
 Napier number 131.D
 Napier rule App. A, Table 2.II
 Nash bargaining solution 173.C
 Nash equilibrium 173.C
 Nash-Moser implicit function theorem 286.J
 nat 213.B
 natural additive functional 261.E
 natural boundary
 (of an analytic function) 198.N
 (of a diffusion process) 115.B
 natural equation
 (of a curve) 111.D
 (of a surface) 110.A
 natural equivalence 52.J
 natural extension (of an endomorphism) 136.E
 natural geometry 110.A
 natural injection (from a subgroup) 190.D
 naturality (of a homotopy operation) 202.O
 natural logarithm 131.D
 natural model (in axiomatic set theory) 33.C
 natural moving frame 417.B
 natural number 294.A,B
 nonstandard 276.E
 Skolem theorem on impossibility of 156.E
 natural positive cone 308.K
 natural scale 260.G
 natural spline 223.F
 natural surjection (to a factor group) 190.D
 natural transformation 52.J
 Navier-Stokes equation(s) 204.B 205.C
 general 204.F
 Navier-Stokes initial value problem 204.B
 nearly Borel measurable set 261.D
 nearly everywhere (in potential theory) 338.F
 necessary (statistic) 396.E
 necessity 411.L
 necklace, Antoine's 65.G
 negation (of a proposition) 411.B
 negative
 (complex) 200.H
 (element of a lattice-ordered group) 243.G
 (element of an ordered field) 149.N
 (rational number) 294.D
 negative binomial distribution 341.D 397.F, App.
 A, Table 22
 negative curvature 178.H
 negative definite (function) 394.C
 negative definite Hermitian form 348.F
 negative definite quadratic form 348.B
 negative half-trajectory 126.D
 negative infinity 87.D 355.C
 negative limit point 126.D
 negatively invariant 126.D
 negatively Lagrange stable 126.E
 negatively Poisson stable 126.E
 negative multinomial distribution 341.D
 negative number 355.A
 negative orientation (of an oriented C^r -manifold)
 105.F
 negative part (of an element of a vector lattice)
 310.B
 negative polynomial distribution App. A, Table
 22
 negative prolongational limit set, first 126.D
 negative resistance 318.B

negative root (of a semisimple Lie algebra) 248.M
negative semidefinite quadratic form 348.C
negative semiorbit 126.D
negative variation
 (of a mapping) 246.H
 (of a real bounded function) 166.B
neighborhood 425.B
 analytic (of a function element in the wider sense) 198.O
 analytic (in a Riemann surface) 367.A
 conoidal 274.D
 convex 364.C
 coordinate (in a fiber bundle) 147.B
 coordinate (in a topological manifold) 105.C
 coordinate, of class C^r 105.D
 derived 65.C
 ε - (of a point) 273.C
 étale 16.AA
 fundamental system of 425.E
 open 425.E
 open tubular 114.B
 regular 65.C
 regular, theorem 65.C
 relative 425.J
 tubular 105.L 114.B 364.C
neighborhood deformation retract 202.D
neighborhood retract 202.D
 absolute 202.D
 fundamental absolute (FANR) 382.C
neighborhood system 425.B
 base for 425.E
 uniform 436.D
 uniform family of 436.D
Nelson formula, Feynman-Kac- 150.F
Nelson symmetry 150.F
Nernst postulate 419.A
Néron minimal model (of an Abelian variety) 3.N
Néron-Severi group
 (of a surface) 15.D
 (of a variety) 16.P
nerve (of a covering) 70.C
nested intervals, principle of (for real numbers) 87.C 355.B
net (in a set) 87.H
 Cauchy (in a uniform space) 436.G
 square 304.E
 universal (in a set) 87.H
net premium 214.A
network(s) 282 425.F
 bilateral 282.C
 capacitated 281.C
 contact 282.B
 electric 282.B
 linear 282.C
 M-port 282.C
 passive 282.C
 reciprocal 282.C
 time-invariant 282.C
 two-terminal 281.C
network flow model 307.C
network flow problem 281 282.B
network programming 264.C
network scheduling 307.C
Neumann function 39.B 188.H, App. A, Table 19.III
Neumann polynomial App. A, Table 19.IV
Neumann problem 193.F 323.F
Neumann series 217.D
neutral element (in a lattice) 243.E

neutral type (of functional differential equation) 163.B
Nevanlinna exceptional value 272.E
Nevanlinna first fundamental theorem 272.B
Nevanlinna second fundamental theorem 272.E
Nevanlinna theory
 (of meromorphic functions) 124.B 272.B
 (for several complex variables) 21.N
Newton, I. 329
Newton backward interpolation formula 223.C
Newton boundary 418.D
 nondegenerate 418.D
Newton-Cotes formula (in numerical integration) 299.A
Newton diagram 254.D
Newton first law 271.A
Newton formula (on interpolation) App. A, Table 21
Newton formula (on symmetric functions) 337.I
Newton forward interpolation formula 223.C
Newtonian capacity 48.B
Newtonian exterior capacity 48.H
Newtonian fluid 205.C
Newtonian inner capacity 48.F
Newtonian interior capacity 48.F
Newtonian mechanics 271.A
Newtonian outer capacity 48.H
Newtonian potential 271.C 338.A
Newton interpolation formula App. A, Table 21
Newton interpolation polynomial 336.G
Newton iterative process 301.D
Newton law (on frictional stresses) 205.C
Newton law of universal gravitation 271.B
Newton-Raphson method 301.D
Newton second law 271.A
Newton third law 271.A
Newton three laws of motion 271.A
Neyman factorization theorem 396.F
Neyman-Pearson lemma 400.B
Neyman structure 400.D
nice function (on a C^∞ -manifold) 114.F
Nicholson formula App. A, Table 19.IV
Nicholson formula, Watson- App. A, Table 19.III
Nickel method, Dejon- 301.G
Nicomedes conchoid 93.H
Nijenhuis tensor 72.B
Nikodým derivative, Radon- 270.L 380.C
Nikodým property, Radon- 443.H
Nikodým theorem, Radon- 270.L 380.C
Nikodým theorem for vector measures, Radon- 443.H
nilalgebra 231.A
nilmanifold 178.D
nilpotent
 (Lie algebra) 248.C
 (Lie group) 249.D
 (subset of a ring) 368.B
 (zero-divisor) 284.A
 generalized (linear operator) 251.F
nilpotent algebraic group 13.F
nilpotent component (of a linear transformation) 269.L
nilpotent element
 (of a ring) 368.B
 generalized (in a Banach algebra) 36.E
nilpotent group 190.J
 finite 151.C
 generalized 190.K
nilpotent ideal
 (of a Lie algebra) 248.C

- largest (of a Lie algebra) 248.D
- nilpotent matrix 269.F
- nilpotent radical (of a Lie algebra) 248.D
- nilradical
 - (of a commutative ring) 67.B
 - (of a ring) 368.H
- 9j symbol 353.C
- Nirenberg space, John- (= *BMO*) 168.B
- Nishijima–Gell-Mann formula, Nakano– 132.A
- Nishina formula, Klein- 351.G
- Nitsche formula, Gauss-Bonnet-Sasaki- 275.C
- niveau surface 193.J
- Nöbeling embedding theorem, Menger- 117.D
- no cycle condition 126.J
- nodal curve 391.H
- nodal domain 391.H
- nodal point 304.C
- nodal set 391.H
- node
 - (of a curve) 93.G
 - (of a graph) 186.B 282.A
 - (of a plane algebraic curve) 9.B
 - completion 281.D
 - start 281.D
- Noetherian domain 284.A
- Noetherian integral domain 284.A
- Noetherian local ring 284.D
- Noetherian module 277.I
- Noetherian ring(s) 284.A
 - left 368.F
 - right 368.F
- Noetherian scheme 16.D
 - locally 16.D
- Noetherian semilocal ring 284.D
- Noether number, Brill- 9.E
- Noether theorem 150.B
- noise
 - thermal 402.K
 - white 176.D
- noisy channel 213.A
- nomograms 19.A,D
- non-Abelian cohomology 200.M
- nonadaptive scheme 299.C
- nonanticipative 406.D
- non-Archimedean geometry 155.D
- non-Archimedean valuation 14.F 439.C
- nonassociative algebra 231.A
- nonatomic 168.C 443.G
- non-Bayesian approach 401.B
- noncentral (quadric hypersurface) 7.F 350.G
- noncentral chi-square distribution 374.B
- noncentral *F*-distribution 374.B
- noncentral Hotelling T^2 statistic 374.C
- noncentrality (sampling distribution) 374.B,C
- noncentrality matrix 374.C
- noncentral *t*-distribution 374.B
- noncentral Wishart distribution 374.C
 - p*-dimensional 374.C
- noncommutative field 149.A
- noncompact real simple Lie algebra App. A, Table 5.II
- noncompact type (symmetric Riemannian homogeneous space) 412.D
- noncomparable, mutually 379.L
- nonconforming type 304.C
- nonconvex quadratic programming 264.D
- noncooperative (game) 173.A
- nondecreasing function 166.A
- nondegenerate
 - (analytic mapping) 23.C
 - (bilinear form) 256.H
 - (critical point) 106.L 279.B 286.N
 - (function on a Hilbert manifold) 279.E
 - (quadratic form) 348.A
 - (representation) 437.N
 - (sesquilinear form) 256.Q
 - (theta-function) 3.I
- nondegenerate critical manifold 279.D,E
- nondegenerate divisor 3.D 16.N
- nondegenerate hypersurface 344.A
- nondegenerate Newton boundary 418.D
- non-Desarguesian geometry 155.E 343.C
- nondeterministic
 - (Turing machine) 31.B
 - purely (weakly stationary process) 395.D
- nondeterministic linear bounded automaton 31.D
- nonelementary (Kleinian group) 234.A
- non-Euclidean angle (in a Klein model) 285.C
- non-Euclidean distance 285.C
- non-Euclidean geometry 285
- non-Euclidean hypersphere 285.C
- non-Euclidean space 285.A
- nonexpansive mapping 286.B
- nonexpansive operator 37.C
- nonhomogeneous difference equation 104.C
- nonhomogeneous *n*-chain (for a group) 200.M
- nonincreasing function 166.A
- nonlinear differential equation 291.D
- nonlinear filter 405.F,H
- nonlinear functional analysis 286
- nonlinear integral equation 217.M
- nonlinear lattice dynamics 287
- nonlinear mechanics 290.A
- nonlinear ordinary differential equations 313.A
 - (global theory) 288
 - (local theory) 289
- nonlinear oscillation 290
- nonlinear partial differential equations 320.A
- nonlinear problems 291
- nonlinear programming 264.C
- nonlinear semigroup 88.E 378.F
 - of operators 286.X
- nonmeager set 425.N
- nonmetric MDS 346.E
- nonnegative (matrix) 269.N
- nonnegative terms, series of 379.B
- non-Newtonian fluid 205.C
- nonparametric method 371
- nonparametric test 371.A
- nonpositive curvature 178.H
 - G*-space with 178.H
- nonprimitive character 450.C,E
- nonrandomized (decision function) 398.A
- nonrandomized estimate 399.B
- nonrandomized test 400.A
- nonrecurrent (chain) 260.B
- nonrecurrent (transient) 260.B
- nonresidue, quadratic 297.H
- nonsaddle set 126.E
- nonsingular (flow) 126.G
 - (point for a flow) 126.D
 - (point of a variety) 16.F
- nonsingular mapping of class C^1 208.B
- nonsingular matrix 269.B
- nonsingular transformation
 - (of a linear space) 256.B
 - (on a measure space) 136.B
- nonsingular variety 16.F
- nonstandard 33.B
 - (element) 293.B

- nonstandard analysis 293
- nonstandard natural number 276.E
- nonstandard real number 276.E
- nonstandard set theory 293.E
- nonstationary oscillations 290.F
- nonsymmetric unified field theory 343.C
- nontangential maximal function 168.B
- nontangential path 333.B
- nontrivial (3-manifold) 65.E
- nontrivially (to act on a G -space) 431.A
- nonwandering 126.E
 - set 126.E
- Nörlund method of summation 379.Q
- norm
 - (of an algebraic element) 149.J
 - (of an element of a general Cayley algebra) 54
 - (of an element of a quaternion algebra) 29.D
 - (of an operator) 37.C
 - (of a separable algebraic element) 149.J
 - (of a vector) 37.B
 - absolute (of an integral ideal) 14.C
 - C^* -cross 36.H
 - C^r - 126.H
 - graph 251.D
 - Hilbert-Schmidt 68.I
 - minimum, property 223.F
 - nuclear 68.K
 - pseudo- (on a topological linear space) 424.F
 - reduced (of an algebra) 362.E
 - relative (of a fractional ideal) 14.I
 - semi- (on a topological linear space) 424.F
 - spinorial 61.D
 - supremum 168.B
 - trace 68.I
 - uniform 168.B
- normal 62.C 110.E 354.F
 - (almost contact structure) 110.E
 - (analytic space) 23.D
 - (current) 275.G
 - (fundamental region) 122.B
 - ($*$ -isomorphism) 308.C
 - (state) 351.B
 - (for a valuation) 439.H
 - (weight on a von Neumann algebra) 308.D
- affine 110.C
- affine principal 110.C
- analytically 284.D
- principal 111.F
- normal algebraic variety 16.F
- normal analytic structure 386.C
- normal basis 172.E
- normal block bundle 147.Q
- normal bundle
 - (of a foliation) 154.B.E
 - (of an immersion) 114.B
 - (of a submanifold) 105.L 274.E 364.C
- normal Cartan connection 80.N
- normal chain
 - (in a group) 190.G
 - (in a Markov chain) 260.D
- normal commutation relation 150.D
- normal connection 365.C
- normal contact Riemannian manifold 110.E
- normal continued fraction 83.E
- normal coordinate(s) 90.C
 - mapping 364.C
- normal covering 425.R
- normal crossings 16.L
 - only 16.L
- normal curvature (of a surface) 111.H
- normal density function 397.D
- normal derivative 106.G
- normal distribution 341.D 397.D, App. A, Table 22
 - k -dimensional 341.D, App. A, Table 22
 - logarithmic App. A, Table 22
 - multidimensional App. A, Table 22
 - standard 341.D
- normal duration 28.I
- normal equation
 - (in the method of least squares) 302.E 403.E
 - (in statistical data analysis) 397.J
- normal estimator, best asymptotically 399.K
- normal estimator, consistent and asymptotically 399.K
- normal extension 149.G 251.K
- normal extension field, strongly 113
- normal family 435.E
- normal fiber space, Spivak 144.J
- normal form
 - (of differential equations) 313.B 324.E
 - (of a surface) 410.B
 - Cantor (for an ordinal number) 312.C
 - Hesse (of a hyperplane) 139.H
 - Jordan (for a matrix) 269.G
 - π -adic (for an ordinal number) 312.C
 - prenex (in predicate logic) 411.J
- normal form theorem, Kleene 356.C
- normal frame 110.B
- normal function (of ordinal numbers) 312.C
- normal g -lattice 27.A
- normal invariant 114.J
- normality, asymptotic 399.K
- normalization
 - (of an analytic space) 23.D
 - (of a variety) 16.F
- normalization theorem
 - for finitely generated rings 369.D
 - for polynomial rings 369.D
- normalized
 - (function) 317.A
 - (into an orthonormal set) 197.C
 - (vector) 139.G
- normalized contrast 102.C
- normalized valuation 439.E
- normalizer 136.F 190.C
- normal j -algebra 384.C
- normal k -vector bundle 114.J
- normal line 93.G, App. A, Table 4.I
- normal linear model 403.C
- normally cobordant 114.J
- normally distributed, asymptotically 399.K
- normally flat along a subscheme (a scheme) 16.L
- normal mapping (map) 114.J
- normal matrix 269.I
- normal model, derived (of a variety) 16.F
- normal Moore space problem 425.AA
- normal number 354.F
- normal operator 390.E
 - (of Sario) 367.G
- normal PL microbundle 147.P
- normal plane 111.F
- normal point 16.F 23.D
- normal polygon 234.C
- normal process 176.C
- normal real form (of a complex semisimple Lie algebra) 248.Q
- normal representation 308.C
- normal ring 67.I
- normal score test, Fisher-Yates-Terry 371.C
- normal section 410.B

- normal sequence (of coverings) 425.R
 normal simple algebra 29.E
 normal space 425.Q
 collectionwise 425.AA
 completely 425.Q
 fully 425.X
 hereditarily 425.Q
 perfectly 425.Q
 normal sphere bundle 274.E
 normal stress 271.G
 normal structure 276.D
 normal subgroup 190.C
 admissible 190.E
 normal system (of E -functions) 430.D
 normal transformation (of a sequence) 379.L
 normal valuation 439.E,H
 normal variety 16.F
 normal vector 105.L 111.H 364.A
 normal vector bundle 105.L
 normal vibration 318.B
 normed linear space 37.B
 normed ring 36.A
 normed space, countably 424.W
 normed vector lattice 310.F
 norm form 118.D
 normic form 118.F
 norm-residue 14.P
 norm-residue symbol 14.Q
 (in local class field theory) 257.F
 Hilbert 14.R
 Hilbert-Hasse 14.R
 norm resolvent convergence 331.C
 northern hemisphere 140
 north pole 74.D 140
 notation
 full international 92.E
 Kendall 260.H
 Schoenflies (for crystal classes) 92.E, App. B,
 Table 6.IV
 short international 92.E
 system of (for ordinal numbers) 81
 notion, common 35.A
 Novikov closed leaf theorem 154.D
 nowhere dense set 425.N
 NP 71.E
 co- 71.E
 NP-complete 71.E
 NP-completeness 71.E
 NP-hard 71.E
 NP-space 71.E
 NP-time 71.E
 NR (neighborhood retract) 202.D
 nuclear (C^* -algebra) 36.H
 nuclear class 68.I
 nuclear norm 68.K
 nuclear operator 68.I,K
 nuclear space 424.S
 nucleolus 173.D
 null (vector in the Minkowski space-time) 359.B
 null-bicharacteristic 320.B
 null boundary, open Riemann surface of 367.E
 null cobordant 235.G
 null function 310.I
 null geodesic 399.D
 null homotopic (continuous mapping) 202.B
 null hypothesis 400.A
 nullity
 (of a critical point) 279.B
 (of a graph) 186.G
 (of a linear mapping) 256.F
 (of a linear operator) 251.D
 (of a matrix) 269.D
 column (of a matrix) 269.D
 of relative 365.D
 row (of a matrix) 269.D
 null recurrent (point) 260.D
 null sequence (in α -adic topology) 284.B
 null set 270.D 310.I 381.A
 of class $N_{\mathfrak{A}}$ 169.E
 function-theoretic 169
 null space 251.D
 null system 343.E
 number(s) 294
 A- 430.C
 abundant 297.D
 algebraic 14.A
 amicable 297.D
 average sample 404.C
 azimuthal quantum 315.E
 Bell 177.D
 Bernoulli 177.B
 Betti 200.K 201.B
 Brill-Noether 9.E
 calculable 22.G
 Cantor's theory of real 294.E
 cardinal 49.A 312.D
 Cayley 54
 characteristic (of a compact operator) 68.I
 characteristic (of a manifold) 56.F
 Chern 56.F
 chromatic 157.E 186.I
 class (of an algebraic number field) 14.E
 class (of a Dedekind domain) 67.K
 class (of a simple algebra) 27.D
 Clifford 61.A
 coincidence (of a mapping) 153.B
 of colors 92.D
 completeness of real 294.E
 complex 74.A 294.F
 composite 297.B
 condition 302.A
 connectedness of real 294.E 355.B
 continuity of real 294.E
 cyclomatic 186.G
 decomposition (of a finite group) 362.I
 Dedekind's theory of real 294.E
 deficient 297.D
 of denominator 186.I
 Euler 177.C 201.B, App. B, Table 4
 Fermat 297.F
 Froude 116.B
 generalized decomposition (of a finite group)
 362.I
 geometry of 182
 Gödel 185 356.C,E
 Grashoff 116.B
 imaginary 74.A
 incidence 146.B 201.B
 of independence 186.I
 initial 312.D
 intersection (of divisors) 15.C
 intersection (of homology classes) 65.B 201.O
 intersection (of sheaves) 16.E
 irrational 294.E 355.A
 irrational real 294.E
 of irregularity (of an algebraic variety) 16.P
 Kullback-Leibler information 398.G
 Lebesgue 273.F
 Lefschetz 153.B
 Lefschetz (of a variety) 16.P

- linking 99.C
 - Liouville 430.B
 - Lyapunov characteristic 314.A
 - Mach 116.B 205.B
 - magnetic Reynolds 259
 - mean (of sheets) 272.J
 - Mersenne 297.E, App. B, Table 1
 - Milnor 418.D
 - modulus 418.E
 - mole 419.A
 - n -gonal 296.A
 - Napier 131.D
 - natural 294.A,B
 - negative 355.A
 - negative rational 294.D
 - normal 354.F
 - Nusselt 116.B
 - orbital magnetic quantum 315.E
 - ordinal 312.B
 - p -adic 439.F
 - of partitions 177.D 328
 - Péclet 116.B
 - pentagonal 4.D
 - perfect 297.D
 - perfect, of the second kind 297.D
 - Picard (of a variety) 15.D 16.P
 - Poisson 271.G
 - polygonal, of order κ 4 5
 - Pontryagin 56.F
 - positive 355.A
 - positive rational 294.D
 - Prandtl 116.B
 - prime 297.B
 - principal quantum 315.E
 - pseudorandom 354.B
 - Pythagorean 145
 - ramification 14.K
 - random 354
 - rational 294.D,E
 - rational real 294.E
 - real 294.E 355.A,D
 - real, mod 1 355.O
 - relatively prime 297.A
 - of replications 102.B
 - Reynolds 116.B 205.C
 - rotation 99.D 111.E 126.I
 - S - 430.C
 - S^* - 430.C
 - self-intersection 15.C
 - of sheets (of an analytic covering space) 23.E
 - of sheets (of covering surface) 367.B
 - Stiefel-Whitney 56.F
 - Stirling, of the second kind 66.D
 - T - 430.C
 - T^* - 430.C
 - Tamagawa 13.P
 - transcendental 430.A
 - translation 18.B,D
 - of treatment combinations 102.L
 - type 314.A
 - U - 430.C
 - U^* - 430.C
 - wave (of a sine wave) 446
 - wave, vector (of a sine wave) 205.F
 - weakly compact cardinal 33.E
 - weakly inaccessible cardinal 33.E
 - Weil 3.C
 - number field 149.C
 - algebraic 14.B
 - p -adic 439.F
 - p -adic 257.A 439.F
 - relative algebraic 14.I
 - numbering, Gödel 185.A
 - number operator 377.A
 - number system, point range of 343.C
 - number-theoretic function(s) 295.A 356.A
 - additive 295.B
 - completely additive 295.B
 - completely multiplicative 295.B
 - multiplicative 295.B
 - number theory 296
 - analytic 296.B
 - consistency proof for pure 156.E
 - elementary 297
 - fundamental theorem of elementary 297.C
 - geometric 296.B
 - pure 156.E
 - numerals, Arabic 26
 - numerator, partial (of an infinite continued fraction) 83.A
 - numerical analysis 300
 - numerical differentiation 299.E
 - numerical integration 299
 - numerically connected (divisor) 232.D
 - numerically equivalent (cycles) 16.Q
 - numerically semipositive 15.D
 - numerical method 300
 - numerical range (of a linear operator) 251.E
 - numerical solution
 - of algebraic equations 301
 - of integral equations 217.N
 - of linear equations 302
 - of ordinary differential equations 303
 - of partial differential equations 304
 - numerical tensor App. A, Table 4.II
 - Nusselt number 116.B
 - notation 392
 - Nyquist criterion 86.A
 - Nyquist theorem 402.K
- O**
- $\mathcal{O}(\Omega)$ (space of holomorphic functions in Ω) 168.B
 - $\mathcal{O}_p(\Omega)$ 168.B
 - $O(n)$ (orthogonal group) 60.I
 - ω -connected space 79.C
 - locally 79.C
 - ω -consistent (system) 156.E
 - ω -limit point 126.D
 - ω -limit set 126.D
 - Ω -conjugate 126.H
 - Ω -equivalent 126.H
 - Ω -explosion 126.J
 - Ω -group 190.E
 - Ω -homomorphism (between Ω -groups) 190.E
 - Ω -isomorphism (between Ω -groups) 190.E
 - Ω -modules, duality theorem for 422.L
 - Ω -stability theorem 126.J
 - Ω -stable, C^r - 126.H
 - Ω -subgroup (of an Ω -group) 190.E
 - \mathfrak{o} -ideal
 - integrated two-sided 27.A
 - two-sided 27.A
 - \mathfrak{o}_l -ideal, left 27.A
 - \mathfrak{o}_r -ideal, right 27.A
 - \mathfrak{D} -differential (on an algebraic curve) 9.F
 - \mathfrak{D} -genus (of an algebraic curve) 9.F
 - \mathfrak{D} -linearly equivalent divisors (on an algebraic curve) 9.F

\mathfrak{S} -specialty index (of a divisor of an algebraic curve) 9.F
 \mathcal{O} -module 383.I
 (o) -convergent 87.L
 (o) -star convergent 87.L
 OA (orthogonal array) 102.L
 Ob (object) 52.A
 object 52.A 411.G
 cofinal 52.D
 final 52.D
 graded 200.B
 group (in a category) 52.M
 initial 52.D
 injective 200.I
 isomorphic 52.D
 mathematical 52.A
 in predicate logic 411.G
 projective 200.I
 quotient 52.D
 S -, category of 52.G
 of type $j + 1$ 356.F
 of type O 356.F
 zero 52.N
 object domain 411.G
 objective function 264.B 307.C
 objective probability 401.B
 object variable 411.G
 oblate App. A, Table 3.V
 oblique circular cone 350.B
 oblique coordinates (in a Euclidean space) 90.B
 observability 86.C
 observables 351.B
 observation
 complete 405.C
 cost of 398.F
 partial 405.C
 observation process 405.F
 observation vector 102.A
 observer, Luenburger 86.E
 obstacle, Dirichlet problem with 440.B
 obstruction(s) 305
 to an n -dimensional homotopy 305.B
 to an $(n + 1)$ -dimensional extension 305.B
 primary 147.L 305.C
 secondary 305.D
 surgery 114.J
 tertiary 305.D
 obstruction class 56.E
 obstruction cocycle 147.L 305.B
 obtuse angle (in Euclidean geometry) 139.D
 OC-curve (operative characteristic curve) 404.C
 octahedral group 151.G
 octahedron 357.B
 odd element (of a Clifford algebra) 61.B
 odd function 165.B
 odd half-spinor 61.E
 odd half-spin representation 61.E
 odd permutation (in a symmetric group) 151.G
 odd ratio 397.K
 odd state 315.H
 of bounded variation 443.G
 Ohm's law (for a moving medium) 130.B 259
 Oka's principle 147.O
 Oka's theorem 72.E
 1-complete manifold, weakly 21.L
 one cycle 16.R
 one-dimensional diffusion processes 115.A
 one-dimensional lattice 287.A
 one-dimensional probability distribution (of random variables) 342.C

one-dimensional statistic 396.B
 $100\alpha\%$ -point 396.C
 1-1 (mapping) 381.C
 one-parameter group
 local (of local transformations) 105.N
 of transformations 105.N 126.B
 one-parameter semigroup of class (C^0) 378.B
 one-parameter subgroup (of a Lie group) 249.Q
 one-parameter variation 178.A
 one-point compactification 425.T
 one-point union 202.F
 one-sided (surface) 410.B
 one-sided stable for exponent $1/2$ App. A, Table 22
 one-sided stable process (of the exponent α) 5.F
 one-step-two-half-steps errors estimate 303.D
 one-to-one correspondence 358.B
 one-to-one mapping 381.C
 only normal crossings 16.L
 Onsager reciprocity relation 402.K
 onto mapping 381.C
 open
 (Riemann surface) 367.A
 (system) 419.A
 (topological manifold) 105.B
 finely 261.D
 Zariski 16.A
 open arc 93.B
 open ball 140
 open base 425.F
 open circle 140
 open continuous homomorphism 423.J
 open covering (of a set) 425.R
 open disk 140
 open formulas 199.A
 opening 186.E
 open interval 140 355.C
 open mapping 425.G
 open mapping theorem
 (in Banach space) 37.I
 (in topological linear spaces) 424.X
 open n -ball 140
 open n -cell 140
 open n -disk 140
 open n -sphere 140
 open neighborhood 425.E
 open parallelotope (in an affine space) 7.D
 open set 425.B
 basic 425.F
 relative 425.J
 system of 425.B
 open simplex 7.D 70.C
 open sphere 140
 open star (in a complex) 70.B,C
 open subgroup (of a topological group) 423.D
 open surface 410.B
 open system entropy 402.G
 open tubular neighborhood 105.L 114.B
 operate
 (in a function algebra) 192.N
 from the left (on a set) 362.B
 from the right (on a set) 362.B
 operating characteristic 404.C
 operating function 192.N
 operating systems 75.C
 operation(s)
 (of an operator domain on a module) 277.C
 (on a set) 409.A
 Adams 237.E
 Bokshtein 64.B
 Boolean 42.A

Operation A (in set theory)

- cohomology 64
- compatible with 277.C
- four arithmetic 294.A
- functional cohomology 202.S
- functional Φ - 202.S
- glide 92.E
- homotopy 202.O
- left 409.A
- primary cohomology 64.B
- Pontryagin (pth) power 64.B
- primitive (of a group) 362.B
- rational 294.A
- reduced square 64.B
- right 409.A
- ring 368.A
- stable cohomology 64.B
- stable primary cohomology 64.B
- stable secondary cohomology 64.C
- Steenrod (pth) power 64.B
- Steenrod square 64.B
- transitive (of a group) 362.B
- operation A (in set theory) 22.B
- operational calculus 251.G 306, App. A, Table 12.II
- operator
 - (in functional analysis) 162 251.A
 - (on a set) 409.A
 - Abelian 308.E
 - accretive (in a Hilbert space) 286.C
 - additive 251.A
 - adjoint (in Banach spaces) 37.D 251.D
 - adjoint (in Hilbert spaces) 251.E
 - adjoint (of a linear partial differential operator) 322.E
 - adjoint (of a microdifferential operator) 274.F
 - adjoint (of a microlocal operator) 274.F
 - amplification (of the scheme) 304.F
 - angular momentum 258.D
 - annihilation 377.A
 - Beltrami differential, of the first kind App. A, Table 4.II
 - Beltrami differential, of the second kind App. A, Table 4.II
 - boundary 200.C 201.B
 - with a boundary condition 112.F
 - bounded linear 37.C
 - Calderón-Zygmund singular integral 217.J 251.O
 - Cartier 9.E
 - channel wave 375.F
 - closable 251.D
 - closed 39.I 251.D
 - closure 425.B
 - coboundary 200.F
 - compact 68
 - completely continuous 68.B
 - conjugate (in Banach spaces) 37.D
 - conjugate (of a differential operator) 125.F
 - conjugate (of a linear operator) 251.D
 - conjugation (in function algebras) 164.K
 - creation 377.A
 - decomposable (on a Hilbert space) 308.G
 - degeneracy (in a semisimplicial complex) 70.E
 - diagonalizable (in an Abelian von Neumann algebra) 308.G
 - differential 112 223.C 306.B
 - differential, of the k th order 237.H
 - differentiation 223.C
 - dissipative 286.C
 - domain (of an Ω -group) 190.E
 - domain of 409.A
 - down-ladder 206.B
 - dual (in Banach spaces) 37.D
 - dual (of a differential operator) 125.F
 - dual (of a linear operator) 251.D
 - elliptic 112.A
 - energy-momentum 258.D
 - evolution 378.G
 - exponential function of 306.C
 - face (in a semisimplicial complex) 70.E
 - formal adjoint 322.E
 - 4-momentum 258.D
 - Fourier integral 274.C
 - Fredholm 68.F 251.D
 - fundamental 163.E
 - generalized wave 375.B
 - Green's 189.A, B 194.C
 - Hamiltonian 351.D
 - Hecke 32.D
 - Hermitian 251.E
 - Hilbert's ε - 411.J
 - holomorphic evolution 378.I
 - identity (on a Banach space) 37.C
 - incoming wave 375.B
 - with index 68.F
 - integral 68.N 100.E 251.O 306.B
 - integral, of Hilbert-Schmidt type 68.C
 - interior 425.B
 - inverse 37.C 251.B
 - isometric 251.E
 - Laplace 323.A 442.D
 - Laplace-Beltrami 194.B
 - linear 251
 - linear (in Banach spaces) 37.C
 - linear (in linear spaces) 256.B
 - linear boundary 315.B
 - linearized 286.E
 - local 125.DD
 - logical 411.E
 - Markov 136.B
 - maximal (of a differential operator) 112.E
 - maximal dissipative 251.J
 - microdifferential 274.F
 - microlocal 274.F
 - microlocally elliptic 345.A
 - Mikusinski's 306.B
 - minimal (of a differential operator) 112.E
 - modified wave 375.B
 - modular 308.H
 - monotone (in a Hilbert space) 286.C
 - nonlinear semigroups of 286.X
 - nonnegative 251.E
 - normal 390.E
 - normal (of Sario) 367.G
 - normal linear 251.E
 - nuclear 68.I, K
 - number 377.A
 - ordinary differential 112.A
 - outgoing wave 375.B
 - partial differential 112.A
 - positive (in vector lattices) 310.E
 - positive semidefinite 251.E
 - projection (in a Hilbert space) 197.E
 - pseudodifferential 251.O 345
 - pseudodifferential (in microlocal analysis) 274.F
 - resolvent (of a Markov process) 261.D
 - ring of 308.C
 - S- 150.D
 - scalar 390.K
 - scattering 375.F, H

- Schrödinger 351.D
- self-adjoint 251.E 390.E
- shift 223.C 251.O 306.C
- spectral 390.K
- Steenrod App. A, Table 6.II
- step-down 206.B
- step-up 206.B
- strongly elliptic 112.G 323.H
- Sturm-Liouville 112.I
- of summable p th power 68.K
- symmetric 251.E
- system of differential 112.R
- T - 375.C
- TCP 150.D
- Toeplitz operator 251.O
- total boundary 200.E
- trace 168.B
- translation 306.C
- transposed 112.E 189.C 322.E
- unilateral shift 390.I
- unitary 390.E
- up-ladder 206.B
- Volterra 68.J
- wave 375.B,H
- operator algebra 308.A
- operator convex 212.C
- operator domain 277.C
 - module with 277.C
- operator homomorphism
 - (of A -modules) 277.E
 - (of Ω -groups) 190.E
- operator isomorphism 190.E
- operator monotone 212.C
- operator topology
 - strong 251.C
 - uniform 251.C
 - weak 251.C
- operator-valued distribution 150.D
- opposite
 - (simplex) 201.C
 - orientation 105.F
 - root 13.R
- optical axis 180.B
- optical direction cosines 180.A
- optical distance 180.A
- optical theorem 386.B
- optics, geometric 180
- optimal
 - (design) 102.E
 - asymptotically 354.O
- optimal control 46.D 86.B,C 405.A
- optimal control problem, time 86.E
- optimality
 - A - 102.E
 - D - 102.E
 - E - 102.E
 - principle of 127.A
- optimal policy 127.A
- optimal regular problem 86.F
- optimal solution 255.A 264.B 292.A
 - basic 255.A
- optimal stopping 405.E
- optimization model 307.C
- optimum allocation 373.E
- optimum predictor, linear 395.D
- optional (stochastic process) 407.B
- optional σ -algebra 407.B
- optional sampling 262.C
- optional sampling theorem 262.A
- orbit
 - (of a dynamical system) 126.B
 - (of a permutation group) 151.H
 - (= system of transitivity) 362.B
 - (of a topological transformation group) 110.A 431.A
 - closed 126.D
 - exceptional 431.C
 - principal 431.C
 - pseudo-, α - 126.J
 - pseudo-, tracing property 126.J
 - singular 431.C
- orbital angular momentum 351.E
- orbital elements, Kepler's 309.B
- orbitally stable 126.F
- orbital stability (of a solution of a differential equation) 394.D
- orbit determination 309.A
- orbit space (of a topological group) 431.A
- orbit type 431.A
 - principal 431.C
- order
 - (of an algebraic number field) 14.B
 - (of a covering) 425.R
 - (of a differential equation) 313.A 320.A
 - (of a differential operator) 112.A
 - (of an element of a group) 190.C
 - (of an elliptic function) 134.E
 - (of a function defined by a Dirichlet series) 121.C
 - (of a function on an algebraic curve) 9.C
 - (of a generating point of a simple maximally overdetermined system) 274.H
 - (of a group) 190.C
 - (of a homomorphism of Abelian varieties) 3.C
 - (of an infinitesimal) 87.G
 - (of an infinity) 87.G
 - (of a Lie algebra) 191.D
 - (of a meromorphic function) 272.C
 - (of a microdifferential operator) 274.F
 - (of a multistep method) 303.E
 - (= order relation) 311.A
 - (of a plane algebraic curve) 9.B
 - (of a point in an ordinary curve) 93.C
 - (of a point with respect to a cycle) 99.D
 - (of a pole of a complex function) 198.D
 - (of the precision of numerical solution) 303.B
 - (= a subring) 27.A
 - (of a system of differential equations) 313.B
 - (of a transcendental entire function) 429.B
 - (of a zero point of a complex function) 198.C
 - d'Alembert's method of reduction of 252.F
 - derivatives of higher App. A, Table 9.III
 - difference of the n th 104.A
 - finite (distribution) 125.J
 - γ -point of the k th (of a holomorphic function) 198.C
 - of higher 87.G
 - infinite (element in a group) 190.C
 - left (of a g -lattice) 27.A
 - of lower 87.G
 - maximal (of a g -lattice) 27.A
 - at most (a function) 87.G
 - of the n th 87.G
 - principal (of an algebraic number field) 14.B
 - principal (fundamental theorem of) 14.C
 - right (of a g -lattice) 27.A
 - of the same 87.G
 - small set of 436.G
 - space of line elements of higher 152.C
 - surface of the second 350.A

Order α

- zero point of the k th (of a holomorphic function) 198.C
- zero point of the $-k$ th (of a complex function) 198.D
- order α
 - capacity of 169.C
 - Cesàro method of summation of 379.M
 - Hölder condition of 84.A
 - Lipschitz condition of 84.A
 - potential of 338.B
 - summable by Cesàro's method of 379.M
- order k
 - coefficient of 110.A
 - converge in the mean of 173.B 342.D
 - invariants of 110.A
 - population moment of 396.C
 - principal components of 110.A
 - quantile of 341.H
 - Riesz method of summation 379.R
 - summable by Hölder's method of 379.M
 - summable by M. Riesz's method of 379.R
- order p
 - contravariant tensor field of 105.O
 - jet of 105.X
- order s , covariant tensor field of 105.O
- order 0, frame of 110.C
- order 1
 - family of frames of 110.B
 - frame of 110.C
- order 2, frame of 110.B,C
- order 3, frame of 110.B,C
- order 4, frame of 110.B
- order bounded 310.B
- order convergent sequence (in a vector lattice) 310.C
- order-disorder transition 402.F
- ordered additive group 439.B
 - totally 439.B
- ordered complex (of a semisimplicial complex) 70.E
- ordered field 149.N
 - Archimedean 149.N
 - Pythagorean 60.O
- ordered group 243.G
 - lattice- 243.G
 - totally 243.G
- ordered linear spaces 310.B
 - lattice- 310.B
- ordered pair 33.B 381.B
- ordered set 311.A
 - inductively 34.C
 - lattice- 243.A
 - linearly 311.A
 - partially 311.A
 - semi- 311.A
 - totally 311.A
- ordered simplex (in a simplicial complex) 70.E
- ordered simplicial complex 70.C
- order function (meromorphic function) 272.B
- order homomorphic (ordered sets) 311.E
- order homomorphism 311.E
- order ideal (of a vector lattice) 310.B
- ordering 96.C 311.A
 - dual 311.A
 - duality principle for 311.A
 - lexicographic 311.G
 - lexicographic linear 248.M
 - linear 311.A
 - partial 311.A
 - pre- 311.H
 - total 311.A
 - well- 311.C
- order isomorphic (ordered sets) 311.E
- order isomorphism 311.E
- order limit (in a vector lattice) 310.C
- order-preserving mapping 311.E
- order-preserving semigroup 286.Y
- order relation 311.A
- order statistic 396.C
- order topology 425.C
- order type 312.A
- ordinal numbers 312.B
 - admissible 356.G
 - cardinality of 49.E
 - constructive 81.B
 - countable 49.E
 - finite 312.B
 - of the first, second, or third number class 312.D
 - of a higher number class 312.D
 - hyperconstructive 81.E
 - initial 49.E
 - isolated 312.B
 - limit 312.B
 - strongly inaccessible 312.E
 - transfinite 312.B
 - transfinite initial 49.E
 - weakly inaccessible 312.E
- ordinal product (of a family of ordered sets) 311.G
- ordinal scale 397.M
- ordinal sum (of a family of ordered sets) 311.G
- ordinary curve 93.C
- ordinary derivative (of a set function) 380.D
- ordinary differential equation(s) 313, App. A, Table 14
 - (asymptotic behavior of solutions) 314
 - (boundary value problems) 315
 - (initial value problems) 316
 - Euler linear App. A, Table 14.I
 - higher-order App. A, Table 14.I
 - homogeneous App. A, Table 14.I
 - homogeneous (of higher order) App. A, Table 14.I
 - linear 252 313.A
 - linear (with constant coefficients) App. A, Table 14.I
 - linear (of the first order) App. A, Table 14.I
 - linear (global theory) 253
 - linear (of higher order) App. A, Table 14.I
 - linear (local theory) 254
 - nonlinear 313.A
 - nonlinear (global theory) 288
 - nonlinear (local theory) 289
 - system of 313.B
- ordinary differential operator 112.A
- ordinary Dirichlet series 121.A
- ordinary double point (of a plane algebraic curve) 9.B
- ordinary element 191.I
- ordinary helicoid 111.I
- ordinary helix 111.F 114.F
- ordinary integral element 428.E
- ordinary integral manifold (of a differential ideal) 428.E
- ordinary lower derivative (of a set function) 380.D
- ordinary point
 - (of an analytic set) 23.B
 - (of a curve) 93.G

(in hyperbolic geometry) 285.C
 (of an ordinary curve) 93.C
 (on a Riemann surface) 11.D
 ordinary representation (of a finite group) 362.G
 ordinary sense, derivable in the 380.D
 ordinary singularity
 (of an analytic function) 198.P
 in the wider sense 198.P
 ordinary solution (of a differential ideal) 428.E
 ordinary upper derivative (of a set function) 380.D
 ordinate set 221.E
 orientable
 (manifold) 105.F 201.N
 (pseudomanifold) 65.B
 transversely 154.B
 orientable fiber bundle 147.L
 orientation
 (of an affine space) 139.B
 (of a contact element) 110.A
 (of a manifold) 105.F 201.N
 local (in an oriented manifold) 201.N
 negative (of an oriented manifold) 105.F
 opposite (of oriented atlases) 105.F
 positive (of an oriented manifold) 105.F
 same (of oriented atlases) 105.F
 orientation cohomology class 201.N
 orientation manifold 201.N
 orientation sheaf 201.R
 orientation-preserving mapping 99.A
 orientation-reversing mapping 99.A
 oriented atlas (of an orientable differentiable manifold) 105.F
 oriented cobordism
 class 114.H
 group 114.H
 oriented differentiable structures, group of (on the combinatorial sphere) 114.I
 oriented element (in a covering manifold) 110.A
 oriented G -manifold 431.E
 oriented graph 186.B
 oriented manifold 105.F 201.N
 integrals over 105.T
 oriented pseudomanifold 65.B
 coherently 65.B
 oriented q -simplex 201.C
 oriented real hypersphere 76.A
 oriented segment 442.A
 oriented simplicial chain complex 201.C
 oriented singular r -simplex of class C^∞ 105.T
 oriented tangent line 76.B
 origin
 (of an affine space) 7.C
 (of a Euclidean space) 140
 (of a projective frame) 343.C
 Orlicz-Pettis theorem 443.D
 Orlicz space 168.B
 Ornstein-Uhlenbeck Brownian motion 45.I
 orthant, positive 89.G
 orthochronous 258.A
 orthocomplement (of a subspace of a linear space) 139.G
 orthogonal
 (block design) 102.J
 (elements of a ring) 368.B
 (in Euclidean geometry) 139.E,G
 (functions) 317.A
 (in a Hilbert space) 197.C
 (linear subspaces) 256.G
 mutually (latin squares) 241.B
 orthogonal array 102.L

Subject Index

Orthonomic system, passive

orthogonal complement (of a subset of a Hilbert space) 197.E
 orthogonal component (of an element of a linear space) 139.G
 orthogonal coordinate system adapted to (a flag) 139.E
 orthogonal curvilinear coordinates 90.C
 orthogonal curvilinear coordinate system App. A, Table 3.V
 orthogonal expansion 317.A
 orthogonal for a finite sum 19.G
 orthogonal fractional factorial design 102.I
 orthogonal frame 111.B 139.E
 orthogonal frame bundle 364.A
 tangent 364.A
 orthogonal function(s) 317, App. A, Table 20
 Haar system of 317.C
 Rademacher system of 317.C
 Walsh's system of 317.C
 orthogonal group 60.I 139.B 151.I
 (over a noncommutative group) 60.O
 complex 60.I
 complex special 60.I
 infinite 202.V
 over K with respect to Q 60.K
 pair 422.I
 proper 60.I 258.A
 reduced 61.D
 special 60.I
 orthogonality for a finite sum 317.D, App. A, Table 20.VII
 orthogonality relation
 (on irreducible characters) 362.G
 (for square integrable unitary representations) 437.M
 orthogonalization
 Gram-Schmidt 317.A
 Schmidt 317.A
 orthogonal k -frame (in \mathbf{R}^n) 199.B
 orthogonal matrix 269.J
 complex 269.J
 proper 269.J
 orthogonal measure 164.C
 orthogonal polynomial(s) 19.G, App. A, Table 20.VII
 Chebyshev 19.G
 simplest 19.G
 system of 317.D
 orthogonal projection
 (in Euclidean geometry) 139.E,G
 (in a Hilbert space) 197.E
 method of 323.G
 orthogonal series (of functions) 317.A
 orthogonal set
 (of functions) 317.A
 (of a Hilbert space) 197.C
 (of a ring) 368.B
 orthogonal system
 (of functions) 317.A
 (of a Hilbert space) 197.C
 complete 217.G
 orthogonal trajectory 193.J
 orthogonal transformation 139.B 348.B
 (over a noncommutative field) 60.O
 (with respect to a quadratic form) 60.K
 orthogonal transformation group 60.I
 over K with respect to Q 60.K
 orthomodular 351.L
 orthonomic system, passive (of partial differential equations) 428.B

orthonormal basis 197.C
 orthonormalization 139.G
 orthonormal moving frame 417.D
 orthonormal set
 (of functions) 317.A
 (of a Hilbert space) 197.C
 complete (of a Hilbert space) 197.C
 orthonormal system
 complete 217.G
 complete (of fundamental functions) 217.G
 orthorhombic system 92.E
 oscillate (for a sequence) 87.D
 oscillating (series) 379.A
 oscillating motion 420.D
 oscillation(s) 318
 (of a function) 216.A
 bounded mean 168.B
 damped 318.B
 equation of App. A, Table 15.VI
 forced 318.B
 harmonic 318.B
 nonlinear 290.A
 relaxation 318.C
 nonstationary 290.F
 oscillator process 351.F
 oscillatory 314.F
 osculating circle 111.F
 osculating elements 309.D
 osculating plane 111.F
 osculating process 77.B
 Oseen approximation 205.C
 Osgood theorem, Hartogs- 21.H
 O-S positivity 150.F
 Osterwalder-Schrader axioms 150.F
 Ostrogradskii formula 94.F
 outdegree 186.B
 outer area 216.F 270.G
 outer automorphisms
 group of (of a group) 190.D
 group of (of a Lie algebra) 248.H
 outer capacity, Newtonian 48.H
 outer function 43.F
 outer harmonic measure 169.B
 outer measure 270.E,G
 Carathéodory 270.E
 Lebesgue 270.G
 outer solution 25.B
 outer variable 25.B
 outer volume 270.G
 outgoing subspace 375.H
 outgoing wave operator 375.B
 outlier test 397.Q
 out-state 150.D 386.A
 oval 89.C 111.E
 Cassini 93.H
 mean (of two ovals) 89.D
 width of the 111.E
 ovaloid 89.C 111.I
 overall approximation formula 303.C
 overconvergence 339.E
 overcrossing point 235.A
 overdetermined system
 (of differential operators) 112.I
 (of partial differential equations) 320.F
 maximally (= holonomic) 274.H
 overfield 149.B
 overidentified 128.C
 overrelaxation
 successive (SOR) 302.C

P

$PSL(n, k)$ (projective special linear group) 60.B
 $PS_p(n, k)$ (projective symplectic group over K) 60.L
 $PU(n)$ (projective unitary group) 60.F
 $P^n(K)$ (projective space) 343.H
 φ -subsequence 354.E
 π -group 151.F
 π -length (of a group) 151.F
 π -manifold 114.I
 π -series (of a group) 151.F
 π -solvable group 151.F
 π theorem 116
 π topology 424.R
 Π_1^1 set 22.A
 Π_n^1 set 22.D
 p -adic exponential valuation 439.F
 p -adic extension (of the field of quotients of a Dedekind domain) 439.F
 p -index (of a central simple algebra over a finite algebraic number field) 29.G
 p -invariant (of a central simple algebra over a finite algebraic number field) 29.G
 p -primary ideal 67.F
 \mathfrak{P} -function, Weierstrass 134.F, App. A, Table 16.IV
 \mathcal{P} -acyclic 200.Q
 p -adic integer(s) 439.F
 ring of 439.F
 p -adic L -function 450.J
 p -adic number 439.F
 p -adic number field 257.A 439.F
 p -adic regulator 450.J
 p -adic valuation 439.F
 p -ary matroid 66.H
 p -atom 168.B
 p -covector 256.O
 p -dimensional noncentral Wishart distribution 374.C
 p -extension (of a field) 59.F
 p -factor (of an element of a group) 362.I
 p -fold exterior power
 (of a linear space) 256.O
 (of a vector bundle) 147.F
 p -form
 tensorial 417.C
 vectorial 417.C
 p -group 151.B
 Abelian 2.A
 complete (Abelian) 2.D
 divisible (Abelian) 2.D
 p -parabolic type 327.H
 p -rank (of a torsion-free additive group) 2.E
 p -regular (element of a finite group) 362.I
 p -Sylow subgroup 151.B
 p th power, operator of summable 68.K
 p th power operation
 Pontryagin 64.B
 Steenrod 64.B
 p -torsion group of an exceptional group App. A, Table 6.IV
 p -valent (function) 438.E
 absolutely 438.E
 circumferentially mean 438.E
 locally 438.E
 locally absolute 438.E
 mean 438.E
 quasi- 438.E
 p -vector 256.O

bundle of 147.F
 (p, q) -ball knot 235.G
 (p, q) -knot 235.G
 $(p + 1)$ -stage method 303.D
 P -convex (for a differential operator) 112.C
 strongly 112.C
 P -function, Riemann 253.B, App. A, Tables
 14.II 18.I
 P -projective resolution 200.Q
 P -wave 351.E
 P_n set 22.D
 P^r -figure 343.B
 p -space 425.Y
 P -space 425.Y
 Padé approximation 142.E
 Padé table 142.E
 Painlevé equation 288.C
 Painlevé theorem 198.G
 Painlevé transcendental function 288.C
 pair 381.B
 (in axiomatic set theory) 33.B
 ball 235.G
 BN - 13.R
 contact (in circle geometry) 76.C
 group (of topological Abelian groups) 422.I
 order 381.B
 ordered (in axiomatic set theory) 33.B
 orthogonal group 422.I
 Poincaré, of formal dimension n 114.J
 simplicial 201.L
 sphere 65.D 235.G
 topological 201.L
 unordered 381.B
 unordered (in axiomatic set theory) 33.B
 paired comparison 346.C
 pairing
 (of linear spaces) 424.G
 axiom of 381.G
 pair test 346.D
 pairwise sufficient (statistic) 396.F
 Palais-Smale condition (C) 279.E 286.Q
 Paley theorem 317.B
 Paley theory, Littlewood- 168.B
 Paley-Wiener theorem 125.O, BB
 pantograph 19.E
 paper
 binomial probability 19.B
 functional 19.D
 logarithmic 19.F
 probability 19.F
 semilogarithmic 19.F
 stochastic 19.B
 Pappus theorem
 (on conic sections) 78.K
 (in projective geometry) 343.C
 parabola(s) 78.A
 family of confocal 78.H
 parabolic
 (differential operator) 112.A
 (Riemann surface) 367.D, E
 (simply connected domain) 77.B
 (visibility manifold) 178.F
 parabolic coordinates 90.C, App. A, Table 3.V
 parabolic cusp (of a Fuchsian group) 122.C
 parabolic cylinder 350.B
 parabolic cylinder function 167.C
 parabolic cylindrical coordinates 167.C, App. A,
 Table 3.V
 parabolic cylindrical equation App. A, Table 14.II

parabolic cylindrical surface 350.B
 parabolic-elliptic motion 420.D
 parabolic geometry 285.A
 parabolic motion 420.D
 parabolic point (on a surface) 110.B 111.H
 parabolic quadric hypersurface 350.I
 parabolic subalgebra (of a semisimple Lie algebra)
 248.O
 parabolic subgroup
 (of an algebraic group) 13.G
 (of the BN -pair) 13.R
 (of a Lie group) 249.J
 cuspidal 437.X
 minimal k - 13.Q
 standard k - 13.Q
 parabolic transformation 74.F
 parabolic type
 (equation of evolution) 378.I
 partial differential equation of 327
 paraboloid
 elliptic 350.B
 elliptic, of revolution 350.B
 hyperbolic 350.B
 paracompact (space) 425.S
 countably 425.Y
 strongly 425.S
 paracompact C' -manifold 105.D
 paradox(es) 319
 Burali-Forti 319.B
 d'Alembert 205.C
 Richard 319.B
 Russel 319.B
 Skolem 156.E
 Zeno 319.C
 parallax
 annual 392
 geocentric 392
 parallel(s)
 (affine subspaces) 7.B
 (lines) 139.A 155.B
 (lines in hyperbolic geometry) 285.B
 (tensor field) 364.B
 axioms of 139.A
 in the narrow sense (in an affine space) 7.B
 in the sense of Levi-Civita 111.H
 in the wider sense (in an affine geometry) 7.B
 parallel coordinates (in an affine space) 7.C
 parallel displacement
 (in an affine connection) 80.H
 (in a connection) 80.C
 (in the Riemannian connection) 364.B
 parallelepiped, rectangular 14.O
 parallelism, absolute 191.B
 parallelizable
 (flow) 126.E
 (manifold) 114.I
 almost 114.I
 s - 114.I
 stably 114.I
 parallelogram, period 134.E
 parallelootope 425.T
 (in an affine space) 7.D
 open (in an affine space) 7.D
 parallel projection (in an affine space) 7.C
 parallel translation 80.C 364.B
 parameter(s) 165.C
 (of an elliptic integral) 134.A
 (in a population distribution) 401.F
 (of a probability distribution) 396.B

- acceleration 302.C
- canonical (of an arc) 111.D
- design for estimating 102.M
- distinct system of 284.D
- estimable 403.E
- isothermal 334.B
- isothermal (for an analytic surface) 111.I 334.B
- Lagrange's method of variation of 252.D
- linear 102.A
- linearly estimable 403.E
- local (Fuchsian groups) 32.B
- local (of a nonsingular algebraic curve) 9.C
- local (of a Riemann surface) 367.A
- local canonical (for power series) 339.A
- local uniformizing (of a Riemann surface) 367.A
- location 396.I 400.E
- one- (group of transformations) 105.N
- one- (subgroup of a Lie group) 249.Q
- one-, semigroup of class (C^0) 378.B
- of regularity (of a Lebesgue measurable set) 380.D
- regular system of 284.D
- scale 396.I 400.E
- secondary 110.A
- selection 396.F
- system of 284.D
- time (of a stochastic process) 407.A
- transformation 396.I
- transformation of 111.D
- true value of 398.A
- parameter space
 - (of a family of compact complex manifolds) 72.G
 - (of a family of probability measures) 398.A
 - (of a probability distribution) 396.B
- parametrically sustained vibration 318.B
- parametric function 102.A 399.A
- parametric programming 264.C
- parametric representation 165.C
 - (of Feynman integrals) 146.B
 - (of a subspace of an affine space) 7.C
- parametrix 189.C
 - left 345.A
 - right 345.A
- parametrized, effectively (at 0) 72.G
- paraxial ray 180.B
- parity check matrix 63.C
- parity check polynomial 63.D
- parity transformation 359.B
- Parreau-Widom type 164.K
- Parseval equality 18.B 197.C
- Parseval identity 18.B 159.A 160.C 192.K 220.B,C,E
- parsing 31.E
- part(s)
 - (for a function algebra) 164.F
 - connected 150.D
 - cyclic (of an ergodic class) 260.B
 - dissipative (of a state space) 260.B
 - essential 260.I
 - finite (of an integral) 125.C
 - Gleason (for a function algebra) 164.F
 - holomorphic (in a Laurent expansion) 198.D
 - homogeneous (of a formal power series) 370.A
 - imaginary 74.A
 - integration by (for the D-integral) 100.G
 - integration by (for the Stieltjes integral) 94.C
 - integration by (for the Riemann integral) 216.C
 - negative (of an element of a vector lattice) 310.B
 - positive (of an element of a vector lattice) 310.B
 - principal (of a differential operator) 112.A
 - principal (of a Laurent expansion) 198.D
 - principal (of a partial differential operator) 320.B
 - purely contractive 251.N
 - real 74.A
 - semisimple (of an algebraic group) 13.E
 - semisimple (of a nonsingular matrix) 13.E
 - singular (of a Laurent expansion) 198.D
 - unipotent (of an algebraic group) 13.E
 - unipotent (of a nonsingular matrix) 13.E
- partial boundary operator 200.E
- partial capture 420.D
- partial correlation coefficient 397.J
 - sample 280.E
- partial denominator (of an infinite continued fraction) 83.A
- partial de Rham system 274.G
- partial derivative 106.F,K
 - n th-order 106.H
- partial derived functor 200.I
- partial differential 200.H
- partial differential coefficient 106.E
- partial differential equation(s) 313.A 320
 - (initial value problems) 321
 - (method of integration) 322
 - of elliptic type 323, App. A, Table 15.VI
 - of the first order 324
 - Fokker-Planck 115.A
 - hyperbolic 325
 - of hyperbolic type 325
 - of mixed type 326
 - of parabolic type 327
 - solution, of the first order App. A, Table 15.II
 - solution, of the second order App. A, Table 15.III
 - system of, of order 1 (on a differentiable manifold) 428.F
- partial differential operator 112.A
- partial differentiation 106.F
- partial fraction App. A, Table 10.V
- partial function 356.E
- partial graph 186.C
- partially balanced incomplete block design 102.J 406.J
- partially confounded (with blocks) 102.J
- partially conserved axial-vector currents 132.G
- partially differentiable (function) 106.F
- partially isometric (operator) 251.E
- partially ordered set 311.A
- partial mapping (of a mapping) 381.C
- partial numerator (of an infinite continued fraction) 83.A
- partial observation 405.C
- partial ordering 311.A
- partial pivoting 302.B
- partial product 379.G
- partial quotient, n th 83.A
- partial recursive (in a partial recursive function) 356.E,F
- partial sum (of a series) 379.A
 - diagonal (of a double series) 379.E
- partial summation, Abel 379.D
- partial wave 386.B
- partial wave expansion 375.E 386.B
- partial wave scattering amplitude 375.E

- particle(s)
 - Bose 132.A
 - composite 132.A
 - elementary 132
 - Fermi 132.A
- particular solution
 - (of a differential equation) 313.A
 - (of partial differential equations) 320.C
 - (for a system of differential equations) 313.C
- particular transformation (of h_R^*) 248.R
- partition(s)
 - (in ergodic theory) 136.E
 - (of an interval) 216.A
 - (of a set) 381.D
 - (of a space) 425.L
 - entropy of 136.E
 - ε -independent 136.E
 - independent sequence of 136.E
 - Markov (for an automorphism) 136.C,G
 - number of 177.D 328
 - of numbers 328
 - Pinsker 136.E
 - principal 66.H
 - of unity 425.R
 - of unity of class C^∞ 105.S
 - of unity subordinate to a covering 425.R
 - upper semicontinuous 425.L
- partition function 402.D
 - grand 402.D
- partitioning algorithm 215.E
- Pascal, B. 329
 - limaçon of 93.H
- Pascal configuration 78.K
- Pascal line 78.K
- Pascal theorem
 - (on conic sections) 78.K
 - (in geometry) 155.E
 - (in projective geometry) 343.E
- Pascal triangle 330
- Pasch's axiom (in geometry) 155.B
- passive
 - (state) 402.G
 - completely 402.G
- passive boundary point 260.I
 - dual 260.I
- passive network 282.C
- passive orthonomic system (of partial differential equations) 428.B
- past cone 258.A
- past history, independent of the 406.D
- past together the boundaries 114.F
- path
 - (in a Finsler space) 152.C
 - (in a graph) 186.F
 - (of a Markov process) 261.B
 - (of a stochastic process) 407.A
 - (in a topological space) 148.C 170
 - asymptotic (for a meromorphic function) 272.H
 - closed (in a graph) 186.F
 - closed (in a topological space) 170
 - closed, space of 202.C
 - critical 376
 - direct 186.F
 - direct closed 186.F
 - Euler (in a graph) 186.F
 - general geometry of 152.C
 - Hamilton 186.F
 - of an integration (curvilinear integral) 94.D
 - inverse 170
 - nontangential 333.B
 - projective geometry of 109
 - quasi-independent of (response probability) 346.G
 - sample 407.A
 - simple 186.F
 - Stolz (in a plane domain) 333.B
- path-component 79.B
- path-connected 79.B
- path-dependent, d -trial 346.G
- path-independent (response probability) 346.G
- path integral 351.F
- pathological (space) 65.F
- path space 148.C 261.B
- pathwise uniqueness of solution 406.D
- pattern formation 263.D
- Pauli approximation 351.G
- Pauli-Lubanski vector 258.D
- Pauli principle 351.G
- Pauli spin matrix 258.A 351.G
- payoff 108.B,C 173.B
- payoff function 173.C
- PBIBD (partially balanced incomplete block design) 102.J
- PC (predictor-corrector) method 303.E
- PCT invariance 386.B
- PCT theorem 386.B
- peak point 164.D
 - generalized 164.D
- peak set 164.D
 - generalized 164.D
- Peano area (of a surface) 246.F
- Peano continuum 93.D
- Peano curves 93.J
- Peano postulates 294.B
- Pearson distribution 397.D
- Pearson lemma, Neyman- 400.B
- Péclet number 116.B
- pedal curve 93.H
- Peierls-Bogolyubov inequality 212.B
- Peirce decomposition (of a Jordan algebra) 231.B
- Peirce left decomposition (in a unitary ring) 368.F
- Peirce right decomposition (in a unitary ring) 368.F
- Peirce space 231.B
- Pełczyński theorem, Bessaga- 443.D
- Pell equation 118.A
- penalized problems 440.B
- penalty method 292.E
- penalty term 440.B
- pencil
 - algebraic 15.C
 - of conics 343.E
 - of hyperplanes (in a projective space) 343.B
 - Lefschetz 16.U
 - linear 16.N
 - of lines (in a projective plane) 343.B
 - of planes (in a 3-dimensional projective space) 343.B
 - of quadric hypersurfaces 343.E
 - of quadrics 343.E
- peninsula (in a Riemann surface) 272.J
- pentagram function 174.B
- pentagon 155.F
- pentagonal number 4.D
- pentagonal number theorem 328
- pentaspherical coordinates 90.B
- percolation process 340.D
 - bond 340.D
 - site 340.D

- perfect
 - (image) 180.A
 - α -(graph) 186.K
 - γ -(graph) 186.K
- perfect additive functional 261.E
- perfect code 63.B
- perfect delay convention 51.F
- perfect field 149.H
- perfect fluid 205.B
- perfect image 425.CC
- perfect inverse image 425.CC
- perfect kernel (in potential theory) 338.E
- perfectly normal space 425.Q
- perfectly separable space 425.P
- perfect mapping 425.W
 - quasi- 425.CC
- perfectness theorem 186.K
- perfect number 297.D
 - of the second kind 297.D
- perfect set 425.O
- perigon, straight 139.D
- perihelion distance 309.B
- period
 - (of an Abelian differential form) 11.C
 - (of an ergodic class) 260.B
 - (of a marked K3 surface) 72.K
 - (of an orbit) 126.D
 - (of an oscillation) 318.A
 - (of a periodic continued fraction) 83.C
 - (of a periodic function) 134.B
 - (of a wave) 446
 - fundamental 134.E
- periodogram 421.C
- periodic (trajectory) 126.D
 - almost 126.F
- periodic continued fraction 83.C
- periodic endomorphism (at a point) 136.E
- periodic function 134.E
 - almost (on a graph) 18.E
 - almost (with respect to p) 18.C
 - almost (in the sense of Bohr) 18.B
 - analytic almost 18.D
 - doubly 134.E
 - simply 134.E
 - uniformly almost 18.B
- periodic group 2.A
 - maximally almost 18.I
 - minimally almost 18.I
- periodic inequality, Riemann 3.L
- periodicity 390.J
- periodicity modulus (of an elliptic integral) 134.A
- periodicity theorem, Bott 202.V 237.D, App. A, Table 6.VII
- periodic solution (of Hill's equation) 268.E
- period matrix
 - (of a closed Riemann surface) 11.C
 - (of a complex torus) 3.H
- period parallelogram 134.E
 - fundamental 134.E
- period relation, Riemann's 3.L 11.C
- peripheral devices 75.B
- peripheral system 235.B
- permeability, magnetic 130.B
- permeable membrane 419.A
- permutation 190.B
 - (in a symmetric group) 151.G
 - even 151.G
 - k- 330
 - odd 151.G
- permutation group 190.B
 - of degree n 151.G
 - imprimitive 151.G
 - intransitive 151.H
 - K-transitive 151.H
 - K-ply transitive 151.H
 - multiply transitive 151.H
 - primitive 151.H
 - regular 151.H
 - transitive 151.H
- permutation representation (of a group) 362.B
 - degree of 362.B
 - faithful 362.B
 - primitive 362.B
 - reciprocal 362.B
 - similar 362.B
- perpendicular
 - (to a hyperplane) 139.E
 - foot of the 139.E
- perpetual motion 402.G
- Perron-Brelot solution (of Dirichlet problem) 120.C
- Perron-Frobenius theorem 310.H
- Perron integrable (function) 100.F
- Perron method (in Dirichlet problem) 120.C
- Perron theorem
 - (on linear transformations of sequences) 379.L
 - (on ordinary differential equations) 316.E
 - (on positive matrices) 269.N
- Perron-Wiener-Brelot solution (of Dirichlet problem) 120.C
- persistent 260.J
- perspective 343.B
- perspective mapping (in projective geometry) 343.B
- PERT 307.C 376
- perturbation(s)
 - analytic 331.D
 - asymptotic 331.D
 - general theory of 420.E
 - Kato 351.D
 - of linear operators 331
 - method 25.A
 - regular 331.D
 - secular 55.B
 - singular 289.E
 - special theory of 420.E
- Petersson conjecture, Ramanujan- 32.D
- Petersson metric 32.B
- Peter-Weyl theory
 - (on compact groups) 69.B
 - (on compact Lie groups) 249.U
- Petrovskii, hyperbolic in the sense of 325.F
- Petrovskii theorem 112.D
- Pettis completely additivity theorem 443.G
- Pettis integrable 443.F
 - Gel'fand- 443.F
- Pettis integral 443.F
 - Gel'fand- 443.F
- Pettis measurability theorem 443.B
- Pettis theorem
 - Dunford- 68.M
 - Orlicz- 443.D
- Petryashvili equation, Kadomtsev- 387.F
- Pfaffian 103.G
- Pfaffian equation(s) 428.A
 - system of 428.A
- Pfaffian form 428.A
- Pfaff problem 428.A
 - generalized 428.B
- Pfluger extremal length, Hersch- 143.A
- phase

- initial (of a simple harmonic motion) 318.B
- pure 402.G
- phase average 402.C
- phase constant (of a sine wave) 446
- phase function (of a Fourier integral operator) 274.C 345.B
- phase portrait 126.B
- phase shift 375.E 386.B
- phase space
 - (of a dynamical system) 126.C 290.C
 - (for functional-differential equation) 163.C
 - (in statistical mechanics) 402.C
 - momentum 126.L
 - velocity 126.L
- phase transition 340.B
- phase velocity (of a sine wave) 446
- phenomenon
 - Gibbs 159.D
 - Runge 223.A
 - Stokes 254.D
- photon 132.B 377.B
- Phragmén-Lindelöf theorem 43.C
- physical Hilbert space 150.G
- physically contains 351.K
- PI-algebra (algebra with polynomial identities) 29.J
- Picard exceptional value 272.E
- Picard group (of a commutative ring) 237.J
- Picard-Lefschetz formula 418.F
- Picard-Lefschetz transformation 16.U
- Picard number (of a variety) 16.P
- Picard scheme 16.P
- Picard theorem
 - (on transcendental entire functions) 429.B
 - (on transcendental meromorphic functions) 272.E
- Picard variety 16.P
 - (of a compact Kähler manifold) 232.C
- Picard-Vessiot extension field 113
- Picard-Vessiot theory 113
- picture
 - Heisenberg 351.D
 - Schrödinger 351.D
- piecewise affine mapping 192.Q
- piecewise continuous function 84.B
- piecewise linear mapping 65.A 70.C
- piecewise smooth curve 364.A
- Pincherle-Goursat kernel 217.F
- Pinching problem (differentiable) 178.E
- Pinsker partition 136.E
- Pitman estimator 399.G
- pivot 302.B
- pivoting
 - complete 302.B
 - partial 302.B
- PL category 65.A
- PL embedding 65.D
- PL homeomorphism 65.A
- PL isomorphism 65.A
- PL k -ball 65.C
- PL $(k - 1)$ -sphere 65.C
- PL mapping (map) 65.A
- PL microbundle 147.P
- PL (n, m) -ball knot 65.D
- PL (n, m) -knot 65.D
- PL normal 147.P
- PL structure 65.C
- PL tangent 147.P
- PL topology 65.A
- place (of a field) 439.J
- placement problem 235.A
- planar 367.G
- planar character 367.G
- planar curvilinear coordinates App. A, Table 3.V
- planar graph 186.H
- planarity (of a graph) 186.H
- Plancherel formula (on a unimodular locally compact group) 437.L
- Plancherel measure (of a locally compact group) 437.L
- Plancherel theorem 160.H 192.A, K
 - (with respect to the Radon transform) 218.G
- Planck constant 351.A
- Planck (partial differential) equation, Fokker-115.A 402.I
- plane(s) 155.B
 - (as an affine space) 7.A
 - (in a projective space) 343.B
 - Cayley projective 54
 - complex 74.C
 - conjugate (with respect to a quadric surface) 350.C
 - coordinates (of a plane) 343.C
 - finite projective 241.B
 - Gauss-Argand 74.C
 - Gaussian 74.C
 - half- 155.B 333.A
 - hodograph 205.B
 - hyperbolic 122.C
 - normal 111.F
 - osculating 111.F
 - pencil of (in a 3-dimensional projective space) 343.B
 - polar (with respect to a quadric surface) 350.C
 - principal (of a quadric surface) 350.B
 - projective 343.B
 - rectifying 111.F
 - tangent 111.H, App. A, Table 4.I
 - w - 74.D
 - z - 74.D
- plane algebraic curve 9.B
- plane coordinates (of a plane) 343.C
- plane curve App. A, Table 4.I
 - continuous 93.B
- plane domains 333
 - closed 333.A
 - multiply connected 333.A
 - n -ply connected 333.A
- plane geometry 181
- plane polygon 155.F
- plane triangle App. A, Table 2.II
- plane trigonometry 432.A
- plane wave 446
- plane wave decomposition 125.CC
- planimeter 19.A
- planning
 - production 376
 - statistical 102.A
- plasticity, theory of 271.G
- Plateau problem 334
- playable 108.B
- Pleijel asymptotic expansion, Minakshisundaram-391.B
- PLK (Poincaré-Lighthill-Kuo) method 25.B
- Plotkin bound 63.B
- plots 102.B
- Plücker coordinates (in a Grassman manifold) 90.B
- Plücker formulas (on plane algebraic curves) 9.B
- Plücker relations (on Plücker coordinates) 90.B

- plurigenera 15.E
- pluriharmonic distribution 21.C
- plurisubharmonic function 21.G
- plus infinity 87.D
- Pochhammer differential equation, Tissot- 206.C
- Poincaré, H. 335
 - last theorem of 153.B
 - theta-Fuchsian series of 32.B
- Poincaré-Birkhoff fixed-point theorem 153.B
- Poincaré-Birkhoff-Witt theorem (on Lie algebras) 248.J
- Poincaré-Bruns theorem 420.A
- Poincaré characteristic, Euler- 16.E 201.B
- Poincaré class
 - Euler- 56.B,F
 - universal Euler- 56.B
- Poincaré complete reducibility theorem 3.C
- Poincaré complex 114.J
- Poincaré condition (in Dirichlet problem) 120.A
- Poincaré conjecture 65.C
 - generalized 65.C
- Poincaré differential invariant 74.G
- Poincaré duality 201.O 450.Q
- Poincaré formula (in integral geometry) 218.C
- Poincaré formula, Euler- 201.B,F
- Poincaré group 170 258.A
- Poincaré-Lefschetz duality theorem 201.O
- Poincaré-Lighthill-Kuo (PLK) method 25.B
- Poincaré manifold 105.A
- Poincaré mapping (map) 126.C,G
- Poincaré method 25.B
- Poincaré method, Lindstedt- 290.E
- Poincaré metric 74.G
- Poincaré model (of geometry) 285.D
- Poincaré pair (of formal dimension n) 114.J
- Poincaré polynomial (of a finite simplicial complex) 201.B
- Poincaré series 32.B
- Poincaré series, Eisenstein- 32.F
- Poincaré theorem 383.E
 - (on Abelian varieties) 3.D
- Poincaré-Volterra theorem 198.J
- Poinot representation 271.E
- point(s)
 - (of an affine space) 7.A
 - (in the foundations of geometry) 155.B
 - (of a graph) 186.B
 - (in projective geometry) 343.B
 - accessible boundary (of a plane domain) 333.B
 - accumulation 425.O
 - accumulation (of a sequence of real numbers) 87.C
 - adherent 425.B
 - algebraic (over a field) 369.C
 - algebraic branch (of a Riemann surface) 367.B
 - almost all, of a variety 16.A
 - α - (of a meromorphic function) 272.B
 - α -limit 126.D
 - ambiguous 62.D
 - antipodal (on a sphere) 140
 - apparent singular 254.C
 - base (of a linear system) 16.N
 - base (of a loop) 170
 - base (of a topological space) 202.B
 - bifurcation 126.M 217.M 286.R
 - boundary (of a subset) 425.N
 - branch (of a covering surface) 367.B
 - branch (of a harmonic mapping) 275.B
 - branch (of an ordinary curve) 93.C
 - catastrophe 51.F
 - cluster 425.O
 - coincidence (of maps) 153.B
 - collinear (in projective geometry) 343.B
 - complete accumulation 425.O
 - condensation 425.O
 - conjugate 46.C 364.C
 - conjugate (in a projective space) 343.E
 - corresponding (with respect to confocal quadrics) 350.E
 - critical (of a C^∞ -function) 279.B
 - critical (of a function) 106.L
 - critical (of a mapping) 105.J 193.J 208.B
 - critical (of a trajectory) 126.D
 - cut (on a geodesic) 178.A
 - degenerate critical 106.L 279.B
 - degree of 99.D
 - of density (of a measurable set of the real line) 100.B
 - dependent (in an affine space) 7.A
 - dependent (in a projective space) 343.B
 - deviation 336.B
 - discontinuity 84.B
 - discontinuity, of the first kind 84.B
 - discontinuity, of the second kind 84.B
 - dual passive boundary 260.I
 - elliptic (of a Fuchsian group) 122.C
 - elliptic (on a surface) 111.H
 - end (of an ordinary curve) 93.C
 - entrance boundary 260.I
 - equianharmonic range of 343.D
 - equilibrium 108.C
 - equilibrium (in the theory of games) 173.C
 - equilibrium (of a trajectory) 126.D
 - equivariant (of a mapping) 153.B
 - equivariant, index (of a mapping) 153.B
 - essentially singular (with respect to an analytic set) 21.M
 - estimation 399.B 401.C
 - exit boundary 260.I
 - exterior (of a subset) 425.N
 - externally irregular 338.L
 - extreme (of a convex set) 89.A
 - extreme (of a subset of a linear space) 424.T
 - fixed (of a discontinuous transformation group) 122.A
 - fixed (of a flow) 126.D
 - fixed (of a mapping) 153.A
 - fixed (of a mapping in a topological linear space) 153.D
 - fixed (method of roundoff) 138.B
 - fixed (of a topological transformation group) 431.A
 - fixed, of discontinuity 5.B 407.A
 - fixed, index (of a mapping) 153.B
 - fixed branch (of an algebraic differential equation) 288.A
 - fixed, theorem 153
 - flat (of a surface) 111.H
 - focal (of a submanifold of a Riemannian manifold) 364.C
 - frontier (of a subset) 425.N
 - functions 380.A
 - fundamental (of a projective space) 343.C
 - fundamental (with respect to a birational mapping) 16.I
 - γ -, of the k th order (of a holomorphic function) 198.C
 - generalized peak 164.D
 - generic 16.A
 - geodesic 111.H 365.D

- geometric (of a scheme) 16.D
- harmonic range of 343.D
- homoclinic 126.J
- hyperbolic (on a surface) 111.H
- hyperbolic fixed 126.G
- hyperbolic singular 126.G
- ideal (in hyperbolic geometry) 285.C
- independent (in an affine space) 7.A
- independent (in a projective space) 343.B
- at infinity (in affine geometry) 7.B
- at infinity (of a Gaussian plane) 74.D
- at infinity (in hyperbolic geometry) 285.C
- at infinity (of a Riemann manifold) 178.F
- of inflection (of a curve of class C^2) 93.G
- of inflection (of a plane algebraic curve) 9.B
- initial (of a curvilinear integral) 94.D
- initial (of a path) 170
- initial (of a position vector) 7.A
- initial (of a vector) 442.A
- integral 428.E,F
- interior 425.B
- internally irregular 338.L
- irregular (of an analytic set) 45.D
- irregular (of a Markov process) 261.D
- irregular boundary 120.D
- irregular singular 254.B
- isolated 425.O
- isolated (of a curve) 93.G
- isolated fixed 126.G
- isotropic 365.D
- k -rational (of an algebraic variety) 16.A 369.C
- left singular (of a diffusion process) 115.B
- limit (of a discontinuous group) 122.C
- limit (of a sequence of points) 87.E
- limit (of a sequence of real numbers) 87.B
- limit, type 112.I
- logarithmic branch (of a Riemann surface) 367.B
- middle (of two points of an affine space) 7.C
- movable branch (of an algebraic differential equation) 288.A
- multiple (on an arc) 93.B
- multiple (of a plane algebraic curve) 9.B
- multiple (on a variety) 16.F
- negative limit 126..D
- nodal 304.C
- nondegenerate critical 106.L 279.B
- nondegenerate critical (of a function on a Hilbert manifold) 286.N
- nonrecurrent 260.B
- nonsingular (of an algebraic variety) 16.F
- normal (of an analytic space) 23.D
- normal (of a variety) 16.F
- null recurrent 260.B
- ω -limit 126.D
- 100%- 396.C
- order of (with respect to a cycle) 99.D
- order of (in an ordinary curve) 93.C
- ordinary (of an analytic set) 23.B
- ordinary (of a curve of class C^k) 93.G
- ordinary (in hyperbolic geometry) 285.C
- ordinary (of an ordinary curve) 93.C
- ordinary (of a plane algebraic curve) 9.B
- ordinary (on a Riemann surface) 11.D
- overcrossing 235.A
- parabolic (on a surface) 110.B 111.H
- passive boundary 260.I
- peak 164.D
- positive limit 126.D
- positive recurrent 260.B
- principal 180.B
- r -ple (of a plane algebraic curve) 9.B
- ramification (of an analytic covering space) 23.E
- rational 118.E
- rational double 418.C
- recurrent (of a Markov process) 261.B
- reflection (with respect to a circle) 74.E
- regular (of an analytic set) 23.B 45.D
- regular (with respect to an analytic set) 21.M
- regular (in catastrophe theory) 51.F
- regular (of a differentiable mapping) 105.J
- regular (of a diffusion process) 115.B
- regular (with respect to the Dirichlet problem) 207.B
- regular (of a flow) 126.D
- regular (of a Hunt process) 261.D
- regular (of a polyhedron or cell complex) 65.B
- regular (of a surface in E^3) 111.J
- regular boundary 120.D
- regular singular 254.B
- rest (of a trajectory) 126.D
- right singular (of a diffusion process) 115.B
- saddle (of a function) 255.B 292.A
- saddle (on a surface) 111.H
- saddle (of a system of ordinary differential equations) 126.G
- saddle (of two-person games) 108.B
- saddle, method 25.C
- sample 342.B 396.B 398.A
- Schwinger 150.F
- semiregular (of a surface in E^3) 111.J
- simple (of an analytic set) 23.B 418.A
- simple (on a variety) 16.F
- singular (of an analytic set) 23.B 418.A
- singular (of a continuous vector field) 153.B
- singular (of a curve of class C^k) 93.G
- singular (of a linear difference equation) 104.D
- singular (of a plane algebraic curve) 9.B
- singular (of a polyhedron or cell complex) 65.B
- singular (of a quadric hypersurface) 343.E
- singular (of a surface in E^3) 111.J
- singular (of a system of linear ordinary differential equations) 254.A
- singular (of a system of ordinary differential equations) 289.A
- singular (of a trajectory) 126.D,G
- singular (on a variety) 16.F
- smooth (of variety) 16.F
- stable 16.W
- stationary (of an arc of class C^n) 111.D
- successive minimum 182.C
- supporting (of a convex set) 89.G
- supporting (of a projective frame) 343.C
- symmetric (with respect to a circle) 74.E
- terminal (of a curvilinear integral) 94.D
- terminal (of a Markov process) 261.B
- terminal (of a path) 176
- terminal (of a vector) 442.A
- transient 260.B
- transition 254.F
- transversal homoclinic 126.J
- turning 25.B 254.F
- ultrafinite (in hyperbolic geometry) 285.C
- umbilical (of a surface) 111.H 365.D
- undercrossing 235.A
- unit (of an affine frame) 7.C
- unit (of a projective frame) 343.C
- unit (of a projective space) 343.C

- w- (of an entire function) 429.B
- wandering (of a trajectory) 126.E
- Weierstrass 11.D
- zero (of a holomorphic function) 198.C
- zero (of a polynomial) 337.B 369.C
- zero (of a subset of a polynomial ring) 369.C
- zero, of the $-k$ th order (of a complex function) 198.D
- zero, of the k th order (of a holomorphic function) 198.C
- point(s) at infinity 74.D 285.C
 - regular at the 193.B
- pointed coalgebra 203.F
- pointed set 172.J
 - morphism of 172.J
- pointed shape category 382.A
- pointed topological spaces, category of 202.B
- pointer 96.B
- point estimation 399.B 401.C, App. A, Table 23
- point-finite covering (of a set) 425.R
- point function 380.A 407.D
- point group (of a crystallographic group) 92.A
- point hypersphere 76.A
- point process 407.D
- point range (in projective geometry) 343.B
 - of the number system (in projective geometry) 343.C
- point set 381.B
- points of indeterminacy, set of 23.D
- point spectrum 390.A
 - pure 136.E
- pointwise convergent sequence 435.B
- pointwise ergodic theorem 136.B
- Poisson bracket 82.B 271.F 324.C,D
 - (of two vector fields) 105.M
- Poisson differential equation 323.A, App. A, Table 15.III
- Poisson distribution 341.D 397.F, App. A, Table 22
- Poisson equation 338.A
- Poisson formula App. A, Table 19.III
 - of $T^m = E^m/\Gamma$ 391.J
- Poisson input 260.H
- Poisson integral 168.B 193.G
- Poisson integral formula 198.B
- Poisson integration formula App. A, Table 15.VI
- Poisson kernel 159.C
- Poisson number 271.G
- Poisson point process, stationary 407.D
- Poisson process 5.D
 - compound 5.F
- Poisson random measure 407.D
- Poisson ratio 271.G
- Poisson solution 325.D
- Poisson stable 126.E
 - negatively 126.E
 - positively 126.E
- Poisson summation formula 192.C
 - (of Fourier transforms) 192.C
 - (on a locally compact Abelian group) 192.I.
- polar
 - (with respect to a conic) 78.J
 - (in projective geometry) 343.E
 - (relative to pairing) 424.H
- polar coordinates 90.C
 - geodesic 90.C
 - tangential 90.C
- polar decomposition 251.E
- polar element
 - (a function element in the wider sense) 198.O
 - (of an integral element) 428.E
- polar form (of a complex number) 74.C
- polarity (with respect to a quadric hypersurface) 343.E
- polarization (on an Abelian variety) 3.G
 - electric 130.A
 - inhomogeneous 3.G
 - magnetic 130.A
 - principal 3.G
- polarized
 - (Hodge structure) 16.V
 - (wave) 446
- polarized Abelian variety 3.G
- polarized Jacobian variety, canonically 3.G 9.E
- polar plane (with respect to a quadric surface) 350.C
- polar set (in potential theory) 261.D 338.H
- polar space 191.I
- polar system (in projective geometry) 343.D
- polar tetrahedron 350.C
 - self- 350.C
- polar triangle 78.J
 - self- 78.J
- pole
 - (of a complex function) 198.D
 - (of a function on an algebraic curve) 9.C
 - (of a function on an algebraic variety) 16.M
 - (of a polar with respect to a conic) 78.J
 - (of a polar plane) 350.C
 - (of a polar of a quadric hypersurface) 158.E
 - (of a roulette) 93.H
 - north (of a complex sphere) 74.D
 - north (of a sphere) 140
 - order of 198.D
 - Regge 132.C 386.C
 - resonance 331.F
 - south (of a complex sphere) 74.D
 - south (of a sphere) 140
- pole divisor (of a function on an algebraic variety) 16.M
- policy 127.A 405.C
 - Markovian 405.C
 - optimal 127.A
- Polish space 22.I 273.J
- Pólya's enumeration theorem 66.E
- Pólya type 374.J
 - strictly of 374.J
- polychromatic group 92.D
- polydisk 21.B
- polygamma functions 174.B, App. A, Table 17.I
- polygon(s) 155.F
 - Cauchy 316.C
 - decomposition-equal 155.F
 - force 19.C
 - normal 234.C
 - plane 155.F
 - regular 357.A
 - simple 155.F
 - supplementation-equal 155.F
- polygonal number of order r 4.D
- polyharmonic 193.O
- polyhedral, convex rational 16.Z
- polyhedral angle, regular 357.B
- polyhedral cone, convex 89.F
- polyhedral group, regular 151.G
- polyhedron (polyhedra)
 - (in an affine space) 7.D
 - (of a simplicial complex) 65.A 70.C
- analytic 21.G
- convex 89.A

corner 215.C
 Euclidean 70.B
 Euler theorem on 201.F
 integer 215.C
 regular 357.B
 topological 65.A
 polymatroid 66.F
 polynomial(s) 337
 Alexander (of a knot) 235.C,D
 alternating 337.I
 associated Laguerre 317.D
 Bernoulli 177.B
 Bernshtein 336.A
 Bernshtein (generalized) 418.H
 characteristic (of a differential operator) 112.A
 321.A
 characteristic (of a linear mapping) 269.L
 characteristic (of a matrix) 269.F
 Chebyshev 317.D 336.H, App. A, Table 20.II
 Chebyshev orthogonal 19.G
 cyclotomic 14.L
 differential 113
 Euler 177.C
 Fourier-Hermite 176.I
 Galois group of the 172.G
 Gegenbauer 317.D 393.E, App. A, Table 20.I
 generalized trigonometric 18.B
 Hermite 317.D
 Hermite interpolation 223.E
 Hilbert (of an algebraic curve) 9.F
 Hilbert (of a graded R -module) 369.F
 Hilbert (of a sheaf) 16.E
 homogeneous of degree n 337.B
 Hosokawa 235.D
 inseparable 337.G
 irreducible 337.F
 isobaric 32.C
 Jacobi 317.D, App. A, Table 20.V
 Lagrange interpolation 223.A 336.G, App. A,
 Table 21
 Laguerre 317.D, App. A, Table 20.VI
 Legendre 393.B, App. A, Table 18.II
 link 235.D
 Lommel App. A, Table 19.IV
 in m variables 337.B
 minimal (of an algebraic element) 149.E
 minimal (of a linear mapping) 269.L
 minimal (of a matrix) 269.F
 monic 337.A
 Neumann App. A, Table 19.IV
 Newton interpolation 336.G
 orthogonal 19.G, App. A, Table 20.VII
 parity check 63.E
 Poincaré 201.B
 primitive 337.D
 reduced link 235.D
 reducible 337.F
 ring of 337.A 369
 ring of differential 113
 Sato-Bernshtein 125.EE
 Schläfli App. A, Table 19.IV
 separable 337.G
 simplest orthogonal 19.G
 Snapper 16.E
 Sonine 317.D, App. A, Table 20.VI
 symmetric 337.I
 system of orthogonal 317.D
 trigonometric interpolation 336.E
 ultraspherical 317.D
 zonal 374.C

Subject Index

Positive cycle (on an algebraic variety)

polynomial approximation 336
 best (in the sense of Chebyshev) 336.H
 polynomial approximation theorem (for C^∞ -
 functions) 58.E
 polynomial distribution App. A, Table 22
 negative App. A, Table 22
 polynomial extrapolation method 303.F
 polynomial identity (on an algebra) 29.J
 polynomially transformable 71.E
 polynomial representation (of $GL(V)$) 60.D
 polynomial ring 337.A 369
 of m variables 337.B
 polynomial time 71.B
 polytropic differential equation 291.F
 Pomeranchuk theorem 386.B
 Pontryagin class(es)
 (of an R^n -bundle) 56.D
 combinatorial 56.H
 of a manifold 56.F
 rational 56.F
 total 56.D
 universal 56.D
 Pontryagin duality theorem (on topological Abelian
 groups) 192.K 422.C
 Pontryagin multiplication 203.D
 Pontryagin number 56.F
 Pontryagin p th power operation 64.B
 Pontryagin product 203.D
 Popov ghost, Faddeev- 132.C 150.G
 population (in statistics) 397.B 401.E
 finite 373.A
 infinite 401.E
 population characteristic 396.C
 population correlation coefficient 396.D
 population covariance 396.D
 population distribution 396.B 401.F
 hypothetical infinite 397.P
 population kurtosis 396.C
 population mean 396.C
 population moment of order k 396.C
 population standard deviation 396.C
 population variance 396.C
 port-admittance matrix 282.C
 porter 168.C
 port-impedance matrix 282.C
 port network, M - 282.C
 portrait, phase 126.B
 position
 general (complexes) 70.B
 general (of a PL mapping) 65.D
 general (in a projective space) 343.B
 general, theorem 65.D
 hyperboloid 350.D
 method of false 301.C
 position representation 351.C
 position vector 442.A
 (of a point of an affine space) 7.A
 positive
 (chain complex) 200.C
 (class of vector bundles) 114.D
 (complex) 200.H
 (functional on a C^* -algebra) 36.G
 (function on a C^* -algebra) 308.D
 (Hermitian operation) 308.A
 (square matrix) 310.H
 completely (linear mapping between C^* -
 algebras) 36.H
 positive boundary, open Riemann surface of 367.E
 positive cone, natural 308.K
 positive cycle (on an algebraic variety) 16.M

- positive definite
 - (function) 192.B,J 394.C
 - (Hermitian form) 348.F
 - (matrix) 269.I
 - (potential) 338.D
 - (sequence) 192.B
 - (on a topological group) 36.L 437.B
- positive definite kernel 217.H
- positive definite quadratic form 348.C
- positive direction (in a curvilinear integral) 198.B
- positive distribution 125.C
- positive divisor
 - (of an algebraic curve) 9.C
 - (on a Riemann surface) 11.D
- positive element
 - (in a lattice-ordered group) 243.G
 - (of an ordered field) 149.N
 - strictly 310.H
 - totally 14.G
- positive entropy, completely 136.E
- positive half-trajectory 126.D
- positive infinity 87.D 355.C
- positive kernel 217.H
- positive limit point 126.D
- positively invariant 126.D
- positively Lagrange stable 126.E
- positively Poisson stable 126.E
- positively regular process 44.C
- positive matrix 269.N
- positive number 355.A
- positive operator (in vector lattices) 310.E
- positive orientation (of an oriented C^* -manifold) 105.F
- positive orthant 89.G
- positive part (of an element of a vector lattice) 310.B
- positive prolongational limit set, first 126.D
- positive Radon measure 270.I
- positive real function 282.C
- positive recurrent ergodic class 260.B
- positive recurrent point 260.B
- positive root (of a semisimple Lie algebra) 248.M
- positive semidefinite (operator) 251.E
- positive semidefinite kernel 217.H
- positive semidefinite matrix 269.I
- positive semidefinite quadratic form 348.C
- positive semiorbit 126.D
- positive system, symmetric 112.S 326.D
- positive terms, series of 379.B
- positive type
 - (function of) 192.B,J
 - (sequence of) 192.B
 - (symmetric kernel of) 338.D
- positive variation
 - (of a mapping) 246.H
 - (of a real bounded function) 166.B
- positive Weyl chamber 248.R
- positivity
 - O-S 150.F
 - reflection 150.F
 - T- 150.F
- possibility 411.L
- possible construction problem 179.A
- posterior density 401.B
- posterior distribution 398.B 401.B 403.G
- posterior risk 399.F
- postliminal C^* -algebra 36.H
- Postnikov complex 70.G
- Postnikov system (of a CW complex) 148.D
- Post problem 356.D
- Post theorem 356.H
- postulate(s) 35.A
 - fifth (in Euclidean geometry) 139.A
 - Nernst 419.A
 - Peano 294.B
- potency of a set 49.A
- potential 338.A
 - (of a force) 271.C
 - (of a Hamiltonian) 375.B
 - (for a lattice spin system) 402.G
 - (in a Markov chain) 260.D
 - (on a network) 281.B
 - central 351.E
 - chemical 402.D 419.B
 - of a double distribution 338.A
 - of a double layer 338.A
 - equilibrium 260.D
 - finite-band 387.E
 - finite-gap 387.E
 - logarithmic 338.A
 - Newtonian 271.C 338.A
 - of order α 338.B
 - reflectionless 387.D
 - Riesz 338.B
 - scalar 130.A 442.D
 - of a simple distribution 338.A
 - of a single layer 338.A
 - vector 130.A 442.D
 - velocity 205.B
 - Yukawa 338.M
- potential energy 271.G
- potential good reduction (of an Abelian variety) 3.N
- potential kernel, weak 260.D
- potential stable reduction (of an Abelian variety) 3.N
- potential theory 338
- power
 - (of a cardinal number) 49.C
 - (of an ordinal number) 312.C
 - (of a test) 400.A
 - of a with exponent x 131.B,G
 - fractional 378.D
 - p -fold exterior (of a linear space) 256.O
 - p -fold exterior (of a vector bundle) 147.F
 - Pontryagin (pth) operation 64.B
 - residue of the n th 14.M
 - of a set 49.A
 - Steenrod (pth) operation 64.B
- power associative algebra 231.A
- power dilation 251.M
- powerful invariant, uniformly most 399.Q
- powerful unbiased, uniformly most 399.Q
- power function 400.A
 - envelope 400.F
- power method 298.C
- power-residue symbol 14.N
- power series 21.B 339 370
 - with center at the point of infinity 339.A
 - convergent 370.B
 - field of, in one variable 370.A
 - formal 370.A
 - formal, field in one variable 370.A
 - ring of 370.A
 - ring of convergent 370.B
 - ring of formal 370.A
- power series space
 - finite type 168.B
 - infinite type 168.B
- power set

(of a set) 381.B
 axiom of 33.B 381.G
 power sum theorem 123.D
 Powers factor 308.I
 Powers-Størmer inequality 212.B
 Poynting vector 130.A
 Prandtl boundary layer equation 205.C
 Prandtl-Glauert approximation 250.B
 Prandtl-Glauert law of similarity 205.D
 Prandtl integrodifferential equation 222.C
 Prandtl number 116.B
 prealgebraic variety 16.C
 precession 392
 precompact
 (metric space) 273.B
 (set in a metric space) 273.B
 precompact uniform space 436.H
 preconditioned (in numerical solution of linear equations) 302.D
 predator relation, prey- 263.B
 predecessor (of an element in an ordered set) 311.B
 predetermined variables 128.C
 predicate 411.G
 analytic 356.H
 arithmetical 356.H
 complete 356.H
 decidable (number-theoretic) 356.C
 enumerating 356.H
 first-order 411.K
 general recursive 356.C
 hyperarithmetical 356.H
 of n -argument 411.G
 n -ary 411.G
 primitive recursive 356.B
 second-order 411.K
 predicate (object) 156.B
 predicate calculus 411.J
 with equality 411.J
 predicate logic 411.J
 with equality 411.J
 first-order 411.K
 higher-order 411.K
 second-order 411.K
 third-order 411.K
 predicate symbol 411.H
 predicate variable 411.G,H
 predictable (σ -algebra) 407.B
 prediction sufficiency 396.J
 prediction theory 395.D
 linear 395.D
 predictive distribution 403.G
 predictor 303.E
 (in a multistep method) 303.E
 linear 395.D
 Milne's 303.E
 optimal linear 395.D
 predictor-corrector (PC) method 303.E
 predual 308.D
 prefix condition code 213.D
 pre-Hilbert space 197.B
 prehomogeneous vector space 450.V
 zeta function associated with 450.V
 premium 214.A
 net 214.A
 risk 214.B
 savings 214.B
 prenex normal form (in predicate logic) 411.J
 preordering 311.H
 preparation theorem

Subject Index

Prime formula

for C^∞ -functions 58.C
 Weierstrass 21.E 370.B
 Weierstrass type (for microdifferential operators) 274.F
 presentation 235.B
 of finite (\mathcal{O} -Modules) 16.E
 Wirtinger (of a knot group) 235.B,D
 presheaf 383.A
 sheaf associated 383.C
 presheaf on a site 16.AA
 pressure 402.G 419.A
 pressure, topological 136.H
 pressure equation 205.B
 prestratification, Whitney 418.G
 preventive maintenance model 307.C
 prey-predator relation 263.B
 price
 imputed 292.C
 shadow 255.B
 primal problem 255.B
 primary Abelian group 2.A
 primary cohomology operation 64.B
 stable 64.B
 primary component
 (of an ideal) 67.F
 embedded (of an ideal) 67.F
 isolated (of an ideal) 67.F
 primary difference 305.C
 primary ideal 67.F
 \mathfrak{p} 67.F
 primary obstruction 147.L 305.C
 primary problem 255.B
 primary ring 368.H
 completely 368.H
 semi- 368.H
 primary solution (of a homogeneous partial differential equation) 320.E
 primary submodule 284.A
 prime(s)
 (3-manifold) 65.E
 over an element (in a lattice) 243.F
 under an element (in a lattice) 243.F
 Mersenne 297.E
 relatively (fractional ideals) 14.E
 relatively (numbers) 297.A
 twin 123.C
 prime differential ideal (of a differential ring) 113
 prime divisor
 (of an algebraic function field of dimension 1) 9.D
 (of an algebraic number field or an algebraic function field of one variable) 439.H
 (of an ideal) 67.F
 (on a Riemann surface) 11.D
 embedded (of an ideal) 67.F
 finite 439.H
 imaginary infinite 439.H
 infinite 439.H
 isolated (of an ideal) 67.F
 maximal (of an ideal) 67.F
 minimal (of an ideal) 67.F
 real 439.H
 real (infinite) 439.H
 prime element
 (of a ring) 67.H
 (for a valuation) 439.E
 prime field 149.B
 prime formula 411.D
 (of a language) 276.A

- prime ideal 67.C
 - (of a maximal order) 27.A
 - associated (of an ideal) 67.F
 - ramified 14.I
 - unramified 14.I
- prime ideal theorem 123.F
- prime knot 235.A
- prime number(s) 297.B
 - regular 14.L
 - relatively 297.A
- prime number theorem 123.B
 - for arithmetic progression 123.D
- prime quotient (in a lattice) 243.F
- prime rational divisor over a field (on an algebraic curve) 9.C
- prime spot (of an algebraic number field or an algebraic function field) 439.H
- primitive 427.B
 - (differential form) 232.B
 - (element of coalgebra) 203.I
 - (element of an extension of a field) 149.D
 - (generator of the cohomology algebra of a compact Lie group) 427.B
- primitive binary quadratic form 348.M
- primitive character 295.D 450.C,E
 - non- 450.C,E
- primitive equation 172.G
- primitive form 232.B
- primitive function(s) 216.C
 - derivatives and App. A, Table 9.I
- primitive hyperbolic type (reduced basis of) 92.C
- primitive hypercubic type (reduced basis of) 92.C
- primitive ideal (of a Banach algebra) 36.D
- primitive idempotent element (of a ring) 368.B
- primitive lattice 92.E
- primitive operation (of a group) 362.B
- primitive permutation group 151.H
- primitive permutation representation (of a group) 362.B
- primitive polynomial 337.D
- primitive recursive 356.B
- primitive recursive function 356.A,B,F
 - uniformly 356.B
- primitive recursive in ψ_1, \dots, ψ_l 356.B
- primitive recursive predicate 356.B
- primitive root of unity 14.L
 - modulo m 297.G
- primitive solution (of a partial differential equation) 320.E
- principal adèle (of an algebraic number field) 6.C
- principal analytic set 23.B
- principal antiautomorphism (of a Clifford algebra) 61.B
- principal automorphism (Clifford algebras) 61.B
- principal axis (axes)
 - (of a central conic) 78.C
 - (of inertia) 271.E
 - (of a parabola) 78.C
 - (of a quadric surface) 350.B
 - transformation to 390.B
- principal bundle 147.C
 - associated 147.D
 - reduced 147.J
 - reducible 147.J
- principal character 295.D
 - (of an Abelian group) 2.G
- principal component(s)
 - (in principal component analysis) 280.F
 - of order p 110.A
- principal component analysis 280.F
- principal congruence subgroup of level N 122.D
- principal convergent (of an irrational number) 83.B
- principal curvature
 - (of a surface) 111.H 365.C
 - radius of (of a surface) 111.H
- principal directions (of a surface) 111.H
- principal discrete series 258.C
- principal divisor
 - (on an algebraic curve) 9.C
 - (on a Riemann surface) 11.D
- principal fiber bundle 147.C
- principal formula of integral geometry 218.C
- principal fractional ideal 67.K
- principal genus
 - (for an ideal group) 59.E
 - of a quadratic field 347.F
- principal half-space (of a flag) 139.B
- principal H -series 437.X
- principal ideal 67.K
 - of an algebraic number field 14.E
- principal ideal domain 67.K
- principal ideal ring 67.K
- principal ideal theorem (in class field theory) 59.D
- principal idele (of an algebraic number field) 6.C
- principal isotropy group(s) 431.C
- principal matrix (belonging to a Riemann matrix) 3.I
- principal minor (of a matrix) 103.D
- principal moment of inertia 271.E
- principal normal 111.F
- principal orbit(s) 431.C
 - type 431.C
- principal order (of an algebraic number field) 14.B
 - fundamental theorem of 14.C
- principal part
 - (of a differential operator) 112.A
 - (of a Laurent expansion) 198.D
 - (of a partial differential operator) 320.B
- principal partition 66.H
- principal plane (of a quadric surface) 350.B
- principal point(s)
 - (for a Gauss mapping) 180.B
- principal polarization (of an Abelian variety) 3.G
- principal quantum number 315.E
- principal series 258.C
 - (in an Ω -group) 190.G
 - (of unitary representations of a complex semi-simple Lie group) 437.W
 - (of unitary representations of a real semisimple Lie group) 437.X
- principal solution 104.B
- principal space (of a flag) 139.B
- principal subspace (of a linear operator) 390.B
- principal symbol 237.H
 - (of a microdifferential operator) 274.F
 - (of a simple holonomic system) 274.H
- principal theorem, Ahlfors 367.B
- principal value
 - (of inverse trigonometric functions) 131.E
 - Cauchy (of an improper integral) 216.D
 - Cauchy (of the integral on infinite intervals) 216.E
 - of $\log Z$ 131.G
- principle(s)
 - argument 198.F
 - balayage 338.L
 - Bellman 405.B
 - Cartan maximum 338.L
 - complete maximum 338.M
 - of condensation of singularities 37.H

- of conditionality 401.C
- continuity 21.H
- continuity (in potential theory) 338.C
- contraction 286.B
- correspondence 351.D
- of counting constants 16.S
- Dedekind (in a modular lattice) 243.F
- of depending choice (DC) 33.F
- dilated maximum (in potential theory) 338.C
- Dirichlet 120.A 323.E
- Dirichlet drawer 182.F
- domination 338.L
- Donsker invariance 250.E
- duality (for closed convex cones) 89.F
- duality, for ordering 311.A
- of duality (in projective geometry) 343.B
- embedding (in dynamic programming) 127.B
- energy 338.D
- energy minimum 419.A
- enthalpy minimum 419.C
- entropy maximum 419.A
- of equal weight 402.E
- equilibrium 338.K
- of equivalence (in insurance mathematics) 214.A 359.D
- Fermat 180.A 441.C
- first maximum (in potential theory) 338.C
- Fisher three 102.A
- Frostman maximum 338.C
- general, of relativity 359.D
- Gibbs free energy minimum 419.C
- Hamilton 441.B
- Hasse 348.G
- Helmholtz free energy minimum 419.C
- Huygens 325.B 446
- Huygens, in the wider sense 325.D
- invariance 375.B 400.E
- of invariance of speed of light 359.B
- inverse domination 338.L
- of least action 441.B
- limiting absorption 375.C
- of linearized stability 286.S
- of localization (on convergence tests of Fourier series) 159.B
- local maximum modulus 164.C
- lower envelope 338.M
- Maupertuis 180.A
- maximal 193.E
- maximum (for control theory) 86.F
- maximum (for a holomorphic function) 43.B
- maximum (for minimal surfaces) 275.B
- maximum modulus (for a holomorphic function) 43.B
- minimax (for eigenvalues of a compact operator) 68.H
- minimax (for λ_k) 391.G
- minimax (for statistical decision problem) 398.B
- minimum (for λ) 391.D
- minimum (for λ_k) 391.G
- of nested intervals (for real numbers) 87.C 355.B
- Oka 21.K 147.O
- of optimality 127.A
- Pauli 351.H
- quasicontinuity (in potential theory) 338.I
- Rayleigh 68.H
- reflection 45.E
- of reflection 74.E
- Schwarz, of reflection 198.G
- separation 405.C
- special, of relativity 359
- stochastic maximum 405.D
- stored program 75.B
- Strassen invariance 250.E
- of sufficiency 401.C
- of superposition 252.B 322.C
- sweeping-out 338.L
- Ugaheri maximum 338.C
- uniqueness (in potential theory) 338.M
- upper boundedness (in potential theory) 338.C
- variational 441
- variational (in statistical mechanics) 340.B 402.G
- variational (in the theory of elasticity) 271.G
- variational, with relaxed continuity requirement 271.G
- variational, for topological pressure 136.H
- Pringsheim theorem 58.E
- prior density 401.B
- prior distribution 401.B 403.G
- probabilistic model 397.P
- probability 342
 - additivity of 342.B
 - a posteriori 342.F
 - a priori 342.F
 - binomial, paper 19.B
 - conditional 342.E
 - continuous in 407.A
 - converge in 342.D
 - converge with, 1 342.D
 - critical percolation 340.D
 - error 213.D
 - of an event 342.B
 - that event ε occurs 342.B
 - event with, 1 342.B
 - extinction 44.B
 - geometric 218.A
 - hitting, for single points 5.G
 - objective 401.B
 - regular conditional 342.E
 - ruin 214.C
 - standard transition 260.F
 - subjective 401.B
 - theory of 342.A
 - transition 260.A 261.A 351.B
- probability amplitude 351.D
- probability density 341.D
- probability distribution(s) 342.B, App. A, Table 22
 - (one-dimensional, of random variable) 342.C
 - (of random variables) 342.C
 - conditional 342.E
 - n -dimensional 342.C
- probability generating function 341.F 397.G
- probability integral App. A, Table 19.II
- probability measure 341 342.B
- probability of loss 307.C
- probability paper 19.F
 - binomial 19.B
- probability ratio test, sequential 400.L
- probability space 342.B
- probable cause, most 401.E
- probable value, most 401.E
- problem(s)
 - Abel 217.L
 - abstract Cauchy 286.X
 - acoustic 325.L
 - adjoint boundary value 315.B
 - all-integer programming 215.A
 - Appolonius (in geometric construction) 179.A

- Behrens-Fisher 400.G
- Bernshtein, generalized 275.F
- boundary value (of ordinary differential equations) 303.H 315.A
- Burnside (in group theory) 161.C
- Cauchy (for ordinary differential equations) 316.A
- Cauchy (for partial differential equations) 320.B 321.A 325.B
- class field tower 59.F
- combinatorial App. A, Table 17.II
- combinatorial triangulation 65.C
- concave programming 292.A
- conditional, in the calculus of variations 46.A
- connection 253.A
- construction 59.F
- convex programming 292.A
- corona 43.G
- correctly posed (for partial differential equations) 322.A
- Cousin, first 21.K
- Cousin, second 21.K
- Cramer-Castillon (in geometric construction) 179.A
- critical inclination 55.C
- decision 71.B 97 186.J
- Delos (in geometric construction) 179.A
- Dido 228.A
- differentiable pinching 178.E
- Dirichlet 120 193.F 323.C
- Dirichlet, with obstacle 440.B
- Dirichlet divisor 242.A
- dual 255.B 349.B
- du Bois Reymond 159.H
- eigenvalue 390.A
- exterior (Dirichlet problem) 120.A
- first boundary value 193.F 323.C
- flow-shop scheduling 376
- four-color 157
- Gauss circle 242.A
- Gauss variational 338.J
- general boundary value 323.H
- generalized eigenvalue 298.G
- generalized isoperimetric 46.A 228.A
- generalized Pfaff 428.B
- Geöcze 246.D
- geometric construction 179.A
- Goldbach 4.C
- group-minimization 215.C
- Hamburger moment 240.K
- Hausdorff moment 240.K
- Hersch 391.E
- Hilbert (in calculus of variations) 46.A
- Hilbert fifth 423.N
- homeomorphism 425.G
- homogeneous boundary value (of ordinary differential equations) 315.B
- Hukuhara 315.C
- of identification (in econometrics) 128.C
- impossible construction 179.A
- inconsistent (of geometric construction) 179.A
- inhomogeneous boundary value (of ordinary differential equations) 315.B
- initial value (for functional differential equations) 163.D
- initial value (of ordinary differential equations) 313.C 316.A
- initial value (for partial differential equations) 321.A
- initial value, for a hyperbolic partial differential equation App. A, Table 15.III
- interior (Dirichlet problem) 120.A
- interpolation 43.F
- invariant measure 136.C
- inverse (in potential scattering) 375.G
- isomorphism (for graphs) 186.J
- isomorphism (for integral group algebra) 362.K
- isoperimetric 111.E 228.A
- Jacobi inverse 3.L
- job-shop scheduling problem 376
- k-sample 371.D
- Lagrange (in calculus of variations) 46.A
- LBA 31.D
- Levi 21.I
- linear least squares 302.E
- linear programming 255.A
- local (on the solutions of differential equations) 289.A
- machine scheduling 376
- machine sequencing 376
- Malfatti (in geometric construction) 179.A
- many-body 402.F 420.A
- martingale 115.C 261.C 406.A
- maximum flow 281.C
- minimum-cost flow 281.C
- mixed integer programming 215.A
- multicommodity flow 281.C
- multiprocessor scheduling 376
- n-body 420.A
- n-decision 398.A
- network-flow 281 282.B
- Neumann (for harmonic functions) 193.F
- Neumann (for partial differential equations of elliptic type) 323.F
- nonlinear 291
- normal Moore space 425.AA
- optimal regulator 86.F
- penalized 440.B
- Pfaff 428.A
- placement 235.A
- Plateau 334.A
- possible construction 179.A
- primal 255.B
- primary 255.B
- properly posed 322.A
- pure integer programming 215.A
- quadratic programming 292.A 349.A
- random walk 260.A
- representation (on surface) 246.I
- restricted Burnside (in group theory) 161.C
- restricted three-body 420.F
- Riemann 253.D
- Riemann-Hilbert (for integral equations) 217.J
- Riemann-Hilbert (for ordinary differential equations) 253.D
- Robin 323.F
- of satisfiability (of a proposition) 97
- Schöenflies 65.G
- second boundary value (for harmonic functions) 193.F
- second boundary value (for partial differential equations of elliptic type) 323.F
- second Cousin 21.K
- self-adjoint boundary value 315.B
- sequential decision 398.F
- shortest path 281.C
- single-commodity flow 281

- singular initial value (for partial differential equations of mixed type) 326.C
- smoothing 114.C
- special isoperimetric 228.A
- of specification 397.P
- statistical decision 398.A
- Steiner (in geometric construction) 179.A
- Stieltjes moment 240.K
- Sturm-Liouville 315.B
- third boundary value (for harmonic functions) 193.F
- third boundary value (for partial differential equations of elliptic type) 323.F
- three big 187
- three-body 420.A
- Thues (general) 31.B
- time optimal control 86.F
- transformation (in a finitely presented group) 161.B
- transient 322.D
- transportation 255.C
- transportation, on a network 255.C
- Tricomi 326.C
- two-body 55.A
- two-point boundary value (of ordinary differential equations) 315.A
- two-terminal 281
- type (for Riemann surfaces) 367.D
- of universal validity of a proposition 97
- Waring 4.E
- weak form of the boundary value 304.B
- well-posed (in general case) 322.A
- word (in a finitely presented group) 161.B
- 0-1 integer programming 215.A
- procedure
 - classification 280.I
 - exploratory 397.Q
 - Lyapunov-Schmidt 286.V
 - random sampling 373.A
 - sampling 373.A
 - shortest-path 281.C
 - statistical decision 398.A
- process
 - (in catastrophe theory) 51.F
 - (on a measure space) 136.E
 - (= stochastic process) 407.A
 - additive 5 342.A
 - age-dependent branching 44.E
 - asymmetric Cauchy 5.F
 - autoregressive 421.D
 - autoregressive integrated moving average 421.G
 - autoregressive moving average 421.D
 - Bernoulli 136.E
 - Bernoulli, very weak 136.E
 - Bernoulli, weak 136.E
 - birth 260.G
 - birth and death 260.G
 - bond percolation 340.D
 - branching 44 342.A
 - branching Markov 44.E
 - Cauchy 5.F
 - centered 5.B
 - compound Poisson 5.F
 - contact 340.C
 - continuous-state branching 44.E
 - death 260.G
 - diffusion 115
 - dual 261.F
 - exponent of the stable 5.F
 - Feller 261.B
 - finitely determined (F.D.) 136.E
 - Galton-Watson branching 44.B
 - Gaussian 176 342.A
 - Gaussian, complex 176.C
 - generalized stochastic 407.C
 - homogeneous Markov 5.H
 - Hunt 261.B
 - increasing 262.D
 - independent 136.E
 - with independent increments 5.B
 - integrable, of bounded variation 406.B
 - integrable increasing 406.B
 - invariant Markov 5.H
 - irreversible 402.A
 - isothermal 419.B
 - Itô 406.B
 - Lévy 5.B
 - linear stationary iterative 302.C
 - Markov 261 342.A
 - Markov branching 44.D
 - Markovian decision 127.E
 - moving average 421.D
 - multistage allocation 127.A
 - multistage choice 127.A
 - multitype Galton-Watson 44.C
 - multitype Markov branching 44.E
 - Newton iterative 301.D
 - normal 176.C
 - observation 405.F
 - one-sided stable, of the exponent α 5.F
 - oscillator 315.F
 - osculating 77.B
 - percolation 340.D
 - point 407.D
 - Poisson 5.D
 - positive regular 44.C
 - progressive 407.B
 - quadratic variation 406.B
 - quasistatic adiabatic 419.B
 - recurrent 261.B
 - reversed 261.F
 - sample 407.A
 - shift associated with the stationary 136.D
 - σ - (of a complex manifold) 72.H
 - signal 405.F
 - site percolation 340.D
 - spatially homogeneous 261.A
 - stable 5.F
 - stationary 342.A 395.A
 - stationary Gaussian 176.C
 - strictly stable 5.F
 - strictly stationary 395.A
 - stochastic 342.A 407
 - stochastic, with stationary increments of order n 395.I
 - strongly stationary 395.A, F
 - strong Markov 261.B
 - subadditive 136.B
 - sweeping-out 338.L
 - symmetric Cauchy 5.F
 - symmetric stable 5.F
 - system 405.F
 - temporally homogeneous 261.A
 - temporally homogeneous additive 5.B
 - transient 261.B
 - very weak Bernoulli (V.W.B.) 136.E
 - weak Bernoulli (W.B.) 136.E

- weakly stationary 395.A
- weakly stationary, of degree k 395.I
- Wiener 5.D 45.B
- processing, data 96
- processor, central 75.B
- producer's risk 404.C
- product(s)
 - (of algebraic varieties) 16.A
 - (of cardinal numbers) 49.C
 - (of completely additive classes) 270.H
 - (of elements of a graded algebra) 203.B
 - (of elements of a group) 190.A
 - (of hyperfunctions) 125.X 274.E
 - (of ideals) 67.B
 - (of knots) 235.A
 - (of linear operators) 37.C 251.B
 - (of matrices) 269.B
 - (of objects) 52.E
 - (of ordinal numbers) 312.C
 - (of paths) 170
 - (in quadrangular set of six points) 343.C
 - (of real numbers) 355.B
 - (of sets) 381.B
 - (of tensors) 256.K
 - amalgamated (of a family of groups) 190.M
 - Blaschke 43.F
 - bracket (in a Lie algebra) 248.A
 - cap (in (co)homology groups of a space) 201.K
 - cap (in homological algebra) 200.K 201.K
 - cardinal (of a family of ordered sets) 311.F
 - Cartesian (of a family of sets) 381.E
 - Cartesian (of mappings) 381.C
 - Cartesian (of ordered simplicial complexes) 70.C
 - Cartesian (of s.s. complexes) 70.E
 - Cartesian (of sets) 381.B
 - Cauchy (of series) 379.F
 - complex (of cell complexes) 70.D
 - cross- (in cohomology groups of a space) 201.I
 - cross- (in homology groups of a space) 201.J
 - cross- (of vector bundles) 237.C
 - crossed (of a C^* -algebra) 36.I
 - crossed (of a commutative ring and a group) 29.D
 - crossed (in von Neumann algebra theory) 308.I
 - cup (of cohomology classes) 201.I
 - cup (in K -theory) 237.C
 - cup, reduction theorem 200.M
 - cup of derived functors 200.K
 - difference 337.I
 - direct \rightarrow direct product
 - divergent infinite 374.F
 - exterior (of differential forms) 105.Q
 - exterior (of elements of a linear space) 256.O
 - exterior (of a p -vector and q -vector) 256.O
 - exterior (of two vectors) 442.C
 - external (of derived functors) 200.K
 - Euler 450.V
 - fiber, over S 52.G
 - free (of groups) 190.M
 - Hermitian inner 256.Q
 - of ideals of a commutative ring 67.B
 - of inertia 271.E
 - infinite 379.G
 - infinite App. A, Table 10.VI
 - inner (in a Hermitian linear space) 256.Q
 - inner (in a Hilbert space) 197.B
 - inner (in a metric vector space) 256.H
 - inner (for a pairing) 424.G
 - inner (with respect to a linear space and its dual space) 256.G
 - inner (of two hyperspheres) 76.A
 - inner (of two n -tuples) 256.A
 - inner (of two vectors) 442.B
 - inner, space 442.B
 - internal (of derived functors) 200.K
 - interior (of a differential form with a vector field) 105.Q
 - intersection (of homology classes) 201.O
 - intersection (of two subvarieties) 16.Q
 - Kronecker (of matrices) 269.C
 - logical (of propositions) 411.B
 - ordinal (of a family of ordered sets) 311.G
 - partial 379.G
 - Pontryagin 203.D
 - projective C^* -tensor 36.H
 - proper (of two normal g -lattices) 27.A
 - restricted direct 6.B
 - Riemannian (of Riemannian manifolds) 364.A
 - scalar (of linear operators) 37.C
 - scalar (of two vectors) App. A, Table 3.I
 - scalar triple (of three vectors) 442.C
 - skew (of measurable transformations) 136.D
 - slant 201.K
 - smash 202.F
 - spatial tensor 36.H
 - sum of 216.A
 - symmetric (of a topological space) 70.F
 - tensor (of A -homomorphisms) 277.J
 - tensor (of algebras) 29.A
 - tensor (of A -modules) 277.J
 - tensor (of chain complexes) 201.J
 - tensor (of cochain complexes) 201.J
 - tensor (of distributions) 125.K
 - tensor (of Hilbert spaces) 308.C
 - tensor (of linear mappings) 256.I
 - tensor (of linear representations) 362.C
 - tensor (of linear spaces) 256.I
 - tensor (of locally convex spaces) 424.R
 - tensor (of sheaves) 383.I
 - tensor (of vector bundles) 147.F
 - tensor (of von Neumann algebras) 308.C
 - torsion (in a category) 200.K
 - torsion (of two A -modules) 200.D,K
 - vector 442.C, App. A, Table 3.I
 - vector triple 442.C, App. A, Table 3.I
 - wedge (of derived functors) 200.K
 - Weierstrass canonical 429.B
 - Whitehead 202.P
 - product algebraic variety 16.A
 - product bundle 147.E
 - product category 52.B
 - product complex 200.H 350.D
 - product decomposition, dual direct 422.H
 - product double chain complex 200.E
 - product event 342.B
 - product formula
 - (for the Hilbert norm-residue symbol) 14.R
 - (on invariant Haar measures) 225.E
 - (for the norm-residue symbol) 14.Q
 - (on valuations) 439.H
 - Trotter 315.F
 - production planning 376
 - production rule 31.B
 - product mapping 425.K
 - product measure 270.H
 - product measure space 270.H

- complete 270.H
- product metric space 273.B
- product of inertia 271.E
- product rule 299.D
- product space 425.L
 - reduced 202.Q
- product theorem for dimension 117.C
- product topological space 425.L
- product topology 425.L
- product uniformity 436.E
- product uniform space 436.E
- profinite groups 210.C
- program 75.C
 - machine-language 75.C
- program evaluation and review technique 376
- programming 75.C 385.B
 - bilinear 364.D
 - chance-constrained 408.A
 - convex 264.C
 - disjunctive 264.C
 - dynamic 127.A 264.C
 - fractional 264.C
 - geometric 264.D
 - integer 264.C
 - linear 264.C
 - linear mathematical 264.D
 - mathematical 264.A
 - multiobjective 264.C
 - multistage 264.C
 - network 264.C
 - nonconvex 264.D
 - parametric 264.C
 - stochastic 264.C 408.A
 - stochastic, model 307.C
 - two-stage linear, under uncertainty 255.F
 - two-stage stochastic 408.A
- programming problem
 - all-integer 215.A
 - concave 292.A
 - convex 292.A
 - linear 255.A
 - mathematical 264.B
 - mixed integer 215.A
 - nonlinear 264.C
 - pure-integer 215.A
 - quadratic 292.A 349.A
 - 0-1 integer 215.A
- progression
 - arithmetic 379.I, App. A, Table 10.I
 - geometric 379.I, App. A, Table 10.I
- progressive (set) 407.B
- progressively measurable (stochastic process) 407.B
- progressive process 407.B
- projecting (in a projective space) 343.B
- projection
 - (of a covering space) 367.B
 - (from a direct product set) 381.E
 - (of a fiber bundle) 147.B
 - (of a fiber space) 148.B
 - (of a Hilbert space) 197.E
 - (onto a homogeneous space) 199.A
 - (in a projective space) 343.B
 - (to a quotient set defined by an equivalence relation) 135.B
 - (on a tangent bundle of a Banach manifold) 286.K
 - canonical (on modules) 277.F
 - canonical (onto a quotient set) 135.B
 - center of (in projective geometry) 343.B
 - Lie minimal 76.B
 - method of orthogonal (of H. Weyl) 323.G
 - orthogonal 139.E, G
 - orthogonal (on a Hilbert space) 197.E
 - parallel (in an affine space) 7.C
 - regular knot 235.A
 - relaxation with 440.E
 - stereographic 74.D
 - unramified (of a covering surface) 367.B
- projection matrix 269.I
- projection method, Rosen's gradient 292.E
- projection operator (in a Hilbert space) 197.E
- projective
 - (Banach space) 37.M
 - (object in an Abelian category) 200.I
- projective (object) 200.I
- projective algebraic variety 16.A
 - fundamental theorems of 72.F
 - quasi- 16.C
- projective A -module 277.K
- projective approximation method 304.B
- projective C^* -tensor product 36.H
- projective class 200.Q
- projective class group 200.K
- projective collineation 343.D
 - in the wider sense 343.D
- projective connection 80.O
- projective coordinates 343.C
- projective coordinate system 343.C
- projective curvature tensor App. A, Table 4.II
- projective deformation (between surfaces) 110.B
- projective determinacy 22.H
- projective differential geometry 110.B
- projective dimension (of a module) 200.K
- projective frame (in projective geometry) 343.C
- projective general linear group 60.B
 - of degree n over K 60.B
- projective geometry 343
 - finite-dimensional 343.B
 - fundamental theorem of 343.D
 - general 343.B
 - of paths 109
- projective limit
 - (in a category) 210.D
 - (of a projective system of sets) 210.B
 - (of a projective system of topological groups) 423.K
- projective limit group 210.C
- projective limit space 210.C
- projective line 343.B
- projective line element 110.B
- projectively flat space App. A, Table 4.II
- projectively related (fundamental figures) 343.B
- projective mapping (in projective geometry) 343.B
- projective module, (R, S) - 200.K
- projective morphism 16.E
 - quasi- 16.E
- projective plane 343.B
 - Cayley 54
 - finite 241.B
- projective representation
 - (of a group) 362.J
 - irreducible 362.J
 - similar 362.J
- projective resolution
 - (in an Abelian category) 200.I
 - left (of an A -module) 200.C
 - \mathfrak{P} - 200.Q
- projective scheme 16.E
 - quasi- 16.E
- projective set of class n 22.D

- projective space 343.B
 - complex 343.D
 - infinite-dimensional complex 56.C
 - infinite-dimensional real 56.B
 - over Λ 147.E
 - left 343.F
 - real 343.D
 - right 343.F
- projective special linear group 60.B
 - (over a noncommutative field) 60.O
- projective special unitary group over K 60.H
- projective symplectic group over K 60.L
- projective system
 - (in a category) 210.D
 - (of groups) 210.C
 - (of sets) 210.B
 - (of topological groups) 423.K
 - (of topological spaces) 210.C
- projective topology 424.R
- projective transformation 343.D 364.F
 - group of 343.D
 - regular 343.D
 - singular 343.D
 - singular, of the k th species 343.D
- projective transformation group 343.D
- projective unitary group 60.F
- projective variety 16.A
- prolate App. A, Table 3.V
- proliferation (of errors) 138.D
- prolongable (Riemann surface) 367.F
- prolongation
 - (along a curve in a covering surface) 367.B
 - (of a Riemann surface) 367.F
 - (of a solution of an ordinary differential equation) 316.C
 - (of a system of partial differential equations) 428.B,F
 - (of a valuation) 439.B
 - analytic 198.G
 - first (of P) 191.E
 - k th (of G) 191.D
 - k th (of a Lie subalgebra) 191.D
 - k th (of P) 191.E
- prolongational limit set
 - first negative 126.D
 - first positive 126.D
- proof, consistency 156.D
 - for pure number theory 156.E
- proof theory 156.D
- propagation
 - of chaos 340.F
 - equation of sound 325.A
 - of errors 138.C
 - of singularities 325.M
 - wave 446
- proper
 - (continuous mapping) 425.W
 - (equivalence relation in an analytic space) 23.E
 - (leaf) 154.D
 - (Lorentz group) 258.A
 - (morphism of scheme) 16.D
 - (PL embedding) 65.D
- proper affine transformation 7.E
- proper class (in set theory) 381.G
- proper complex Lorentz group 258.A
- proper component (of an intersection of subvarieties) 16.G
- proper convex function 88.D
- proper factor (of an element of a ring) 67.H
- proper flag manifold 199.B
- proper function (of a boundary value problem) 315.B
- proper hypersphere (in hyperbolic geometry) 285.C
- proper Lorentz group 60.J
- properly convex (subset of a sphere) 274.E
- properly discontinuous transformation group 122.A
- properly divergent 379.A
- properly equivalent (binary quadratic forms) 348.M
- properly infinite 308.E
- properly intersect (on a variety) 16.G
- properly ($n - 1$)-dimensional quadric hypersurface 350.G
- properly posed
 - (initial value problem) 321.E
 - (problems for partial differential equations) 322.A
- proper mapping(s) 425.W
 - fundamental theorem of 16.X
- proper meromorphic mapping (between analytic spaces) 23.D
- proper modification (of an analytic space) 23.D
- proper motion
 - in Euclidean geometry 139.B
 - of a star 392
- proper orthogonal group 60.I 258.A
- proper orthogonal matrix 269.J
- proper product (of two normal g -lattices) 27.A
- proper quadric surface 350.B
- proper rotation group 258.A
- proper subset 381.A
- proper time 258.A
- proper transform (of a subvariety) 16.I
- property (properties) 411.G
 - approximation 37.L
 - asymptotic (of solutions of a system of linear ordinary differential equations) 314.A
 - Baire 425.N
 - basic (of a structure) 409.B
 - bounded approximation 37.L
 - clustering 402.G
 - combinatorial 65.A
 - continuity, for Čech theory 201.M
 - of continuity (in a continuous geometry) 85.A
 - countably productive 425.Y
 - covering homotopy 148.B
 - duality (of linear spaces) 256.G
 - equivalence 135.A
 - finite intersection 425.S
 - finite subset 396.F
 - global (in differential geometry) 109
 - homotopy extension 202.E
 - in the large (in differential geometry) 109
 - local (in differential geometry) 109
 - local (of a pseudodifferential operator) 345.A
 - Markov 261.B
 - micro-pseudolocal (of a pseudodifferential operator) 345.A
 - minimum curvature 223.F
 - minimum norm 223.F
 - P conjecture 235.B
 - pseudolocal (of a pseudodifferential operator) 345.A
 - pseudo-orbit tracing 126.J
 - reproducing (of a probability distribution) 341.E, App. A, Table 22
 - in the small (in differential geometry) 109

spectral 136.E
 star-finite 425.S
 strong Markov 261.B
 topological 425.G
 uniformity 399.N
 universal mapping 52.L
 proper value
 (of a boundary value problem) 315.B
 (of a linear mapping) 269.L
 (of a linear operator) 390.A
 (of a matrix) 269.F
 proper variation 279.F
 proper vector
 (belonging to an eigenvalue) 269.F
 (of a linear operator) 390.A
 (of a linear transformation) 269.L
 proposition(s)
 existential 411.B
 modal 411.L
 universal 411.B
 variables 411.E
 propositional calculus 411.F
 propositional connectives 411.E
 propositional function 411.C
 propositional logic 411.E
 provable (formula) 411.I
 proximity function (of a meromorphic function)
 272.B
 Prüfer ring 200.K
 pseudoanalytic function, K - 352.B
 pseudo-arc 79.D
 pseudocompact (space) 425.S
 pseudoconformal geometry 344.A
 pseudoconformally equivalent 344.A
 pseudoconformal transformation 344.A
 pseudoconvex (domain) 21.G
 Cartan 21.I
 d - 21.G
 Levi 21.I
 locally Cartan 21.I
 locally Levi 21.I
 strictly 344.A
 strongly 21.G
 pseudodifferential operator 251.O 274.F 345
 pseudodistance
 Carathéodory 21.O
 Kobayashi 21.O
 pseudodistance function 273.B
 pseudofunction 125.C
 pseudogeometric ring 284.F
 pseudogroup (of topological transformations)
 105.Y
 of transformations (on a topological space)
 90.D
 pseudogroup structure 105.Y
 pseudo-Hermitian manifold 344.F
 pseudointerior 382.B
 pseudo-isotopic 65.D
 pseudo-isotopy 65.D
 pseudolocal property (of a pseudodifferential opera-
 tor) 345.A
 micro- 345.A
 pseudomanifold 65.B
 pseudometric 273.B
 pseudometric space 273.B
 indiscrete 273.B
 pseudometric uniformity 436.F
 pseudometrizable 436.F
 pseudonorm (on a topological linear space) 37.O
 424.F

Subject Index

q -expansion formula

pseudo-orbit 126.J
 α - 126.J
 tracing property 126.J
 pseudo-ordering 311.H
 pseudopolynomial, distinguished 21.E
 pseudorandom numbers 354.B
 pseudo-Riemannian metric 105.P
 pseudo-Runge-Kutta method 303.D
 pseudosphere 111.I 285.E
 pseudotensorial form 80.G
 pseudovaluation 439.K
 ψ -collective 354.E
 psi function 174.B
 psychometrics 346
 Puiseux series 339.A
 pullback
 (of a differential form) 105.Q
 (of a distribution) 125.Q
 (of a divisor) 16.M
 Puppe exact sequence 202.G
 pure
 (continued fraction) 83.C
 (differential form) 367.H
 (state) 351.B
 pure geometry 181
 pure ideal 284.D
 pure integer programming problem 215.A
 purely contractive 251.N
 purely contractive part 251.N
 purely d -dimensional analytic set 23.B
 (at a point) 23.B
 purely discontinuous distribution 341.D
 purely imaginary number 74.A
 purely infinite (von Neumann algebra) 308.E
 purely inseparable
 (extension of a field) 149.H
 (rational mapping) 16.I
 purely inseparable element (of a field) 149.H
 purely n -codimensional 125.W
 purely nondeterministic 395.D
 purely transcendental extension 149.K
 pure number theory 156.E
 pure periodic continued fraction 83.C
 pure phase 402.G
 pure point spectrum 136.E
 pure strategy 173.B
 pursuit, curve of 93.H
 push-down automaton 31.D
 push-down storage 96.E
 Putnam's theorem 251.K
 Pyatetskii-Shapiro reciprocity law, Gel'fand-
 437.DD
 Pythagorean closure (of a field) 155.C
 Pythagorean extension (of a field) 155.C
 Pythagorean field 139.B 155.C
 Pythagorean number 145
 Pythagorean ordered field 60.O
 Pythagorean theorem 139.D

Q
 Q (rational numbers) 294.A,D
 q -block bundle 147.Q
 q -block structure 147.Q
 q -boundary 201.B
 q -chains 201.B
 q -cochains, singular 201.H
 q -cycle 201.B
 q -dimensional homology classes 201.B
 q -expansion formula 134.I

q-face

- q*-face 70.B
- q*-function
 - Chebyshev 19.G, App. A, Table 20.VII
 - simplest Chebyshev 19.G
- q*-numbers 130.A
- q*-representation 351.C
- q*-simplex
 - oriented 201.C
 - singular 201.E
 - standard 201.E
- q*th homology group 201.B
- Q*-manifold 382.D
- Q*-matrix 260.F
- Q*-spaces 425.BB
- QR* method 298.E
- QZ* method 298.C
- QCD (= quantum chromodynamics) 132.D
- q.e. (= quasi-everywhere) 338.F
- QFD (= quantum flavor dynamics) 132.D
- quadrangles 155.F
 - complete 343.C
- quadrangular set of six points 343.C
- quadrant, first (of a spectral sequence) 200.J
- quadratic differential (on a Riemann surface) 11.D
- quadratic equation 10.D, App. A, Table 1
- quadratic field(s) 347
 - complex 347.A
 - imaginary 347.A
 - real 347.A
- quadratic form(s) 348
 - (on a linear space) 256.H
 - bilinear form associated with 256.H
 - binary 348.M
 - complex 348.A,B
 - definite 348.C
 - equivalent 348.A
 - indefinite 348.C
 - matrix of 348.A
 - negative definite 348.C
 - negative semidefinite 348.C
 - nondegenerate 348.A
 - positive definite 348.C
 - positive semidefinite 348.C
 - primitive binary 348.M
 - properly equivalent binary 348.M
 - real 348.A,C
 - reduced 348.I
 - Siegel zeta function of indefinite 450.K
- quadratic irrational number, irreducible 83.C
- quadratic loss function 398.A 399.E
- quadratic nonresidue 297.H
- quadratic programming 349
 - nonconvex 264.D
- quadratic programming problem 292.A 349.A
- quadratic reciprocity of Jacobi symbol, law of 297.I
- quadratic reciprocity of Legendre symbol, law of 297.I
- quadratic residue 297.H
- quadratic transformation 16.I
 - locally 16.K
 - locally (of an algebraic surface) 15.G
 - locally (of a complex manifold) 72.H
- quadratrix 187
- quadrature 107.A, App. A, Table 15.III
 - of a circle 179.A
 - method of 313.D
 - solution by App. A, Table 14.I
 - spectral density 397.N
- quadric(s) 350.A
 - confocal, family of 350.E
 - Darboux 110.B
 - pencil of 343.E
- quadric cone 350.B,G
- quadric conical hypersurface 350.G
- quadric conical surface 350.B
- quadric cylindrical hypersurface 350.G
- quadric hypersurface 350.G,I
 - (in a projective space) 343.D 350.I
 - central 350.G
 - elliptic 350.G
 - of the *h*th species, singular (in a projective space) 343.E
 - hyperbolic 350.G
 - noncentral 350.G
 - parabolic 350.G
 - pencil of 343.E
 - properly ($n - 1$)-dimensional 350.G
 - regular (in a projective space) 343.E
- quadric surface(s) 350
 - canonical form of the equation of 350.B
 - central 350.B
 - degenerate 350.B
 - proper 350.B
- quadrivium 187
- qualification
 - Gaugnard's constraint 292.B
 - Slater's constraint 292.B
- qualitative (data) 397.A
- quality 404.A
- quality characteristic 404.A
- quality control 404.A
 - statistical 404.A
- quantifier 411.C
 - bounded 356.B
 - existential 411.C
 - Hilbert's ε - 411.J
 - universal 411.C
- quantile
 - α - 396.C
 - of order p 341.H
 - restricted 33.B
- quantitative (data) 397.A
- quantity (quantities)
 - analog 138.B
 - digital 138.B
 - first fundamental (of a surface) 111.H
 - second fundamental (of a surface) 111.H
 - thermodynamical 419.A
- quantization 351.D
 - second 377.B
- quantized contact transformation 274.F
- quantum chromodynamics (QCD) 132.C,D
- quantum electrodynamics 132.C
- quantum field theory 132.C 150.C
- quantum flavor dynamics (QFD) 132.D
- quantum logic 351.L
- quantum mechanics 351
- quantum number
 - azimuthal 351.E
 - orbital magnetic 351.E
 - principal 351.E
- quantum statistical mechanics 402.A
- quartic equation 10.D, App. A, Table 1
- quartile(s) 396.C
 - first 396.C
 - third 396.C
- quasi-affine (algebraic variety) 16.C
- quasi-algebraically closed field 118.F
- quasi-analytic function 58.F

family of 58.A
 in the generalized sense 58.F
 set of 58.F
 quasi-Banach space 37.O
 quasibarreled (locally convex space) 424.I
 quasibounded harmonic function 193.G
 quasicomplete (locally convex space) 424.F
 quasiconformal mappings 352
 extremal 352.C
 quasiconformal reflection, theorem of 352.C
 quasicontinuity principle (in potential theory) 338.I
 quasicontinuous function 338.I
 quasidiscrete spectrum 136.E
 quasidual space (of a locally compact group) 437.I
 quasi-equivalent unitary representation 437.C
 quasi-everywhere (in potential theory) 338.F
 quasi-Frobenius algebra 29.H
 quasi-Fuchsian group 234.B
 quasigroup 190.P 241.C
 quasi-independent of path (a response probability) 346.G
 quasi-invariant measure 225.J
 quasi-inverse (in a Banach algebra) 36.C
 quasi-inverse element (of an element of a ring) 368.B
 quasi-invertible element (of a ring) 368.B
 quasilinear
 (operator) 224.E
 (partial differential equation) 320.A 323.D 326.A
 quasilocal ring 284.D
 quasinilpotent (operator) 251.F
 quasinorm (of a vector) 37.O
 quasinormal family (of analytic functions) 435.E
 quasinormed linear space 37.O
 quasi-perfect mapping 425.CC
 quasiperiodic (translational flow) 126.L
 quasiperiodic motion 136.G 404.F
 quasiperiodic solution (of Hill's differential equation) 268.B
 quasiprojective algebraic variety 16.C
 quasiprojective morphism 16.E
 quasiprojective scheme 16.E
 quasi-*p*-valent 438.E
 quasiregular element (of a ring) 368.B
 quasiregular function, *K*- 352.B
 quasisemilocal ring 284.D
 quasisimple ring 368.E
 quasisplit algebraic group, *k*- 13.O
 quasistable distribution 341.G
 quasistatic adiabatic process 419.B
 quasistationary electric circuit 130.B
 quasisymmetric 384.E
 quasivariational inequalities 440.D
 quaternion 29.B
 quaternion algebra 29.D
 generalized 29.D
 Hamilton 29.B
 total definite 27.D
 quaternion field 29.B
 quaternion group 151.B
 generalized 151.B
 quaternion hyperbolic space 412.G
 quaternion unimodular group 412.G
 quaternion vector bundle 147.F
 query 96.F
 questions 351.L
 queue 96.E
 length 260.H
 queuing model 260.H 307.C

queuing theory 260.H 307.C
 quotient(s)
 (of an ideal and a subset of a commutative ring) 67.B
 (in a lattice) 243.F
 (of numbers) 297.A
 (of an ordered set) 311.B
 difference 104.A
 differential (at a point) 106.A
 field of 67.G
 geometric 16.W
 group of (of a commutative semigroup) 190.P
 Herbrand 200.N
 integral (in the division algorithm of polynomials) 337.C
 module of, of an *R*-module with respect to *S* 67.G
 prime (in a lattice) 243.F
 Rayleigh 68.H 304.B
 ring of, of a ring with respect to a prime ideal 67.G
 ring of, of a ring with respect to *S* 67.G
 ring of total 67.G
 quotient bialgebra 203.G
 quotient bundle 147.F
 (of a vector bundle on an algebraic variety) 16.Y
 quotient category 52.N
 quotient chain complex 200.C
 quotient coalgebra 203.F
 quotient complex 201.L
 quotient group
 (of a group) 190.C
 (of a topological group) 423.E
 quotient *G*-set 362.B
 quotient lattice 243.C
 quotient Lie algebra 248.A
 quotient Lie group 249.G
 quotient (linear) space
 (by a linear subspace) 256.F
 (with respect to an equivalence relation) 256.F
 quotient mapping 425.G
 hereditarily 425.G
 quotient measure 225.H
 quotient object 52.D
 quotient representation (of a linear representation) 362.C
 quotient set (with respect to an equivalence relation) 135.B
 quotient singularity 418.C
 quotient space
 (by a discontinuous transformation group) 122.A
 (of a linear space) 256.F
 (of a topological space) 425.L
 left (of a topological group) 423.E
 right (of a topological group) 423.E
 quotient system (of an algebraic system) 409.C
 quotient topological space 425.L
 quotient topology 425.L

R

R (real numbers) 294.A 355.A
ρ-set 308.I
r-closed space 425.U
r-frame, tangent 105.H
r-ple point (of a plane algebraic curve) 9.B
r-section
 (of a Euclidean (simplicial) complex) 70.B

r-skeleton (of a Euclidean complex)

- (of a simplicial complex) 70.C
- r*-skeleton (of a Euclidean complex) 70.B
- r*th differential 286.E
- r*th syzygy 369.F
- R*-estimator 371.H
- R*-progenerator 29.K
- (*R*, *k*)-summable 379.S
- (*R*, *S*)-exact sequence (of modules) 200.K
- (*R*, *S*)-injective module 200.K
- (*R*, *S*)-projective module 200.K
- R**-action (continuous) 126.B
- R**^{*n*}-valued random variable 342.C
- Raabe criterion App. A, Table 10.II
- Racah algebra 353.A
- Racah coefficient 353.B
- Rademacher-Men'shov theorem 317.B
- Rademacher system of orthogonal functions 317.C
- radial equation 351.E
- radial maximal function 168.B
- radian 139.D
- radiation condition, Sommerfeld 188.D
- radical(s)
 - (of an algebraic group) 13.I
 - (of a Banach algebra) 36.D
 - (of a commutative Banach algebra) 36.E
 - (of a commutative ring) 67.B
 - (of an ideal) 67.B
 - (of a Jordan algebra) 231.B
 - (of a Lie algebra) 248.D
 - (of a ring) 368.H
 - Jacobson (of a ring) 67.D
 - nilpotent (of a Lie algebra) 248.D
 - solution by (of an algebraic equation) 10.D
 - solvable by 172.H
 - unipotent 13.I
- radius (radii)
 - (of a solid sphere) 140
 - (of a sphere) 139.I
 - associated convergence 21.B
 - of convergence (of a power series) 339.A
 - of curvature (of a plane curve) 111.E
 - of curvature (of a space curve) 111.F
 - injectivity 178.C
 - of meromorphy (of a power series) 339.D
 - of principal curvature (of a surface) 111.H
 - spectral 126.K 251.F 390.A
 - of torsion (of a space curve) 111.F
- Radó solution, Douglas- (to Plateau problem) 275.C
- Radon, decomposition formula of 125.CC
- Radon integral, Lebesgue- 94.C
- Radon measure 270.I
 - positive 270.I
- Radon-Nikodým derivative 270.L 380.C
- Radon-Nikodým property 443.H
- Radon-Nikodým theorem 270.L 380.C
 - for vector measures 443.H
- Radon transform 218.F
 - conjugate 218.F
- raising the subscripts (for tensor fields) 417.D
- Ramanujan conjecture 32.D
- Ramanujan-Petersson conjecture 32.D
- Ramanujan sum 295.D
- ramification, degree of (of a branch point) 367.B
- ramification field (of a prime ideal) 14.K
 - nth* 14.K
- ramification group
 - (of a finite Galois extension) 257.D
 - (of a prime ideal) 14.K
- nth* 14.K
- ramification index
 - (of an algebraic function) 17.C
 - (of a finite extension) 257.D
 - (of a prime ideal over a field) 14.I
 - (of a valuation) 439.I
 - relative (of a prime ideal over a field) 14.I
- ramification numbers (of a prime ideal) 14.K
- ramification point (of an analytic covering space) 23.E
- ramification theorem (in the theory of algebraic functions) 17.C
 - conductor- (in class field theory) 59.C
- ramified (prime ideal) 14.I
- ramified covering space 23.B
- ramified element 198.O
- ramified type theory 411.K
- random, at 401.F
- random current 395.I
- random distribution 395.H 407.C
 - with independent values at every point 407.C
 - strictly stationary 395.H
 - strongly stationary 395.H
 - weakly stationary 395.C
 - in the wider sense 395.C
 - in the wide sense 407.C
- random effect 102.A
- random-effects model 102.A 403.C
- random event 342.B
- random field 407.B
- randomization 102.A
- randomized (decision function) 398.A
- randomized block design 102.B
- randomized design, completely 102.A
- randomized estimator 399.B
- randomized test 400.A
- random measure
 - Gaussian 407.D
 - Poisson 407.D
- random numbers 354
 - pseudo- 354.B
- random sample 374.A 396.B 401.F
- random sampling procedure 373.A
- random Schrödinger equations 340.E
- random sequence 354.E
- random tensor field 395.I
- random variable(s) 342.C
 - distribution of 342.C
 - independent 342.C
 - joint 342.C
 - measurable with respect to a family of 342.C
 - n*-dimensional 342.C
 - one-dimensional probability distribution of 342.C
 - probability distribution of 342.C
 - R**^{*n*}-valued 342.C
 - (*S*, **E**)-valued 342.C
- random walk 260.A
 - general 260.A
 - standard 260.A
- range
 - (of a correspondence) 358.B
 - (of a linear operator) 37.C
 - (of a mapping) 381.C
 - (of a population characteristic) 396.C
 - (of statistical data) 397.C
 - closed, theorem 37.J
 - equianharmonic (of points) 343.D
 - harmonic (of points) 343.D

- interquartile 397.C
- long 375.B
- metastable (of embeddings) 114.D
- numerical (of a linear operator) 251.E
- point (in projective geometry) 343.B
- point, of the number system (in projective geometry) 343.C
- sample 396.C
- short 375.B
- stable (of embeddings) 114.D
- of values (of a meromorphic function) 62.A
- rank
 - (of an analytic mapping) 23.C
 - (of a bilinear mapping) 256.H
 - (of a complex) 13.R
 - (of a connected compact Lie group) 248.X
 - (of an element of a complex) 13.R
 - (of an elliptic curve over \mathbb{Q}) 118.D
 - (of first-order predicates) 411.K
 - (of a free Abelian group) 2.C
 - (of a free group) 161.A
 - (of a free module) 277.G
 - (of a graph) 186.G
 - (of a Lie algebra) 248.K
 - (of a linear mapping) 256.F
 - (of a matrix) 269.D
 - (of a module) 2.E
 - (of a normal j -algebra) 384.C
 - (of a prime ideal) 67.E
 - (of a quadratic form) 348.A
 - (of a sesquilinear form) 256.Q
 - (of a symmetric Riemannian homogeneous space) 412.D
 - (of a Tits system) 151.J
 - (of a valuation) 439.B
 - bispinor 258.B
 - of finite (operator) 68.C
 - k - 13.Q
 - p - (of a torsion-free additive group) 2.E
 - at a point (of an analytic mapping) 23.C
 - rational (of a valuation) 439.B
- rank correlation
 - Kendall 371.K
 - Spearman 371.K
- rank function 66.F
- Rankine-Hugoniot relation 204.G 205.B
- rank k , irreducible tensor of 353.C
- rank test
 - signed 371.B
 - Wilcoxon signed 371.B
- Rao inequality, Cramér- 399.D
- Raphson method, Newton- 301.D
- rapidly decreasing C^∞ -function 168.B
- rapidly decreasing distribution 125.O
- rapidly decreasing sequence 168.B
- rarefied gas, equation of 41.A
- rate
 - coding 213.D
 - hazard 397.O
 - infinitesimal birth 260.G
 - infinitesimal death 260.G
 - of interest, assumed 214.A
 - transmission 213.A
- ratio
 - anharmonic 343.D
 - of the circumference of a circle to its diameter 332
 - cross 343.D
 - damping (of a damped oscillation) 318.B
 - direction (of a line in an affine space) 7.F
 - double 343.D
 - likelihood 400.I
 - likelihood, test 400.I
 - monotone likelihood 374.J
 - odds 397.K
 - Poisson 271.G
 - sequential probability, test 400.L
 - stiffness 303.G
 - ratio ergodic theorem 136.B
 - ratio estimator 373.C
 - rational action 226.B
 - rational cohomology group 200.O
 - rational curve 9.C 93.H
 - rational differential equation 288.A
 - rational divisor
 - k - (on an algebraic curve) 9.C
 - prime 9.C
 - rational double point 418.C
 - rational element 198.O
 - rational entire function 429.A
 - rational expression 337.H
 - field of 337.H
 - rational extrapolation method 303.F
 - rational function(s)
 - field of 337.H
 - generalized 142.B
 - on a variety 16.A
 - rational function field in n variables 149.K
 - rational function matrix 86.D
 - rational homomorphism 3.C 13.A
 - rational injectivity 200.O
 - rational integer 294.C
 - rationally equivalent cycles 16.R
 - rational mapping 16.I
 - defined along a subvariety 16.I
 - purely inseparable 16.I
 - separable 16.I
 - rational number(s) 294.D
 - denseness of 355.B
 - rational operation 294.A
 - rational point 118.E
 - over a field 369.C
 - k' - (of an algebraic variety) 16.A
 - rational polyhedral, convex 16.Z
 - rational Pontryagin class 56.F
 - rational rank (of a valuation) 439.B
 - rational real number 294.E
 - rational representation
 - (of $GL(V)$) 60.D
 - (of a matrix group) 226.B
 - rational singularity 418.C
 - rational surface 15.E
 - rational variety 16.J
 - uni- 16.J
 - ratio set 136.F
 - asymptotic 308.I
 - ray
 - (in affine geometry) 7.D
 - (in foundation of geometry) 155.B
 - (modulo m^*) 14.H
 - (in a Riemannian manifold) 178.F
 - asymptotic 178.F
 - grazing 325.L
 - paraxial 180.B
 - unit 351.B
 - Rayleigh principle 68.H
 - Rayleigh quotient 68.H 298.C 304.B
 - Rayleigh-Ritz method 46.F 271.G
 - Rayleigh-Schrödinger series 331.D
 - reaction, law of 271.A

Real analytic (at a point)

- real analytic (at a point) 106.K
- real analytic fiber bundle 147.O
- real analytic foliation 154.H
- real analytic function 106.K 198.H
 - exponentially decreasing 125.BB
- real analytic manifold 105.D
- real analytic structure 105.D
- real axis 74.C
- real closed field 149.N
- real-compact space 425.BB
- real field 149.N
 - formally 149.N
 - totally 14.F
- real form 412
 - (of a complex algebraic group) 60.O
 - (of a complex Lie algebra) 248.P
 - normal (of a complex semisimple Lie algebra) 248.Q
- real function 165.B
- real Grassmann manifold 199.B
- real Hilbert space 197.B
- real hyperbolic space 412.G
- real hypersphere 76.A
 - oriented 76.A
- real hypersurface, spherical 344.C
- real immersion, totally 365.M
- real infinite prime divisor 439.H
- real interpolation space 224.C
- realizable
 - (for a linear representation) 362.F
 - (by a submanifold) 114.G
- realization
 - (of a linear time-varying system) 86.D
 - (of an s.s. complex) 70.E
 - (of an s.s. mapping) 70.E
 - minimal 86.D
- realization theorem (of a homotopy group) 202.N
- realization theory 86.D
- real Lie algebra 248.A
 - compact 248.P
- real line 355.E
- real linear space 256.A
- real monoidal transform 274.E
- real number(s) 294.E 355
 - Cantor's theory of 294.E
 - completeness of 294.E 355.B
 - connectedness of 294.E
 - continuity of 294.E
 - Dedekind's theory of 294.E
 - extended 87.E
 - infinitesimal 276.E
 - irrational 294.E
 - mod 1 355.D
 - nonstandard 276.E
 - rational 294.E
- real part 74.A
- real prime divisor 439.H
- real projective space 343.D
 - infinite-dimensional 56.B
- real quadratic field 347.A
- real quadratic form 348.A.C
- real representation (of a Lie group) 249.O
- real root (of an algebraic equation) 10.E
- real simple Lie algebra
 - classical compact 248.T
 - exceptional compact 248.T
- real spectral measure 390.D
- real Stiefel manifold
 - of k -frames 199.B
 - of orthogonal k -frame 199.B
- real submanifold, totally 365.M
- real-time (computation) 19.E
- real topological vector space 424.A
- real-valued functions 165.B
- real-valued measurable (cardinal) 33.F
- real variable 165.C
- rearrangement 168.B
- rearrangement invariant 168.B
- reciprocal equation 10.C
- reciprocal linear representation (of an algebra) 362.C
- reciprocal network 282.C
- reciprocal permutation representation (of a group) 362.B
- reciprocal spiral 93.H
- reciprocity
 - of annihilators (in topological Abelian groups) 422.E
 - Artin's general law of 59.C
 - complementary law of 14.O
 - Fourier 160.C
 - general law of 14.O
 - law of quadratic, of Jacobi symbol 297.I
 - law of quadratic, of Legendre symbol 297.I
 - relations, Onsager's 402.K
- reciprocity law 297.I
 - for Dedekind sums 328
 - explicit (for Hilbert norm-residue symbol) 14.R
 - Gel'fand–Pyatetskii–Shapiro (on unitary representation) 437.DD
 - Shafarevich 257.H
- record 96.B
- rectangle 140
 - latin 241.E
- rectangular coordinates (in a Euclidean space) 90.B
- rectangular distribution App. A, Table 22
- rectangular hyperbola 78.E
- rectangular hyperbolic coordinates 90.C
- rectangular matrix 269.A
- rectangular parallelepiped 140
- rectifiable
 - (current) 275.G
 - (curve) 93.F 246.A
 - locally 143.A 246.A
- rectifying plane 111.F
- rectifying surface 111.F
- rectilinear complex 70.B
- recurrence formulas for indefinite integrals App. A, Table 9.II
- recurrence theorem 136.A.C
- recurrence time 260.C
 - mean 260.C
- recurrent
 - (Lévy process) 5.G
 - (Markov chain) 260.B
 - (Markov process) 261.B
 - (nonsingular measurable transformation) 136.C
 - (point of a dynamical system) 126.E
 - chain 126.E
 - infinitely (measurable transformation) 136.C
 - linear (sequence) 295.A
 - non- (Markov chain) 260.B
 - null (point) 260.B
 - positive (ergodic class) 260.B
 - positive (point) 260.B
 - regionally (flow) 126.E
 - strongly (measurable transformation) 136.C

- recurrent chain 260.B
- recurrent event 250.D 260.C
 - delayed 260.C
- recurrent point
 - (of a Markov chain) 260.B
 - (of a Markov process) 261.B
- recurrent sequence of order r 295.A
- recurrent set 260.E
 - chain 126.E
- recursive function(s) 356
 - general 356.C,F
 - partial 356.E,F
 - primitive 356.A,B,F
 - uniformly primitive 356.B
- recursively
 - (define a partial recursive function) 356.E
 - uniformly in Ψ 356.E
- recursively enumerable predicate 356.D
- recursively enumerable set 356.D
- recursive predicate
 - general 356.C
 - primitive 356.B
- recursive set 97 356.D
 - general 97
- reduced
 - (a closed linear subspace) 251.L
 - (latin square) 241.A
 - (scheme) 16.D
- reduced Abelian group 2.D
- reduced algebra 231.B
- reduced basis (of a lattice) 92.C
- reduced bundle (of a principal G -bundle) 147.J
- reduced character (of an algebra) 362.E
- reduced Clifford group 61.D
- reduced cone (of a topological space) 202.F
- reduced dual 437.L
- reduced extremal distance 143.B
- reduced form (of a linear structural equation system) 128.C
- reduced homology exact sequence 201.F
- reduced homology group 201.E
- reduced join
 - (of homotopy classes) 202.Q
 - (of mappings) 202.F
 - (of topological spaces) 202.F
- reduced link polynomial 235.D
- reduced mapping cone 202.F
- reduced norm (of an algebra) 362.E
- reduced orthogonal group 61.D
- reduced product space 202.Q
- reduced quadratic form 348.I
- reduced representation (of an algebra) 362.E
- reduced residue system modulo m 297.G
- reduced square, Steenrod 64.B
- reduced square operation, Steenrod 64.B
- reduced suspension
 - (of a topological space) 202.F
 - n -fold 202.F
- reduced trace (of an algebra) 362.E
- reduced von Neumann algebra 308.C
- reducibility, axiom of 156.B 411.K
- reducible
 - (algebraic equation) 10.B
 - (algebraic variety) 16.A
 - (continuous geometry) 85.A
 - (fiber bundle) 147.J
 - (in four color problem) 157.D
 - (germ of an analytic set) 23.B
 - (linear system) 16.N
 - (linear system in control theory) 86.C
 - (polynomial) 337.F
 - (positive matrix) 269.N
 - (representation) 362.C
 - (Riemannian manifold) 364.E
 - completely (A -module) 277.H
 - completely (group) 190.L
 - completely (representation) 362.C
- reductio ad absurdum 156.C 411.I
- reduction
 - d'Alembert method of, of order 252.F
 - good (of an Abelian variety) 3.N
 - modulo \mathfrak{A} (of a representation) 277.L
 - modulo m (of a linear representation) 362.F
 - potential good (of an Abelian variety) 3.N
 - potential stable (of an Abelian variety) 3.N
 - stable (of an Abelian variety) 3.N
 - stable (of a curve) 9.K
- reduction formula (of a surface) 110.A
- reduction theorem, cup product (on cohomology or homology of groups) 200.M
- reduction theory, Minkowski (on fundamental regions) 122.E
- reductive
 - (algebraic group) 13.I
 - (homogeneous space) 199.A
 - (Lie algebra) 248.G
- reductive action 226.B
 - defined by a rational representation 226.B
 - geometrically 226.B
 - linearly 226.B
 - semi- 226.B
- reductive stabilizer 199.A
- Reeb component 154.B
- Reeb foliation 154.B
- Reeb stability theorems 154.D
- Ree group 151.I
- Reeh-Schlieder theorem 150.E
- Rees lemma, Artin- 284.A
- Ree type
 - group of 151.J
 - group of Janko- 151.J
- reference edge 281.C
- refinement
 - (of a covering) 425.R
 - (of a descending chain in a lattice) 243.F
 - (of a normal chain in a group) 190.G
 - barycentric 425.R
 - cushioned 425.X
 - Δ - (of a covering) 425.R
 - star (of a covering) 425.R
- reflected wave 325.L
- reflecting barrier 115.B,C
- reflection
 - (associated with Φ) 13.R
 - (of a principal space) 139.B
 - glide 92.E
 - Schwartz's principle of 74.E 198.G
 - space 359
 - theorem of quasiconformal 352.C
- reflection coefficient 387.D
- reflectionless potential 387.D
- reflection points (with respect to a circle) 74.E
- reflection positivity 150.F
- reflection principle 45.E
- reflexive
 - (locally convex space) 424.O
 - (relation) 358.A
 - Banach space 37.G
- reflexive law
 - (for an equivalence relation) 135.A

- (on ordering) 311.A
- refraction, atmospheric 392
- Regge behavior 386.C
- Regge poles 132.C 386.C
- regime, local 51.B
- region 79.A
 - acceptance 400.A
 - confidence 399.Q
 - confidence, uniformly most powerful 399.Q
 - confidence, uniformly most powerful unbiased 399.Q
 - critical 400.A
 - Dirichlet 234.C
 - of discontinuity 234.A
 - estimation 399.Q
 - feasible 264.B 292.A
 - Ford fundamental 234.C
 - fundamental (of a discrete transformation group) 122.B
 - invariance of a confidence 399.Q
 - of relative stability 303.G
 - star 339.D
 - tolerance 399.R
 - unbiased confidence 399.Q
- regionally recurrent (flow) 126.E
- regionally recurrent on an invariant set 126.E
- region of absolute stability (of the Runge-Kutta (P, p) method) 303.G
- regression, line of 111.F.I
- regression analysis 403.D
- regression coefficient 397.H,J 403.D
- regression function 397.I
 - linear 397.H 403.D
- regression hyperplane 403.D
- regression line 403.D
- regula falsi 301.C
- regular
 - (almost contact manifold) 110.E
 - (almost periodic system) 290.B
 - (boundary point) 120.D
 - (cell complex) 70.D
 - (closed set) 125.J
 - (coherent \mathcal{E} -module) 274.G
 - (differential form on an algebraic variety) 16.O
 - (Dirichlet form) 261.C
 - (element of a connected Lie group) 249.P
 - (element of a real Lie algebra) 248.B
 - (estimator) 399.N
 - (Green line) 193.J
 - (kernel) 125.L
 - (left ideal of a Banach algebra) 36.D
 - (ordinal number) 312.E
 - (permutation group) 151.H
 - (point for an additive process) 5.G
 - (point of an analytic set) 23.B 45.D
 - (point with respect to an analytic set) 21.M
 - (point with respect to the Dirichlet problem) 207.B
 - (point of a flow) 126.D
 - (prime number) 14.L
 - (sampling procedure) 373.A
 - (spectral sequence) 200.J
 - (submartingale) 262.D
 - (at a subvariety) 16.B
 - homogeneously 275.C
 - of the h th species 343.E
 - at the point at infinity (for a harmonic function) 193.B
 - along a subvariety (for a rational mapping) 16.I
 - regular affine transformation 7.E
 - regular Banach space 37.G
 - regular boundary
 - (of a diffusion process) 115.B
 - domain with (in a C^∞ -manifold) 105.U
 - regular chain (of integral elements) 428.E
 - regular conditional probability 342.E
 - regular cone 384.A
 - self-dual 384.E
 - regular covering (space) 91.A
 - regular element
 - (of a ring) 368.B
 - p - (of a finite group) 362.I
 - regular embedding 105.K
 - regular extension (of a field) 149.K
 - regular factorization 251.N
 - regular first integral 126.H
 - regular form 16.O
 - regular function(s) 198.A
 - on an open set (of a variety) 16.B
 - sheaf of germs of 16.B
 - at a subvariety 16.B
 - regular grammar 31.D
 - regular graph 186.C
 - regular integral element 191.I 428.E
 - regular integral manifold (of a differential ideal) 428.E
 - regularity
 - abscissa of (of a Dirichlet series) 121.B
 - axiom of (in axiomatic set theory) 33.B
 - up to a boundary 112.F
 - parameter of (of a Lebesgue measurable set) 380.D
 - regularization (of a distribution) 125.M
 - regularizing (kernel) 125.L
 - regular knot projection 235.A
 - regular local equation (at an integral point) 428.E
 - regular local ring 284.D
 - regularly convex set 89.G
 - regularly homotopic (immersion) 114.D
 - regularly hyperbolic (partial differential equation) 325.A,F
 - regular mapping
 - (between prealgebraic varieties) 16.C
 - of class C^1 208.B
 - regular matrix 269.B
 - regular measure 270.F
 - \mathcal{B} - 270.F
 - K - 270.F
 - regular n -gon 357.A
 - regular neighborhood 65.C
 - regular neighborhood system 65.C
 - regular outer measure 270.E
 - regular perturbation 331.D
 - regular point
 - (in catastrophe theory) 51.F
 - (of a differentiable mapping) 105.J
 - (of a diffusion process) 115.B
 - (for a Hunt process) 261.D
 - (of a polyhedron or cell complex) 65.B
 - (of a surface in E^3) 111.J
 - semi- (of a surface in E^3) 111.J
 - regular polygon 357.A
 - regular polyhedra 357.B
 - regular polyhedral angle 357.B
 - regular polyhedral group 151.G
 - regular positive Radon measure 270.H
 - regular process, positively 44.C
 - regular projective transformation 343.D
 - regular representation

(of a group) 362.B
 (of a locally compact group) 69.B
 (of a topological transformation group) 437.A
 left (of an algebra) 362.C
 left (of a group) 362.C
 right (of an algebra) 362.E
 right (of a group) 362.E
 regular ring 284.D
 regular ring (continuous geometry) 85.B
 regular sequence (of Lebesgue measurable sets) 380.D
 regular singularity (of a coherent \mathcal{E} -module) 274.H
 regular singular point 254.B
 regular solution (of a differential ideal) 428.E
 regular space 425.Q
 completely 425.Q
 regular submanifold (of a C^∞ -manifold) 105.L
 regular system
 of algebraic equations 10.A
 of parameters (of a local ring) 284.D
 regular transformation
 (of a linear space) 256.B
 (of a sequence) 379.L
 totally (of a sequence) 379.L
 regular tube 193.K
 regular value 105.J
 regulator (of an algebraic number field) 14.D
 p -adic 450.J
 regulator problem, optimal 80.F
 Reinhardt domain 21.B
 complete 21.B
 reiteration theorem 224.D
 rejection 400.A
 related differential equation 254.F
 relation(s) 358
 (among elements of a group) 190.C
 (among the generators of a group) 161.A
 Adem (for Steenrod p th power operations) 64.B
 Adem (for Steenrod square operations) 64.B
 analytic, invariance theorem of 198.K
 antisymmetric 358.A
 binary 358.A 411.G
 canonical anticommutation 377.A
 canonical commutation 351.C 377.A,C
 coarser 135.C
 defining (among the generators of a group) 161.A
 dispersion 132.C
 equivalence 135.A 358.A
 Euler 419.B
 finer 135.C
 Fuchsian 253.A
 functional (among components of a mapping) 208.C
 functional, of class C^r 208.C
 fundamental (among the generators of a group) 161.A 419.A
 Gibbs-Duhem 419.B
 Heisenberg uncertainty 351.C
 Hurwitz (on homomorphisms of Abelian varieties) 3.K
 identity 102.I
 incidence 282.A
 inverse 358.A
 Legendre 134.F, App. A, Table 16.I
 Maxwell 419.B
 n -ary 411.G
 normal commutation 150.D
 order 311.A

Subject Index

Relative nullity, index of

orthogonality (on irreducible characters) 362.G
 orthogonality (for square integrable unitary representations) 437.M
 period 11.C
 Plücker (on Plücker coordinates) 90.B
 prey-predator 263.B
 proper equivalence (in an analytic space) 23.E
 Rankine-Hugoniot 204.G 205.B
 reciprocity, Onsager's 402.K
 reflexive 358.A
 Riemann-Hurwitz 367.B
 Riemann period 3.L 11.C
 stronger 135.C
 symmetric 358.A
 transitive 358.A
 weaker 135.C
 relationship algebra 102.J
 relative Alexander cohomology group 201.M
 relative algebraic number field 14.I
 relative boundary 367.B
 relative Bruhat decomposition 13.Q
 relative Čech cohomology group 201.M
 relative Čech homology group 201.M
 relative chain complex 200.C
 relative cochain complex 200.F
 relative cohomology group 215.W
 relative complement (at two sets) 381.B
 relative components (of a Lie transformation group) 110.A
 relative consistency 156.D
 relative degree
 (of a finite extension) 257.D
 (of a prime ideal over a field) 14.I
 relative derived functor 200.K
 relative different 14.J
 relative discriminant 14.J
 relative entropy 212.B
 relative extremum, conditional 106.L
 relative frequency (of samples) 396.C
 relative homological algebra 200.K
 relative homotopy group 202.K
 relative integral invariant 219.A
 Cartan's 219.B
 relative invariant 12.A 226.A
 relative invariant measure 225.H
 relatively ample sheaf 16.E
 relatively bounded (with respect to a linear operator) 331.B
 relatively closed set 425.J
 relatively compact
 (with respect to a linear operator) 331.B
 (maximum likelihood method) 399.M
 (set) 425.S
 (subset) 273.F
 relatively dense 126.E
 relatively invariant measure 225.H
 relatively minimal 15.G 16.I
 relatively minimal model 15.G
 relatively open set 425.J
 relatively prime
 (fractional ideals) 14.E
 (numbers) 297.A
 relatively stable 303.G
 relative maximum (of a function) 106.L
 relative Mayer-Vietoris exact sequence 201.L
 relative minimum (of a function) 106.L
 relative neighborhood 425.J
 relative norm (of a fractional ideal) 14.I
 relative nullity, index of 365.D

relative open set 425.J
relative ramification index (of a prime ideal over a field) 14.I
relative singular homology group 201.L
relative stability 303.G
 interval of 303.G
 region of 303.G
relative topology 425.J
relative uniformity 436.E
relative uniform star convergence 310.F
relativistically covariant 150.D
relativity
 general principle of 359
 general theory of 359.A
 special principle of 359
 special theory of 359.A
relativization
 (of a definition of primitive recursive functions) 356.B
 (of a topology) 425.J
 (of a uniformity) 436.E
relativized 356.F
relaxation 215.A
 with projection 440.E
relaxation oscillation 318.C
relaxed continuity requirements, variational principles with 271.G
Rellich-Dixmier theorem 351.C
Rellich-Kato theorem 331.B
Rellich lemma 68.C
Rellich theorem 323.G
Rellich uniqueness theorem 188.D
remainder 297.A 337.C
 (in Taylor's formula) 106.E
 Cauchy App. A, Table 9.IV
 Lagrange App. A, Table 9.IV
 Roche-Schlömilch App. A, Table 9.IV
remainder theorem 337.E
 Chinese 297.G
Remak-Schmidt theorem, Krull- (in group theory) 190.L
Remmert-Stein continuation theorem 23.B
Remmert theorem 23.C
removable (set for a family of functions) 169.C
removable singularity
 (of a complex function) 198.D
 (of a harmonic function) 193.L
Renaissance mathematics 360
renewal
 equation 260.C
 theorem 260.C
renormalizable 111.B 132.C 150.C
 super 150.C
renormalization
 constant 150.C
 equation 111.B
 group 111.A
 method 111.A
Rényi theorem 123.E
reoriented graph 186.B
repeated integral
 (for the Lebesgue integral) 221.E
 (for the Riemann integral) 216.G
repeated series
 by columns 379.E
 by rows 379.E
replacement, axiom of 33.B 381.G
replacement, model 307.C
replica 13.C
replication 102.A

 number of 102.B
represent
 (a functor) 52.L
 (an ordinal number) 81.B
representable
 (functor) 52.L
 linearly (matroid) 66.H
representation(s) 362.A
 (of an algebraic system) 409.C
 (of a Banach algebra) 36.D
 (of a Jordan algebra) 231.C
 (of a knot group) 235.E
 (of a lattice) 243.E
 (of a Lie algebra) 248.B
 (of a mathematical system) 362.A
 (of a vector lattice) 310.D
 absolutely irreducible 362.F
 adjoint (of a Lie algebra) 248.B
 adjoint (of a Lie group) 249.P
 adjoint (of a representation) 362.E
 analytic (of $GL(V)$) 60.B
 in terms of arc length (of a continuous arc) 246.A
 canonical (of Gaussian processes) 176.E
 completely reducible 362.C
 complex (of a Lie group) 249.O
 complex conjugate 362.F
 conjugate 362.F
 contragredient 362.E
 coregular (of an algebra) 362.E
 cyclic (of a C^* -algebra) 36.G
 cyclic (of a topological group) 437.A
 differential (of a unitary representation of a Lie group) 437.S
 direct sum of 362.C
 double-valued 258.B
 dual 362.E
 equivalent 362.C
 factor (of a topological group) 437.E
 factor, of type I, II, or III 308.M 437.E
 faithful 362.B
 Fock 150.C
 Gel'fand (of a commutative Banach algebra) 36.E
 generalized canonical (of Gaussian processes) 176.E
 generating (of a compact Lie group) 249.U
 half-spin (even, odd) 61.E
 Herglotz's integral 43.I
 induced 362.G
 induced (of a finite group) 362.G
 induced (of a unitary representation of a subgroup) 437.O
 integral (of a group) 362.G,K
 integral, Cauchy's 21.C
 irreducible (of an algebra or a group) 362.C
 irreducible (of a Banach algebra) 36.D
 irreducible projective 362.J
 isomorphic 362.C
 isotropy 431.C
 Källén-Lehmann 150.D
 kernel (of a Green's operator) 189.B
 l -adic 3.E
 Lax 287.B,C 387.C
 left regular (of a group) 362.B
 linear \rightarrow linear representation
 list 186.D
 Mandelstam 132.C
 matrix 362.D
 modular (of a finite group) 362.G

module of 69.D
 momentum 351.C
 without multiplicity 437.G
 normal (of a von Neumann algebra) 308.C
 ordinary (of a finite group) 362.G
 parametric 165.C
 parametric (of Feynman integrals) 146.B
 parametric (of a subspace of an affine space) 7.C
 permutation (of a group) 362.B
 permutation, reciprocal (of a group) 362.B
 polynomial (of $GL(V)$) 60.D
 position 351.C
 projective (of a group) 362.J
 projective, irreducible 362.J
 quotient (of a linear representation) 362.C
 rational (of $GL(V)$) 60.D
 rational (of a matrix group) 226.B
 real (of a Lie group) 349.O
 reciprocal linear (of an algebra) 362.C
 reciprocal permutation (of a group) 362.B
 reduced (of an algebra) 362.F
 reducible 362.C
 regular (of a locally compact group) 69.B
 regular (of a topological transformation group) 437.A
 regular, left (of an algebra) 362.C
 regular, left (of a group) 362.B
 regular, right (of an algebra) 362.E
 regular, right (of a group) 362.B
 Schrödinger 351.C
 semisimple 362.C
 similar 362.C
 similar matrix (semilinear mapping) 256.D
 similar projective 362.J
 simple 362.C
 slice 431.C
 special (of a Jordan algebra) 231.C
 spectral 390.E
 spherical (of a differentiable manifold) 111.G
 spherical (of a space curve) 111.F
 spherical (of a unimodular locally compact group) 437.Z
 spin 61.E
 spin (of $SO(n)$) 60.J
 spinor, of rank 258.B
 strongly continuous (of a topological group) 69.B
 sub- 362.C
 sub- (of a projective representation) 362.J
 tensor (of a general linear group) 256.M
 tensor product of 362.C
 translation, theorem 375.H
 transposed 362.E
 tree 96.D
 unit (of a group) 362.C
 unitary \rightarrow unitary representation
 vector (of a Clifford group) 61.D
 weakly continuous (of a topological group) 69.B
 zero (of an algebra) 362.C
 representation module
 (of a linear representation) 362.C
 faithful 362.C
 representation problem (on surfaces) 246.I
 representation ring 237.H
 representation space
 (of a Banach algebra) 36.D
 (of a Lie algebra) 248.B
 (of a Lie group) 249.O

Subject Index

Resolvent set (of a linear operator)

(of a unitary representation) 437.A
 representative (of an equivalence class) 135.B
 representative function (of a compact Lie group) 249.U
 representative ring (of a compact Lie group) 249.U
 representing function (of a predicate) 356.B
 representing function (of a subset) 381.C
 representing measure 164.C
 reproducing kernel 188.G
 reproducing property (of a probability distribution) 341.E, App. A, Table 22
 reproduction function 263.A
 requirements, variational principles with relaxed continuity 271.G
 reserve, liability 214.B
 residual (subset of a directed set) 311.D
 residual limit set 234.E
 residual set 126.H 425.N
 residual spectrum 390.A
 residue(s)
 (of a complex function) 198.E
 calculus of 198.F
 of the n th power (modulo p) 14.M
 norm- (modulo p) 14.P
 norm- (symbol) 14.Q 257.F
 power- (symbol) 14.N
 quadratic 297.H
 residue character 295.D
 residue class (modulo an ideal in a ring) 368.F
 residue (class) algebra 29.A
 residue (class) field 149.C 368.F 439.B
 residue (class) ring (modulo an ideal) 368.F
 residue system modulo m
 complete 297.G
 reduced 297.G
 residue theorem 198.E
 (on a nonsingular curve) 9.E
 resistance, negative 318.B
 resistance, specific 130.B
 resolution 200.H
 complete free (of \mathbf{Z}) 200.N
 complex spectral 390.E
 flabby 125.W
 of the identity 390.D
 injective (in an Abelian category) 200.I
 $2I$ 102.I
 $2I + 1$ 102.I
 minimal 418.C
 projective (in an Abelian category) 200.I
 right (of an A -module) 200.F
 right injective (of an A -module) 200.F
 of singularities 16.L
 of singularities (of an analytic space) 23.D 418.B
 spectral 390.E
 standard (of \mathbf{Z}) 200.M
 resolutive 207.B
 resolutive compactification 207.B
 resolvent
 (of a kernel) 217.D
 (of a linear operator) 251.F
 (operator of a Markov process) 261.D
 cubic App. A, Table 1
 resolvent (operator of a Markov process) 261.D
 resolvent convergence
 norm 331.C
 strong 331.C
 resolvent equation 251.F
 resolvent set (of a linear operator) 251.F 390.A

Resonance model, dual

- resonance model, dual 132.C
- resonance pole 331.F
- resonance theorem 37.H
- response 405.A
- response surface 102.M
 - designs for exploring 102.M
- rest energy 359.C
- restitutive force 318.B
- rest point (of a trajectory) 126.D
- restricted (Lorentz group) 258.A
- restricted Burnside problem (in group theory) 161.C
- restricted differential system 191.I
- restricted direct product 6.B
 - (of an infinite number of groups) 190.L
 - (of locally compact groups) 6.B
- restricted holonomy group 80.D 364.E
- restricted homogeneous holonomy group 364.E
- restricted homotopy 202.B
- restricted Lie algebra 248.V
- restricted minimal condition (in a commutative ring) 284.A
- restricted quantifier 33.B
- restricted three-body problem 420.F
- restriction
 - (of a connection) 80.F
 - (of a continuous flow) 126.D
 - (of a distribution) 125.D
 - (of a mapping) 381.C
 - (in a presheaf) 383.A
 - crystallographic 92.A
 - scalar (of a B -module) 277.L
 - unitary (of a semisimple Lie algebra) 248.P
- resultant(s) 369.E
 - system of 369.E
- retardation 163.A
- retarded differential equation 163.A
- retarded type (functional differential equation) 163.A
- retract 202.D
 - absolute 202.D
 - absolute neighborhood 202.D
 - deformation 202.D
 - fundamental 382.C
 - fundamental absolute (FAR) 382.C
 - fundamental absolute neighborhood (FANR) 382.C
 - neighborhood 202.D
 - neighborhood deformation 202.D
 - strong deformation 202.D
- retraction 202.D
- retrieval, information (system) 96.F
- retrospective study 40.E
- return 127.C
 - first- (mapping, map) 126.C
 - maximum 127.B
- Reuleaux triangle 89.E 111.E
- reversal, time 258.A
- reversed process 261.F
- review technique, program evaluation and 376
- revolution
 - ellipsoid of 350.B
 - elliptic paraboloid of 350.B
 - hyperboloid of, of one sheet 350.B
 - hyperboloid of, of two sheets 350.B
 - surface of 111.I
- Reynolds law of similarity 205.C
- Reynolds number 116.B 205.C
- Reynolds number, magnetic 259
- Riccati differential equation App. A, Table 14.I
 - generalized App. A, Table 14.I
 - matrix 86.E
- Riccati equation, matrix 405.G
- Ricci curvature 364.D
- Ricci equation 365.C
- Ricci formula 417.B, App. A, Table 4.II
- Ricci tensor 364.D 417.B, App. A, Table 4.II
- Richard paradox 319.B
- Richardson method 302.C
- Riemann, G. F. B. 363
 - P -function of 253.B
- Riemann bilinear relations, Hodge- 16.V
- Riemann continuation theorem 21.F
- Riemann differential equation App. A, Table 18.I
- Riemann (differential) equation, Cauchy- 198.A 274.G
 - (for a holomorphic function of several complex variables) 21.C
 - (for a holomorphic function of two complex variables) 320.F
- Riemann function (of a Cauchy problem) 325.D
- Riemann-Hilbert problem
 - (for integral equations) 217.J
 - (for linear ordinary differential equations) 253.D
- Riemann-Hurwitz formula (on coverings of a non-singular curve) 9.I
- Riemann-Hurwitz relation 367.B
- Riemann hypothesis 450.A, I, P, Q
- Riemannian connection 80.K 364.B
 - coefficients of 80.L
- Riemannian curvature 364.D
- Riemannian foliation 154.H
- Riemannian geometry 137, App. A, Table 4.II
- Riemannian homogeneous space 199.A
 - symmetric 412.B
- Riemannian manifold(s) 105.P 286.K 364
 - flat 364.E
 - irreducible 364.E
 - isometric 364.A
 - locally flat 364.E
 - normal contact 110.E
 - reducible 364.E
- Riemannian metric 105.P
 - pseudo- 105.P
 - volume element associated with 105.W
- Riemannian product (of Riemannian manifolds) 364.A
- Riemannian space 364.A
 - irreducible symmetric App. A, Table 5.III
 - locally symmetric App. A, Table 4.II
 - symmetric \rightarrow symmetric Riemannian space
- Riemannian submanifold 365
- Riemann integrable (function) 216.A
- Riemann integral 37.K 216.A
- Riemann-Lebesgue theorem 159.A 160.A
- Riemann lower integral 216.A
- Riemann mapping theorem 77.B
- Riemann matrix 3.I
- Riemann method of summation 379.S
- Riemann non-Euclidean geometry 285.A
- Riemann period inequality 3.L
- Riemann period relation 3.L
- Riemann P -function App. A, Tables 14.II 18.I
- Riemann problem 253.D
- Riemann-Roch group 366.D
- Riemann-Roch inequality (on algebraic surfaces) 15.D

- Riemann-Roch theorem(s) 366
 - (on algebraic surfaces) 15.D
 - (for compact complex surface) 366.C
 - (on nonsingular algebraic curves) 9.C
 - (on Riemann surfaces) 11.D
 - for an adjoint system 15.D
 - for differentiable manifolds 237.G
 - generalized (on algebraic curves) 9.F
 - for a line bundle 366.C
- Riemann-Roch type
 - Grothendieck theorem of 366.D
 - Hirzebruch theorem of 366.B
- Riemann sphere 74.D
- Riemann-Stieltjes integral 94.B 166.C
- Riemann structure, Cauchy- 344.A
- Riemann sum 216.A
- Riemann surface(s) 367
 - abstract 367.A
 - classification theory of 367.E
 - closed 367.A
 - elliptic 367.D
 - hyperbolic 367.D
 - maximal 367.F
 - open 367.A
 - open, of null boundary 367.E
 - open, of positive boundary 367.E
 - parabolic 367.D
 - prolongable 367.F
- Riemann theorem
 - (on removable singularities) 198.D
 - (on series with real terms) 379.C
- Riemann Θ function 3.L
- Riemann upper integral 216.A
- Riemann ζ function 450.V
- Riesz convexity theorem 88.C
- Riesz decomposition
 - (in a Markov chain) 260.D
 - of a superharmonic or subharmonic function 193.S
- Riesz-Fischer theorem 168.B 317.A
- Riesz group 36.H
- Riesz method of order k , summable by 379.R
- Riesz method of summation of the k th order 379.R
- Riesz potential 338.B
- Riesz-Schauder theorem 68.E
- Riesz space 310.B
- Riesz (F.) theorem
 - (on L_p functions) 317.B
 - (representation) 197.F
- Riesz (F. and M.) theorem 168.C
 - (on bounded holomorphic functions on a disk) 43.D
- Riesz-Thorin theorem 224.A
- Riesz transform 251.O
- right, limit on the 87.F
- right-adjoint (linear mapping) 256.Q
- right adjoint functor 52.K
- right A -module 277.D
- right angle 151.D
- right annihilator (of a subset of an algebra) 29.H
- right Artinian ring 368.F
- right balanced (functor) 200.I
- right circular cone 350.B
- right conoid 111.I
- right continuous (function) 84.B
- right coset (of a subgroup of a group) 190.C
- right coset space (of a topological group) 423.E
- right decomposition, Peirce (in a unitary ring) 368.F
- right derivative 106.A
- right derived functor 200.I
- right differentiable 106.A
- right endpoint (of an interval) 355.C
- right equivalent 51.C
- right exact (functor) 200.I
- right global dimension (of a ring) 200.K
- right G -set 362.B
- right helicoid 111.I
- right ideal
 - (of a ring) 368.F
 - integral 27.A
- right injective resolution (of an A -module) 200.F
- right invariant Haar measure 225.C
- right invariant tensor field (on a Lie group) 249.A
- right inverse (of $df_1(0)$) 286.G
- right inverse element (of an element of a ring) 368.B
- right linear space 256.A
- right majorizing function 316.E
- right Noetherian ring 368.F
- right \mathfrak{o} , ideal 27.A
- right operation (of a set to another set) 409.A
- right order (of a g -lattice) 27.A
- right parametrix 345.A
- right projective space 343.H
- right quotient space (of a topological group) 423.E
- right regular representation
 - (of an algebra) 362.E
 - (of a group) 362.B
- right resolution (of an A -module) 200.F
- right satellite 200.I
- right semihereditary ring 200.K
- right semi-integral 68.N
- right shunt 115.B
- right singular point (of a diffusion process) 115.B
- right superior function 316.E
- right translation 249.A 362.B
- right uniformity (of a topological group) 423.G
- rigid
 - (characteristic class of a foliation) 154.G
 - (isometric immersion) 365.E
- rigid body 271.E
- rigidity
 - (of a sphere) 111.I
 - modulus of 271.G
- rigidity theorem 178.C
 - strong 122.G
- ring(s) 368
 - adele (of an algebraic number field) 6.C
 - affine 16.A
 - anchor 410.B
 - Artinian 284.A
 - associated graded 284.D
 - basic (of a module) 277.D
 - Boolean 42.C
 - Burnside 431.F
 - category of 52.B
 - category of commutative 52.B
 - Chow (of a projective variety) 16.R
 - cobordism 114.H
 - coefficient (of an algebra) 29.A
 - coefficient (of a semilocal ring) 284.D
 - coherent sheaf of 16.E
 - cohomology 201.I
 - cohomology, of compact connected Lie groups App. A, Table 6.IV

cohomology, of an Eilenberg-MacLane complex App. A, Table 6.III
commutative 67 368.A
complete local 284.D
completely integrally closed 67.I
completely primary 368.H
complete Zariski 284.C
completion, with respect to an ideal 16.X
complex cobordism 114.H
of convergent power series 370.B
coordinate (of an affine variety) 16.A
correspondence (of a nonsingular curve) 9.H
de Rham cohomology (of a differentiable manifold) 105.R 201.I
differential 113
differential extension 113
of differential polynomials 113
discrete valuation 439.E
division 368.B
endomorphism (of an Abelian variety) 3.C
endomorphism (of a module) 277.B 368.C
of endomorphisms (of an Abelian variety) 3.C
factor, modulo an ideal 368.F
form 284.D
of formal power series 370.A
of fractions 67.G
generalized Boolean 42.C
Gorenstein 200.K
graded 369.B
ground (of an algebra) 29.A
ground (of a module) 277.D
group (of a compact group) 69.A
Hecke 32.D
Hensel 370.C
Henselian 370.C
hereditary 200.K
homogeneous 369.B
homogeneous coordinate 16.A
integrally closed 67.I
Krull 67.J
left Artinian 368.F
left hereditary 200.K
left Noetherian 368.F
left semihereditary 200.K
local 284.D
local (of a subvariety) 16.B
locally Macaulay 284.D
Macaulay 284.D
Macaulay local 284.D
Noetherian 284.A
Noetherian local 284.D
Noetherian semilocal 284.D
normal 67.I
normed 36.A
of operators 308.C
of p -adic integers 439.F
polynomial 337.A 369.A
polynomial, in m variables 337.B
of polynomials 337.A 369
power series 370.A
of power series 370
primary 368.H
primitive 368.H
principal ideal 67.K
Prüfer 200.K
pseudogeometric 284.F
quasilocal 284.D
quasisemilocal 284.D
quasisimple 368.E
quotient 368.E

of quotients of a ring with respect to a prime ideal 67.G
of quotients of a ring with respect to a subset of the ring 67.G
regular 85.B 284.D
regular local 284.D
representation 237.H
representative (of a compact Lie group) 249.U
residue class, modulo an ideal 368.F
right Artinian 368.F
right hereditary 200.K
right Noetherian 368.F
right semihereditary 200.K
of scalars (of a module) 277.D
semihereditary 200.K
semilocal 284.D
semiprimary 368.H
semiprimitive 368.H
semisimple 368.G
simple 368.G
splitting 29.K
topological 423.P
of total quotients 67.G
unitary 368.A 409.C
universally Japanese 284.F
of a valuation 439.B
of valuation vectors 6.C
Zariski 284.C
zero 368.A
ringed space 383.H
 local 383.H
ring homomorphism 368.D
ring isomorphism 368.D
ring operations 368.A
ripple 205.F
risk
 Bayes 398.B
 consumer's 404.C
 posterior 399.F
 producer's 404.C
risk function 398.A
risk premium 214.B
risk theory 214.C
 classical 214.C
 collective 214.C
 individual 214.C
Ritt basis theorem (on differential polynomials) 113
Ritz method 46.F 303.I 304.B
 Rayleigh- 46.F 271.G
Robbins-Kiefer inequality, Chapman- 399.D
Robertson-Walker metrics 359.E
Robin constant 48.B
Robin problem 323.F
robust and nonparametric method 371
robust estimation 371.A
robust method 371.A
Roch \rightarrow Riemann-Roch
Roche-Schlömilch remainder App. A, Table 9.IV
Rodrigues formula 393.B
Roepstorff-Araki-Sewell inequality 402.G
Roepstorff-Fannes-Verbeure inequality 402.G
Rogers theorem, Dvoretzky- 443.D
Rokhlin theorem 114.K
Rolle theorem 106.E
rolling curve (of a roulette) 93.H
Roman and medieval mathematics 372
Romberg integration 299.C
Room square 241.D
root

(of a chamber complex) 13.R
 (of a polynomial) 337.B
 characteristic (of an autonomous linear system) 163.F
 characteristic (of a linear mapping) 269.L
 characteristic (for a linear partial differential equation with variable coefficients) 325.F
 characteristic (of a matrix) 269.F
 co- 13.J
 imaginary (of an algebraic equation) 10.E
 k - 13.Q
 m th 10.C
 multiple (of an algebraic equation) 10.B
 negative (of a semisimple Lie algebra) 248.M
 of a polynomial 337.B
 positive (of a semisimple Lie algebra) 248.M
 primitive, modulo m 297.G
 primitive, of unity 14.L
 real (of an algebraic equation) 10.E
 of a semisimple algebraic group 13.J
 of a semisimple Lie algebra 248.K
 simple (of an algebraic equation) 10.B
 simple (in a root system) 13.J
 simple (of a semisimple Lie algebra) 248.M
 root extraction 10.C
 root subspace (of a linear operator) 390.B
 root subspace (of a semisimple Lie algebra) 248.K
 root system
 (of a semisimple algebraic group) 13.J
 (of a semisimple Lie algebra) 248.K
 (of a symmetric Riemannian space) 413.F
 (in a vector space over \mathbb{Q}) 13.J
 fundamental (of a semisimple Lie algebra) 248.N
 irreducible 13.L
 root vector 390.B
 Rosen gradient projection method 292.E
 rot (rotation) 136.D 442.D, App. A, Table 3.II
 rotatable 102.M
 rotation App. A, Table 3.II
 (of a differentiable vector field) 442.D
 (on a locally compact Abelian group) 136.D
 axis of (of a surface of revolution) 111.I
 rotational 205.B
 rotational coordinates App. A, Table 3.V
 rotation group 60.I
 rotation number 99.D 111.E 126.I
 rotation theorem 438.B
 Roth theorem 118.D 182.G
 Rouché theorem 10.E 99.D 198.F
 roulette 93.H
 roundoff error 138.B
 global 303.B
 row (of a matrix) 269.A
 iterated series by (of a double series) 379.E
 repeated series by (of a double series) 379.E
 row finite matrix 269.K
 row nullity (of a matrix) 269.D
 row vector 269.A
 Royden compactification 207.C
 Royden theorem, Arens- 36.M
 Rückert zero-point theorem 23.B
 Ruelle scattering theory, Haag- 150.D
 ruin probability 214.C
 rule
 Adler-Weisberger sum 132.C
 chain (on the differentiation of composite functions) 106.C
 Cramer 269.M
 Feynman 146.A,B

Subject Index

σ -discrete (covering of a set)

formation 411.D
 of inference 411.I
 midpoint 303.E
 Napier App. A, Table 2.II
 product 299.D
 projection 31.B
 selection 351.H
 sequential decision 398.F
 Simpson's $1/3$ 299.A
 Simpson's $3/8$ 299.A
 slide 19.A
 stopping 398.F
 terminal decision 398.F
 trapezoidal (of numerical integration) 299.A
 trapezoidal (of numerical solution of ordinary differential equations) 303.E
 univalence superselection 351.K
 ruled surface
 (algebraic surface) 15.E
 (in differential geometry) 111.I
 criterion of 15.E
 ruler 155.G 179.A
 run (in a sequence of Bernoulli trials) 396.C
 Runge-Kutta-Gill method 303.D
 Runge-Kutta method 303.D
 explicit 303.D
 general 303.D
 implicit 303.D
 pseudo- 303.D
 semi-explicit 303.D
 semi-implicit 303.D
 Runge-Kutta (P, p) method, region of absolute stability of 303.G
 Runge phenomenon 223.A
 Runge theorem (on polynomial approximation) 336.F
 Russell paradox 319.B
 Ryser-Chowla theorem, Bruck- 102.E

S

 \mathcal{S} (the totality of rapidly decreasing C^∞ -functions) 168.B
 \mathcal{S}' (the totality of tempered distributions) 125.N
 S , space of type 125.T
 $SL(n, K)$ (special linear group) 60.B
 $Sp(A)$ (Spur of a matrix A) 269.F
 $SP(n, K)$ (symplectic group) 60.L
 $SU(n)$ (special unitary group) 60.F
 $S(\Omega)$ (totality of measurable functions on Ω that take finite value almost everywhere) 168.B

 * \rightarrow also star
 *-automorphism group 36.K
 *-derivation 36.K
 *-homomorphism 36.F
 *-representation (of a Banach *-algebra) 36.F
 *-subalgebra 443.C
 σ -additive measure 270.D
 σ -additivity 270.D
 σ -algebra 270.B
 optional 407.B
 predictable 407.B
 tail 342.G
 topological 270.C
 well-measurable 407.B
 σ -compact space 425.V
 σ -complete (vector lattice) 310.C
 σ -complete lattice 243.D
 conditionally 243.D
 σ -discrete (covering of a set) 425.R

σ -field σ -field

- Bayer sufficient 396.J
- boundedly complete 396.E
- complete 396.E
- D-sufficient 396.J
- decision theoretically sufficient 396.J
- minimal sufficient 396.F
- pairwise sufficient 396.F
- test sufficient 396.J
- σ -finite (measure space) 270.D
- σ -function, of Weierstrass 134.F, App. A, Table 16.IV
- σ -locally finite covering (of a set) 425.R
- σ -process (of a complex manifold) 72.H
- σ -space 425.Y
- σ -subfield
 - necessary 396.E
 - sufficient 396.E
- σ -weak topology 308.B
- Σ_1^1 set 22.A
- Σ_n^1 set 22.D
- Σ -space 425.Y
- s-cobordism 65.C
- s-cobordism theorem 65.C
- s-field 149.A
- s-handle 114.F
- s^h -factorial experiment 102.H
- S-admissible (lattice in \mathbf{R}^n) 182.B
- S-flow 136.D
- S-levels 102.K
- S-matrix 150.D
- S-matrix theory 386.C
- S-morphism 52.G
- S-number 430.C
- S-object 52.G
- S-operator 150.D 386.B
- S-scheme 16.D
- S-set 308.I
- S-topology (on a linear space) 424.K
- S-wave 351.E
- (S)-space 424.S
- S*-number 430.C
- (S, \mathbb{C})-valued random variable 342.C
- s-parallelizable (manifold) 114.I
- Sacks bound, Varshamov-Gilbert- 63.B
- saddle point
 - (differential game) 108.B
 - (of a dynamical system) 126.G
 - (of a function) 255.B
 - (in nonlinear programming) 292.A
 - (on a surface) 111.H
- saddle-point method 25.C
- saddle set 126.E
- Sakata model 132.D
- Salam model, Glashow-Weinberg- 132.D
- same kind (of mathematical systems) 409.B
- same orientation (for oriented atlases) 105.F
- same shape 382.A
- sample(s) 373.A 401.E
 - Bernoulli 396.B
 - random 374.A 396.B 401.F
 - small 401.F
- sample autocovariance 421.B
- sample characteristic 396.C
- sample characteristic value 396.C
- sample correlation coefficient 396.D
- sample covariance 396.D
- sample covariance function 395.G
- sample function 407.A
- sample generalized variance 280.E
- sample mean 396.C
- sample median 396.C
- sample mode 396.C
- sample moment of order k 396.C
- sample multiple correlation 280.E
- sample number, average 404.C
- sample partial correlation coefficient 280.E
- sample path 407.A
- sample point 342.B 396.B 398.A
- sample problem, k - 371.D
- sample process 407.A
- sample range 396.C
- sample size 373.A
- sample space 342.B 396.B 398.A
- sample standard deviation 396.C
- sample survey 373
- sample theory, large 401.E
- sample value 396.B
- sample variance 396.C
- sampling
 - exact, theory 401.F
 - multistage 373.E
 - optional 262.C
 - optional, theorem 262.A
 - stratified 373.E
 - two-stage 373.E
- sampling distribution 374.A
- sampling inspection 404.C
 - with adjustment 404.C
 - by attributes 404.C
 - double 404.C
 - multiple 404.C
 - with screening 404.C
 - sequential 404.C
 - single 404.C
 - by variables 404.C
- sampling inspection plan 404.C
- sampling inspection tables 404.C
- sampling procedure 373.A
 - invariant 373.C
 - random 373.A
 - regular 373.A
 - uniform 373.A
- Sard theorem 105.J 208.B
- Sard-Smale theorem 286.P
- Sasakian manifold 110.E
- Sasaki-Nitsche formula, Gauss-Bonnet- 275.C
- Satake diagram App. A, Table 5.II
 - (of a compact symmetric Riemannian space) 437.AA
 - (of a real semisimple Lie algebra) 248.U
- satellite
 - left 200.I
 - right 200.I
- satisfiability, problem of (of a proposition) 97
- satisfiable (formula) 276.C
- Sato-Bernsteĭn polynomial 125.EE
- Sato conjecture 450.S
- saturated
 - ((B, N)-pair) 151.J
 - (fractional factorial design) 102.I
- saturated model, κ - 293.B
- Savage theorem, Girshick- 399.F
- Savage zero-one law, Hewitt- 342.G
- savings premium 214.B
- Sazonov topology 341.J
- SC^p -manifold 178.G
- scalar(s)
 - (in a linear space) 256.A.J
 - (of a module over a ring) 277.D

field of (of a linear space) 256.A
 ring of (of a module) 277.D
 scalar change (of a B -module) 277.L
 scalar curvature 364.D, App. A, Table 4.II
 scalar extension
 (of an algebra) 29.A
 (of an A -module) 277.L
 (of a linear representation) 362.F
 scalar field 105.O
 (in a 3-dimensional Euclidean space) 442.D
 free 377.C
 scalar integral 443.F,I
 scalarly integrable 443.F,I
 scalarly measurable 443.B,I
 scalar matrix 269.A
 scalar multiple
 (of an element of a module) 277.D
 (of a linear operator) 37.C
 (in a linear space) 256.A
 (of a vector) 442.A
 scalar multiplication
 (in a module) 277.D
 (on vectors) 442.A
 scalar operator 390.K
 scalar potential 130.A 442.D
 scalar product 442.B, App. A, Table 3.I
 scalar restriction (of a B -module) 277.L
 scalar sum (of linear operators) 37.C
 scalar triple product 442.C
 scale
 of Banach space 286.Z
 canonical 115.B
 natural 260.G
 ordinal 397.M
 two-sided 19.D
 scaled, u_i - 19.D
 scale matrix 374.C
 scale parameter 396.I 400.E
 scaling, metric multidimensional 346.E
 scaling, multidimensional 346.E
 scaling method 346.E
 scatter diagram 397.H
 scattered (sheaf) 383.E
 scattered set 425.O
 scattered zeros, function with 208.C
 scattering 375.A
 data 387.D
 elastic 375.A
 inelastic 375.A
 scattering amplitude 375.C 386.B
 partial wave 375.E
 scattering cross section 375.A
 scattering data 387.D
 scattering operator 375.B,F,H
 scattering state 375.B
 completeness of the 150.D
 scattering theory, Haag-Ruelle 150.D
 Schafheitlin formula, Sonine- App. A, Table 19.III
 Schauder basis 37.L
 Schauder degree, Leray- 286.D
 Schauder estimate 323.C
 Schauder fixed-point theorem 153.D 286.D
 Leray- 286.D 323.D
 Schauder theorem, Riesz- 68.E
 scheduling 376
 job-shop 307.C
 model 307.C
 network 307.C
 scheduling and production planning 376
 scheduling problem

Subject Index

Schmidt theorem, Knopp-

flow-shop 376
 job-shop 376
 machine 376
 multiprocessor 376
 Scheffé model 346.C
 Scheffé theorem, Lehmann- 399.C
 Scheja theorem 21.M
 schema of Souslin 22.B
 scheme 16.D
 adaptive 299.C
 affine 16.D
 Aitken's interpolation 223.B
 algebraic 16.D
 automatic integration 299.C
 coarse moduli 16.W
 complete 16.D
 consistent-mass 304.D
 deformation of X over a connected 16.W
 difference 304.E
 difference, of backward type 304.F
 difference, of forward type 304.F
 explicit 304.F
 fine moduli 16.W
 formal 16.X
 Friedrichs 304.F
 group 16.H
 Hilbert 16.S
 implicit 304.F
 integral 16.D
 inverted filing 96.F
 irreducible 16.D
 K -complete 16.D
 Lax-Wendroff 304.F
 locally Noetherian formal 16.X
 moduli 16.W
 morphism of 16.D
 Noetherian 16.D
 nonadaptive 299.C
 Picard 16.P
 projective 16.E
 quasiprojective 16.E
 S - 16.D
 over S 16.D
 separated 16.D
 separated formal 16.X
 Scherk's surface 275.A
 Schläfli diagram (of a complex semisimple Lie algebra) 248.S
 Schläfli formula App. A, Table 19.III
 Schläfli integral representation 393.B
 Schläfli polynomial App. A, Table 19.IV
 Schlesinger equations 253.E
 schlicht 438.A
 schlicht Bloch constant 77.F
 schlichtartig 367.G
 Schlieder theorem, Reeh- 150.E
 Schlömilch criterion App. A, Table 10.II
 Schlömilch remainder, Roche- App. A, Table 9.N
 Schlömilch series 39.D, App. A, Table 19.III
 generalized 39.D
 Schmidt class, Hilbert- 68.I
 Schmidt condition 379.M
 Schmidt expansion theorem, Hilbert- 217.H
 Schmidt norm, Hilbert- 68.I
 Schmidt orthogonalization 317.A
 Gram- 317.A
 Schmidt procedure, Lyapunov- 286.V
 Schmidt theorem 118.D
 Schmidt theorem, Knopp- 208.C

Schmidt theorem, Krull-Remak-

- Schmidt theorem, Krull-Remak- (in group theory) 190.L
- Schmidt type, integral operator of Hilbert- 68.C
- Schmidt type, kernel of Hilbert- 217.I
- Schnee theorem, Knopp- (on method of summation) 379.M
- Schoenflies notation (for crystal classes) 92.E, App. B, Table 6.IV
- Schoenflies problem 65.G
- Schoenflies theorem 65.G
- Schottky group 234.B
- Schottky theorem 43.J
- Schottky uniformization 367.C
- Schrader axioms, Osterwalder- 150.F
- Schreier conjecture (on simple groups) 151.I
- Schreier extension, Artin- (of a field) 172.F
- Schröder equation, Königs- 44.B
- Schröder functional equation 388.D
- Schrödinger equation 351.D
- 1-body 351.E
- random 340.E
- time-dependent 351.D
- time-independent 351.D
- Schrödinger operator 351.D
- Schrödinger picture 351.D
- Schrödinger representation 351.C
- Schrödinger series, Rayleigh- 331.D
- Schubert cycle 56.E
- Schubert variety 56.E
- Schur index
- (of a central simple algebra) 29.E
- (of an irreducible representation) 362.F
- Schur lemma
- (on linear representations) 362.C
- (on simple modules) 277.H 368.G
- (on unitary representations) 437.D
- Schur subgroup 362.F
- Schur theorem (on linear transformations of sequences) 379.L
- Schur theorem, Kojima- (on linear transformations of sequences) 379.L
- Schur-Zassenhaus theorem (on Hall subgroups) 151.E
- Schwartz-Christoffel transformation 77.D, App. A, Table 13
- Schwartz-Christoffel transformation formula 77.D
- Schwartz integral, Bartle-Dunford- 443.G
- Schwartz space 424.S
- Schwarzian derivative App. A, Table 9.III
- Schwarz inequality 211.C
- Cauchy- 211.C, App. A, Table 8
- Schwarz lemma 43.B
- Schwarz principle of reflection 198.G
- Schwinger equation, Lippmann- 375.C
- Schwinger function 150.F
- Schwinger points 150.F
- sciences, information 75.F
- scores
- canonical 397.M
- factor 280.G 346.F
- score test, Fisher-Yates-Terry normal 371.C
- scoring method 397.M
- screening, sampling inspection with 404.C
- seasonal adjustment 397.N
- sec (secant) 131.E
- secant 432.A
- hyperbolic 131.F
- sech (hyperbolic secant) 131.F
- second (unit of an angle) 139.D
- secondary cohomology operation, stable 64.C
- secondary components (of a homogeneous space) 110.A
- secondary composition 202.R
- secondary obstruction 305.D
- secondary parameters 110.A
- second axiom, Tietze's 425.Q
- second barycentric derived neighborhood 65.C
- second boundary value problem
- (for harmonic functions) 193.F
- (of partial differential equations of elliptic type) 323.F
- second category, set of 425.N
- second classification theorem (in the theory of obstructions) 305.C
- second complementary law (of Legendre symbols) 297.I
- second countability axiom 425.P
- second Cousin problem 21.K
- second definition (of an algebraic K -group) 237.J
- second difference 104.A
- second extension theorem (in the theory of obstructions) 305.C
- second factor (of a class number) 14.L
- second fundamental form (of an immersion of a manifold) 111.G 365.C, App. A, Table 4.I
- second fundamental quantities (of a surface) 111.H
- second fundamental tensor 417.F
- second fundamental theorem (in Morse theory) 279.D
- second homotopy theorem 305.C
- second incompleteness theorem 185.C
- second isomorphism theorem (on topological groups) 423.J
- second kind
- (Abelian differential of) 11.C
- (Abelian integral of) 11.C
- (Fuchsian group of) 122.C
- (integral equations of Fredholm type of) 217.A
- perfect number of 297.D
- Stirling number of 66.D
- second law of cosines 432.A, App. A, Table 2.II
- second law of thermodynamics 419.A
- second mean value theorem
- (for the D -integral) 100.G
- (for the Riemann integral) 216.B
- (for the Stieltjes integral) 94.C
- second-order asymptotic efficiency 399.O
- second-order design 102.M
- second-order efficiency 399.O
- second-order predicate 411.K
- second-order predicate logic 411.K
- second quantization 377
- second separation axiom 425.Q
- second variation formula 178.A
- section
- (of a finite group) 362.I
- (of a sheaf space) 383.C
- circular 350.F
- conic 78
- cross- 126.C 286.H
- cross (of a fiber bundle) 147.L
- cross (of a fiber space) 148.D
- cross-, for a closed orbit 126.G
- differential cross 375.A 386.B
- local 126.E
- local cross (of a fiber bundle) 147.E
- n - (in a cell complex) 70.D
- normal (of a surface) 410.B
- r - (of a Euclidean complex) 70.B

- r - (of a simplicial complex) 70.C
- scattering cross 375.A
- set of (of a sheaf) 383.C
- total (elastic) cross 386.B
- zero- (of a block bundle) 147.Q
- sectional curvature 364.D
 - holomorphic 364.D
- section graph 186.C
- sectors, superselection 150.E 351.K
- secular equation 55.B 269.F
- secular perturbation 55.B
- sedenion 29.D
- segment 155.B 178.H
 - (in affine geometry) 7.D
 - (in an ordered set) 311.B
 - oriented 442.A
- Seidel method, Gauss- 302.C
- Seifert conjecture 126.N 154.D
- Seifert matrix 235.C
- Seifert surface 235.A
- Selberg sieve 123.E
- Selberg theorem, Evans- 48.E 338.H
- Selberg zeta function 450.T
- selection, measurable 443.I
- selection, model 401.D
- selection function 354.E
- selection parameter 396.F
- selection rule 351.H
- selection statistic 396.F
- self-adjoint
 - (linear homogeneous ordinary differential equation) 315.B
 - essentially 251.E 390.I
- self-adjoint differential equation 252.K
- self-adjoint differential operator, formally 112.I
- self-adjoint operator 251.E 390.F
- self-adjoint system of differential equations 252.K
- self-commutator 251.K
- self-dual (linear space) 256.H
- self-dual (regular cone) 384.E
- self-dual, anti- (G -connection) 80.Q
- self-excited vibration 318.B
- self-information 213.B
- self-intersection number 15.C
- self-loop 186.B
- self-polar tetrahedron 350.C
- self-polar triangle 78.J
- self-reciprocal function 220.B
- semicontinuity, lower (of length) 246.A
- semicontinuous 84.C
 - (mapping in a topological linear space) 153.D
 - lower 84.C
 - upper 84.C
- semicontinuous function 84.C
- semicontinuous partition, upper 425.L
- semidefinite Hermitian form 348.F
- semidefinite kernel, positive 217.H
- semidefinite matrix, positive 269.I
- semidefinite operator, positive 251.E
- semidefinite quadratic form, positive or negative 348.C
- semidirect product (of two groups) 190.N
- semidiscrete approximation 304.B
- semiexact (differential on an open Riemann surface) 367.I
- semi-explicit 303.D
- semifinite (von Neumann algebra) 308.E
- semifinite (weight on a von Neumann algebra) 308.D
- semiflow 126.B
- of class C^r 126.B
- continuous 126.B
- discrete 126.B
- discrete, of class C^r 126.B
- semigroup 88.E 190.P 409.A
 - (of a Markov process) 261.B
 - of class (C^0) 378.B
 - differentiable 378.F
 - distribution 378.F
 - dual 378.F
 - equicontinuous, of class (C^0) 378.B
 - free 161.A
 - holomorphic 378.D
 - locally equicontinuous 378.F
 - nonlinear 378.F
 - nonlinear, of operators 286.X
 - of operators 378
 - order-preserving 286.Y
 - unitary 409.C
- semigroup algebra 29.C
 - large 29.C
- semigroup bialgebra 203.G
- semihereditary ring 200.K
 - left 200.K
 - right 200.K
- semi-implicit 303.D
- semi-integral, left 68.N
- semi-integral, right 68.N
- semi-intuitionism 156.C
- semi-invariant 226.A
 - G - 226.A
 - of a probability distribution 341.C
- semilattice 243.A
 - lower 243.A
 - upper 243.A
- semilinear (partial differential equations of elliptic type) 323.D
- semilinear mapping 256.P 277.L
- semilinear transformation 256.P
- semilocal ring 284.D
 - analytically unramified 284.D
 - Noetherian 284.D
 - quasi- 284.D
- semilogarithmic paper 19.F
- semimartingale 262.E 406.B
 - continuous 406.B
- semimartingale decomposition 406.B
- seminorm (on a topological linear space) 424.F
- semiorbit 126.D
 - negative 126.D
 - positive 126.D
- semiordered set 311.A
- semiordering 311.A
- semipolar set 261.D
- semiprimary ring 368.H
- semiprime differential ideal (of a differential ring) 113
- semiprime ideal (of a differential ring) 113
- semiprimitive ring 368.H
- semireductive (action defined by a rational representation) 226.B
- semireflexive (locally convex space) 424.O
- semiregular point (of a surface in E^3) 111.J
- semiregular transformation (of a sequence) 379.L
- semisimple
 - (algebraic group) 13.I
 - (Banach algebra) 36.D
 - (Jordan algebra) 231.B
 - (Lie algebra) 248.E
 - (Lie group) 249.D

- (matrix) 269.G
- semisimple algebra 29.A
- semisimple A -module 277.H
- semisimple component (of a linear transformation) 269.L
- semisimple linear representation 362.C
- semisimple linear transformation 269.L
- semisimple part
 - of an algebraic group 13.E
 - of a nonsingular matrix 13.E
- semisimple ring 368.G
- semisimplicial complex 70.E
- semisimplicity, Cartan's criterion of 248.F
- semistable (coherent sheaf) 16.Y
- semistable distribution 341.G
- semistable reduction theorem 16.Z
- semistable vector bundle (algebraic) 16.Y
- semivariation 443.G
- sensitive grammar, context- 31.D
- sensory test 346.B
- separable
 - (function in nomograms) 19.D
 - (polynomial) 337.G
 - (rational mapping) 16.I
 - (stochastic process) 407.A
 - (topological space) 425.P
 - perfectly 425.P
- separable algebra 29.F,K 200.L
 - central 29.K
- separable element (of a field) 149.H
- separable extension
 - (of a field) 149.H,K
 - maximal (of a field) 149.H
- separable metric space 273.E
- separably generated extension (of a field) 149.K
- separated
 - (formal scheme) 16.X
 - (morphism) 16.D
- separated convex sets, strongly 89.A
- separated kernel 217.F
- separated scheme 16.D
- separated space 425.Q
- separated S -scheme 16.D
- separated topological group 423.B
- separated type App. A, Table 14.I
- separated uniform space 436.C
- separated variable type App. A, Table 15.II
- separately continuous (bilinear mapping) 424.Q
- separating family 207.C
- separating transcendence basis (of a field extension) 149.K
- separation
 - axioms of (in set theory) 33.B
 - of variables 322.C
- separation axioms 425.Q
 - the first 425.Q
 - the fourth 425.Q
 - the second 425.Q
 - the third 425.Q
 - Tikhonov 425.Q
- separation cochain 305.B
- separation cocycle 305.B
- separation principle 405.C
- separation theorem (on convex sets) 89.A
- separator 186.F
- sequence(s) 165.D
 - admissible (in Steenrod algebra) 64.B, App. A, Table 6.III
 - asymptotic 30.A
 - of Bernoulli trials 396.B
 - Blaschke 43.F
 - Cauchy (in a -adic topology) 284.B
 - Cauchy (in a metric space) 273.J
 - Cauchy (of rational numbers) 294.E
 - Cauchy (of real numbers) 355.B
 - Cauchy (in a uniform space) 436.G
 - cohomology exact 201.L
 - cohomology spectral 200.J
 - connected, of functors 200.I
 - convergent (of real numbers) 87.B 355.B
 - divergent (of real numbers) 87.B
 - double 379.E
 - exact (of A -homomorphisms of A -modules) 277.E
 - exact, of cohomology 200.F
 - exact, of Ext 200.G
 - exact, of homology 200.C
 - exact, of Tor 200.D
 - of factor groups (of a normal chain) 190.G
 - Farey 4.B
 - Fibonacci 295.A
 - finite 165.D
 - of functions 165.B,D
 - fundamental (in a metric space) 273.J
 - fundamental (of rational numbers) 294.E
 - fundamental (of real numbers) 355.B
 - fundamental (in a uniform space) 436.G
 - fundamental, of cross cuts (in a simply connected domain) 333.B
 - fundamental exact (on cohomology of groups) 200.M
 - Gysin exact (of a fiber space) 148.E
 - Hodge spectral 16.U
 - homology exact (of a fiber space) 148.E
 - homology exact (for simplicial complexes) 201.L
 - homotopy exact 202.L
 - homotopy exact (of a fiber space) 148.D
 - homotopy exact (of a triad) 202.M
 - homotopy exact (of a triple) 202.L
 - independent, of partitions 136.E
 - infinite 165.D
 - interpolating 43.F
 - Jordan-Hölder (in a group) 190.G
 - linear recurrent 295.A
 - Mayer-Vietoris exact 201.C
 - minimizing 46.E
 - monotone (of real numbers) 87.B
 - monotonically decreasing (of real numbers) 87.B
 - monotonically increasing (of real numbers) 87.B
 - normal (of open coverings) 425.R
 - null (in a -adic topology) 284.B
 - of numbers 165.D
 - (o)-convergent 87.L
 - (o)-star convergent 87.L
 - order-convergent (in a vector lattice) 310.C
 - oscillating (of real numbers) 87.D
 - of points 165.D
 - pointwise convergent 435.B
 - positive definite 192.B
 - of positive type 192.B
 - Puppe exact 202.G
 - random 354.E
 - rapidly decreasing 168.B
 - recurrent, of order r 295.A
 - reduced homology exact 201.F
 - regular (of Lebesgue measurable sets) 380.D
 - regular spectral 200.J

- relative Mayer-Vietoris exact 201.L
- (R, S) -exact (of modules) 200.K
- of sets 165.D
- short exact 200.I
- simply convergent 435.B
- slowly increasing 168.B
- spectral 200.J
- spectral (of singular cohomology of a fiber space) 148.E
- standard 400.K
- symbol (in the theory of microdifferential operators) 274.F
- of Ulm factors (of an Abelian p -group) 2.D
- uniformly convergent 435.A
- Wang exact (of a fiber space) 148.E
- sequencing problem, machine 376
- sequential decision function 398.F
- sequential decision problem 398.F
- sequential decision rule 398.F
- sequentially compact (space) 425.S
- sequential probability ratio test 400.L
- sequential sampling inspection 404.C
- sequential space 425.CC
- sequential test 400.L
- serial correlation coefficient 397.N 421.B
- serial cross correlation coefficient 397.N
- series 379, App. A, Table 10
 - absolutely convergent 379.C
 - absolutely convergent double 379.E
 - allied (of a trigonometric series) 159.A
 - alternating 379.C
 - ascending central (of a Lie algebra) 248.C
 - asymptotic 30
 - asymptotic power 30.A
 - binomial App. A, Table 10.IV
 - binomial coefficient 121.E
 - characteristic (in a group) 190.G
 - commutatively convergent 379.C
 - complementary (of unitary representations of a complex semisimple Lie group) 437.W
 - complementary degenerate (of unitary representations of a complex semisimple Lie group) 437.W
 - composition (in a group) 190.G
 - composition (in a lattice) 243.F
 - composition factor (of a composition series in a group) 190.G
 - conditionally convergent 379.C
 - conditionally convergent double 379.E
 - conjugate (of a trigonometric series) 159.A
 - convergent 379.A
 - convergent double 379.E
 - convergent power 370.B
 - convergent power, ring 370.B
 - degenerate (of unitary representations of a complex semisimple Lie group) 437.W
 - derived (of Lie algebra) 248.C
 - descending central (of a Lie algebra) 248.C
 - Dini 39.D
 - Dirichlet 121
 - Dirichlet, of the type $\{\lambda_n\}$ 121.A
 - discrete (of unitary representations of a semisimple Lie group) 437.X
 - divergent 379.A
 - divergent double 379.E
 - double 379.E
 - Eisenstein 32.C
 - Eisenstein-Poincaré 32.F
 - exponential 131.D
 - factorial 104.F 121.E
 - field of formal power, in one variable 370.A
 - finite 379.A, App. A, Table 10.I
 - formal power 370.A
 - formal power, field in one variable 370.A
 - formal power, ring 370.A
 - Fourier 159 197.C, App. A, Table 11.I
 - Fourier (of an almost periodic function) 18.B
 - Fourier (of a distribution) 125.P
 - Fourier-Bessel 39.D
 - Fourier cosine App. A, Table 11.I
 - Fourier sine App. A, Table 11.I
 - Gauss 206.A
 - generalized Eisenstein 450.T
 - generalized Schlömilch 39.D
 - generalized trigonometric 18.B
 - geometric 379.B, App. A, Table 10.I
 - Heine 206.C
 - hypergeometric 206.A
 - infinite 379.A, App. A, Table 10.III
 - iterated, by columns (of a double series) 379.E
 - iterated, by rows (of a double series) 379.E
 - Kapteyn 39.D, App. A, Table 19.III
 - Lambert 339.C
 - Laurent 339.A
 - logarithmic 131.D
 - lower central (of a group) 190.J
 - majorant 316.G
 - majorant (of a sequence of functions) 435.A
 - Neumann 217.D
 - of nonnegative terms 379.B
 - ordinary Dirichlet 121.A
 - orthogonal (of functions) 317.A
 - oscillating 379.A
 - π - (of a group) 151.F
 - Poincaré 32.B
 - of positive terms 379.B
 - power 21.B 339 370.A, App. A, Table 10.IV
 - power (in a complete ring) 370.A
 - power, with center at the point at infinity 339.A
 - power, ring 370.A
 - principal (in an Ω -group) 190.G
 - principal (of unitary representations of a complex semisimple Lie group) 258.C 437.W
 - principal (of unitary representations of a real semisimple Lie group) 258.C 437.X
 - principal H - 437.X
 - properly divergent 379.A
 - Puiseux 339.A
 - repeated, by columns (of a double series) 379.E
 - repeated, by rows (of a double series) 379.E
 - ring of convergent power 370.B
 - ring of formal power 370.A
 - ring of power 370.A
 - Schlömilch 39.D, App. A, Table 19.III
 - simple 379.E
 - singular 4.D
 - supplementary 258.C
 - Taylor 339.A
 - termwise integrable 216.B
 - theta 348.L
 - theta-Fuchsian, of Poincaré 32.B
 - time 397.A 421.A
 - trigonometric 159.A
 - unconditionally convergent 379.C
 - uniformly absolutely convergent 435.A
 - upper central (of a group) 190.J
 - Serre conjecture 369.F
 - Serre \mathcal{C} -theory 202.N

- Serre duality theorem
 - (on complex manifolds) 72.E
 - (on projective varieties) 16.E
- Serre formulas, Frenet- (on curves) 111.D, App. A, Table 4.I
- Serre theorem (for ample line bundles) 16.E
- sesquilinear form
 - (on a linear space) 256.Q
 - (on a product of two linear spaces) 256.Q
 - matrix of 256.Q
 - nondegenerate 256.Q
- set(s) 381
 - A- 22.A 409.A
 - absolutely convex (in a linear topological space) 424.E
 - α -limit 126.D
 - analytic 22.A.I
 - analytic (in the theory of analytic spaces) 23.B
 - analytically thin (in an analytic space) 23.D
 - of analyticity 192.N
 - analytic wave front 274.D
 - of antisymmetry 164.E
 - arbitrary 381.G
 - asymptotic 62.A
 - asymptotic ratio 308.I
 - axiom of power 33.B 381.G
 - B_n 22.D
 - \mathfrak{B} -measurable 270.C
 - Baire 126.H 270.C
 - bargaining 173.D
 - basic (for an Axiom A flow) 126.J
 - basic (of a structure) 409.B
 - basic open 425.F
 - bifurcation 51.F 418.F
 - border 425.N
 - Borel (in a Euclidean space) 270.C
 - Borel (in a topological space) 270.C
 - Borel in the strict sense 270.C
 - boundary 425.N
 - boundary cluster 62.A
 - bounded (in an affine space) 7.D
 - bounded (in a locally convex space) 424.F
 - bounded (in a metric space) 273.B
 - C_n 22.D
 - CA 22.A
 - Cantor 79.D
 - capacity of 260.D
 - catastrophe 51.F
 - category of 52.B
 - chain recurrent 126.E
 - characteristic (of an algebraic family on a generic component) 15.F
 - characteristic (of a partial differential operator) 320.B
 - choice 34.A
 - closed 425.B
 - cluster 62.A
 - coanalytic 22.A
 - compact (in a metric space) 273.F
 - compact (in a topological space) 425.S
 - complementary 381.B
 - complementary analytic 22.A
 - complete 241.B
 - complete orthonormal (of a Hilbert space) 197.C
 - connected 79.A
 - constraint (of a minimization problem) 292.A
 - convex 7.D 89
 - countably equivalent (under a nonsingular bimeasurable transformation) 136.C
 - curvilinear cluster 62.C
 - cylinder 270.H
 - of degeneracy (of a holomorphic mapping between analytic spaces) 23.C
 - Δ_n^1 22.D
 - dense 425.N
 - dependent 66.G
 - derived 425.O
 - determining (of a domain in \mathbb{C}^n) 21.C
 - difference (of blocks) 102.E
 - directed 311.D
 - discrete 425.O
 - disjoint 381.B
 - dominating 186.I
 - empty (\emptyset) 381.A
 - externally stable 186.I
 - equipollent 49.A
 - equipotent 49.A
 - F_σ 270.C
 - factor (of a crossed product) 29.D
 - factor (of an extension of groups) 190.N
 - factor (of a projective representation) 362.J
 - family of 165.D 381.B,D
 - family of (indexed by Λ) 381.D
 - final (of a correspondence) 358.B
 - final (of a linear operator) 251.E
 - finite 49.F 381.A
 - finitely equivalent (under a nonsingular bimeasurable transformation) 136.C
 - of the first category 425.N
 - of the first kind 319.B
 - first negative prolongational limit 126.D
 - first positive prolongational limit 126.D
 - function 380.A
 - function-theoretic null 169.A
 - fundamental (of a transformation group) 122.B
 - fundamental open (of a transformation group) 122.B
 - G_δ 270.C
 - general Cantor 79.D
 - generalized peak 164.D
 - (general) recursive 97
 - germ of an analytic 23.B
 - homotopy 202.B
 - idempotent (of a ring) 368.B
 - increasing directed 308.A
 - independent 66.G 186.I
 - index 102.L
 - index (of a family) 165.D
 - index (of a family of elements) 381.D
 - indexing (of a family of elements) 381.D
 - infinite 49.F 381.A
 - information 173.B
 - initial (of a correspondence) 358.B
 - initial (of a linear operator) 251.E
 - interior cluster 62.A
 - internally stable 186.I
 - interpolating (for a function algebra) 164.D
 - Kronecker 192.R
 - lattice of 243.E
 - lattice-ordered 243.A
 - Lebesgue measurable 270.G
 - Lebesgue measurable (of \mathbb{R}^n) 270.G
 - level 279.D
 - limit 234.A
 - locally closed 425.J
 - M- 159.J
 - meager 425.N
 - minimal 126.E

μ -measurable 270.D
 μ -null 270.D
 of multiplicity 159.J
 n -cylinder 270.H
 nilpotent (of a ring) 368.B
 nodal 391.H
 nonmeager 425.N
 nonsaddle 120.E
 nonwandering 126.E
 nowhere dense 425.N
 null (in a measure space) 270.D 310.I
 null, of class N_α 169.E
 null (\emptyset) 381.A
 ω -limit 126.D
 open 425.B
 ordered \rightarrow ordered set
 ordinate 221.E
 orthogonal (of functions) 317.A
 orthogonal (of a Hilbert space) 197.C
 orthogonal (of a ring) 368.B
 orthonormal (of functions) 317.A
 orthonormal (of a Hilbert space) 197.C
 P_n 22.D
 P -convex (for a differential operator) 112.C
 peak 164.D
 perfect 425.O
 Π_1^1 22.A
 Π_n^1 22.D
 point 381.B
 of points of indeterminacy (of a proper meromorphic mapping) 23.D
 polar (in potential theory) 261.D 338.H
 power 381.B
 precompact (in a metric space) 273.B
 principal analytic 23.B
 projective, of class n 22.D
 purely d -dimensional analytic 23.B
 of quasi-analytic functions 58.F
 quotient (with respect to an equivalence relation) 135.B
 ratio 136.F
 recurrent 260.E
 recursive 356.D
 recursively enumerable 356.D
 regularly convex 89.G
 relative closed 425.J
 relatively compact 425.S
 relatively compact (in a metric space) 273.F
 relatively open 425.J
 removable (for a family of functions) 169.C
 residual 126.H 425.N
 resolvent (of a closed operator) 251.F
 resolvent (of a linear operator) 390.A
 ρ - 308.I
 S - 308.I
 saddle 126.E
 scattered 425.O
 of the second category 425.N
 of the second kind 319.B
 semipolar 261.D
 Sidon 192.R 194.R
 sieved 22.B
 Σ_1^1 22.A
 Σ_n^1 22.D
 singularity (of a proper meromorphic mapping) 23.D
 stable 173.D
 stable, externally 186.I
 stable, internally 186.I

Subject Index

Sheaf (sheaves)

standard 22.I
 strongly P -convex 112.C
 strongly separated convex 89.A
 system of closed 425.B
 system of open 425.B
 ternary 79.D
 thin (in potential theory) 261.D
 totally bounded (in a metric space) 273.B
 totally bounded (in a uniform space) 436.H
 U - 159.J
 of uniqueness 159.J
 universal (for the projective sets of class n) 22.E
 universal (of set theory) 381.B
 wandering (under a measurable transformation) 136.C
 wave front 274.B 345.A
 wave front, analytic 274.D
 weakly wandering 136.C
 weakly wandering (under a group) 136.F
 well-ordered 311.C
 Z - 382.B
 Zariski closed 16.A
 Zariski dense 16.A
 Zariski open 16.A
 set function(s) 380
 additive 380.C
 of bounded variation 380.B
 completely additive 380.C
 finitely additive 380.B
 monotone decreasing 380.B
 monotone increasing 380.B
 μ -absolutely continuous additive 380.C
 μ -singular additive 380.C
 set-theoretic formula 33.B
 set-theoretic topology 426
 set theory 381.F
 axiomatic 33 156.E
 Bernays-Gödel 33.A.C
 Boolean-valued 33.E
 classical descriptive 356.H
 effective descriptive 356.H
 general 33.B
 Gödel 33.C
 Zermelo 33.B
 Zermelo-Fraenkel 33.A.B
 Severi group, Néron-
 (of a surface) 15.D
 (of a variety) 16.P
 Sewell inequality, Roepstorff-Araki- 402.G
 $\operatorname{sgn} P$ (sign) 103.A
 shadow costs 292.C
 shadow price 255.B
 Shafarevich group, Tate- 118.D
 Shafarevich reciprocity law 257.H
 shallow water wave 205.F
 shape
 pointed 382.A
 same 382.A
 shape category 382.A
 shape dominate 382.A
 shape function 223.G
 shape group 382.C
 shape invariant(s) 382.C
 shape morphism 382.A
 shape theory 382
 Shapiro-Lopatinskiĭ condition 323.H
 Shapley value 173.D
 sheaf (sheaves) 383

Sheaf space

- (in étale (Grothendieck) topology) 16.AA
- of Abelian groups 383.B
- analytic 72.E
- associated with a presheaf 383.C
- Čech cohomology group with coefficient 383.F
- coherent, of rings 16.E
- coherent algebraic 16.E 72.F
- coherent analytic 72.E
- constructible 16.AA
- cohomology group with coefficient 383.E
- constant 383.D
- derived 125.W
- flabby 383.E
- of germs of analytic functions 383.D
- of germs of analytic mapping 383.D
- of germs of continuous functions 383.D
- of germs of differentiable sections of a vector bundle 383.D
- of germs of differential forms of degree of r 383.D
- of germs of functions of class C^r 383.D
- of germs of holomorphic functions (on an analytic manifold) 383.D
- of germs of holomorphic functions (on an analytic set) 23.C
- of germs of holomorphic functions (on an analytic space) 23.C
- of germs of regular functions 16.B
- of germs of sections of a vector bundle 383.D
- of groups 383.C
- of ideals of a divisor (of a complex manifold) 72.F
- invertible 16.E
- locally constructible (constant) 16.AA
- of \mathcal{O} -modules 383.I
- orientation 201.R
- pre- 383.A
- pre-, on a site 16.AA
- of rings 383.C
- scattered 383.E
- structure (of a prealgebraic variety) 16.C
- structure (of a ringed space) 383.H
- structure (of a variety) 16.B
- trivial 383.D
- sheaf space 383.C
- shear, modules of elasticity in 271.G
- shearing strain 271.G
- shearing stress 271.G
- shear viscosity, coefficient of 205.C
- sheet(s)
 - hyperboloid of one 350.B
 - hyperboloid of revolution of one 350.B
 - hyperboloid of revolution of two 350.B
 - hyperboloid of two 350.B
 - mean number of (of a covering surface of a Riemann sphere) 272.J
 - number of (of an analytic covering space) 23.E
 - number of (of a covering surface) 367.B
- sheeted, n - 367.B
- Shelah isomorphism theorem, Keisler- 276.E
- Shields-Zeller theorem, Brown- 43.C
- shift 251.O
 - associated with the stationary process 136.D
 - automorphism 126.J
 - Bernoulli 136.D
 - generalized Bernoulli 136.D
 - Markov 136.D
 - phase 375.E 386.B
 - shift operator 223.C 251.O 306.C
 - unilateral 390.I
 - shift transformation 136.D
- Shilov boundary
 - (of a domain) 21.D
 - (for a function algebra) 164.C
 - (of a Siegel domain) 384.D
- Shilov generalized function, Gel'fand- 125.S
- Shmul'yan theorem 424.V
 - Eberlein- 37.G
 - Krein- 37.E 424.O
- Shnirel'man theory, Lyusternik- 286.Q
- shock wave 205.B 446
- shortening 186.E
- shortest-path problem 281.C
- shortest representation (of an ideal) 67.F
- short exact sequence 200.I
- short international notation 92.E
- short range 375.B
- Shrikhande square 102.K
- shrinking (a space to a point) 202.E
- shunt
 - left 115.B
 - right 115.B
- SI (international system of units) 414.A
- side 155.B,F
 - (of an angle) 139.D 155.B
 - (of a complete quadrangle) 343.C
 - (on a line) 155.B
 - (on a plane) 155.B
 - (of a point with respect to a hyperplane) 7.D
 - (of a polygon) 155.F
 - (of a spherical triangle) 432.B
- side cone 258.A
- Sidon set 192.R
- Siegel domain(s) 384
 - of the first kind 384.A
 - generated 384.F
 - irreducible 384.E
 - of the second kind 384.A
 - of the third kind 384.A
- Siegel mean value theorem 182.E
- Siegel modular form of weight k (or of dimension $-k$) 32.F
- Siegel modular function of degree n 32.F
- Siegel modular group of degree n 32.F
- Siegel space of degree n 32.F
- Siegel theorem
 - (on Diophantine equations) 118.D
 - (on positive definite forms) 348.K
- Siegel upper half-space of degree n 32.F
- Siegel zero 123.D
- Siegel zeta function of indefinite quadratic forms 450.K
- sieve 16.AA 22.B
 - Eratosthenes 297.B
 - large 123.E
 - large, method 123.D
 - Selberg 123.E
- sieved set 22.B
- sieve method 4.A
 - large 123.D
- sign (of a permutation) 103.A
- signal process 405.F
- signature
 - (of a Hermitian form) 348.F
 - (of an irreducible representation of $GL(V)$) 60.D
 - (of a knot) 235.C
 - (of a manifold) 56.G

- (of a quadratic form) 348.C
- Hirzebruch, theorem 72.K
- signed Lebesgue-Stieltjes measure 166.C
- signed measure 380.C
- signed rank test 371.B
- signed rank test, Wilcoxon 371.B
- sign test 371.B
- similar
 - (central simple algebra) 29.E
 - (linear representation) 362.C
 - (matrix representation of a semilinear mapping) 256.P
 - (permutation representation) 362.B
 - (projective representation) 362.J
 - (square matrices) 269.G
- similar central simple algebras 29.E
- similar correspondence (between surfaces) 111.I
- similarity
 - (of an affine space) 7.E
 - Prandtl-Glauert law of 205.D
 - Reynolds law of 205.C
 - von Kármán transonic 205.D
- similarly isomorphic (ordered fields) 149.N
- similar mathematical systems 409.B
- similar test 400.D
- similar unitary representations 437.A
- simple
 - (A -module) 277.H
 - (Abelian variety) 3.B
 - (algebraic group) 13.L
 - (eigenvalue) 390.A,B
 - (function) 438.A
 - (Lie algebra) 248.E
 - (Lie group) 249.D
 - (linear representation) 362.C
 - (polygon) 155.F
 - (subcoalgebra) 203.F
 - absolutely (algebraic group) 13.L
 - algebraically (eigenvalue) 390.B
 - almost (algebraic group) 13.L
 - geometrically (eigenvalue) 390.A
 - k - (algebraic group) 13.O
 - k -almost (algebraic group) 13.O
- simple algebra 29.A
 - central 29.E
 - normal 29.E
 - zeta function of 27.F
- simple arc 93.B
- simple Bravais lattice 92.E
- simple character (of an irreducible representation) 362.E
- simple closed curve 93.B
- simple component (of a semisimple ring) 368.G
- simple continued fraction 85.A
- simple convergence, abscissa of (of a Dirichlet series) 121.B
- simple distribution, potential of 338.A
- simple extension (of a field) 149.D
- simple function 221.B 443.B
- simple group 190.C
 - linear 151.I
 - Tits 151.I
- simple harmonic motion 318.B
- simple holonomic system 274.H
- simple homotopy equivalence 65.C
- simple homotopy equivalent 65.C
- simple homotopy theorem 65.C
- simple hypothesis 400.A
- simple Lie algebra 248.E
 - classical compact real 248.T
 - classical complex 248.S
 - exceptional compact real 248.T
 - exceptional complex 248.S
- simple Lie group 249.D
 - classical compact 249.L
 - classical complex 249.M
 - exceptional compact 249.L
 - exceptional complex 249.M
- simple loss function 398.A
- simple model 403.F
- simple pair (of an H -space and an H -subspace) 202.L
 - n - (of topological spaces) 202.L
- simple path 186.F
- simple point
 - (on an algebraic variety) 16.F
 - (of an analytic set) 23.B 418.A
- simple ring 368.G
 - quasi- 368.E
- simple root
 - (of an algebraic equation) 10.B
 - (in a root system) 13.J
 - (of a semisimple Lie algebra) 248.M
- simple series 379.E
- simple spectrum 390.G
- simplest alternating polynomial 337.I
- simplest Chebyshev q -function 19.G
- simplest orthogonal polynomial 19.G
- simple type theory 411.K
- simplex
 - (in an affine space) 7.D
 - (of a complex) 13.R
 - (in a locally convex space) 424.U
 - (in a polyhedron of a simplicial complex) 70.C
 - (in a simplicial complex) 70.C
 - (of a triangulation) 70.C
 - degenerate (in a semisimplicial complex) 70.D
 - n - (in a Euclidean simplicial complex) 70.B
 - n - (in a semisimplicial complex) 70.E
 - n - (in a simplicial complex) 70.C
 - open (in an affine space) 7.D
 - open (in the polyhedron of a simplicial complex) 70.C
 - ordered (in a semisimplicial complex) 70.E
 - ordered (in a simplicial complex) 70.C
 - oriented q - 201.C
 - oriented singular r -, of class C^∞ 105.T
 - singular n - (in a topological space) 70.E
- simplex method 255.C
 - two-phase 255.C
- simplex tableau 255.C
- simplicial approximation (to a continuous mapping) 70.C
- simplicial approximation theorem 70.C
- simplicial chain complex, oriented 201.C
- simplicial complex(es) 65.A 70.C
 - abstract 70.C
 - countable 70.C
 - Euclidean 70.B
 - finite 70.C
 - isomorphic 70.C
 - locally countable 70.C
 - locally finite 70.C
 - ordered 70.C
- simplicial decomposition (of a topological space) 70.C
- simplicial division 65.A
- simplicial homology group 201.D

- ul style="list-style-type: none; padding-left: 0;">
- simplicial mapping (map) 70.C
 - (between polyhedra) 70.C
 - (relative to triangulations) 70.C
- simplicial pair 201.L
- simply connected (space) 79.C 170
- simply connected covering Lie group (of a Lie algebra) 249.C
- simply connected group (isogenous to an algebraic group) 13.N
- simply convergent sequence 435.B
- simply elliptic (singularity) 418.C
- simply invariant (subspace) 164.H
- simply periodic function 134.E
- simply transitive (G-set) 362.B
- Simpson formula, Milne- 303.E
- Simpson $\frac{1}{3}$ rule 299.A
- Simpson $\frac{3}{8}$ rule 299.A
- simulation 307.C 385
 - analog 385.A
 - in the narrow sense 385.A
 - system 385.A
- simultaneous distribution 342.C
- simultaneous equations 10.A
- sin (sine) 131.E
- \sin^{-1} 131.E
- sine(s) 432.A
 - hyperbolic 131.F
 - integral 167.D
 - laws of 432.A, App. A, Tables 2.II 2.III
 - laws of (on spherical trigonometry) 432.B
- sine curve 93.H
- Sine-Gordon equation 387.A
- sine integral 167.D, App. A, Table 19.II
- sines and cosines, law of App. A, Table 2.III
- sine transform 160.C, App. A, Table 11.II
- sine wave 446
- Singer fixed point theorem, Atiyah- 153.C
- Singer index theorem
 - Atiyah- 237.H
 - equivariant Atiyah- 237.H
- single-address instruction 75.C
- single-commodity flow problem 281.F
- single integral theorem, Fourier 160.B
- single layer, potential of a 338.A
- single-objective model 307.C
- single sampling inspection 404.C
- single-valued function 165.B
- singular
 - (distribution) 374.C
 - (element of a connected Lie group) 249.P
 - (element of a real Lie algebra) 248.B
 - (element with respect to a quadratic form) 348.E
 - (Galton-Watson process) 44.C
 - (harmonic function) 193.G
 - (mapping) 208.B
 - (ordinal number) 270.I
 - (set function) 380.C
 - essentially (with respect to an analytic set) 21.M
 - of the h th species 343.D.E
 - μ - 380.C
 - relative, homology group 201.L
- singular cardinal problem 33.F
- singular chain complex (of a topological space) 201.E
- singular cochain complex 201.H
- singular cohomology group 201.H
- singular cohomology ring 201.I
- singular complex (of a topological space) 70.E
- singular fiber 72.K
- singular homology group 201.E,G,L,R
 - integral 201.E
- singular initial value problem (of a partial differential equation of mixed type) 326.C
- singular inner function 43.F
- singular integral 217.J
- singular integral equation 217.J
- singular integral manifold (of a differential ideal) 428.E
- singular integral operator, Calderón-Zygmund 217.J 251.O
- singularity (singularities) 51.C 198.M
 - algebraic 198.M
 - of an analytic function 198.M
 - cusp 418.C
 - direct transcendental (of an analytic function in the wider sense) 198.P
 - elliptic 418.C
 - essential (of a complex function) 198.D
 - fixed (of an algebraic differential equation) 288.A
 - indirect transcendental (of an analytic function in the wider sense) 198.P
 - isolated (of an analytic function) 198.D
 - isolated (of a complex function) 198.M
 - logarithmic (of an analytic function) 198.M
 - logarithmic (of an analytic function in the wider sense) 198.P
 - movable (of an algebraic differential equation) 288.A
 - ordinary (of an analytic function in the wider sense) 198.P
 - principle of condensation of 37.H
 - propagation of 325.M
 - quotient 418.C
 - rational 418.C
 - regular (of a coherent \mathcal{E} -module) 274.H
 - removable (of a complex function) 198.D
 - removable (of a harmonic function) 193.L
 - resolution of 16.L 23.D 418.B
 - space of 390.E
 - theory of 418
 - transcendental (of an analytic function in the wider sense) 198.P
 - two-dimensional 418.C
- singularity set (of a proper meromorphic mapping) 23.D
- singularity spectrum (of a hyperfunction) 125.CC 274.E
- singularity theorem (in physics) 359.F
- singular kernel 217.J
- singular locus (of a variety) 16.F
- singular n -simplex (in a topological space) 70.E
- singular orbit 431.C
- singular part (of a Laurent expansion) 198.D
- singular perturbation 289.E
- singular point
 - (of an algebraic variety) 16.F
 - (of an analytic set) 23.B 418.A
 - (of a continuous vector field) 153.B
 - (of a curve of class C^k) 93.G
 - (of a flow) 126.D
 - (of a linear difference equation) 104.D
 - (of a plane algebraic curve) 9.B
 - (of a polyhedron) 65.B
 - (of a quadratic hypersurface) 343.E
 - (of a surface in E^3) 111.J
 - (of a system of linear ordinary differential equations) 254.A

- (of a system of ordinary differential equations) 126.G 289.A
- apparent (of a system of linear ordinary differential equations) 254.C
- hyperbolic 126.G
- irregular (of a solution) 254.B
- irregular (of a system of linear ordinary differential equations) 254.B
- isolated 198.D
- left (of a diffusion process) 115.B
- regular (of a solution) 254.B
- regular (of a system of linear ordinary differential equations) 254.B
- right (of a diffusion process) 115.B
- singular projective transformation 343.D
 - of the h th species 343.D
- q -cochain 201.H
- q -simplex 201.E
- singular quadric hypersurface of the h th species (in a projective space) 343.E
- singular r -chain of class C^∞ 105.T
- singular r -cochain of class C^∞ 105.T
- singular r -simplex of class C^∞ , oriented 105.T
- singular series 4.D
- singular solution
 - (of a differential ideal) 428.E
 - (of a general partial differential equation) 320.C
 - (of an ordinary differential equation) 313.A, App. A, Table 14.I
 - (of a partial differential equation) 320.C
 - totally (with respect to a quadratic form) 348.E
- singular spectrum 345.A 390.E
 - (of a hyperfunction) 125.CC 274.E
- singular subspace 343.D
- singular subspace, totally 348.E
- singular support
 - (of a distribution) 112.C
 - (of a hyperfunction) 125.W
- singular value 302.A
- singular value decomposition (SVD) 302.E
- \sinh (hyperbolic sine) 131.F
- sink 126.G 281.C
- sinusoid 93.D
- sinusoidal wave 446
- site 16.AA
 - étale 16.AA
 - flat 16.AA
 - presheaf on 16.AA
 - Zariski 16.AA
- site percolation process 340.D
- $6j$ -symbol 353.B
- size
 - (of a balanced array) 102.L
 - (complexity of computation) 71.A
 - (of a population) 397.B
 - (of a random sample) 396.B
 - (of a sample) 401.E
 - (of a test) 400.A
 - block 102.B
 - sample 373.A
 - step 303.B
- skeleton
 - (of a domain in C^n) 21.C
 - r - (of a Euclidean complex) 70.B
- skew field 149.A 368.B
- skew-Hermitian form 256.Q
- skew-Hermitian matrix 269.I
- skew h -matrix 269.I
- skewness 396.C 397.C
 - coefficient of 341.H
- skew product (of measure-preserving transformations) 136.D
- skew surface 111.I
- skew-symmetric (multilinear mapping) 256.H
- skew-symmetric matrix 269.B
- skew-symmetric tensor 256.N
- Skitovich-Darmois theorem 374.H
- Skolem-Löwenheim theorem 156.E
- Skolem paradox 156.E
- Skolem theorem on the impossibility of characterizing the system of natural numbers by axioms 156.E
- slackness, Tucker theorem on complementary 255.B
- slack variable 255.A
- slant product
 - (of a cochain and a chain) 201.K
 - (of a cohomology class and a homology class) 201.K
- Slater constraint qualification 292.B
- slender body theorem 205.B
- slice knot 235.G
- slice representation 431.C
- slice theorem, differentiable 431.C
- slicing theorem, watermelon- 125.DD
- slide rule 19.A
- sliding block code 213.E
- slit (of a plane domain) 333.A
- slit domain 333.A
- slit mapping
 - extremal horizontal 367.G
 - extremal vertical 367.G
- slope function 46.C
- slowly increasing C^∞ -function 125.O
- slowly increasing distribution 125.N
- slowly increasing function in the sense of Deny 338.P
- slowly increasing sequences 168.B
- slow wave 259
- Smale condition C, Palais- 279.E 286.Q
- Smale diffeomorphism, Morse- 126.J
- Smale flow, Morse- 126.J
- Smale theorem, Sard- 286.P
- Smale vector field, Morse- 126.J
- small-displacement theory of elasticity 271.G
- smaller topology 425.H
- small inductive dimension (ind) 117.B
- small numbers, law of 250.B
- small sample 401.F
- small set of order U 436.G
- smashing (a space to a point) 202.E
- smash product 202.F
- Smirnov test, Kolmogorov- 317.F
- Smirnov test statistic, Kolmogorov- 374.E
- Smirnov theorem 250.F
- Smith conjecture 235.E
- Smith convergence, Moore- 87.H
- Smith theorem 431.B
- smooth
 - (function) 106.K
 - (measure for a Riemann metric) 136.G
 - (morphism of schemes) 16.F
 - (point of a variety) 16.F
 - piecewise (curve) 364.A
 - in the sense of A. Zygmund 168.B
 - uniformly (normed linear space) 37.G
- smooth boundary, domain with (in a C^∞ -manifold) 105.U

Smooth characteristic class of foliations

- smooth characteristic class of foliations 154.G
 smoothing (of a combinatorial manifold) 114.C
 smoothing problem 114.C
 smooth invariant measure 126.J
 smooth manifold 105.D 114.B
 smooth structure 114.B
 smooth variety 16.F
 sn 134.J, App. A, Table 16.III
 Snapper polynomial 16.E
 Sobolev-Besov embedding theorem 168.B
 Sobolev inequality, Hardy-Littlewood- 224.E
 Sobolev space 168.B
 software 75.C
 sojourn time density 45.G
 solenoidal (vector field) 442.D
 solid geometry 181
 solid harmonics 393.A
 solid n -sphere 140
 solid sphere 140
 topological 140
 solitary wave 387.B
 soliton 387.B
 solution
 (of equations of neutral type) 163.H
 (of a functional-differential equation) 163.C
 (of an inequality) 211.A
 (of an ordinary differential equation) 313.A
 (of a partial differential equation) 320.A
 (of partial differential equations of first order)
 App. A, Table 15.II
 (of partial differential equations of second
 order) App. A, Table 15.III
 (of a system of differential equations) 313.B
 (of a system of linear equations) 269.M
 (of a system of partial differential equations)
 428.B
 algebraic (of an algebraic equation) 10.D
 asymptotic 325.L
 basic 255.A
 basic feasible 255.A
 basic optimal 255.A
 Bayes 398.B
 Bayes, in the wider sense 398.B
 of boundary value problems App. A, Table
 15.VI
 of the Cauchy problem 325.D
 classical (to Plateau's problem) 275.C
 complete (of partial differential equations)
 320.C
 d'Alembert 325.D
 Douglas-Radó (to Plateau's problem) 275.C
 elementary (of a differential operator) 112.B
 elementary (of a linear partial differential
 operator) 320.H
 elementary (of partial differential equations
 of elliptic type) 323.B
 elementary (of a partial differential operator)
 App. A, Table 15.V
 equilateral triangle 420.B
 feasible (of a linear equation in linear program-
 ming) 264.A
 formal (for a system of ordinary differential
 equations) 289.C
 fundamental (of a Cauchy problem) 325.D
 fundamental (of a differential operator) 112.B
 fundamental (of an evolution equation) 189.C
 fundamental (of a linear parabolic equation
 with boundary conditions) 327.F
 fundamental (of a linear partial differential
 operator) 320.H
 fundamental (of a partial differential equation
 of parabolic type) 327.D
 fundamental (of partial differential equations
 of elliptic type) 323.B
 fundamental (of a partial differential operator
 with C^∞ -coefficients) 189.C
 fundamental system of (of a homogeneous
 linear ordinary differential equation) 252.B
 fundamental system of (of a homogeneous
 system of linear differential equations of first
 order) 252.H
 general (of a differential equation) 313.A
 general (of a general partial differential equa-
 tion) 320.C
 general (of a nonhomogeneous linear difference
 equation) 104.D
 general (of partial differential equations)
 320.C
 general (of a system of differential equations)
 313.C
 general (of a system of partial differential
 equations) 428.B
 generalized Bayes 398.B
 genuine 323.G
 half-periodic (of the Hill equation) 268.E
 Hill's method of 268.B
 Hopf's weak 204.C
 inner 25.B
 Kirchhoff 325.D
 to the martingale problem 115.C
 maximum 316.E
 minimax 398.B
 minimum 316.E
 Nash bargaining 173.C
 numerical (of algebraic equations) 301
 numerical (of integral equations) 217.N
 numerical (of linear equations) 302
 numerical (of ordinary differential equations)
 303
 numerical (of partial differential equations)
 304
 optimal (of a linear programming problem)
 255.A
 optimal (of a nonlinear programming problem)
 292.A
 ordinary (of a differential ideal) 428.E
 outer 25.B
 particular (of a differential equation) 313.A
 particular (of partial differential equations)
 320.C
 particular (for a system of differential equations)
 313.C
 pathwise uniqueness of 406.D
 periodic (of the Hill equation) 268.E
 Perron-Brelet (of the Dirichlet problem)
 120.C
 Perron-Wiener-Brelet (of the Dirichlet problem)
 120.C
 Poisson 325.D
 primary (of a homogeneous partial differential
 equation) 320.E
 primitive (of a partial differential equation)
 320.E
 principal 104.B
 by quadrature App. A, Table 14.I
 quasiperiodic (of the Hill equation) 268.B
 by radicals (of an algebraic equation) 10.D
 regular (of a differential ideal) 428.E
 singular App. A, Table 14.I
 singular (of a differential ideal) 428.E

singular (of a general partial differential equation) 320.C
 singular (of an ordinary differential equation) 313.A, App. A, Table 14.I
 singular (of partial differential equations) 320.C
 stable (of the Hill equation) 268.E
 straight line 420.B
 strong (of Navier-Stokes equation) 204.C
 strong (of stochastic differential equations) 406.D
 system of fundamental (of a system of linear homogeneous equations) 269.M
 trivial (of a system of linear homogeneous equations) 269.M
 unique strong 406.D
 uniqueness theorem of (of systems of linear differential equations of the first order) 316.D.G
 unstable (of the Hill equation) 268.E
 von Neumann-Morgenstern 173.D
 weak 204.C 323.G 378.I
 solution curve (of ordinary differential equations) 316.A
 solution operator 163.E
 solvability, Cartan's criterion of 248.F
 solvable
 (ideal of a Lie algebra) 248.C
 (Lie algebra) 248.C
 (Lie group) 249.D
 (by a Turing machine) 71.B
 by radicals 172.H
 solvable algebra 231.A
 solvable algebraic group 13.F
 k - 13.F
 solvable group 190.I
 finite 151.D
 generalized 190.K
 π - 151.F
 solve
 (a conditional inequality) 211.A
 (by means of a Turing machine) 71.E
 (an ordinary differential equation) 313.A
 (a partial differential equation) 320.A
 (a system of algebraic equations) 10.A
 (a triangle) 432.A
 Sommerfeld formula App. A, Table 19.III
 Kneser- App. A, Table 19.III
 Sommerfeld radiation condition 188.D
 Sonine formula, Weber- App. A, Table 19.III
 Sonine polynomials 317.D, App. A, Table 20.VI
 Sonine-Schafheitlin formula App. A, Table 19.III
 SOR (successive overrelaxation) 302.C
 sorting 96.C
 soudure 80.N
 sound propagation, equation of 325.A
 source 126.G 281.C
 (of a jet) 105.X
 autoregressive Gaussian 213.E
 ergodic information 213.C
 information 213.A
 stationary 213.C
 without (vector field) 442.D
 source branch 282.C
 source coding theorem 213.D
 with a fidelity criterion 213.F
 noiseless 213.D
 source coding theory 213.A
 southern hemisphere 140

Subject Index

Space(s)

south pole 74.D 140
 space(s) 381.B
 of absolute continuity 390.E
 absolutely closed 425.U
 abstract 381.B
 abstract L 310.G
 abstract L_p 310.G
 abstract M 310.G
 action 398.A
 adjoint (of a topological linear space) 424.D
 affine 7.A
 affine locally symmetric 80.J
 affine symmetric 80.J
 N_0 - 425.Y
 algebraic 16.W
 algebraic fiber 72.I
 analytic 23.C
 analytic, in the sense of Behnke and Stein 23.E
 analytically uniform 125.S
 analytic covering 23.E
 analytic measurable 270.C
 arcwise connected 79.B
 attaching 202.E
 Baire 425.N
 Baire zero-dimensional 273.B
 Banach 37.A,B
 Banach analytic 23.G
 base (of a fiber bundle) 147.B
 base (of a fiber space) 148.B
 base (of a Riemann surface) 367.A
 base for 425.F
 basic (of a probability space) 342.B
 Besov 168.B
 bicomact 408.S
 biprojective 343.H
 Boolean 42.D
 Borel 270.C
 boundary 112.E
 bundle (of a fiber bundle) 147.B
 C-analytic 23.E
 C-covering 23.E
 Cartan 152.C
 Cartesian 140
 Čech-complete 436.I
 classifying (of a topological group) 147.G,H
 closed half- 7.D
 of closed paths 202.C
 co-echelon 168.B
 collectionwise Hausdorff 425.AA
 collectionwise normal 425.AA
 comb 79.A
 compact 425.S
 compact metric 273.F
 complete 436.G
 completely normal 425.Q
 completely regular 425.Q
 complete measure 270.D
 complete product measure 270.H
 complete uniform 436.G
 complex, form 365.L
 complex Hilbert 197.B
 complex interpolation 224.B
 complexity 71.A
 complex projective 343.D
 concentrically flat App. A, Table 4.II
 configuration 126.L 402.G
 conformal 76.A
 conformally flat App. A, Table 4.II
 conjugate (of a normed linear space) 37.D

- conjugate (of a topological linear space) 424.D
- connected 79.A
- of constant curvature 364.D, App. A, Table 4.II
- of continuous mapping 435.D
- contractible 79.C
- control (in catastrophe theory) 51.B
- countable paracompact 425.Y
- countably compact 425.S
- countably Hilbertian 424.W
- countably normed 424.W
- covering 91.A
- crystallographic, group 92.A
- decision 398.A
- of decision functions 398.A
- de Sitter 359.D
- developable 425.AA
- (DF)- 424.P
- Dieudonné complete topological 435.I
- Dirichlet 338.Q
- discrete metric 273.B
- discrete topological 425.C
- Douady 23.G
- dual (of a C^* -algebra) 36.G
- dual (of a linear space) 256.G
- dual (of a locally compact group) 437.J
- dual (of a normed linear space) 37.D
- dual (of a projective space) 343.B
- dual (of a topological linear space) 424.D
- \mathcal{E} - 193.N
- echelon 168.B
- eigen- 269.L 390.A
- Eilenberg-MacLane 70.F
- Einstein 364.D, App. A, Table 4.II
- of elementary events 342.B
- elliptic 285.C
- error 403.E
- estimation 403.E
- Euclidean 140
- external (in static model in catastrophe theory) 51.B
- (F)- 424.I
- fiber 72.I 148.B
- finite type power series 168.B
- Finsler 152.A
- Fock (antisymmetric) 377.A
- Fock (symmetric) 377.A
- Fréchet 37.O 424.I 425.CC
- Fréchet, in the sense of Bourbaki 37.O 424.I
- Fréchet L - 87.K
- Fréchet-Uryson 425.CC
- fully normal 425.X
- function 168.A 435.D
- fundamental 125.S
- G - 178.H 431.A
- general analytic 23.G
- generalized topological 425.D
- generating (of a quadric hypersurface) 343.E
- globally symmetric Riemannian 412.A
- Green 193.N
- group 92.A
- H - 203.D
- Haar 142.B
- half- (of an affine space) 7.D
- Hardy 168.B
- Hausdorff 425.Q
- Hausdorff uniform 436.C
- H -closed 425.U
- hereditarily normal 425.Q
- Hermitian hyperbolic 412.G
- Hilbert 173.B 197.B
- Hilbert, adjoint 251.E
- Hilbert, exponential 377.D
- Hölder 168.B
- holomorphically complete 23.F
- homogeneous \rightarrow homogeneous space
- hyperbolic 285.C 412.H
- identification (by a partition) 425.L
- indiscrete pseudometric 273.B
- inductive limit 210.C
- infinite-dimensional 117.B
- infinite lens 91.C
- infinite type power series 168.B
- at infinity (in affine geometry) 7.B
- inner product 442.B
- internal (in static model in catastrophe theory) 51.B
- interpolation 224.A
- of irrational numbers 22.A
- irreducible symmetric Hermitian 412.A
- isometric 273.B
- John-Nirenberg (= BMO) 168.B
- k - 425.CC
- k' - 425.CC
- K -complete analytic 23.F
- Kawaguchi 152.C
- Kolmogorov 425.Q
- Köthe 168.B
- Kuranishi 72.G
- Kuratowski 425.Q
- L - 87.K
- L^* - 87.K
- Lashnev 425.CC
- lattice ordered linear 310.B
- Lebesgue (= $L_p(\Omega)$) 168.B
- Lebesgue measure, with (σ -) finite measure 136.A
- left coset (of a topological group) 423.E
- left projective 343.H
- left quotient (of a topological group) 423.E
- lens 91.C
- (LF)- 424.W
- of line elements of higher order 152.C
- linear \rightarrow linear space
- linear topological 424.A
- Lindelöf 425.S
- Lipschitz 168.B
- locally arcwise connected 79.B
- locally compact 425.V
- locally connected 79.A
- locally contractible 79.C 204.C
- locally convex Fréchet 424.I
- locally Euclidean 425.V
- locally n -connected 79.C
- locally ω -connected 79.C
- locally symmetric 364.D
- locally symmetric Riemannian 412.A
- locally totally bounded uniform 436.H
- locally trivial fiber 148.B
- local moduli, of a compact complex manifold 72.G
- local ringed 383.H
- Loeb 293.D
- loop 202.C
- Lorentz 168.B
- Luzin 22.I 422.CC
- M - 425.Y
- (M)- 424.O
- Mackey 424.N
- mapping 202.C 435.D

- maximal ideal (of a commutative Banach space) 36.E
- measurable 270.C
- measure 270.D
- metric \rightarrow metric space
- metric vector 256.H
- metrizable topological 273.K
- metrizable uniform 436.F
- Minkowski 258.A
- moduli 16.W 72.G
- Moishezon 16.W
- momentum phase 126.L
- Montel 424.O
- Moor 273.K 425.AA
- n -classifying (of a topological group) 147.G
- n -connected 79.C 202.L
- n -connective fiber 148.D
- n -dimensional 117.B
- n -simple 202.L
- non-Euclidean 285.A
- normal 425.Q
- normal analytic 23.D
- normed linear 37.B
- NP- 71.E
- nuclear 424.S
- null 251.D
- ω -connected 79.C
- orbit (of a G -space) 431.A
- ordered linear 310.B
- Orlicz 168.B
- P- 425.Y
- p- 425.Y
- paracompact 425.S
- parameter (of a family of compact complex manifolds) 72.G
- parameter (for a family of probability measures) 398.A
- parameter (of a probability distribution) 396.B
- partition of a 425.L
- path 148.C
- path (of a Markov process) 261.B
- path-connected 79.B
- pathological 65.F
- Peirce 231.B
- perfectly normal 425.Q
- perfectly separable 425.P
- phase 126.B 163.C 402.C
- physical Hilbert 150.G
- pinching a set to a point 202.E
- polar 191.I
- Polish 22.I 273.J
- precompact metric 273.B
- precompact uniform 436.H
- pre-Hilbert 197.B
- principal (of a flag) 139.B
- principal half- 139.B
- probability 342.B
- product 425.K
- product measure 270.H
- product metric 273.B
- product topological 425.K
- product uniform 436.E
- projective, over Λ 147.E
- projective limit 210.C
- projectively flat App. A, Table 4.II
- pseudocompact 425.S
- pseudometric 273.B
- pseudometrizable uniform 436.F
- Q- 425.BB
- quasi-Banach 37.O
- quasicompact 408.S
- quasidual (of a locally compact group) 437.I
- quasinormed linear 37.O
- quaternion hyperbolic 412.G
- quotient 425.L
- quotient (by a discrete transformation group) 122.A
- quotient (of a linear space with respect to an equivalence relation) 256.F
- quotient (by a transformation group) 122.A
- quotient topological 425.L
- r -closed 425.U
- ramified covering 23.B
- real-compact 425.BB
- real Hilbert 197.B
- real hyperbolic 412.G
- real interpolation 224.C
- real linear 256.A
- real projective 343.D
- reduced product 202.Q
- reflexive Banach 37.G
- regular 425.Q
- regular Banach 37.G
- representation (for a Banach algebra) 36.D
- representation (of a representation of a Lie algebra) 248.B
- representation (of a representation of a Lie group) 249.O
- representation (of a unitary representation) 437.A
- Riemannian 364.A
- Riesz 310.B
- right coset (of a topological group) 423.E
- right projective 343.H
- right quotient (of a topological group) 423.E
- ringed 383.H
- (S)- 424.S
- sample 342.B 396.B 398.A
- scale of Banach 286.Z
- Schwartz 424.S
- separable 425.P
- separable metric 273.E
- separated 425.Q
- separated uniform 436.C
- sequential 425.CC
- sequentially compact 425.S
- sheaf 383.C
- shrinking, to a point 202.E
- Siegel, of degree n 32.F
- Siegel upper half-, of degree n 32.F
- Σ - 425.Y
- σ - 425.Y
- σ -compact 425.V
- σ -finite measure 270.D
- simply connected 79.C 170
- of singularity 390.E
- smashing, to a point 202.E
- Sobolev 168.B
- Spanier cohomology theory, Alexander-Kolmogorov- 201.M
- spherical 285.D
- Spivak normal fiber 114.J
- standard Borel 270.C
- standard measurable 270.C
- standard vector (of an affine space) 7.A
- state (of a dynamical system) 126.B
- state (of a Markov process) 261.B
- state (in static model in catastrophe theory) 51.B
- state (of a stochastic process) 407.B

- Stein 23.F
- stratifiable 425.Y
- strongly paracompact 425.S
- structure (of a Banach algebra) 36.D
- subbase for 425.F
- Suslin 22.I 425.CC
- symmetric Hermitian 412.E
- symmetric homogeneous 412.B
- symmetric Riemannian 412
- symmetric Riemannian homogeneous 412.B
- T_0 - 425.Q
- T_1 - 425.Q
- T_1 -uniform 436.C
- T_2 - 425.Q
- T_3 - 425.Q
- T_4 - 425.Q
- T_5 - 425.Q
- T_6 - 425.Q
- tangent 105.H
- tangent vector 105.H
- Teichmüller 416
- tensor, of degree k 256.J
- tensor, of type (p, q) 256.J
- test function 125.S
- Thom 114.G
- Tikhonov 425.Q
- time parameter 260.A
- topological \rightarrow topological space
- topological complete 436.I
- topological linear 424.A
- topological vector 424.A
- total (of a fiber bundle) 147.B
- total (of a fiber space) 148.B
- totally bounded metric 273.B
- totally bounded uniform 436.H
- totally disconnected 79.D
- transformation (of an algebraic group) 13.G
- of type S 125.T
- underlying topological (of a complex manifold) 72.A
- underlying topological (of a topological group) 423.A
- uniform \rightarrow uniform space
- uniformizable topological 436.H
- uniformly locally compact 425.V
- uniform topological 436.C
- unisolvant 142.B
- universal covering 91.B
- universal Teichmüller 416
- vector, over K 256.A
- velocity phase 126.L
- weakly symmetric Riemannian 412.J
- well-chained metric 79.D
- wild 65.F
- space complexity 71.A
- space form 285.E 412.H
 - Euclidean 412.H
 - hyperbolic 412.H
 - spherical 412.H
- space geometry 181
- space group 92.A
 - crystallographic 92.A
 - equivalent 92.A
- spacelike 258.A 359.B
- space reflection 359.B
- space-time, Minkowski 359.B
- space-time Brownian motion 45.F
- space-time inversion 258.A
- space-time manifold 359.D
- span
 - (of a domain) 77.E
 - (a linear subspace by a set) 256.F
 - (of a Riemann surface) 367.G
- spanning tree 186.G
- sparse 302.C
- Späth type division theorem (for microdifferential operators) 274.F
- spatial (\ast -isomorphism on von Neumann algebras) 308.C
- spatially homogeneous (process) 261.A
- spatially isomorphic (automorphisms on a measure space) 136.E
- spatial tensor product 36.H
- Spearman rank correlation 371.K
- Spec (spectrum) 16.D
- special Clifford group 61.D
- special divisor 9.C
- special flow 136.D
- special function(s) 389, App. A, Table 14.II
 - of confluent type 389.A
 - of ellipsoidal type 389.A
 - of hypergeometric type 389.A
- special functional equations 388
- special isoperimetric problem 228.A
- speciality index
 - (of a divisor of an algebraic curve) 9.C
 - (of a divisor on an algebraic surface) 15.D
 - α - (of a divisor of an algebraic curve) 9.F
- specialization 16.A
 - (in étale topology) 16.AA
- special Jordan algebra 231.A
- special linear group 60.B
 - (over a noncommutative field) 60.O
 - of degree n over K 60.B
 - projective 60.B
 - projective (over a noncommutative field) 60.O
- special orthogonal group 60.I
 - complex 60.I
 - over K with respect to Q 60.K
- special principle of relativity 359.B
- special relativity 359.B
- special representation (of a Jordan algebra) 231.C
- special surface 110.A
- special theory of perturbations 420.E
- special theory of relativity 359.A
- special unitary group 60.F
 - (relative to an ε -Hermitian form) 60.O
 - over K 60.H
 - projective, over K 60.H
- special universal enveloping algebra (of a Jordan algebra) 231.C
- special valuation 439.B
- species
 - ellipsoidal harmonics of the first, second, third or fourth 133.C
 - Lamé functions of the first, second, third or fourth 133.C
 - singular projective transformation of the h th 343.D
 - singular quadric hypersurface of the h th (in a projective space) 343.E
- specification 401.A
 - problem of 397.P
- specific heat
 - at constant pressure 419.B
 - at constant volume 419.B
- specificity 346.F
- specific resistance 130.B
- spectral analysis 390.A
- spectral concentration 331.F

- spectral decomposition 126.J 395.B
- spectral density, quadrature 397.N
- spectral functor 200.J
- spectral geometry 391.A
- spectral integral 390.D
- spectral invariant 136.E
- spectrally isomorphic (automorphisms on a measure space) 136.E
- spectral mapping theorem 251.G
- spectral measure 390.B,K 395.B,C
 - complex 390.D
 - maximum 390.G
 - real 390.D
- spectral method 304.B
- spectral operator 390.K
- spectral property 136.E
- spectral radius 126.K 251.F 390.A
- spectral representation 390.E
 - complex 390.E
- spectral resolution 390.E
 - complex 390.E
- spectral sequence 200.J
 - (of a fiber space) 148.E
 - cohomology 200.J
 - Hodge 16.U
- spectral synthesis 36.L
- spectral theorem 390.E
- spectrum 390.A
 - (of a commutative ring) 16.D
 - (of a domain in a Riemannian manifold) 391.A
 - (of an element of a Banach algebra) 36.C
 - (in homotopy theory) 202.T
 - (of a hyperfunction) 274.E
 - (of an integral equation) 217.J
 - (of a linear operator) 251.F 390.A
 - (of a spectral measure) 390.C
 - absolutely continuous 390.E
 - continuous (of a linear operator) 390.A
 - continuous (of an integral equation) 217.J
 - countable Lebesgue 136.E
 - discrete 136.E 390.E
 - Eilenberg-MacLane 202.T
 - essential 390.E,I
 - formal (of a Noetherian ring) 16.X
 - intermittent 433.C
 - joint 36.M
 - Kolmogorov 433.C
 - for p -forms 391.B
 - point 390.A
 - pure point 136.E
 - quasidiscrete 136.E
 - residual 390.A
 - simple 390.G
 - singular 125.CC 345.A 390.A
 - singular (of a hyperfunction) 274.E
 - singularity (of a hyperfunction) 125.CC 274.E
 - sphere 202.T
 - stable homotopy group of the Thom 114.G
 - Thom 114.G 202.T
- spectrum condition 150.D
- speed measure 115.B
- Speiser theorem, Hilbert- 172.J
- Spencer mapping (map), Kodaira- 72.G
- sphere(s) 139.I 150
 - circumscribing (of a simplex) 139.I
 - combinatorial, group of oriented differentiable structures on the 114.I
 - complex 74.D
 - ε - (of a point) 273.C
 - exotic 114.B
 - homotopy n - 65.C
 - homotopy n -, h -cobordism group of 114.I
 - horned, Alexander's 65.G
 - open n - 140
 - open 140
 - PL ($k - 1$)- 65.C
 - pseudo- 111.I
 - Riemann 74.D
 - solid 140
 - solid n - 140
 - topological 140
 - topological solid 140
 - unit 140
 - w - 74.D
 - z - 74.D
- sphere bundle n - 147.K
 - cotangential 274.E
 - normal 274.E
 - tangential 274.E
 - unit tangent 126.L
- sphere geometry 76.C
- sphere pair 235.G 65.D
- sphere spectrum 202.T
- sphere theorem
 - (characterization of a sphere) 178.C
 - (embedding in a 3-manifold) 65.E
- spherical
 - (real hypersurface) 344.C
 - (space form) 412.H
- spherical astronomy 392
- spherical Bessel function 39.B
- spherical coordinates 90.C, App. A, Table 3.V
- spherical derivative (for an analytic or meromorphic function) 435.E
- spherical excess 432.B
- spherical Fourier transform 437.Z
- spherical function(s) 393
 - (on a homogeneous space) 437.X
 - Laplace 393.A
 - zonal (on a homogeneous space) 437.Y
- spherical G -fiber homotopy type 431.F
- spherical geometry 285.D
- spherical harmonic function 193.C
- spherical harmonics, biaxial 393.D
- spherical indicatrix (of a space curve) 111.F
- spherical modification 114.F
- spherical representation
 - of a differentiable manifold 111.G
 - of a space curve 111.F
 - of a unimodular locally compact group 437.Z
- spherical space 285.D
- spherical triangle 432.B, App. A, Table 2.III
- spherical trigonometry 432.B
- spherical type 13.R
- spherical wave 446
- spheroidal coordinates 133.D, App. A, Table 3.V
- spheroidal wave function 133.E
- spin 132.A 258.A 415.G
 - continuous 258.A
- spin and statistics, connection of 132.A 150.D
- spin ball 351.L
- spin bundle 237.F
- Spin^c bundle 237.F
- spin-flip model 340.C
- spin mapping (map) 237.G
- spin matrix, Pauli 258.A 415.G
- spinor(s) 61.E
 - contravariant 258.A
 - covariant 258.A
 - dotted 258.B

Spinor group

- even half- 61.E
- mixed, of rank (k, n) 258.A
- odd half- 61.E
- undotted 258.B
- spinor group 60.I 61.D
 - complex 61.E
- spinorial norm 61.D
- spinor representation (of rank k) 258.A
- spin representation
 - (of $SO(n)$) 60.J
 - (of $Spin(n, \mathbb{C})$) 61.E
 - even half- 61.E
 - half- 61.E
 - odd half- 61.E
- spin-structure 237.F 431.D
- spin systems, lattice 402.G
- spiral 93.H
 - Archimedes 93.H
 - Bernoulli 93.H
 - Cornu 93.H
 - equiangular 93.H
 - hyperbolic 93.H
 - logarithmic 93.H
 - reciprocal 93.H
- Spivak normal fiber space 114.J
- spline 223.F
 - natural 223.F
- spline interpolation 223.F
- split
 - $((B, N)$ -pair) 151.J
 - (cocycle in an extension) 257.E
 - (exact sequence) 277.K
 - k - (algebraic group) 13.N
 - K - (algebraic torus) 13.D
 - k -quasi- (algebraic group) 13.O
 - maximal k -, torus 13.Q
- split extension (of a group) 190.N
- splitting, Heegaard 65.C
- splitting field
 - for an algebra 362.F
 - for an algebraic torus 13.D
 - minimal (of a polynomial) 149.G
 - of a polynomial 149.G
- splitting ring 29.K
- split torus, maximal k - 13.Q
- spot prime 439.H
- Spur 269.F
- square(s)
 - Euler 241.B
 - latin 241
 - Latin 102.K
 - least, approximation 336.D
 - matrix of the sum of, between classes 280.B
 - matrix of the sum of, within classes 280.B
 - method of least 303.I
 - middle-, method 354.B
 - Room 241.D
 - Shrikhande 102.K
 - Youden 102.K
 - Youden, design 102.K
- square-free integer 347.H
- square integrable 168.B
- square integrable unitary representation 437.M
- square matrix 269.A
- square net 304.E
- square numbers 4.D
- s.s. complex(es) (semisimplicial complex) 70.E
 - geometric realization of 70.E
 - isomorphic 70.E
- s.s. mapping (semisimplicial mapping) 70.E
 - realization of 70.E
- stability 286.S 303.E 394
 - A - 303.G
 - A_0 - 303.G
 - $A(\alpha)$ - 303.G
 - absolute 303.G
 - conjecture 126.J
 - exchange of 286.T
 - interval of absolute 303.G
 - interval of relative 303.G
 - orbital (of a solution of a differential equation) 394.D
 - principle of linearized 286.S
 - region of absolute (of the Runge-Kutta (P, p) method) 303.G
 - region of relative 303.G
 - relative 303.G
 - stiff- 303.G
 - structural 290.A
 - structural, theorem 126.J
- stability group 362.B
- stability subgroup (of a topological group) 431.A
- stability theorem
 - Ω - 126.J
 - structural 126.J
- stabilizer
 - (in an operation of a group) 362.B
 - (in a permutation group) 151.H
 - (in a topological transformation group) 431.A
 - reductive 199.A
- stable 394.A
 - (coherent sheaf on a projective variety) 241.Y
 - (compact leaf) 154.D
 - (discretization, initial value problems) 304.D
 - (equilibrium solution) 286.S
 - (initial value problem) 304.F
 - (invariant set) 126.F
 - (linear function) 163.H
 - (manifold) 126.G
 - (minimal submanifold) 275.B
 - (static model in catastrophe theory) 51.E
 - absolutely 303.G
 - asymptotically 126.F 286.S 394.B
 - in both directions (Lyapunov stable) 394.A
 - C^r - Ω - 126.H
 - C^r -structurally 126.H
 - conditionally 394.D
 - exponentially 163.G 394.B
 - externally, set 186.I
 - globally asymptotically 126.F
 - internally, set 186.I
 - Lagrange 126.E
 - Lyapunov 126.F
 - Lyapunov, in the positive or negative direction 394.A
 - negatively Lagrange 126.E
 - negatively Poisson 126.E
 - one-side, for exponent $\frac{1}{2}$ App. A, Table 22
 - orbitally 126.F
 - Poisson 126.E
 - positively Lagrange 126.E
 - positively Poisson 126.E
 - relatively 303.G
 - uniformly 394.B
 - uniformly asymptotically 163.G 394.B
 - uniformly Lyapunov 126.F
- stable cohomology operation 64.B
- stable curve 9.K

stable distribution 341.G
 quasi- 341.G
 semi- 341.G
 stable homotopy group 202.T, App. A, Table 6.VII
 (of Thom spectrum) 114.G
 of classical groups 202.V
 of k -stem 202.U
 stable manifold 126.G,J
 stable point 16.W
 stable primary cohomology operation 64.C
 stable process 5.F
 exponent of 5.F
 one-sided, of the exponent α 5.F
 strictly 5.F
 symmetric 5.F
 stable range (of embeddings) 114.D
 stable reduction
 (of an Abelian variety) 3.N
 (of a curve) 9.K
 potential (of an Abelian variety) 3.N
 stable reduction theorem 3.N 9.K
 stable secondary cohomology operation 64.C
 stable set 173.D
 externally 186.I
 internally 186.I
 stable solution (of the Hill equation) 268.E
 stable state 260.F 394.A 404.A
 stable vector bundle
 (algebraic) 16.Y
 (topological) 237.B
 stably almost complex manifold 114.H
 stably equivalent (vector bundles) 237.B
 stably fiber homotopy equivalent 237.I
 stably parallelizable (manifold) 114.I
 stack 96.E
 stage method, $(P + 1)$ - 303.D
 stalk (of a sheaf over a point) 16.AA 383.B
 standard
 (in nonstandard analysis) 293.B
 (transition probability) 260.F
 standard Borel space 270.C
 standard complex (of a Lie algebra) 200.O
 standard defining function 125.Z
 standard deviation
 (characteristics of the distribution) 397.C
 (of a probability distribution) 341.B
 (of a random variable) 342.C
 population 396.C
 sample 396.C
 standard form 241.A
 (of a difference equation) 104.C
 of the equation (of a conic section) 78.C
 Legendre-Jacobi (of an elliptic integral) 134.A,
 App. A, Table 16.I
 standard Gaussian distribution 176.A
 standard Kähler metric (of a complex projective
 space) 232.D
 standard measurable space 270.D
 standard normal distribution 341.D
 standard parabolic k -subgroup 13.Q
 standard part (in nonstandard analysis) 293.D
 standard q -simplex 201.E
 standard random walk 260.A
 standard resolution (of Z) 200.M
 standard sequence 400.K
 standard set 22.I
 standard vector space (of an affine space) 7.A
 star \rightarrow also *
 (in a complex) 13.R
 (in a Euclidean complex) 70.B

Subject Index

Stationary random distribution

 (in a projective space) 343.B
 (in a simplicial complex) 70.C
 (of a subset defined by a covering) 425.R
 open 70.B,C
 star body, bounded 182.C
 star convergence 87.K
 (o)- 87.L
 relative uniform 310.F
 star-finite (covering of a set) 425.R
 star-finite property 425.S
 star refinement (of a covering) 425.R
 star region 339.D
 starting values (in a multistep method) 303.E
 start node 281.D
 star topology, weak (of a normed linear space)
 37.E 424.H
 state(s)
 (of a C^* -algebra) 308.D
 (in Ising model) 340.B
 (in quantum mechanics) 351.B
 bound 351.D
 ceiling 402.G
 completeness of the scattering 150.D
 equation of 419.A
 equilibrium 136.H 340.B 419.A
 even 415.H
 fictitious 260.F
 final 31.B
 Gibbs 340.B
 ground 402.G
 in- 150.D 386.A
 initial 31.B
 instantaneous 260.F 261.B
 internal 31.B
 odd 415.H
 out- 150.D 386.A
 scattering 395.B
 stable 260.F 394.A 404.A
 stationary 340.C 351.D
 of statistical control 404.A
 sum over 402.D
 unstable 394.A
 state estimator 86.E
 state space 126.B
 (in catastrophe theory) 51.B
 (of a dynamical system) 126.B
 (of a Markov process) 261.B
 (of a stochastic process) 407.B
 state-space approach 86.A
 state variable 127.A
 static model (in catastrophe theory) 51.B
 stable 51.B
 stationary capacity 213.F
 stationary curve
 (of the Euler-Lagrange differential equation)
 324.E
 (of a variation problem) 46.B
 stationary function 46.B
 stationary iterative process, linear 302.C
 stationary phase method 30.B
 stationary point (of an arc of class C^n) 111.D
 stationary Poisson point process 407.D
 stationary process(es) 342.A 395
 shift associated with 136.D
 strictly 395.A
 strongly 395.A
 weakly 395.A
 weakly, of degree k 395.I
 in the wider sense 395.A
 stationary random distribution

- strictly 395.H
- strongly 395.F.H
- weakly 395.C
- stationary source 213.C
- stationary state 340.C 351.D
- stationary value (of a function) 106.L
- stationary variational inequality 440.B
- stationary wave 446
- statistic 396
 - ancillary 396.H 401.C
 - Hotelling's T^2 280.B
 - invariant 396.H
 - Kolmogorov-Smirnov test 374.E
 - maximal invariant 396.I
 - minimal sufficient 396.I
 - n -dimensional 396.B
 - necessary and sufficient 396.E
 - 1-dimensional 396.B
 - order 396.C
 - selection 396.F
 - t - 374.B
 - U - 374.I
- statistical control, state of 404.A
- statistical data analysis 397.A
- statistical decision function 398
- statistical decision problem 398.A
- statistical decision procedure 398.A
- statistical estimation 399, App. A, Table 23
- statistical experiment 398.G
- statistical genetics 40.B
- statistical hypothesis 400.A
- statistical hypothesis testing 400, App. A, Table 23
- statistical inference 401
- statistical model 403
- statistical mechanics 342.A 402
 - classical 402.A
 - equilibrium 402.A
 - of irreversible processes 402.A
 - Markov 340.C
 - quantum 402.A
- statistical planning 102.A
- statistical quality control 404
- statistical structure 396.E
 - dominated 396.F
- statistical thermodynamics 402.A
- statistics 397.C
 - Bose 377.B 402.E
 - Fermi 377.B 402.E
- statistics and spin, connection of 150.D
- Staudt algebra 343.C
- Steenrod algebra 64.B
- Steenrod axioms, Eilenberg- 201.Q
- Steenrod isomorphism theorem, Hurewicz- 148.D
- Steenrod operator App. A, Table 6.II
- Steenrod p th power operation 64.B
- Steenrod square operation 64.B
- steepest descent, curve of 46.A
- steepness, wave 205.F
- Stein, analytic space in the sense of Behnke and 23.E
- Steinberg formula (on representation of compact Lie groups) 248.Z
- Steinberg group 237.J
- Steinberg symbol 237.J
- Steinberg type 13.O
- Stein continuation theorem, Remmert- 23.B
- Stein decomposition, Fefferman- 168.B
- Steiner problem 179.A
- Steiner symmetrization 228.B
- Steinhaus theorem, Banach-
 - (in a Banach space) 37.H
 - (in a topological linear space) 424.J
- Stein lemma, Hunt- 400.F
- Stein manifold 21.L
 - fundamental theorems of 21.L 72.E
- Stein space 23.F
- Stein theorem, Behnke- 21.H
- Stein theorem, Lehmann- 400.B
- step, fractional 304.F
- step-by-step method 163.D
- step-down operator 206.B
- step size (in numerical solution) 303.B
- step-up operator 206.B
- stereographic projection 74.D
- Stiefel manifold 199.B
 - complex 199.B
 - infinite 147.I
 - of k -frames 199.B
 - of orthogonal k -frames 199.B
 - real, of k -frames 199.B
 - real, of orthogonal k -frames 199.B
- Stiefel-Whitney class
 - (of a differentiable manifold) 56.F 147.M
 - (of an $O(n)$ -bundle) 147.M
 - (of an \mathbb{R}^n -bundle) 56.B
 - (of a topological manifold) 56.F
 - total 56.B
 - universal 56.B
- Stiefel-Whitney number 56.F
- Stieltjes integral 94.E
 - Lebesgue- 94.C 166.C
 - Riemann- 94.B 166.C
- Stieltjes measure, Lebesgue- 166.C 270.L
- Stieltjes moment problem 240.K
- Stieltjes theorem 133.C
- Stieltjes transform 220.D
 - Fourier- 192.B, O
 - Laplace- 240.A
- Stiemke theorem 255.B
- stiff 303.G
 - in an interval 303.G
- stiffness matrix 304.C
- stiffness ratio 303.G
- stiff-stability 303.G
- stimulus-sampling model, Estes 342.H 346.G
- Stirling formula 174.A 212.C, App. A, Table 17.I
- Stirling interpolation formula App. A, Table 21
- Stirling number of the second kind 66.D
- stochastically larger (random variable) 371.C
- stochastic calculus 406.A
- stochastic control 342.A 405
- stochastic differential 406.C
- stochastic differential equation 342.A
 - linear (LSDE) 405.G
 - of Markovian type 406.D
- stochastic differential of Stratonovich type 406.C
- stochastic filtering 324.A 405.F
- stochastic inference, graphical method of 19.B
- stochastic integral 261.E 406.B
 - of Itô type 406.C
 - of Stratonovich type 406.C
- stochastic Ising model 340.C
- stochastic matrix 260.A
- stochastic maximum principle 405.D
- stochastic model 264
- stochastic moving frame 406.G
- stochastic paper 19.B
- stochastic process(es) 342.A 407
 - generalized 407.C
 - with stationary increments of order n 395.I

- stochastic programming 264.C 307.C 408
 - two-stage 408.A
- Stoïlow compactification, Kerékjártó- 207.C
- Stoïlow type (compactification) 207.B
- Stokes approximation 205.C
- Stokes assumption 205.C
- Stokes differential equation 167.E 188.E
- Stokes equation, general Navier- 204.F
- Stokes equation, Navier- 204.B 205.C
- Stokes formula 94.F 105.O, App. A, Table 3.II
 - Green- 94.F
- Stokes initial value problem, Navier- 204.B
- Stokes multiplier 254.C
- Stokes phenomenon 254.C
- Stokes theorem App. A, Table 3.III
- Stokes wave 205.F
- Stolz, differentiable in the sense of 106.G
- Stolz path (in a plane domain) 333.B
- Stone-Čech compactification 207.C 425.T
- Stone-Gel'fand theorem 168.B
- Stone integrable function, Daniell- 310.I
- Stone integral, Daniell- 310.I
- Stone theorem 378.C 425.X 437.R
- Stone theorem, Weierstrass- 168.B
- Stone-Titchmarsh-Kodaira theory, Weyl- 112.O
- stopping, optimal 405.E
- stopping rule 398.F
- stopping time 261.B 407.B
- storage, push-down 96.E
- stored program principle 75.B
- Størmer inequality, Powers- 212.B
- straight angle 139.D
- straightening of the angle 114.F
- straight, G -space is 178.H
- straight line(s) 93.A 155.B
- straight line solution 420.B
- strain 271.G
- strain, shearing 271.G
- strain tensor 271.G
- strange attractor 126.N
- Strassen invariance principle 250.E
- strategic variable 264
- strategy (strategies) 33.F 108.B,C 173.C
 - behavior 173.B
 - local 173.B
 - mixed 173.C
 - pair 108.B
 - pure 173
 - winning 33.F
- stratifiable space 425.Y
- stratification, Whitney 418.G
- stratified sampling 373.E
- Stratonovich type
 - stochastic differential of 406.C
 - stochastic integral of 406.C
- stratum (strata) 373.E 418.G
 - μ -constant 418.E
- stream function 205.B
- streamlined (body) 205.C
- stream lines 205.B
- strength 102.L
- stress 271.G
 - normal 271.G
 - shearing 271.G
 - tangential 271.G
- stress tensor 150.B 271.G
 - Maxwell 130.A
- strict Albanese variety 16.P
- strict implication 411.L
- strict localization 16.AA
- strictly concave function 88.A
- strictly convex function 88.A
- strictly decreasing function 166.A
- strictly ergodic (homeomorphism on a compact metric space) 136.H
- strictly G -stationary (system of random variables) 395.I
- strictly increasing function 166.A
- strictly inductive limit (of a sequence of locally convex spaces) 424.W
- strictly monotone function 166.A
 - (of ordinal numbers) 312.C
- strictly of Pólya type (a family of probability densities) 374.J
- strictly positive (element in E^n) 310.H
- strictly pseudoconvex 344.A
- strictly stable process 5.F
- strictly stationary process 395.A
- strictly stationary random distribution 395.H
- strict morphism (between topological groups) 423.J
- string
 - α - 248.L
 - equation of a vibrating 325.A
- string model 132.C
- strip
 - bicharacteristic 320.B
 - characteristic 320.D 324.B
 - Möbius 410.B
- strip condition 320.D
- strong (boundary component) 77.E
- strong convergence (of operators) 251.C
- strong convergence theorem (on distributions) 125.G
- strong deformation retract 202.D
- strong dilation 251.M
- strong dual (space) 424.K
- stronger
 - (equivalence relation) 135.C
 - (method of summation) 379.L
 - (topology) 425.H
 - (uniformity) 436.E
- stronger form of Cauchy's integral theorem 198.B
- strong extension
 - (of a differential operator) 112.E
 - (of a differential operator with boundary condition) 112.F
- strong infinity, axiom of 33.E
- strong integrability 443.I
- strong lacuna 325.J
- strong law of large numbers 250.C
- strong Lefschetz theorem 16.U
- strongly, converge (in a Banach space) 37.B
- strongly acute type 304.C
- strongly closed subgroup 151.J
- strongly compact cardinal number 33.E
- strongly connected (graph) 186.F
- strongly connected components 186.F
- strongly continuous
 - (Banach space-valued function) 37.K
 - (in unitary representations) 437.A
- strongly continuous representation (of a topological space) 69.B
- strongly continuous semigroup 378.B
- strongly distinguished basis 418.F
- strongly elliptic (differential operator) 112.G
- strongly elliptic operator 323.H
- strongly embedded subgroup 151.J
- strongly exposed (of a convex set) 443.H
- strongly hyperbolic differential operator 325.H
- strongly inaccessible 33.F 312.E

Strongly inaccessible cardinal number

- strongly inaccessible cardinal number 33.E
- strongly measurable 443.B.I
- strongly mixing automorphism 136.E
- strongly nonlinear differential equation 290.D
- strongly normal extension field 113
- strongly P -convex set 112.C
- strongly paracompact space 425.S
- strongly pseudoconvex domain 21.G
- strongly recurrent (measurable transformation) 136.C
- strongly separated (convex sets) 89.A
- strongly stationary process 395.A
- strongly stationary random distribution 395.H
- strong Markov process 261.B
- strong Markov property 261.B
- strong maximum principle 323.C
- strong measurability 443.I
- strong operator topology 251.C
- strong rigidity theorem 122.G
- strong solution
 - (of Navier-Stokes equation) 204.C
 - (of a stochastic differential equation) 406.D
 - unique 406.D
- strong topology
 - (on a direct product space) 425.K
 - (on a family of measures) 338.E
 - (on a normed space) 37.E
 - (on a topological linear space) 424.K
- strong transversality condition 126.J
- structural constants (of a Lie algebra) 248.C
- structural equation system, linear 128.C
- structurally stable, C^r - 126.H
- structural stability 290.A
- structural stability theorem 126.J
- structure(s) 409
 - (of a language) 276.B
 - almost complex 72.B
 - almost contact 110.E
 - almost contact metric 110.E
 - almost symplectic 191.B
 - analytic (in function algebras) 164.F
 - analytic (on a Riemann surface) 367.A
 - arithmetically equivalent 276.D
 - Cauchy Riemann 344.A
 - CR 344.A
 - C^r - (of a differentiable manifold) 105.D 114.A
 - C^r -, Haefliger 154.F
 - classifying space for Γ_q 154.E
 - coalition 173.D
 - compatible with C^r - 114.B
 - complex 105.Y
 - complex (in a complex manifold) 72.A
 - complex (on \mathbf{R}^{2n}) 3.H
 - complex (on a Riemann surface) 367.A
 - complex analytic (in a complex manifold) 72.A
 - conformal 191.B
 - conformal (on a Riemann surface) 367.A
 - contact 105.Y
 - contact metric 110.E
 - data 96.B
 - deformation of complex 72.G
 - differentiable 114.B
 - differentiable, of class C^r 105.D
 - elementarily equivalent 276.D
 - equations (of a Euclidean space) 111.B
 - equations of (for relative components) 110.A
 - foliated 105.Y
 - G - 191
 - Γ - (on a differentiable manifold) 105.Y
 - Γ - (on a topological space) 90.D
 - Γ_q - 154.E
 - Γ_φ - 154.H
 - group of oriented differentiable (on a combinatorial sphere) 114.I
 - Hodge (of a vector space) 16.V
 - isomorphic 276.E
 - jumping of 72.G
 - lacunary (of a power series) 339.E
 - level n (on an Abelian variety) 3.N
 - linear 96.C
 - mathematical 409.B
 - mixed 16.V
 - Neyman 400.D
 - normal 276.D
 - normal analytic 386.C
 - PL 65.C
 - pseudogroup 105.Y
 - real analytic 105.D
 - smooth 114.B
 - spin- 237.F 431.D
 - statistical 396.E
 - symplectic 219.C
 - tensor field of almost complex (induced by a complex structure) 72.B
 - topological 425.A,B
 - tree 96.D
 - twinning 92.D
 - uniform 436.B
- structure equation
 - (of an affine connection) 417.B
 - (for a curvature form) 80.G
 - (for a torsion form) 80.H
 - linear, system 128.C
- structure function 191.C
- structure group (of a fiber bundle) 147.B
- structure morphism 52.G
- structure sheaf
 - (of a prealgebraic variety) 16.C
 - (of a ringed space) 383.H
 - (of a variety) 16.B
- structure space (of a Banach algebra) 36.D
- structure theorem
 - (on topological Abelian groups) 422.E
 - of complete local rings 284.D
 - for von Neumann algebras of type III 308.I
- Sturm-Liouville operator 112.I
- Sturm-Liouville problem 315.B
- Sturm method 301.C
- Sturm theorem (on real roots of an algebraic equation) 10.E
- Struve function App. A, Table 19.IV
- Student test 400.G
- subadditive cuts 215.C
- subadditive ergodic theorem 136.B
- subadditive functional 88.B
- subadditive process 136.B
- subalgebra 29.A
 - Borel (of a semisimple Lie algebra) 248.O
 - Cartan (of a Lie algebra) 248.I
 - Cartan (symmetric Riemann space) 413.F
 - closed (of a Banach algebra) 36.B
 - Lie 248.A
 - of a Lie algebra associated with a Lie subgroup 249.D
 - parabolic (of a semisimple Lie algebra) 248.O
 - *- 308.C
- subbase
 - for a space 425.F

- for a topology 425.F
- subbialgebra 203.G
- subbundle
 - (of an algebraic vector bundle) 16.Y
 - (of a vector bundle) 147.F
- subcategory 52.A
 - full 52.A
- subcoalgebra 203.F
- subcomplex
 - (of a cell complex) 70.D
 - (of a chain complex) 200.H 201.B
 - (of a cochain complex) 201.H
 - (of a complex) 13.R
 - (of a Euclidean complex) 70.B
 - (of a simplicial complex) 70.C
 - (of an s.s. complex) 70.E
 - chain 200.C
 - cochain 200.F
- subcontraction 186.E
- subcritical (Galton-Watson process) 44.B
- subdifferential 88.D
- subdivision
 - (of a Euclidean complex) 70.B
 - (of a simplicial complex) 70.C
 - (of a triangulation) 70.C
 - barycentric (of a Euclidean complex) 70.B
 - barycentric (of a simplicial complex) 70.C
 - dual (of a triangulation of a homology manifold) 65.B
- subelliptic 112.D
- subfamily 165.D
- subfield 149.B
 - valuation over a 439.B,C
- subgraph 186.E
- subgroup
 - (of a group) 190.C
 - (of a topological group) 423.D
 - admissible 190.E
 - admissible normal 190.E
 - algebraic 13.A
 - arithmetic 13.P 122.F,G
 - Borel 13.G 249.J
 - Cartan 13.H 249.I
 - Carter 151.D
 - closed 423.D
 - commutator 190.H
 - congruence 122.D
 - connected Lie 249.D
 - cyclic 190.C
 - divisible 422.G
 - Hall 151.E
 - invariant 190.C
 - irreducible discrete 122.F
 - isotropy 431.A
 - Iwahori 13.R
 - k -Borel 13.G
 - Levi- 13.Q
 - Lie 249.D
 - maximal torsion 2.A
 - minimal parabolic k - 13.Q
 - normal 190.G
 - Ω - 190.E
 - one-parameter 249.Q
 - open 423.D
 - parabolic 13.G 249.J
 - parabolic 13.R
 - principal congruence, of level N 122.D
 - p -Sylow 151.B
 - rational 404.B
 - Schur 362.F
 - sequences of 190.F
 - stability 431.A
 - standard parabolic k - 13.Q
 - strongly closed 151.J
 - strongly embedded 151.J
 - subnormal 190.G
 - Sylow 151.B
 - toroidal 248.X
 - torsion 2.A,C
- subharmonic functions 193
 - almost 193.T
- subinvariant measure 261.F
- subjective probability 401.B
- sublattice 243.C
- submanifold
 - (of a Banach manifold) 286.N
 - (of a combinatorial manifold) 65.D
 - (of a C^∞ -manifold) 105.L
 - closed 105.L
 - complex analytic 72.A
 - immersed (of a Euclidean space) 111.A
 - isotropic 365.D
 - Kähler 365.L
 - minimal 275 365.D
 - regular 105.L
 - Riemannian 365.A
 - totally geodesic 365.D
 - totally real 365.M
 - totally umbilical 365.D
- submartingale 262.A
- submedian 193.T
- submersion 105.L
- submodular 66.F
- submodule
 - A - 277.C
 - allowed 277.C
 - complementary 277.H
 - homogeneous A - (of a graded A -module) 200.B
 - primary 284.A
- subnet 87.H
 - cofinal 87.H
- subnormal (operator in a Hilbert space) 251.K
- subnormal subgroup 190.G
- subobject 52.D
- subordinate 105.D 437.T
- subordination 5 261.F
 - of the α th order 261.F
- subordinator of the exponent α 5.F
- subproblems 215.D
- subrepresentation
 - (of a linear representation) 362.C
 - (of a projective representation) 362.J
 - (of a unitary representation) 437.C
- subring 368.E
 - differential 113
- subroutine 75.C
- subscripts, raising 417.D
- subsequence 165.D
 - φ - 354.D
- subset(s) 381.A
 - (in axiomatic set theory) 33.B
 - analytic (of a complex manifold) 72.E
 - axiom of 33.B 381.G
 - Borel 270.C
 - circled (of a linear topological space) 424.E
 - cofinal 311.D
 - G - 362.B
 - k - 330
 - proper 381.A

residual 311.D
subshift 126.J
 of finite type 126.J
 Markov 126.J
subsidiary equation, Charpit 82.C 320.D
subsonic (Mach number) 205.B
subsonic flow 326.A
subspace
 (of an affine space) 7.A
 (of a linear space) 256.F
 (of a projective space) 343.B
 (of a topological space) 425.J
 analytic 23.C,G
 closed linear (of a Hilbert space) 197.E
 complementary (of a linear subspace) 256.F
 horizontal 191.C
 ingoing 375.H
 invariant (of a linear operator) 251.L
 involutive 428.F
 linear (of a linear space) 256.F
 metric 273.B
 n -particle 377.A
 orthogonal (determined by a linear subspace)
 256.G
 orthogonal (of a linear space) 139.G
 outgoing 375.H
 precompact (metric) 273.B
 parallel (in an affine space) 7.B
 parallel, in the narrower sense (in an affine
 space) 7.B
 parallel, in the wider sense (in an affine space)
 7.B
 principal (of a linear operator) 390.B
 root (of a linear operator) 390.B
 root (of a semisimple Lie algebra) 248.K
 singular (of a singular projective transforma-
 tion) 343.D
 totally bounded (metric) 273.B
 totally isotropic (relative to an ε -Hermitian
 form) 60.O
 totally isotropic (with respect to a quadratic
 form) 348.E
 totally singular (with respect to a quadratic
 form) 348.E
 U -invariant (of a representation space of a
 unitary representation) 437.C
 uniform 436.E
substituted distribution 125.Q
substitution
 (of a hyperfunction) 125.X 274.E
 axiom of 381.G
 back 302.B
 Frobenius (of a prime ideal) 14.K
subsystem
 (of an algebraic system) 409.C
 closed (of a root system) 13.L
subtraction 361.B
subtraction terms 361.B
subvariety, Abelian 3.B
successive approximation
 method of (for an elliptic partial differential
 equation) 323.D
 method of (for Fredholm integral equations of
 the second kind) 217.D
 method of (for ordinary differential equations)
 316.D
successive minima (in a lattice) 182.C
successive minimum points 182.C
successive overrelaxation (SOR) 302.C
successor

 (of an element in an ordered set) 311.B
 (of a natural number) 294.B
sufficiency
 prediction 396.J
 principle of 401.C
sufficient (σ -field, statistic)
 Bayes 396.J
 D - 396.J
 decision theoretically 396.J
 minimal 396.E
 pairwise 396.F
 test 396.J
sufficiently many irreducible representations
 437.B
sum
 (of convergent double series) 379.E
 (of a divergent series by a summation) 379.L
 (of elements of a group) 190.A
 (of elements of a linear space) 256.A
 (a function) 104.B
 (of ideals) 67.B
 (of linear operators) 251.B
 (of linear subspaces) 256.F
 (of matrices) 269.B
 (of ordinal numbers) 312.C
 (of potencies) 49.C
 (of real numbers) 355.A
 (of a quadrangular set of six points) 343.C
 (of a series) 379.A
 (of submodules) 277.B
 (= union of sets) 33.B 381.B
 (of vectors) 442.A
 amalgamated 52.G
 Baer (of extensions) 200.K
 cardinal (of a family of ordered sets) 311.F
 Cauchy (of a series) 379.A
 connected (of oriented compact C^∞ -manifolds)
 114.F
 connected (of 3-manifolds) 65.E
 constant- (game) 173.A
 Darboux 216.A
 Dedekind 328.A
 diagonal (of a matrix) 269.F
 diagonal partial (of a double series) 379.E
 direct \rightarrow direct sum
 disjoint 381.B
 fiber 52.G
 Gaussian 295.D 450.C
 general- (game) 173.A
 indefinite (of a function) 104.B
 Kloosterman 32.C
 local Gaussian 450.F
 logical (of propositions) 411.B
 ordinal (of a family of ordered sets) 311.G
 orthogonality for a finite 19.G 317.D
 partial (of a series) 379.A
 of products 216.A
 Ramanujan 295.D
 Riemann 216.A
 scalar (of linear operators) 37.C
 over states 402.D
 topological 425.M
 trigonometric 4.C
 Whitney (of vector bundles) 147.F
 zero (game) 173.A
 zero-, two-person game 108.B
sum event 342.B
summable
 A - 379.N
 by Abel's method 379.N

- absolute Borel 379.O
- \mathfrak{B} - 379.O
- $|\mathfrak{B}|$ - 379.O
- by Borel's exponential method 379.O
- by Borel's integral method 379.O
- by Cesàro's method of order α 379.M
- by Euler's method 379.P
- by Hölder's method of order p 379.M
- (H, p) - 379.M
- by Nörlund's method 379.Q
- (R, k) - 379.S
- by Riesz's method of order k 379.R
- T - 379.L
- summable p th power, operator of 68.K
- summand, direct (of a direct sum of sets) 381.E
- summation
 - Abel's method of 379.N
 - Abel's partial 379.D
 - Borel's method of 379.N
 - (C, α) - 379.M
 - Cesàro's method of, of order α 379.M
 - Euler's method of 379.P
 - of a function 104.B
 - Lebesgue's method of 379.S
 - methods of 379.L
 - Nörlund's method of 379.Q
 - Riemann's method of 379.S
 - Riesz's method of, of the k th order 379.R
- summation convention, Einstein's 417.B
- summation formula
 - Euler 295.E
 - Poisson (on Fourier transforms) 192.C
 - Poisson (on a locally compact Abelian group) 192.L
- summing, absolutely (operator) 68.N
- sum theorem for dimension 117.C
- Sundman theorem 420.C
- sup (supremum) 311.B
- superabundance (of a divisor on an algebraic surface) 15.D
- superadditive 173.D
- superconductivity 130.B
- supercritical (Galton-Watson process) 44.B
- superefficient estimator 399.N
- superharmonic (function) 193.P 260.D
- superharmonic measure 260.I
- superharmonic transformation 261.F
- superior function, right 316.E
- superior limit
 - (of a sequence of real numbers) 87.C
 - (of a sequence of subsets of a set) 270.C
- superior limit event 342.B
- supermartingale 262.A
- supermultiplet theory 351.J
- superposition, principle of 252.B 322.C
- superregular function 260.D
- superrenormalizable 150.C
- superscripts, lowering 417.D
- superselection rule, univalence 351.K
- superselection sector 150.E 351.K
- supersolvable group 151.D
- supersonic 205.B 326.A
- supplementary angles 139.D
- supplementary interval 4.B
- supplementary series 258.C
- supplementation-equal polygons 155.F
- supplemented algebra 200.M
- support
 - (of a coherent sheaf) 16.E
 - (of a differential form) 105.Q
 - (of a distribution) 125.D
 - (of a function) 125.B 168.B 425.R
 - (of a section of a sheaf) 383.C
 - (of a spectral measure) 390.D
 - compact (of a singular q -cochain) 201.P
 - essential (of a distribution) 274.D
 - singular (of a distribution) 112.C
 - singular (of a hyperfunction) 125.W
- supporting function 125.O
- supporting functional (of a convex set) 89.G
- supporting half-space (of a convex set) 89.A
- supporting hyperplane (of a convex set) 89.A
- supporting line (of an oval) 89.C
- supporting line function (of an oval) 89.C
- supporting point
 - (of a convex set) 89.G
 - (of a projective frame) 343.C
- supremum
 - (of an ordered set) 168.B
 - (of a set of Hermitian operators) 308.A
 - (of a subset of a vector lattice) 310.C
 - essential (of a measurable function) 168.B
- supremum norm 168.B
- supremum theorem, Hardy-Littlewood App. A, Table 8
- sure event 342.B
- surely, almost 342.B,D
- surface(s) 111.A 410, App. A, Table 4.I
 - Abelian 15.H
 - abstract Riemann 367.A
 - affine minimal 110.C
 - algebraic 15
 - basic (of a covering surface) 367.B
 - with boundary 410.B
 - branched minimal 275.B
 - center 111.I
 - characteristic 320.B
 - circular cylindrical 350.B
 - closed 410.B
 - closed (in a 3-dimensional Euclidean space) 111.I
 - closed convex 111.I
 - conical 111.I
 - of constant curvature 111.I
 - covering 367.B
 - covering, Ahlfors theory of 367.B
 - covering, with relative boundary 367.B
 - cylindrical 111.I
 - deformation of 110.A
 - degenerate quadric 350.B
 - developable 111.I, App. A, Table 4.I
 - Dini 111.I
 - elliptic 72.K
 - elliptic cylindrical 350.B
 - energy 126.L 402.C,G
 - Enneper 275.B
 - Enriques 72.K
 - enveloping 111.I
 - equipotential 193.J
 - Fréchet 246.I
 - fundamental theorem of the theory of 111.G
 - fundamental theorem of the topology of 410.B
 - G - 178.H
 - of general type 72.K
 - geometry on a 111.G
 - helicoidal 111.I
 - Hilbert modular 15.H
 - Hirzebruch 15.G
 - Hopf 72.K
 - hyperbolic cylindrical 350.B

- hyperelliptic 72.K
- initial 321.A
- K^3 15.H 72.K
- Kummer 15.H
- level 193.J
- marked K^3 72.K
- minimal 111.I 334.B
- niveau 193.J
- one-sided 410.B
- open 410.B
- parabolic cylindrical 350.B
- proper quadric 350.B
- quadric 350.A
- quadric conical 350.B
- rational 15.E
- rectifying 111.I
- response 102.L
- response, design for exploring 102.M
- of revolution 111.I
- Riemann \rightarrow Riemann surface
- ruled 15.E
- ruled (in differential geometry) 111.I
- of the second class 350.D
- of the second order 350.A
- Scherk's 275.A
- Seifert 235.A
- skew 111.I
- special 110.A
- tangent 111.F
- two-sided 410.B
- unbounded covering 367.B
- unirational 15.H
- universal covering 367.B
- unramified covering 367.B
- Veronese 275.F
- W - 111.I
- Weingarten (W) 111.I
- surface area of unit hypersphere App. A, Table 9.V
- surface element 324.B
- surface harmonics 393.A
- surface integral 94.A,E
- (with respect to a surface element) 94.E
- surface wave 446
- surgery 114.F,J
- surgery obstruction 114.J
- surjection 381.C
- (in a category) 52.D
- canonical (on direct products of groups) 190.L
- canonical (to a factor group) 190.D
- canonical (onto a quotient set) 135.B
- natural (to a factor group) 190.D
- surjective mapping 381.C
- survival insurance 214.B
- susceptibility
- electric 130.B
- magnetic 130.B
- suspension
- (of a discrete dynamical system) 126.C
- (of a homotopy class) 202.Q
- (of a map) 202.F
- (of a space) 202.E,F
- n -fold reduced 202.F
- reduced (of a topological space) 202.F
- Suslin
- κ - 22.H
- schema of 22.B
- system of 22.B
- Suslin hypothesis 33.F
- Suslin space 22.I 425.CC
- Suslin theorem 22.C
- suspension isomorphism (on singular (co)homology groups) 201.E
- suspension theorem, generalized 202.T
- Suzuki group 151.I
- SVD (singular value decomposition) 302.E
- sweep (a bounded domain) 384.F
- sweepable (bounded domain) 384.F
- sweeping-out principle 338.L
- sweeping-out process 338.L
- sweep out (a measure to a compact set) 338.L
- Swinnerton-Dyer conjecture, Birch- 118.D 450.S
- Sylow subgroup 151.B
- p - 151.B
- Sylow theorem 151.B
- Sylvester elimination method 369.E
- Sylvester law of inertia (on a quadratic form) 348.B
- Sylvester theorem (on determinants) 103.F
- Symanzik equation, Callan- 361.B
- symbol 369.A
- (of a Fourier integral operator) 274.C
- (of a pseudodifferential operator) 251.O 345.B
- (= Steinberg symbol) 237.J
- (of a vector field) 105.M
- Artin 14.K
- Christoffel 80.L 111.H 417.D, App. A, Table 4.II
- function 411.H
- Gauss 83.A
- Hilbert ε - 411.J
- Hilbert-Hasse norm-residue 14.R
- Hilbert norm-residue 14.R
- individual 411.H
- Jacobi 297.I
- Jacobi, complementary law of 297.I
- Jacobi, law of quadratic reciprocity of 297.I
- Kronecker 347.D
- Landau (O, o) 87.G
- Legendre 297.H
- Legendre, first complementary law of 297.I
- Legendre, law of quadratic reciprocity of 297.I
- Legendre, second complementary law of 297.I
- logical 411.B
- $9j$ - 353.C
- norm-residue 14.Q 257.F
- power-residue 14.N
- predicate 411.H
- principal (of a differential operator) 237.H
- principal (of a microdifferential operator) 274.F
- principal (of a simple holonomic system) 274.H
- $6j$ - 353.B
- Steinberg (in algebraic K -theory) 237.J
- $3j$ - 353.B
- symbolic logic 411
- symbol sequence (in microlocal analysis) 274.F
- symmetric
- (block design) 102.E
- (factorial experiment) 102.H
- (Fock space) 377.A
- (member of a uniformity) 436.B
- (multilinear mapping) 256.H
- (relation) 358.A
- (tensor) 256.N
- symmetric algebra 29.H
- symmetric bilinear form (associated with a quadratic form) 348.A
- symmetric bounded domain 412.F
- irreducible 412.F
- symmetric Cauchy process 5.F

- symmetric difference 304.E
- symmetric distribution function 341.H
- symmetric event 342.G
- symmetric function 337.I
 - elementary 337.I
- symmetric group 190.B
 - of degree n 151.G
- symmetric Hermitian space 412.E
 - irreducible 412.E
- symmetric homogeneous space 412.B
- symmetric hyperbolic system (of partial differential equations) 325.G
- symmetric kernel 217.G 335.D
- symmetric law (in an equivalence relation) 135.A
- symmetric Markov process 261.C
- symmetric matrix 269.B
 - anti- 269.B
 - skew- 269.B
- symmetric multilinear form 256.H
 - anti- 256.H
 - skew- 256.H
- symmetric multilinear mapping 256.H
 - anti- 256.H
 - skew- 256.H
- symmetric multiplication 406.C
- symmetric operator 251.E
- symmetric points (with respect to a circle) 74.E
- symmetric polynomial 337.I
 - elementary 337.I
 - fundamental theorem on 337.I
- symmetric positive system
 - (of differential operators) 112.S
 - (of first-order linear partial differential equations) 326.D
- symmetric product (of a topological space) 70.F
- symmetric Riemannian homogeneous space 412.B
- symmetric Riemannian space(s) 412
 - globally 412.A
 - irreducible 412.C, App. A, Table 5.III
 - locally 412.A, App. A, Table 4.II
 - weakly 412.J
- symmetric space 412.A
 - affine 80.J
 - affine locally 80.J
 - locally 80.J 364.D
- symmetric stable process 5.F
- symmetric tensor 256.N
 - anti- 256.N
 - contravariant 256.N
 - covariant 256.N
 - skew- 256.N
- symmetric tensor field 105.O
- symmetrization
 - (in isoperimetric problem) 228.B
 - Steiner (in isoperimetric problem) 228.B
- symmetrizer 256.N
 - Young 362.H
- symmetry
 - (at a point of a Riemannian space) 412.A
 - (of a principal space) 139.B
 - (in quantum mechanics) 415.H
 - broken 132.C
 - central (of an affine space) 139.B
 - charge 415.J
 - crossing 132.C 386.B
 - degree of 431.D
 - hyperplanar (of an affine space) 139.B
 - internal 150.B
 - law of (for the Hilbert norm-residue symbol) 14.R
 - Nelson 150.F
 - TCP 386.B
- symmetry group, color 92.D
- symmorphic space group 92.B
- symmorphous space group 92.B
- symplectic form 126.L
- symplectic group 60.L 151.I
 - complex 60.L
 - infinite 202.V
 - over a field 60.L
 - over a noncommutative field 60.O
 - projective (over a field) 60.L
 - unitary 60.L
- symplectic manifold 219.C
- symplectic matrix 60.L
- symplectic structure 219.C
- symplectic transformation 60.L
 - (over a noncommutative field) 60.O
- symplectic transformation group (over a field) 60.L
- synchronous (system of circuits) 75.B
- syndrome 63.C
- synthesis (in the theory of networks) 282.C
 - spectral 36.L
- synthetic geometry 181
- system
 - adjoint (of a complete linear system on an algebraic surface) 15.D
 - adjoint, of differential equations 252.K
 - algebraic 409.B
 - algebraic, in the wider sense 409.B
 - ample linear 16.N
 - asynchronous (of circuits) 75.B
 - axiom 35
 - axiom (of a structure) 409.B
 - axiom (of a theory) 411.I
 - of axioms 35.B
 - base for the neighborhood 425.E
 - categorical (of axioms) 35.B
 - C*-dynamical 36.K
 - character (of a genus of a quadratic field) 347.F
 - characteristic linear (of an algebraic family) 15.F
 - Chebyshev (of functions) 336.B
 - classical dynamical 126.L 136.G
 - of closed sets 425.B
 - complete (of axioms) 35.B
 - complete (of independent linear partial differential equations) 324.C
 - complete (of inhomogeneous partial differential equations) 428.C
 - complete (of nonlinear partial differential equations) 428.C
 - complete linear (on an algebraic curve) 9.C
 - complete linear (on an algebraic variety) 16.N
 - complete linear, defined by a divisor 16.N
 - completely integrable (of independent 1-forms) 428.D
 - complete orthogonal 217.G
 - complete orthonormal 217.G
 - complete orthonormal, of fundamental functions 217.G
 - complete residue, modulo m 297.G
 - continuous dynamical 126.B
 - coordinate 90.A
 - coordinate (of a line in a projective space) 343.C
 - crystal 92.B
 - determined (of differential operators) 112.R

- determined (of partial differential equations) 320.F
- differentiable dynamical 126.B
- differential 191.I
- of differential equations of Maurer-Cartan 249.R
- of differential operators 112.R
- direct (of sets) 210.B
- discrete dynamical 126.B
- distinct, of parameters 284.D
- dynamical 126
- of equations 10.A
- equilibrium, transformation to 82.D
- formal 156.D 411.I
- of functional differential equations 163.E
- fundamental (of eigenfunctions to an eigenvalue for an integral equation) 217.F
- fundamental (for a linear difference equation) 104.D
- fundamental (of a root system) 13.J
- fundamental, of irreducible representations (of a complex semisimple Lie algebra) 248.W
- fundamental, of neighborhoods 425.E
- fundamental, of solutions (of a homogeneous linear ordinary differential equation) 252.B
- fundamental, of solutions (of a homogeneous system of first-order linear differential equations) 252.H
- fundamental root (of a semisimple Lie algebra) 248.N
- of fundamental solutions (of a system of linear homogeneous equations) 269.M
- Garnier 253.E
- of generators (of a A -module) 277.D
- of gravitational units 414.B
- group 235.B
- Haar, of orthogonal functions 317.C
- Hamiltonian 126.L
- holonomic 274.H
- holonomic, with regular singularities 274.H
- homotopy equivalent (of topological spaces) 202.F
- of hyperbolic differential equations (in the sense of Petrovskii) 325.G
- incompatible (of partial differential equations) 428.B
- inconsistent (of algebraic equations) 10.A
- indeterminate (of algebraic equations) 10.A
- inductive (in a category) 210.D
- inductive (of sets) 210.B
- inductive, of groups 210.C
- inductive, of topological spaces 210.C
- inertial 271.D 359
- information retrieval 96.F
- integrable 287.A
- international, of units 414.A
- inverse (of sets) 210.B
- involution \rightarrow involutory
- involutionary (of differential forms) 428.F
- involutionary (of nonlinear equations) 428.C
- involutionary (of partial differential equations) 428.F
- involutionary (of partial differential equations of first order) 324.D
- irreducible linear 16.N
- lattice spin 402.G
- linear (on an algebraic variety) 16.N
- of linear differential equations of first order 252.G
- linear dynamical 86.B
- of linear equations 269.M
- of linear homogeneous equations 269.M
- linear structural equation 128.C
- linear time-varying 86.B
- local, of groups (over a topological space) 201.R
- local coordinate (of a manifold) 105.C
- local coordinate (of a topological space) 90.D
- local coordinate, holomorphic 72.A
- mathematical (for a structure) 409.B
- maximal independent (of an additive group) 2.E
- neighborhood 425.B
- of notations (for ordinal numbers) 81.B
- null (in projective geometry) 343.D
- number, point range of (in projective geometry) 343.C
- of open sets 425.B
- operating 75.C
- of ordinary differential equations 313.B
- orthogonal (of functions) 317.A
- orthogonal (of a Hilbert space) 197.C
- of orthogonal functions App. A, Table 20
- of orthogonal polynomials 317.D
- overdetermined (of differential operators) 112.R
- overdetermined (of partial differential equations) 320.F
- of parameters 284.D
- partial de Rham 274.G
- of partial differential equations of order l (on a differentiable manifold) 428.F
- passive orthonomic (of partial differential equations) 428.B
- peripheral 235.B
- of Pfaffian equations 428.A
- polar (in projective geometry) 343.D
- Postnikov (of a CW complex) 148.D
- projective (in a category) 210.D
- projective (of sets) 210.B
- projective, of groups 210.C
- projective, of topological groups 423.K
- projective, of topological spaces 210.C
- quotient (of an algebraic system) 409.C
- Rademacher, of orthogonal functions 317.C
- reduced residue 297.G
- reducible linear 16.N
- regular, of parameters 284.D
- of resultants 369.E
- root (of a symmetric Riemann space) 413.F
- root (in a vector space over \mathbb{Q}) 13.J
- self-adjoint, of differential equations 252.K
- simple holonomic 274.H
- of simultaneous differential equations App. A, Table 14.I
- of Suslin 22.B
- symmetric positive (of differential operators) 112.S
- symmetric positive (of first-order linear partial differential equations) 326.D
- Tits 13.R 151.J 343.I
- of total differential equations 428.A
- of transitivity (of a G -set) 362.B
- trigonometric 159.A
- two-bin 227
- underdetermined (of differential operators) 112.R
- underdetermined (of partial differential equations) 320.F
- uniform covering 436.D

uniform family of neighborhoods 436.D
 uniform neighborhood 436.D
 unisolvent (of functions) 336.B
 of units 414.A
 very ample linear 16.N
 Walsh, of orthogonal functions 317.C
 system process 405.F
 system simulation 385.A
 syzygy 369.F
 first 369.F
 r th 369.F
 syzygy theorem, Hilbert 369.F
 syzygy theory 200.K 369.F
 Szegő kernel function 188.H
 Szemerédi theorem, ergodic 136.C
 Sz.-Nagy-Foiaş model 251.N

T

 τ -function 150.D
 t -distribution 341.D 374.B, App. A, Table 22
 noncentral 374.B
 t -statistic 374.B
 t -test 400.G
 T -bound 331.B
 T -bounded 331.B
 T -compact 68.F 331.B
 T -operator 375.C
 T^2 -statistic
 Hotelling's 280.B
 noncentral Hotelling 374.C
 T -number (transcendental number) 430.C
 T^* -number (transcendental number) 430.C
 T_0 -space 425.Q
 T_1 -space 425.Q
 T_2 -space 425.Q
 T_3 -space 425.Q
 T_4 -space 425.Q
 T_5 -space 425.Q
 T_6 -space 425.Q
 T -positivity 150.F
 T -set 308.I
 T -summable (series) 379.L
 T_1 -uniformity 436.C
 T_1 -uniform space 436.C
 T_2 -topological group 423.B
 table
 analysis-of-variance 400.H
 contingency 397.K 400.K
 difference 223.C
 k -way contingency 397.K
 Padé 142.E
 tableau, simplex 255.C
 table look-up 96.C
 tail event 342.G
 tail σ -algebra 342.G
 Tait algorithm 157.C
 Tait coloring 157.C
 Takagi, Teiji 415
 Takesaki, duality theorem of 308.J
 Takesaki theory, Tomita- 308.H
 Tamagawa number (of an algebraic group) 13.P
 Tamagawa zeta function 450.L
 Tamano product theorem 425.X
 tame (knot) 235.A
 tan (=tangent) 131.E
 \tan^{-1} 131.E
 Tanaka embedding 384.D
 tangent 432.A
 (of pressure) 402.G

Subject Index

Teichmüller space

asymptotic 110.B
 Darboux 110.B
 hyperbolic 131.F
 law of App. A, Table 2.III
 tangent bundle
 (of a Banach manifold) 286.K
 (of a differentiable manifold) 105.H 147.F
 (of a foliation) 154.B
 tangent hyperplane (of a quadric hypersurface) 343.E
 tangential polar coordinates 90.C
 tangential sphere bundle 274.E
 tangential stress 271.G
 tangent line 93.G 111.C, F, App. A, Table 4.I
 oriented 76.B
 tangent orthogonal n -frame bundle 364.A
 tangent plane 111.H, App. A, Table 4.I
 tangent PL microbundle 147.P
 tangent r -frame(s) 105.H
 bundle of 105.H
 tangent r -frame bundle 147.F
 tangent space 105.H
 tangent sphere bundle, unit 126.L
 tangent surface 111.F
 tangent vector 105.H
 holomorphic 72.A
 of type (0, 1) 72.C
 of type (1, 0) 72.C
 tangent vector bundle 105.H 147.F
 tangent vector space 105.H
 tanh (hyperbolic tangent) 131.F
 Taniyama-Weil conjecture 450.S
 Tannaka duality theorem 69.D 249.U
 target (of a jet) 105.X
 target variable 264
 Tate cohomology 200.N
 Tate conjecture 450.S
 Tate-Shafarevich group 118.D
 Tate theorem 59.H
 Tauberian theorem 121.D 339.B
 generalized 36.L 160.G
 generalized, of Wiener 192.D
 Tauberian type, theorem of 339.B
 Tauber theorem 339.B
 tautochrone 93.H
 tautological line bundle 16.E
 tautology 411.E
 Taylor expansion
 (of an analytic function of many variables) 21.B
 (of a holomorphic function of one variable) 339.A
 formal 58.C
 Taylor expansion and remainder App. A, Table 9.IV
 Taylor formula App. A, Table 9.IV
 (for a function of many variables) 106.J
 (for a function of one variable) 106.E
 Taylor series 339.A
 Taylor theorem (in a Banach space) 286.F
 TCP invariance 386.B
 TCP operator 150.D
 TCP symmetry 386.B
 TCP theorem 386.B
 technique, program evaluation and review 376
 TE wave 130.B
 Teichmüller mapping 352.C
 Teichmüller metric 416
 Teichmüller space 416
 universal 416

- telegraph equation 325.A, App. A, Table 15.III
- temperature (of states) 419.A
 - absolute 419.A
- tempered distribution 125.N
- temporally homogeneous
 - (additive process) 5.B
 - (Markov process) 261.A
- TEM wave 130.B
- tension 281.B
 - modulus of elasticity in 271.G
- tension field 195.B
- tensor
 - alternating 256.N
 - angular momentum 258.D
 - antisymmetric 256.N
 - conformal curvature App. A, Table 4.II
 - contracted 256.L
 - contravariant, of degree p 256.J
 - correlation 433.C
 - covariant, of degree q 256.J
 - curvature 80.J,L 364.D 417.B
 - energy-momentum 150.B 359.D
 - energy spectrum 433.C
 - fundamental (of a Finsler space) 152.C
 - fundamental (of a Riemannian manifold) 364.A
 - Green 188.E
 - irreducible, of rank k 353.C
 - Maxwell stress 130.A
 - mixed 256.J
 - Nijenhuis 72.B
 - numerical App. A, Table 4.II
 - projective curvature App. A, Table 4.II
 - Ricci 364.D, App. A, Table 4.II
 - second fundamental 417.F
 - skew-symmetric 256.N
 - strain 271.G
 - stress 150.B 271.G
 - symmetric 256.N
 - torsion App. A, Table 4.II
 - torsion (of an affine connection) 80.J 417.B
 - torsion (of an almost contact structure) 110.E
 - torsion (of a Fréchet manifold) 286.L
 - torsion (of a Riemannian connection) 80.L
 - of type (p, q) 256.J
 - Weyl's conformal curvature 80.P
- tensor algebra
 - (on a linear space) 256.K
 - contravariant 256.K
- tensor bundle (of a differentiable manifold) 147.F
- tensor calculus 417, App. A, Table 4.II
- tensor field 105.O
 - of almost complex structure (induced by a complex structure) 72.B
 - alternating 105.O
 - of class C^r 105.O
 - contravariant, of order r 105.O
 - covariant, of order s 105.O
 - covariant derivative of (in the direction of a tangent vector) 80.I
 - left invariant (on a Lie group) 249.A
 - parallel 364.B
 - random 395.I
 - right invariant (on a Lie group) 249.A
 - symmetric 105.O
 - of type (r, s) 105.O
 - of type (r, s) with value in E 417.E
- tensorial form 80.G
- tensorial p -form 417.C
- tensor product
 - (of A -homomorphisms) 277.J
 - (of algebras) 29.A
 - (of A -modules) 277.J
 - (of chain complexes) 201.J
 - (of cochain complexes) 201.J
 - (of distributions) 125.K
 - (of Hilbert spaces) 308.C
 - (of linear mappings) 256.I
 - (of linear representations) 362.C
 - (of linear spaces) 256.I
 - (of locally convex spaces) 424.R
 - (of sheaves) 383.I
 - (of vector bundles) 147.F
 - (of von Neumann algebras) 308.C
 - continuous 377.D
 - ε 424.R
 - projective C^* - 36.H
 - spatial 36.H
- tensor representation (of a general linear group) 256.M
- tensor space
 - of degree k 256.J
 - of type (p, q) 256.J
- term
 - (of a language) 276.A
 - (of a polynomial) 337.B
 - (in predicate logic) 411.H
 - (of a sequence) 165.D
 - (of a series) 379.A
 - base (of a spectral sequence) 200.J
 - closed (of a language) 276.A
 - constant (of a formal power series) 370.A
 - constant (of a polynomial) 337.B
 - error 403.D
 - fiber (of a spectral sequence) 200.J
 - initial (of an infinite continued fraction) 83.A
 - n th (of sequence) 165.D
 - penalty 440.B
 - subtraction 111.B
 - undefined 35.B
- terminal decision rule 398.F
- terminal point
 - (of a curvilinear integral) 94.D
 - (in a Markov process) 261.B
 - (of a path) 170
 - (of a vector) 442.A
- terminal time 261.B
- terminal vertex 186.B
- termwise differentiable (infinite series with function terms) 379.H
- termwise differentiation, theorem of (on distributions) 125.G
- termwise integrable (series) 216.B
- ternary set 79.D
- Terry model, Bradley- 346.C
- Terry normal score test, Fisher-Yates- 371.C
- tertiary obstruction 305.D
- tertium non datur 156.C
- tesseral harmonics 393.D
- test 400.A
 - Abel 379.D
 - almost invariant 400.E
 - Cauchy condensation 379.B
 - Cauchy integral 379.B
 - chi-square 400.G
 - chi-square, of goodness of fit 400.K
 - comparison 379.B
 - consistent 400.K
 - Dini (on the convergence of Fourier series) 159.B

Dini-Lipschitz (on the convergence of Fourier series) 159.B
 Dirichlet (on Abel's partial summation) 379.D
 Dirichlet (on the convergence of Fourier series) 159.B
 of du Bois-Reymond and Dedekind 379.D
 duo-trio 346.D
F- 400.G
 Fisher-Yates-Terry normal score 371.C
 goodness-of-fit 397.Q 401.E
 invariant 400.E
 Jordan (on the convergence of Fourier series) 159.B
 Kolmogorov 45.F
 Kolmogorov-Smirnov 371.F
 Kruskal-Wallis 371.D
 Lebesgue (on the convergence of Fourier series) 159.B
 Leibniz (for convergence) 379.C
 level α 400.A
 likelihood ratio 400.I
 Mann-Whitney *U*- 371.C
 minimax level α 400.F
 most powerful 400.A
 most stringent level α 400.F
 nonparametric 371.A
 nonrandomized 400.A
 outlier 397.Q
 pair 346.D
 randomized 400.A
 sensory 346.B
 sequential 400.L
 sequential probability ratio 400.L
 sign 371.B
 signed rank 371.B
 similar 400.D
 Student 400.G
t- 400.G
 triangle 346.D
 UMP unbiased level α 400.C
 UMP in variant level α 400.E
 unbiased level α 400.C
 uniformly consistent 400.K
 uniformly most powerful (UMP) 400.A
 uniformly most powerful invariant level α 400.E
 uniformly most powerful unbiased level α 400.C
 van der Waerden, 371.C
 Welch 400.G
 Wiener (for Brownian motion) 45.D
 Wiener (for Dirichlet problem) 338.G
 Wiener (for random walk) 260.E
 Wilcoxon 371.C
 Wilcoxon signed rank 371.B
 test channel 213.E
 test function 400.A
 test function space 125.S
 testing
 hypothesis 401.C
 statistical hypothesis 400
 test statistics, Kolmogorov-Smirnov 374.E
 test sufficient (σ -field) 396.J
 tetracyclic coordinates 90.B
 tetragamma function 174.B
 tetragonal (system) 92.E
 tetrahedral group 151.G
 tetrahedron 7.D 357.B
 polar 350.C
 self-polar 350.C

TE waves 130.B
 Theodorsen function 39.E
 theorem(s) \rightarrow also specific theorems
 of angular momentum 271.E
 Brouwer's, on the invariance of domain 117.D
 of coding 273.D-F
 on complete form 356.H
 of completeness (in geometry) 155.B
 cup product reduction 200.M
 fundamental \rightarrow fundamental theorem(s)
 of identity 21.C
 invariance, of analytic relations 198.K
 on invariance of dimension of Euclidean spaces 117.D
 kernel 125.L 424.S
 of linear ordering (in geometry) 155.B
 local limit 250.B
 of momentum 271.E
 product, for dimension 117.C
 of quasiconformal reflection 352.C
 structure, for von Neumann algebras of type III 308.I
 of Tauberian type 339.B
 of termwise differentiation (of distributions) 125.G
 translation (in class field theory) 59.C
 translation representation 375.H
 transversality 105.L
 triangle comparison 178.A
 Tucker's, on complementary slackness 255.B
 unicursal graph (Euler's) 186.F
 Weierstrass's, of double series 379.H
 Theorem A 21.L 72.E,F
 Theorem B 21.L 72.E,F
 theoretical formula 19.F
 theory
 Ahlfors's, of covering surfaces 272.J 367.B
 of buildings 343.I
 of calculus of variations, classical 46.C
 Cantor's, of real numbers 294.E
 class field 59
 classification, of Riemann surfaces 367.E
 combinatorial 66.A
 complete cohomology 200.N
 constructive field 150.F
 Dedekind's, of real numbers 294.E
 de Rham homotopy 114.L
 dimension 117
 of elasticity 271.G
 of electromagnetic waves 130.B
 of errors 138.A
 Euclidean field 150.F
 Euclidean Markov field 150.F
 exact sampling 401.F
 finite-displacement (of elasticity) 271.G
 of functions 198.Q
 of functions of a complex variable 198.Q
 Galois 172
 Galois, of differential fields 113
 game 173
 of gases, kinetic 402.B
 graph 186
 Haag-Ruelle scattering 150.D
 hidden variables 351.L
 hydromagnetic dynamo 259
 information 213
 Kaluza's 5-dimensional 434.C
 large sample 401.E
 lattice gauge 150.G
 Littlewood-Paley 168.B

- local class field 59.G
- Lyusternik-Shnirel'man 286.Q
- Minkowski reduction (on fundamental regions) 122.E
- Morse 279
- Morse, fundamental theorems of 279.D
- Nevanlinna (of meromorphic functions) 124.B 272.B
- nonsymmetric unified field 434.C
- number, analytic 296.B
- number, elementary 297
- number, geometric 296.B
- of perturbations, general 420.E
- of perturbations, special 420.E
- Peter-Weyl (on compact groups) 69.B
- Peter-Weyl (on compact Lie groups) 249.U
- Picard-Vessiot 113
- of plasticity 271.G
- prediction 395.D
- prediction, linear 395.D
- of probability 342.A
- proof 156.D
- quantum field 150.C
- ramified type 411.K
- realization 86.D
- of relativity, general 359.A
- of relativity, special 359.A
- risk 214.C
- risk, classical 214.C
- risk, collective 214.C
- risk, individual 214.C
- Serre \mathcal{C} - 202.N
- set 381.F (\rightarrow also set theory)
- of singularities 418
- slender body 205.B
- small-displacement, of elasticity 271.G
- S-matrix 386.C
- supermultiplet, Wigner's 351.J
- syzygy 200.K
- thin wing 205.B
- Tomita-Takesaki 308.H
- type 411.K
- unified field 434.A
- unitary field 434.C
- thermal contact 419.A
- thermal expansion, coefficient of 419.A
- thermal noise 402.K
- thermodynamical quantity 419.A
 - extensive 419.A
 - intensive 419.A
- thermodynamic limit 402.G
- thermodynamics 419
 - first law of 419.A
 - second law of 419.A
 - statistical 402.A
 - third law of 419.A
 - 0th law of 419.A
- theta formula (on ideles) 6.F
- theta-Fuchsian series of Poincaré 32.B
- theta function 134.I
 - (on a complex torus) 3.I
 - elliptic 134.I, App. A, Table 16.II
 - graded ring of 3.N
 - Jacobian 134.C
 - nondegenerate 3.I
 - Riemann 3.L
- theta series 348.L
- thick (chamber complex) 13.R
- thickness (of an oval) 89.C
- thin (chamber complex) 13.R
- thin set
 - (in Markov processes) 261.D
 - (in potential theory) 338.G
 - analytically (in an analytic space) 23.D
 - internally 338.G
- thin wing theory 205.B
- third boundary value problem 193.F 323.F
- third classification theorem (in the theory of obstructions) 305.C
- third extension theorem (in the theory of obstructions) 305.C
- third fundamental form App. A, Table 4.I
- third homotopy theorem (in the theory of obstructions) 305.C
- third isomorphism theorem (on topological groups) 423.J
- third kind
 - Abelian differential of 11.C
 - Abelian integral of 11.C
- third law of thermodynamics 419.A
- third-order predicate logic 411.K
- third quartiles 396.C
- third separation axiom 425.Q
- Thom algebra 114.H
- Thom complex 114.G
 - associated with (G, n) 114.G
- Thom first isotropy theorem 418.G
- Thom fundamental theorem 114.H
- Thom-Gysin isomorphism 114.G
 - (on a fiber space) 148.E
- Thompson inequality, Golden- 212.B
- Thompson theorem, Feit- (on finite groups) 151.D
- Thom space 114.G
- Thom spectrum, stable homotopy group of 114.G 202.T
- Thorin theorem, Riesz- 224.A
- thorn (of a convergence domain) 21.B
- three big problems 187
- three-body problem 420.A
 - restricted 420.F
- three-circle theorem, Hadamard 43.E
- 3j-symbol 353.B
- three laws of motion, Newton's 271.A
- three-line theorem, Doetsch 43.E
- three-stage least squares method 128.C
- three-valued logic 411.L
- three principles, Fisher's 102.A
- three-series theorem 342.D
- threshold Jacobi method 298.B
- Thue problem 31.B
- Thue theorem 118.D
- Thullen theorem, Cartan- 21.H
- Thurstone-Mosteller model 346.C
- tieset 186.G
- tieset matrix, fundamental 186.G
- Tietze extension theorem 425.Q
- Tietze first axiom 425.Q
- Tietze second axiom 425.Q
- tight family (of probability measures) 341.F
- tight immersion 365.O
- tightness 399.M
- Tikhonov embedding theorem 425.T
- Tikhonov fixed-point theorem 153.D
- Tikhonov product theorem 425.S
- Tikhonov separation axiom 425.Q
- Tikhonov space 425.Q
- Tikhonov theorem 425.Q
 - Uryson- (on metrizability) 273.K
- Tikhonov-Uryson theorem 425.Q
- time

- exit 261.B
- explosion 406.D
- first splitting 44.E
- hitting 260.B 261.B 407.B
- killing 260.A
- life 260.A 261.B
- local 45.G
- Markov 261.B 407.B
- NP- 71.E
- polynomial 71.E
- proper 258.A
- real- 19.E
- recurrence 260.C
- sojourn, density 45.G
- stopping 261.B 407.B
- terminal 261.B
- waiting 260.H
- waiting, distribution 307.C
- time change
 - (of a Markov process) 261.B
 - (of a semimartingale) 406.B
 - (of a submartingale) 262.C
- time complexity 71.A
- time-dependent Schrödinger equation 351.D
- time-homogeneous Markovian type 406.D
- time-independent Markovian type 406.D
- time-independent Schrödinger equation 351.D
- time-invariant, linear (dynamical systems) 86.B
- time-invariant network 282.C
- timelike
 - (curve) 325.A
 - (vector of a Minkowski space) 258.A 359.B
- time-one mapping (map) 126.C
- time optimal control problem 86.F
- time ordered function 150.D
- time parameter (of a stochastic process) 407.A
- time parameter space 260.A
- time reversal 258.A 359.B
- time series 397.A 421.A
- time series analysis 421
- time series data 397.N
- time-varying system, linear 86.B
- Tissot-Pochhammer differential equation 206.C
- Titchmarsh-Kodaira theory, Weyl- Stone- 112.O
- Titchmarsh theorem 306.B
 - Brun- 123.D
- Tits simple group 151.I
- Tits system 13.R 151.J 343.I
- TM waves 130.B
- Toda bracket 202.R
- Toda lattice 287.A 387.A
- Todd characteristic 366.B
- Todd class (of a complex vector bundle) 237.F
- Toeplitz operator 251.O
- Toeplitz theorem 379.L
- tolerance interval 399.R
- tolerance limits 399.R
- tolerance percent defective, lot 404.C
- tolerance region 399.R
- Tomita-Takesaki theory 308.H
- Tonelli
 - absolutely continuous in the sense of 246.C
 - bounded variation in the sense of 246.C
- topological Abelian group(s) 422
 - dual 422.C
 - elementary 422.E
- topological conjugacy 126.B
- topological entropy 126.K 136.H
- topological equivalence 126.B
- topological field 423.P
- topological generator (of a compact Abelian group) 136.D
- topological group(s) 423
 - completable 423.H
 - complete 423.H
 - Hausdorff 423.B
 - homomorphic 423.J
 - isomorphic 423.A
 - locally isomorphic 423.O
 - metrizable 423.I
 - separated 423.B
 - T_2 - 423.B
- topological groupoid 154.C
- topological index (of an elliptic complex) 237.H
- topological invariance (of homology groups) 201.A
- topological invariant 425.G
- topological linear spaces 424
- topologically complete space 436.I
- topologically conjugate 126.B
- topologically equivalent 126.B,H
- topological manifold 105.B
 - with boundary 105.B
 - without boundary 105.B
- topological mapping 425.G
- topological n -cell 140
- topological pair 201.L
- topological polyhedron 65.A
- topological pressure 136.H
- topological property 425.G
- topological ring 423.P
- topological σ -algebra 270.C
- topological solid sphere 140
- topological space(s) 425
 - category of 52.B
 - category of pointed 202.B
 - complex linear 424.A
 - discrete 425.C
 - generalized 425.D
 - homotopy category of 52.B
 - inductive system of 210.C
 - linear 424.A
 - metrizable 273.K
 - product 425.K
 - projective system of 210.C
 - quotient 425.L
 - real linear 424.A
 - underlying (of a complex manifold) 72.A
 - underlying (of a differentiable manifold) 105.D
 - underlying (of a topological group) 423.A
 - uniform 436.C
 - uniformizable 436.H
- topological sphere 140
- topological structure 425.A,B
- topological sum 425.M
- topological transformation group 431.A
- topological vector space 424.A
- topology 425.B 426
 - α -adic (of an R -module) 284.B
 - algebraic 426
 - base for a 425.F
 - of biquicontinuous convergence 424.R
 - box 425.K
 - coarser 425.H
 - combinatorial 426
 - compact-open 279.C 435.D
 - compact-open C^∞ 279.C
 - differential 114
 - discrete 425.C
 - étale 16.AA
 - fine (on a class of measures) 261.D 338.E

Topology ε

- finer 425.H
- Gel'fand 36.E
- general 426
- Grothendieck 16.AA
- hereditarily weak 425.M
- hull-kernel 36.D
- I -adic (of a ring) 16.X
- indiscrete 425.C
- induced 425.I
- induced by a mapping 425.I
- inner (of a Lie subgroup) 249.E
- Jacobson 36.D
- Krull (for an infinite Galois group) 172.I
- larger 425.H
- leaf 154.D
- of Lie groups and homogeneous spaces 427
- linear 422.L
- Mackey 424.N
- metric 425.C
- order 425.C
- PL 65.A
- product 425.K
- projective 424.R
- quotient 425.L
- relative 425.J
- S - (on a linear space) 424.K
- Sazonov 341.J
- set-theoretic 426
- σ -weak 308.B
- smaller 425.H
- strong (on a class of measures) 338.E
- strong (on a direct product space) 425.K
- strong (on a normed space) 37.E
- strong (on a topological linear space) 424.K
- stronger 425.H
- strong operator 251.C
- subbase for a 425.F
- of surfaces, fundamental theorem of 410.B
- trivial 425.C
- uniform 436.C
- of uniform convergence 424.K
- of the uniformity 436.C
- uniformizable 436.H
- uniform operator 251.C
- vague (on a class of measures) 338.E
- weak (in a cell complex) 70.D
- weak (on a class of measures) 338.E
- weak (on a direct product space) 425.K
- weak (on a direct sum) 425.M
- weak (on a locally convex space) 424.H
- weak (on a normed linear space) 37.E
- weak, relative to the pairing $\langle E, F \rangle$ 424.H
- weak C^∞ 279.C
- weaker 425.H
- weak operator 251.C
- weak* (on a locally convex space) 424.H
- weak* (on a normed space) 37.E
- Zariski (of a spectrum) 16.D
- Zariski (of a variety) 16.A
- topology ε (on the tensor product of locally convex spaces) 424.R
- topology π (on the tensor product of locally convex spaces) 424.R
- Tor 200.D
 - exact sequence of 200.D
- $\text{Tor}_n^R(A, B)$ 200.K
- $\text{Tor}_n^R(M, N)$ 200.D
- Torelli theorem 9.E, 11.C
- Tor groups 200.D
- toric variety 16.Z
- toroidal coordinates App. A, Table 3.V
- toroidal embedding 16.Z
- toroidal subgroup, maximal (of a compact Lie group) 248.X
- torsion
 - (of a curve of class C^n) 111.D
 - affine 110.C
 - analytic 391.M
 - conformal 110.D
 - radius of (of a space curve) 111.F
 - Whitehead 65.C
- torsion A -module 277.D
- torsion Abelian group 2.A
 - bounded 2.F
- torsion coefficients (of a complex) 201.B
- torsion element (of an A -module) 277.D
- torsion form 80.H
- torsion-free A -module 277.D
- torsion-free Abelian group 2.A
- torsion group 2.A
 - (of a complex) 201.B
- torsion product
 - (in a category) 200.K
 - (of A -modules) 200.D
- torsion subgroup
 - (of an Abelian group) 2.C
 - maximal 2.A
- torsion tensor App. A, Table 4.II
 - (of an affine connection) 80.J, L 417.B
 - (of an almost contact structure) 110.E
 - (of a Fréchet manifold) 286.L
- torus
 - (algebraic group) 13.D
 - (compact group) 422.E
 - (surface) 111.I 410.B
 - algebraic 13.D
 - Clifford 275.F
 - complex 3.H
 - generalized Clifford 275.F
 - invariant 126.L
 - K -split 13.D
 - K -trivial 13.D
 - maximal (of a compact Lie group) 248.X
 - maximal K -split 13.Q
 - n - 422.E
- torus embedding 16.Z
- torus function App. A, Table 18.III
- torus group 422.E
- total (set of functions) 317.A
- total boundary operator 200.E
- total Chern class 56.C
- total cross section 386.B
- total curvature
 - (of an immersion) 365.O
 - (of a space curve) 111.F
 - (of a surface) 111.H, App. A, Table 4.I
 - Gaussian 111.H
- total degree 200.J
- total differential
 - (of a complex) 200.H
 - (of a function) 106.G
 - (on a Riemann surface) 367.H
- total differential equation(s) 428, App. A, Table 15.I
- system of 428.A
- total elastic cross section 386.B
- total energy 271.C
- total excess 178.H
- total Gaussian curvature (of a surface) 111.H
- total isotropy, index of (with respect to a quadratic form) 348.E

- totally bounded
 - (metric space) 273.B
 - (subset of a metric space) 273.B
 - (subset of a uniform space) 436.H
 - (uniform space) 436.H
 - locally 436.H
- totally definite quaternion algebra 27.D
- totally degenerate (group) 234.B
- totally differentiable
 - (complex function) 21.C
 - (real function) 106.G
- totally disconnected (topological space) 79.D
- totally geodesic submanifold 365.D
- totally imaginary field 14.F
- totally isotropic (subspace) 60.O 348.E
- totally ordered additive group 439.B
- totally ordered group 243.G
- totally ordered set 311.A
- totally positive (element) 14.G
- totally real field 14.F
- totally real immersion 365.M
- totally real submanifold 365.M
- totally regular transformation (of a sequence) 379.L
- totally singular (subspace) (with respect to a quadratic form) 348.E
- totally umbilical submanifold 365.D
- totally unimodular 186.G
- total matrix algebra 269.B
- total mean curvature 365.O
- total monodromy group 418.F
- total ordering 311.A
- total Pontryagin class 56.D
- total quotients, ring of 67.G
- total space
 - (of a fiber bundle) 147.B
 - (of a fiber space) 148.B
- total Stiefel-Whitney class 56.B
- total transform (of a subvariety) 16.I
- total variation
 - (of a function) 166.B
 - (of a mapping from a plane into a plane) 246.H
 - (of a set function) 380.B
 - (of a vector measure) 443.G
- tower, class field (problem) 59.F
- trace
 - (of an algebraic element) 149.J
 - (of an element of a general Cayley algebra) 54
 - (of a linear system of an algebraic surface) 15.C
 - (of a matrix) 269.F
 - (of a nuclear operator) 68.L
 - (in a von Neumann algebra) 308.D
 - reduced (of an algebra) 362.E
- trace class 68.I
- traced, β - 126.J
- trace form, ε - 60.O
- trace formula (on unitary representations) 437.DD
- trace norm 68.I
- trace operator 168.B
- tracing property, pseudo-orbit 126.J
- tractrix 93.H
- traffic intensity 260.H
- trajectory 126.B,C
 - negative half- 126.D
 - orthogonal 193.J
 - positive half- 126.D
- transcendence basis (of a field extension) 149.K
- transcendence degree (of a field extension) 149.K
- transcendency, degree of (of a field extension) 149.K
- transcendental curve 93.H
- transcendental element (of a field) 149.E
- transcendental entire functions 429
- transcendental extension 149.E
 - purely 149.K
- transcendental function
 - higher 389.A
 - of Painlevé 288.C
- transcendental meromorphic function 272.A
- transcendental numbers 430
- transcendental singularity (of an analytic function in the wider sense), direct, indirect 198.P
- transfer (in group theory) 190.O
- transfer function 86.D
- transfer function matrix 86.B
- transferrer of constant lengths 155.G
- transfinite cardinal number 49.A
- transfinite diameter 48.D
- transfinite induction
 - (in a well-ordered set) 311.C
 - definition by 311.C
- transfinite initial ordinal number 49.E
- transfinite logical choice function 411.J
- transfinite ordinal number 312.B
- transform
 - (of a sequence by a linear transformation) 379.L
 - Cauchy (of a measure) 164.J
 - Cayley (of a closed symmetric operator) 251.I
 - discrete Fourier 142.D
 - fast Fourier 142.D
 - Fourier 160, App. A, Table 11.II
 - Fourier (of a distribution) 125.O
 - Fourier (in topological Abelian groups) 36.L 192.I
 - Fourier, generalized 220.B
 - Fourier, inverse (of a distribution) 125.O
 - Fourier, spherical 437.Z
 - Fourier-Bessel 39.D
 - Fourier cosine 160.C, App. A, Table 11.II
 - Fourier-Laplace 192.F
 - Fourier sine 160.C, App. A, Table 11.II
 - Fourier-Stieltjes 192.B,O
 - Gelfand 36.E
 - Hankel 220.B
 - Hilbert 160.D 220.E
 - integral 220
 - inverse (of an integral transform) 220.A
 - Klein 150.D
 - k th 160.F
 - Laplace 240, App. A, Table 12.I
 - Laplace-Stieltjes 240.A
 - Legendre 419.C
 - Mellin 220.C
 - proper (of a subvariety) 16.I
 - Radon 218.F
 - Radon, conjugate 218.F
 - real monoidal 274.E
 - Riesz 251.O
 - Stieltjes 220.D
 - total (of a subvariety) 16.I
 - Watson 160.C 220.B
- transformable, polynomially 71.E
- transformation(s) 381.C
 - (on a measure space) 136.D

Transformation formula

- affine 7.E
- affine (of a manifold with an affine connection) 80.J
- affine (of a Riemannian manifold) 364.F
- affine, group of 7.E
- affine, proper 7.E
- affine, regular 7.E
- Ampère 82.A
- angular 374.D
- arc sine 374.D
- bimeasurable (on a measure space) 136.B
- birational 16.I
- BRS 150.G
- canonical 82.B 271.F
- canonical, group of 271.F
- Cayley (of a matrix) 269.J
- chain (between complexes) 200.H
- conformal 80.P 364.F
- congruent (in Euclidean geometry) 139.B
- congruent, group of 285.C
- contact 82, App. A, Table 15.IV
- coordinate (of a fiber bundle) 147.B
- coordinate (of a locally free \mathcal{O}_X -Module) 16.E
- covering 91.A 367.B
- Cremona 16.I
- by drift 261.F
- of drift 406.B
- elliptic 74.F
- entire linear 74.E
- to an equilibrium system 82.D
- equilong 76.B
- ergodic (on a measure space) 136.B
- Euler (of infinite series) 379.I
- factor (of a measure preserving transformation) 136.D
- Fisher z- 374.D
- Galilei 359.C
- gauge (in electromagnetism) 130.A
- gauge (in a lattice spin system) 402.G
- gauge (of a principal fiber bundle) 80.Q
- gauge (in unified field theory) 343.B
- gauge, of the first kind 150.B
- Gauss App. A, Table 16.III
- Givens 302.E
- Householder 302.E
- hyperbolic 74.F
- infinitesimal (of a Lie transformation group) 431.G
- infinitesimal (of a one-parameter group of transformations) 105.N
- inner (in the sense of Stollow) 367.B
- Jacobi imaginary 134.I, App. A, Table 16.III
- Kelvin 193.B
- Laguerre 76.B
- Landen 134.B, App. A, Table 16.III
- Legendre 82.A, App. A, Table 15.IV
- Lie (in circle geometry) 76.C
- Lie line-sphere 76.C
- linear (= linear fractional) 74.E
- linear (of a linear space) 251.A 256.B
- linear (of a sequence) 379.L
- linear fractional 74.E
- local, local Lie group of 431.G
- local, local one-parameter group of 105.N
- of local coordinates 90.D
- locally quadratic (of an algebraic surface) 15.G
- locally quadratic (of an algebraic variety) 16.K
- locally quadratic (of a complex manifold) 72.H
- Lorentz 359.B
- loxodromic 74.F
- measurable (on a measure space) 136.B
- measure-preserving 136.B
- Möbius 74.E 76.A
- monoidal (of an analytic space) 23.D
- monoidal (of a complex manifold) 72.H
- monoidal (by an ideal sheaf) 16.K
- monoidal, with center W 16.K
- by a multiplicative functional (in a Markov process) 261.F
- natural 52.J
- nonsingular (of a linear space) 256.B
- nonsingular (on a measure space) 136.B
- normal (of a sequence) 379.L
- one-parameter group of 105.N
- one-parameter group of class C^r 126.B
- orthogonal 139.B 348.B
- orthogonal (over a noncommutative field) 60.O
- orthogonal (with respect to a quadratic form) 60.K
- orthogonal, around the subspace A^k 139.B
- parabolic 74.F
- of the parameter 111.D
- parity 359.B
- particular (of h_R^*) 248.R
- Picard-Lefschetz 16.U
- to principal axes 390.B
- projective 343.D
- projective (of a Riemannian manifold) 367.F
- projective, group of 343.D
- pseudoconformal 344.A
- pseudogroup of (on a topological space) 90.D
- quadratic 16.I, K
- quantized contact 274.F
- regular (of a linear space) 256.B
- regular (of a sequence) 379.L
- regular projective 343.D
- Schwarz-Christoffel 77.D
- semilinear 256.P
- semiregular (of a sequence) 379.L
- shift 136.D
- singular projective 343.D
- singular projective, of the h th species 343.D
- superharmonic 261.F
- symplectic 60.L
- symplectic (over a noncommutative field) 60.O
- totally regular (of a sequence) 379.L
- triangular (linear) 379.L
- unitary 348.F
- unitary (relative to an ε -Hermitian form) 60.O
- weakly equivalent 136.F
- transformation formula
 - (for the generating function of the number of partitions) 328
 - (of a theta function) 3.I
 - (for theta series) 348.L
 - Schwarz-Christoffel 77.D
- transformation group(s) 431, App. A, Table 14.III
 - (of a set) 431.A
 - covering 91.A
 - differentiable 431.C
 - discontinuous 122.A
 - free discontinuous 122.A
 - Lie 431.C
 - Möbius 76.A
 - orthogonal 60.I
 - orthogonal (over a field with respect to a quadratic form) 60.K

properly discontinuous 122.A
 symplectic 60.L
 topological 431.A
 unitary 60.F
 transformation parameter 396.I
 transformation problem (in a finitely presented group) 161.B
 transformation space (of an algebraic group) 13.G
 transgression
 (homomorphism of cohomology groups) 200.M
 (in the spectral sequence of a fiber space) 148.E
 transgressive (element in the spectral sequence of a fiber space) 148.E
 transient
 (Lévy process) 5.G
 (Markov chain) 260.B
 (Markov process) 261.B
 transient problem 322.D
 transition
 order-disorder 402.F
 phase 340.B
 transition function
 (of a fiber bundle) 147.B
 (of a Markov chain) 260.A
 (of a Markov process) 261.B
 Feller 261.B
 transition matrix 126.J 260.A
 transition point 254.F
 transition probability
 (of a diffusion process) 115.B
 (of a Markov chain) 260.A
 (of a Markov process) 261.A
 (in quantum mechanics) 351.B
 standard 260.F
 transitive
 (dynamical system) 126.I,J
 (operation of a group) 362.B
 (permutation representation) 362.B
 (relation) 358.A
 fully (subgroup of an orthogonal group) 92.C
 k - (permutation group) 151.H
 k -ply (G -set) 362.B
 k -ply (permutation group) 151.H
 multiply (permutation group) 151.H
 simply (G -set) 362.B
 transitive extension (of a permutation group) 151.H
 transitive law
 (in an equivalence relation) 135.A
 (on ordering) 311.A
 transitively (act on G -space) 431.A
 transitive permutation group 151.H
 transitivity, system of (of a G -set) 362.B
 translation(s)
 (in an affine space) 7.F
 group of (of an affine space) 7.E
 left 249.A 362.B
 parallel 80.C 364.B
 right 249.A 362.B
 translational flow 126.L 136.G
 frequencies of 126.L 136.G
 translation group (of a Lorentz group) 258.A
 translation number 18.B,D
 translation operator 306.C
 translation representation theorem 375.H
 translation theorem (in class field theory) 59.C
 transmission coefficient 387.D
 transmission rate 213.A

Subject Index

Triangle

transonic flow 205.B
 transonic similarity, von Kármán 205.D
 transportation problem 255.C
 transport coefficient 402.K
 transport equations 325.L
 transpose
 (of a linear mapping) 256.G
 (of a rational homomorphism) 3.E
 transposed integral equation 217.F
 transposed mapping
 (of a diffusion kernel) 338.N
 (of a linear mapping) 256.G
 transposed matrix 269.B
 transposed operator 112.E 189.C 322.E
 transposed representation 362.E
 transposition (in a symmetric group) 151.G
 transvection 60.O
 transversal (matroid) 66.H
 transversal field 136.G
 transversal flow 136.G
 transversal homoclinic point 126.J
 transversality, condition of (in calculus of variations) 46.B
 transversality condition 108.B
 strong 126.J
 transversality theorem 105.L
 transverse
 (foliations) 154.H
 (to a submanifold of a differentiable manifold) 105.L
 to a foliation 154.B
 transverse axis (of a hyperbola) 78.C
 transverse electric waves 130.B
 transverse electromagnetic waves 130.B
 transverse invariant measure 154.H
 transversely (intersect) 105.L
 transversely orientable 154.B
 transverse magnetic waves 130.B
 transverse structure 154.H
 transverse wave 446
 trap
 (of a diffusion process) 115.B
 (of a Markov process) 261.B
 trapezoidal rule
 (of numerical integration) 299.A
 (of numerical solution of ordinary differential equations) 303.E
 treatment 102.B
 connected 102.B
 treatment combinations, number of 102.L
 treatment contrast 102.C
 treatment effect 102.B
 tree 93.C 186.G
 co- 186.G
 derivation 31.E
 spanning 186.G
 tree code 213.E
 tree representation 96.D
 tree structure 96.D
 trefoil knot 235.C
 trellis code 213.E
 trend 397.N
 triad 202.M
 homotopy exact sequence of 202.M
 homotopy group of 202.M
 trial path dependent, d - (response probability) 346.G
 triangle 7.D 155.F 178.H
 geodesic 178.A
 Pascal 330

Triangle comparison theorem

- plane App. A, Table 2.II
- polar 78.J
- Reuleaux 89.E 111.E
- self-polar 78.J
- solving a 432.A
- spherical 432.B, App. A, Table 2.III
- triangle comparison theorem 178.A
- triangle inequality 273.A
- triangle test 346.E
- triangular 65.A
- triangular (linear transformation) 379.L
- triangular element 304.C
- triangular factorization 302.B
- triangular matrix 269.B
 - lower 269.B
 - upper 269.B
- triangular number 4.D
- triangulated manifold 65.B
- triangulation 65.A 70.C
 - C^* - 114.C
 - combinatorial 65.C
 - combinatorial, problem 65.C
 - compatible with 114.C
 - finite 70.C
- trick, Alexander's 65.D
- triclinic (system) 92.E
- Tricomi differential equation 326.C
- Tricomi problem 326.C
- tridiagonal matrix 298.D
- trigamma function 174.B
- trigonal (system) 92.E
- trigonometric function 131.E 432.A, App. A, Table 2
 - inverse 131.E
- trigonometric integral 160.A
- trigonometric interpolation polynomial 336.E
- trigonometric polynomial, generalized 18.B
- trigonometric series 159.A
 - generalized 18.B
- trigonometric sum 4.C
- trigonometric system 159.A
- trigonometry 432, App. A, Table 2
 - plane 432.A
 - spherical 432.B
- trilinear coordinates 90.C
- trimmed mean, α - 371.H
- triple 200.Q,L
 - homotopy exact sequence of 202.L
- triple product, scalar, vector 442.C, App. A, Table 3.I
- triplet, Gel'fand 424.T
- tripolar coordinates 90.C
- trisection of an angle 179.A
- trivalent map 157.B
- trivial
 - (extension) 390.J
 - (knot) 65.D 235.A
 - K- (torus) 13.D
- trivial bundle 147.E
- trivial fiber space, locally 148.B
- trivialization (of a block bundle) 147.Q
- trivially (act on a G -space) 431.A
- trivial sheaf 383.D
- trivial solution (of a system of linear homogeneous equations) 269.M
- trivial topology 425.C
- trivial valuation 439.C,F
- trochoid 93.H
- Trotter product formula 351.F
- true 411.E
- true anomaly 309.B
- true value of a parameter 398.A
- truncated Wightman function 150.D
- truncation error 138.B 303.B
 - local 303.E
- truth definition 185.D
- truth function 411.E
- truth value (of a formula) 411.E
- Tsen theorem 27.E 118.F
- tube 126.E
 - regular 193.K
 - vector 442.D
- tuboid 125.V
- tubular neighborhood 105.L 114.B 364.C
 - open 105.L 114.B
- tubular neighborhood system, controlled 418.G
- Tucker theorem on complementary slackness 255.B
- Tucker theorem, Kuhn- 292.B
- tuple, n - 256.A 381.B
- turbulence 433
 - homogeneous 433.C
 - isotropic 433.C
- turbulent flow 205.E 433
- Turing machine 31.B
 - universal 31.C
- turning point 25.B 254.F
- twinable 92.D
- twinning structure 92.D
- twin primes 123.C
- twisted type, group of 151.I
- two-bin system 227
- two-body interaction 271.C
- two-body problem 55.A
- two-dimensional KdV equation 387.F
- 2-isomorphic 186.H
- two-person game, zero-sum 108.B
- two-phase simplex method 255.C
- two-point boundary value problem (of ordinary differential equations) 315.A
- two sheets
 - hyperboloid of 350.B
 - hyperboloid of revolution of 350.B
- two-sided exponential distribution App. A, Table 22
- two-sided generator (for an automorphism of a measure space) 136.E
- two-sided ideal 368.F
- two-sided \mathfrak{o} -ideal 27.A
 - integral 27.A
- two-sided scale 19.D
- two-sided surface 410.B
- two-stage least squares method 128.C
- two-stage sampling 373.E
- two-stage stochastic programming 408.A
- two-terminal characteristic 281.C
- two-terminal network 281.C
- two-terminal problem 281.C
- two-valued logic 411.L
- two-way elimination of heterogeneity, design for 102.K
- two-way layout 102.H
- type
 - (of an Abelian group) 2.B,D
 - (of an object) 356.F
 - (of a quadratic form) 348.E
 - (of a structure) 409.B
 - (of a transcendental number) 430.C
 - acute 304.C
 - backward 304.D

- Bravais (of lattices) 92.B
 Bravais, of the class of (T, K) 92.B
 compact 412.D
 dimension 117.H
 of finite (graded module) 203.B
 of finite (module) 277.D
 of finite (morphism of schemes) 16.D
 of finite (\mathcal{O} -module) 16.E
 of finite, subshift 126.J
 forward 304.D
 Fredholm, integral equation of 217.A
 Fredholm, integrodifferential equation of 222.A
 Fuchsian (linear ordinary differential equations) 253.A
 Fuchsian (visibility manifold) 178.F
 general 72.I
 general, surface of 72.K
 homotopy 202.F
 homotopy (of a link) 235.D
 homotopy, invariant 202.F
 isotopy (of knots) 235.A
 isotopy (of a transformation group) 431.A
 k -, Markov branching process 44.E
 knot 235.A
 link 235.D
 locally of finite (morphism of schemes) 16.D
 mixed 304.C 326.A
 multi-, Markov branching process 44.E
 multidiagonal 304.C
 noncompact 412.D
 nonconforming 304.C
 orbit (of a G -space) 431.A
 Parreau-Widom 164.K
 positive (symmetric kernel) 338.D
 principal orbit 431.C
 spherical G -fiber homotopy 431.F
 Stoilow 207.B
 twisted, group of 151.I
 Volterra, integral equation of 217.A
 Volterra, integrodifferential equation of 222.A
 Weierstrass-, preparation theorem (for micro-differential operators) 274.F
 type p^∞ , Abelian group of 2.D
 type (p, q)
 (of an operator) 224.E
 tensor of 256.J
 tensor space of 256.J
 weak (of an operator) 224.E
 type (r, s) , differential form of 72.C
 type S (harmonic boundary) 207.B
 type S , space of 125.T
 type number (of a solution of a system of linear ordinary differential equations) 314.A
 type problem (for Riemann surfaces) 367.D
 type theory 411.K
 ramified 411.K
 simple 411.K
 type $(0, 1)$, tangent vector of 72.C
 type $(1, 0)$, tangent vector of 72.C
 type I
 (von Neumann algebra) 308.E
 C^* -algebra of 308.L
 group of 437.E
 type I group 437.E
 type $I_{m,m}$ (irreducible symmetric bounded domain) 412.I
 type I_n (von Neumann algebra) 308.E
 type II (von Neumann algebra) 308.E
 type II_1
 (ergodic countable group) 136.F
 (von Neumann algebra) 308.E,F
 type II_m (irreducible symmetric bounded domain) 412.I
 type II_∞
 (ergodic countable group) 136.F
 (von Neumann algebra) 308.E,F
 type III
 (ergodic countable group) 136.F
 (von Neumann algebra) 308.E
 structure theorem for von Neumann algebras of 308.I
 type III_0 (factor) 308.I
 type III_1 (factor) 308.I
 type III_2 (factor) 308.I
 type III_m (irreducible symmetric bounded domain) 412.I
 type IV_m (irreducible symmetric bounded domain) 412.I
 type AI, AII, AIII, AIV (irreducible symmetric Riemannian space) 412.G
 type BDI, BDII (irreducible symmetric Riemannian space) 412.G
 type CI, CII (irreducible symmetric Riemannian space) 412.G
 type DIII (irreducible symmetric Riemannian space) 412.G

 U

 $U(n)$ (unitary group) 60.F
 u -chain 260.I
 u -curve 111.H
 u_i -scale 19.D
 U -invariant (subspace) 437.C
 U -set 159.J
 U -statistic 274.I
 U test, Mann-Whitney 371.C
 U-number 430.C
 U^* -number 430.C
 UCL (upper control limit) 404.B
 Ugaheri maximum principle 338.C
 Uhlenbeck Brownian motion, Ornstein- 45.I
 Ulam theorem, Borsuk- 153.B
 Ulm factor(s) 2.D
 sequence of 2.D
 ultrabornological (locally convex space) 424.W
 ultradifferentiable function 168.B
 ultradistribution 125.U, BB
 of class $\{M_p\}$ 125.U
 of class $\{M_p\}$ 125.U
 ultrafilter 87.I
 ultrainfinite point 285.C
 ultrapower 276.E
 ultraproduct(s) 276.E
 fundamental theorem of 276.E
 ultraspherical polynomials 317.D
 ultraviolet divergence 132.C 146.B
 umbilical point (of a surface) 111.H 365.D
 umbilical submanifold, totally 365.D
 Umkehr homomorphism 201.O
 UMP (uniformly most powerful) (test) 400.A
 UMP invariant level α test 400.E
 UMP unbiased level α test 400.C
 UMV unbiased estimator 399.C
 unavoidable set 157.D
 unbiased confidence region 399.Q
 uniformly most powerful 399.Q
 unbiased estimator 399.C
 asymptotically (mean) 399.K

Unbiased level α test

- best linear 403.E
- k th order asymptotically median 399.O
- mean 399.C
- median 399.C
- modal 399.C
- UMV 399.C
- uniformly minimum variance 399.C
- unbiased level α test 400.C
 - uniformly most powerful 400.C
- unbiasedness 399.C
- unbounded (covering surface) 367.B
- uncertainty (in observations) 351.C
- uncertainty relation, Heisenberg 351.C
- unconditionally convergent
 - (series) 379.C
 - (series in a Banach space) 443.D
- undecidable proposition, formally 185.C
- undefined concept 35.B
- undefined term 35.B
- undercrossing point 235.A
- underdetermined system
 - of differential operators 112.R
 - of partial differential operators 320.F
- underflow 138.B
- underlying group (of a topological group) 423.A
- underlying topological space
 - (of a complex manifold) 72.A
 - (of a differentiable manifold) 105.D
 - (of a topological group) 423.A
- undirected graph 186.B
- undotted index 258.B
- undotted spinor (of rank k) 258.B
- unfolding
 - (of a germ of an analytic function) 418.E
 - constant 51.D
 - r - 51.D
 - universal 418.E
- unicity theorem, Luzin's 22.C
- unicursal (ordinary curve) 93.C
- unicursal curve 9.C 93.H
- unicursal graph theorem (Euler's) 186.F
- unified field theory 434
 - nonsymmetric 434.C
- uniform
 - (lattice of a Lie group) 122.K
 - (sampling procedure) 373.A
- uniform algebra 164.A
- uniform boundedness theorem 37.H
- uniform continuity
 - lower class with respect to 45.F
 - upper class with respect to 45.F
- uniform convergence 435
 - (of an infinite product) 435.A
 - (of operators) 251.C
 - (of a series) 435.A
 - abscissa of 121.B 240.B
 - on compact sets 435.C
 - Weierstrass criterion for 435.A
- uniform covering system 436.D
- uniform distribution 182.H 341.D, App. A, Table 22
- uniform family of neighborhoods system 436.D
- uniform isomorphism 436.E
- uniformity 436.B
 - base for the 436.B
 - discrete 436.D
 - generated by a family of pseudometrics 436.F
 - generated by a pseudometric 436.F
 - left (of a topological group) 423.G
 - product 436.E
 - pseudometric 436.F
 - relative 436.E
 - right (of a topological group) 423.G
 - stronger 436.E
 - T_1 - 436.C
 - topology of the 436.C
 - weaker 436.E
- uniformizable (topological space) 436.H
- uniformization 367.C
 - (of a set in a product space) 22.F
 - Schottky 367.C
- uniformization theorem
 - general 367.G
 - Kondô 22.F
- uniformized 367.C
 - locally 367.C
- uniformizing parameter, local (of a Riemann surface) 367.A
- uniformly
 - (partial recursive function) 356.E
 - (primitive recursive function) 356.B
- uniformly absolutely convergent (series) 435.A
- uniformly almost periodic function 18.B
- uniformly asymptotically stable
 - (solution of a differential equation) 394.B
 - (solution of a functional differential equation) 163.G
- uniformly best (estimator) 399.C
- uniformly better (decision function) 398.B
- uniformly consistent test 400.K
- uniformly continuous
 - (function) 84.A
 - (mapping) 273.I 436.E
 - on a subset 436.G
- uniformly convergent
 - (sequence) 435.A
 - on a family of sets 435.C
 - in the wider sense 435.C
- uniformly convex (normed linear space) 37.G
- uniformly equivalent (uniform spaces) 436.E
- uniformly integrable (family of random variables) 262.A
- uniformly locally compact (space) 425.V
- uniformly Lyapunov stable 126.F
- uniformly minimum variance unbiased estimator 399.C
- uniformly most powerful
 - (confidence region) 399.Q
 - (test) 400.A
 - invariant 399.Q
 - invariant level α 400.E
 - unbiased 399.Q
 - unbiased level α 400.E
- uniformly recursive in Ψ (define a partial recursive function) 356.E
- uniformly smooth (normed linear space) 37.G
- uniformly stable 394.B
- uniform neighborhood system 436.D
- uniform norm 168.B
- uniform operator topology 251.C
- uniform space(s) 436
 - analytically 125.S
 - complete 436.G
 - Hausdorff 436.C
 - locally totally bounded 436.H
 - metrizable 436.F
 - precompact 436.H
 - product 436.E
 - pseudometrizable 436.F
 - separated 436.C

T_1 - 436.C
 totally bounded 436.H
 uniform star convergence, relative 310.F
 uniform structure 436.B
 uniform subspace 436.E
 uniform topological space 436.B
 uniform topology 436.B
 unilateral constraints 440.A
 unilateral shift operator 390.I
 unimodal (distribution function) 341.H
 unimodular
 (germ of an analytic function) 418.E
 (locally compact group) 225.D
 totally 186.G
 unimodular group 60.B
 quaternion 412.G
 union
 (in axiomatic set theory) 33.B
 (of matroids) 66.H
 (of sets) 381.B
 axiom of 381.G
 disjoint 381.B,D
 of hypersurface elements 82.A
 of surface elements 324.B
 unipotent
 (algebraic group) 13.E
 (linear transformation) 269.L
 unipotent component 269.L
 unipotent matrix 269.F
 unipotent part
 (of an algebraic group) 13.E
 (of a nonsingular matrix) 13.E
 unipotent radical 13.I
 unique continuation theorem 323.J
 unique decomposition theorem (for a 3-manifold)
 65.E
 unique factorization domain 67.H
 unique factorization theorem (in an integral domain)
 67.H
 uniquely ergodic (homeomorphism (on a compact
 metric space) 136.H
 uniqueness
 in the sense of law of solutions 406.D
 set of 159.J
 of solution, pathwise 406.D
 uniqueness condition (for solutions of ordinary
 differential equations) 316.D
 uniqueness principle (in potential theory) 338.M
 uniqueness theorem
 (for analytic functions) 198.C
 (for class field theory) 59.B
 (for differential equations in a complex domain)
 316.G
 (for Fourier transform) 192.I
 (for harmonic functions) 193.E
 (for an initial value problem of ordinary differ-
 ential equations) 316.D
 of the analytic continuation 198.C,I
 Holmgren 321.F
 of homology theory 201.R
 Rellich 188.D
 von Neumann 351.C
 unique strong solution 406.D
 unirational surface 15.H
 unirational variety 16.J
 uniserial algebra 29.I
 absolutely 29.I
 generalized 29.I
 unsolvent space 142.B
 unsolvent system (of functions) 336.B

unit(s)
 (of an algebraic number field) 14.D
 (for measure of length) 139.C
 (in a ring) 368.B
 (of a symmetric matrix with rational coordi-
 nates) 348.J
 (of a vector lattice) 310.B
 Archimedean (of a vector lattice) 310.B
 arithmetic 75.B
 auxiliary 414.A
 base 414.A
 circular 14.L
 control 75.B
 derived 414.D
 fundamental 414.A
 fundamental (of an algebraic number field)
 14.D
 gravitational, system of 414.B
 imaginary 74.A 294.F
 international system of 414.A
 Kakutani 310.G
 matrix 269.B
 memory 75.B
 system of 414
 unital 36.A
 unitarily equivalent (self-adjoint operators) 390.G
 unitary
 (homomorphism between rings) 368.D
 (module) 277.D
 essentially 390.I
 unitary algebra 29.A
 unitary dilation 251.M
 unitary field theory 434.C
 unitary group 60.F 151.I
 (relative to an ε -Hermitian form) 60.O
 infinite 202.V
 over K 60.H
 over K , projective special 60.H
 over K , special 60.H
 projective 60.F
 special 60.F
 special (relative to an ε -Hermitian form) 60.O
 unitary matrix 269.I
 unitary monoid 409.C
 unitary operator 251.E 390.E
 unitary representation(s) 437
 disjoint 437.C
 equivalent 437.A
 induced by a representation of a subgroup
 437.O
 integrable 437.X
 irreducible 437.A
 isomorphic 437.A
 quasi-equivalent 437.C
 similar 437.A
 square integrable 437.M
 sufficiently many irreducible 437.B
 unitary restriction (of a semisimple Lie algebra)
 248.P
 unitary ring 368.A 409.C
 unitary semigroup 409.C
 unitary symplectic group 60.L
 unitary transformation 348.F
 (relative to an ε -Hermitian form) 60.O
 unitary transformation group 60.F
 unit ball
 (of a Banach space) 37.B
 (of a Euclidean space) 140
 unit cell 140
 unit circle 74.C 140

- ul style="list-style-type: none; padding-left: 0;">
- unit cost 281.D
- unit cube 139.F 140
- unit disk 140
- unit distribution 341.D
- unit element
 - (of a field) 149.A
 - (of a group) 190.A
 - (of a ring) 368.A
- unit function 306.B, App. A, Table 12.II
- unit group (of an algebraic number field) 14.D
- unit impulsive function App. A, Table 12.II
- unit mapping 203.F
- unit matrix 269.A
- unit n -cube 140
- unit point
 - (of an affine frame) 7.C
 - (of a projective frame) 343.C
- unit ray 351.B
- unit representation (of a group) 362.C
- unit sphere 140
- unit tangent sphere bundle 126.L
- unit theorem, Dirichlet 14.D
- unit vector 7.C 442.B
- unity
 - (in the axioms for the real numbers) 355.A
 - partition of 425.R
 - partition of, of class C^∞ 105.S
 - partition of, subordinate to a covering 425.R
 - primitive root of 14.L
- unity element
 - (of a field) 149.A
 - (of a ring) 368.A
- univalence superselection rule 351.K
- univalent (analytic function) 438.A
- univalent correspondence 358.B
- univalent function 438
- univariate (statistical data) 397.A
- universal
 - (∂ -functor) 200.I
 - ($*$ -representation of a Banach $*$ -algebra) 36.G
 - (unfolding) 51.D
- universal bundle 147.G,H
 - n - 147.G
- universal Chern class 56.C
- universal coefficient theorem
 - (in Abelian categories) 200.H
 - (for cohomology) 200.G 201.H
 - for homology 200.D 201.G
- universal constants (in the theory of conformal mapping) 77.F
- universal covering group 91.B 423.O
- universal covering space 91.B
- universal covering surface 367.B
- universal curve 93.H
- universal domain 16.A
- universal enveloping algebra
 - (of a Lie algebra) 248.J
 - special (of a Jordan algebra) 231.C
- universal enveloping bialgebra 203.G
- universal Euler-Poincaré class 56.B
- universal gravitation, law of 271.B
- universally Japanese ring 16.Y 284.F
- universally measurable 270.I
- universal mapping property 52.L
- universal net (in a set) 87.H
- universal Pontryagin class 56.D
- universal proposition 411.B
- universal quantifier 411.C
- universal set
 - (for the projective sets of class n) 22.E
 - (in set theory) 381.B
- universal Stiefel-Whitney class 56.B
- universal Teichmüller space 416
- universal Turing machine 31.C
- universal unfolding 418.E
- universal validity of a proposition, problem of 97
- universe
 - (in nonstandard analysis) 293.B
 - (of a structure) 276.B
- unknotted
 - (ball pair) 65.D 235.G
 - (knot) 235.A
 - (sphere pair) 65.D 235.G
- unknotting conjecture 235.G
- unknotting theorem, Zeeman 65.D
- unlabeled graph 186.B
- unmixed ideal 284.D
- unmixedness theorem 284.D
- unordered pair 381.B
 - (in axiomatic set theory) 33.B
 - axiom of 33.B
- unoriented cobordism class 114.H
- unoriented cobordism group 114.H
- unoriented graph 186.H
- unramified
 - (covering surface) 367.B
 - (prime ideal) 14.I
 - (projection of a covering surface) 367.B
 - analytically (semilocal ring) 284.D
- unramified covering (of a nonsingular curve) 9.I
- unramified extension 14.I 257.D
- unrenormalizable 132.C 361.B
- unsolvability
 - degree of 97
 - recursive, arithmetical hierarchy of degrees of 356.H
 - recursive, degree of 97
 - recursive, hyperarithmetical hierarchy of degrees of 356.H
- unstable
 - (boundary component) 77.E
 - (state) 394.A
 - completely (flow) 126.E
- unstable manifold 126.G,J
- unstable solution (of Hill's equation) 264.E
- up-ladder 206.B
- upper bound
 - (of a subset in an ordered set) 311.B
 - least (of an ordered set) 311.B
 - least (of a subset of a vector lattice) 310.C
- upper boundedness principle (in potential theory) 338.C
- upper central series (of a group) 190.J
- upper class
 - with respect to local continuity 45.F
 - with respect to uniform continuity 45.F
- upper control limit 404.B
- upper derivative
 - general (of a set function) 380.D
 - ordinary (of a set function) 380.D
- upper end (of a curvilinear integral) 94.D
- upper envelope (of a family of subharmonic functions) 193.R
- upper half-space of degree n , Siegel 32.F
- upper integral, Riemann 216.A
- upper limit (of a Riemann integral) 216.A
- upper limit function 84.C
- upper semicontinuous
 - (at a point) 84.C
 - (partition) 425.L

in a set 84.C
 upper semilattice 243.A
 upper triangular matrix 269.B
 upper variation (of a set function) 380.B
 Uryson lemma 425.Q
 Uryson space, Fréchet- 425.CC
 Uryson theorem, Tikhonov- 425.Q
 Uryson-Tikhonov theorem (on metrizability) 273.K
 Uzawa gradient method, Arrow-Hurwicz- 292.E

V

v-curve 111.H
 vacuum vector 377.A
 free 150.C
 vague topology (on a class of measures) 338.E
 valid formula 411.G
 valuation(s) 439
 additive 439.B
 Archimedean 14.F 439.C
 complete 439.D
 completion of 439.D
 discrete 439.E
 equivalent 439.B
 exponential 439.B
 generalized 439.B
 multiplicative 439.C
 non-Archimedean 14.F 439.C
 normal 439.E,H
 normalized 439.E
 p-adic 439.F
 p-adic exponential 439.F
 prolongation of 439.B
 pseudo- 439.K
 special 439.B
 over a subfield 439.B,C
 trivial 439.C,F
 valuation ideal (of a valuation) 439.B
 valuation ring 439.B
 completion of 439.D
 discrete 439.E
 valuation vector(s) 6.C
 ring of 6.C
 value(s)
 (of an infinite continued fraction) 83.A
 (of an infinite product) 379.G
 (of a variable) 165.C
 absolute (of a complex number) 74.B
 absolute (of an element of an ordered field) 149.N
 absolute (of an element of a vector lattice) 310.B
 absolute (of a real number) 355.A
 absolute (of a vector) 442.B
 asymptotic (of a meromorphic function) 62.A 272.H
 boundary (of a conformal mapping) 77.B
 boundary (hyperfunction) 125.V
 boundary (relative to a differential operator) 112.E
 boundary, problem \rightarrow boundary value problem
 characteristic (of a linear operator) 390.A
 cluster 62.A
 cluster, theorem 43.G
 critical (in bifurcation theory) 286.R
 critical (of a C^∞ -function on a manifold) 279.B
 critical (of a contact process) 340.C
 critical (of an external magnetic field) 340.B
 critical (of a mapping $u: \mathbb{R}^n \rightarrow \mathbb{R}^m$) 208.B

critical (of a C^∞ -mapping $\varphi: M \rightarrow M'$) 105.J
 exceptional (of a transcendental entire function) 429.B
 exceptional, Borel 272.E
 exceptional, Nevanlinna 272.E
 exceptional, Picard 272.E
 expectation (of an observable) 351.B
 expected (of a random variable) 342.C
 gap (of a point on a Riemann surface) 11.D
 initial (for ordinary differential equations) 316.A
 initial (for partial differential equations) 321.A
 initial (for stochastic differential equations) 406.D
 initial, problem \rightarrow initial value problem
 limit (of a mapping) 87.F
 mean (of a function on a compact group) 69.A
 mean (of a weakly stationary process) 395.C
 mean, theorem \rightarrow mean value theorem
 most probable 401.E
 principal (of inverse trigonometric functions) 131.E
 principal (of $\log z$) 131.G
 principal, Cauchy (of an improper integral) 216.D
 principal, Cauchy (of the integral of a function in $(-\infty, \infty)$) 216.E
 proper (of a boundary value problem) 315.B
 proper (of a linear mapping) 269.L
 proper (of a linear operator) 390.A
 proper (of a matrix) 269.F
 range of (of a meromorphic function) 62.A
 regular 105.J
 sample 396.B
 sample characteristic 396.C
 Shapley 173.D
 singular 302.A
 singular, decomposition (SVD) 302.E
 starting 303.E
 stationary (of a function) 106.L
 true, of parameter 398.A
 truth (of a formula) 411.E

value distribution 124.A
 value function 108.B 405.A
 value group
 (of an additive valuation) 439.B
 (of a multiplicative valuation) 439.C
 Vandermonde determinant 103.G
 van der Pol differential equation 290.C
 van der Waerden-Bortolotti covariant derivative 417.E
 van der Waerden test 371.C
 Vandiver conjecture 14.L
 van Hove sense, limit in 402.G
 vanishing cocycle 16.U
 vanishing cycle 418.F
 vanishing theorem
 (on compact complex manifolds) 194.G
 Kodaira 232.D
 van Kampen theorem (on fundamental groups) 170
 variability, measure of 397.C
 variable(s) 165.C
 (of a polynomial) 369.A
 artificial 255.C
 auxiliary 373.C
 basic 255.A
 bound 411.C
 canonical (in analytical dynamics) 271.F
 change of (in integral calculus) 216.C

Variable component (of a linear system)

- complex 165.C
- complex, theory of functions of 198.Q
- dependent 165.C
- differential (of a differential polynomial) 113
- endogenous 128.C
- exogenous 128.C
- explanatory 403.D
- hidden, theories 351.L
- independent 165.C
- individual 411.H
- inner 25.B
- lagged 128.C
- object 411.G
- outer 25.B
- predetermined 128.C
- predicate 411.G,H
- proposition 411.E
- random 342.C
- random, independent 342.C
- random, joint 342.C
- random, n -dimensional 342.C
- random, \mathbf{R}^n -valued 342.C
- random, (S, \mathfrak{E}) -valued 342.C
- real 165.C
- sampling inspection by 404.C
- separation of 322.C
- slack 255.A
- state 127.A
- variable component (of a linear system) 15.C 16.N
- variable method, discrete 303.A
- variable-step variable-order (VSVO) algorithms 303.E
- variance
 - (of a probability distribution) 341.B
 - (of a random variable) 342.C
 - (of univariate quantitative data) 397.C
 - analysis of 400.H 403.D
 - between-group 397.L
 - generalized 280.E 397.J
 - multivariate analysis of 280.B
 - population 396.C
 - sample 396.C
 - sample generalized 280.E
 - uniformly minimum unbiased estimator 399.C
 - within-group 397.L
- variance-covariance matrix 341.B 397.J
- variance matrix 341.B
- variate
 - canonical 280.E
 - fixed 403.D
- variation(s)
 - (of an integral) 100.E
 - bounded \rightarrow of bounded variation
 - calculus of 46
 - calculus of, classical theory of 46.C
 - calculus of, conditional problems in 46.A
 - calculus of, fundamental lemma in 46.B
 - coefficient of 397.C
 - of constants, Lagrange's method of 252.D
 - of constants, method of 55.B 252.I
 - first 46.B
 - first, formula 178.A
 - geodesic 178.A
 - lower (of a set function) 380.B
 - negative (of a mapping) 246.H
 - negative (of a real bounded function) 166.B
 - one-parameter 178.A
 - of parameters, Lagrange method of 252.D
 - of parameters, method of App. A, Table 14.I
 - positive (of a mapping) 246.H
 - positive (of a real bounded function) 166.B
 - proper 279.F
 - quadratic, process 406.B
 - second formula 178.A
 - total (of a finitely additive vector measure) 443.G
 - total (of a mapping) 246.H
 - total (of a real bounded function) 166.B
 - total (of a set function) 380.B
 - upper 380.B
- variational derivative 46.B
- variational equation 316.F 394.C
- variational formula, constant 163.E
- variational inequality 440
 - of evolution 440.C
 - stationary 440.B
- variational method 438.B
- variational principles(s) 441
 - (in ergodic theory) 136.G
 - (in statistical mechanics) 340.B 402.G
 - (in the theory of elasticity) 271.G
 - with relaxed continuity requirements 271.G
 - for the topological pressure 136.H
- variational problem, Gauss 338.J
- variation curve 178.A
- variation vector field 178.A
- variety
 - (algebraic variety) 16.A
 - (of block design) 102.B
 - Abelian 3
 - Abelian, isogeneous 3.C
 - Abelian, polarized 3.G
 - Abelian, simple 3.B
 - abstract 16.C
 - abstract algebraic 16.C
 - affine 16.A
 - affine algebraic 16.A
 - Albanese 16.P
 - Albanese (of a compact Kähler manifold) 232.C
 - algebraic 16
 - algebraic group 13.B
 - almost all points of a 16.A
 - Brieskorn 418.D
 - characteristic (of a microdifferential equation) 274.G
 - Chow 16.W
 - complex algebraic 16.T
 - function on a 16.A
 - generalized Jacobian 9.F 11.C
 - group 13.B 16.H
 - irreducible 16.A
 - Jacobian 9.E 11.C 16.P
 - Landau 146.C
 - Landau-Nakanishi 146.C 386.C
 - linear (in an Ω -module) 422.L
 - linear, linearly compact 422.L
 - minimal 275.G
 - nonsingular 16.F
 - normal 16.F
 - normal algebraic 16.F
 - Picard 16.P
 - Picard (of a compact Kähler manifold) 232.C
 - prealgebraic 16.C
 - product algebraic 16.A
 - projective 16.A
 - projective algebraic 16.A
 - quasi-affine algebraic 16.C
 - quasiprojective algebraic 16.C
 - rational 16.J

rational function on a 16.A
 reducible 16.A
 Schubert 56.E
 smooth 16.F
 strict Albanese 16.P
 toric 16.Z
 unirational 16.J
 Zariski topology of a 16.A
 varifold 275.G
 Varshamov-Gilbert-Sacks bound 63.B
 vector(s)
 (in a Euclidean space) 442
 (in a linear space) 256.A
 analytic 437.S
 characteristic (of a linear mapping) 269.L
 characteristic (of a linear operator) 390.A
 characteristic (of a matrix) 269.F
 coherent 377.D
 collinear 442.A
 column 269.A
 contravariant 256.J
 coplanar 442.A
 covariant 256.J
 cyclic (of a representation space of a unitary
 representation) 437.A
 effect 102.A
 eigen- (of a linear mapping) 269.L
 eigen- (of a linear operator) 390.A
 eigen- (of a matrix) 269.F
 eigen-, generalized 390.B
 error 102.A
 fixed 442.A
 four- 359.C
 four-, energy-momentum 258.C
 free 442.A
 free vacuum 150.C
 fundamental (in a vector space) 442.A
 horizontal 80.C
 independent 66.F
 integral 428.E
 mean 341.B
 mean curvature 365.D
 normal 105.L 111.H 364.A
 normalized 409.G
 observation 102.A
 orthogonal 139.G
 p - 256.O
 p -, bundle of 147.F
 positive 7.A 442.A
 Poynting 130.A
 proper (of a linear mapping) 269.L
 proper (of a linear operator) 390.A
 proper (of a matrix) 269.F
 root 390.B
 row 269.A
 tangent 105.H
 tangent, holomorphic 72.A
 tangent, of type (0,1) 72.C
 tangent, of type (1,0) 72.C
 unit 442.B
 unit (of an affine frame) 7.C
 vacuum 377.A
 valuation 6.C
 valuation, ring of 6.C
 vertical 80.B
 wave number (of a sine wave) 446
 Witt 449
 Witt, of length n 449.B
 zero 442.A
 vector algebra App. A, Table 3.I

vector analysis and coordinate systems App. A,
 Table 3
 vector bundle 147.F
 (algebraic) 16.Y
 ample 16.Y
 complex 147.F
 cotangent 147.F
 dual 147.F
 indecomposable 16.Y
 normal 105.L
 normal k - 114.J
 quaternion 147.F
 semistable 16.Y
 stable 16.Y 237.B
 stably equivalent 237.B
 tangent 105.H 147.F
 vector field
 (on a differentiable manifold) 105.M
 (in a 3-dimensional Euclidean space) 442.D
 Anosov 126.J
 Axiom A 126.J
 basic 80.H
 of class C^r 105.M
 contravariant 105.O
 covariant 105.O
 differentiation of App. A, Table 3.II
 formal 105.AA
 fundamental 191.A
 G - 237.H
 Hamiltonian 126.L 219.C
 holomorphic 72.A
 integral of App. A, Table 3.III
 irrotational 442.D
 Killing 364.F
 Lagrangian 126.L
 lamellar 442.D
 Morse-Smale 126.J
 solenoidal 442.D
 variation 178.A
 without source 442.D
 without vortex 442.D
 vector flux (through a surface) 442.D
 vector function, measurable 308.G
 vector group 422.E
 vectorial form, canonical 417.C
 vectorial p -form 417.C
 vector integral 443.A
 vector invariant 226.C
 vector lattice 310.B
 Archimedean 310.C
 complete 310.C
 normed 310.F
 σ -complete 310.C
 vector line (of a vector field) 442.D
 vector measure 443.G
 absolutely continuous 443.G
 bounded 443.G
 completely additive 443.G
 finitely additive 443.G
 vector potential 130.A 442.D
 vector product 442.C, App. A, Table 3.I
 vector representation (of a Clifford group) 61.D
 vector space 442.A
 over a field 256.H
 metric 256.H
 prehomogeneous 450.V
 standard (of an affine space) 7.A
 tangent 105.H
 topological 424.A
 vector triple product 442.C, App. A, Table 3.I

Vector tube

- vector tube 442.D
 - vector-valued integral 443
 - velocity
 - group 446
 - phase (of a sine wave) 446
 - velocity phase space 126.L
 - velocity potential 205.B
 - Veneziano model 132.C 386.C
 - Verbeure inequality, Roepstorff- Fannes- 402.G
 - Veronese surface 275.F
 - versal (unfolding) 51.D
 - version (of a stochastic process) 407.A
 - vertex (vertices)
 - (of an angle) 139.D 155.B
 - (in a cell complex) 70.D
 - (of a circular cone) 78.A
 - (of a complete quadrangle) 343.C
 - (of a convex cell in an affine space) 7.D
 - (of a convex polyhedron) 89.A
 - (in a Euclidean (simplicial) complex) 70.B
 - (of a geodesic triangle) 178.A
 - (of a graph) 186.B
 - (of a linear graph) 282.A
 - (of a parabola) 78.C
 - (of a polygon) 155.F
 - (in the polyhedron of a simplicial complex) 70.C
 - (of a simplex in an affine space) 7.D
 - (in a simplicial complex) 70.C
 - (of a spherical triangle) 432.B
 - (of a star region) 339.D
 - adjacent 186.B
 - end 186.B
 - initial 186.B
 - isolated 186.B
 - terminal 186.B
 - vertical angles 139.D
 - vertical component (of a vector field) 80.C
 - vertical slit mapping, extremal 367.G
 - vertical vector 80.B
 - very ample
 - (divisor) 16.N
 - (linear system) 16.N
 - (sheaf) 16.E
 - very weak Bernoulli process 136.E
 - Vessiot extension field, Picard- 113
 - Vessiot theory, Picard- 113
 - Vey classes, Godbillon- 154.G
 - vibrating membrane, equation of a 325.A
 - vibrating string, equation of a 325.A
 - vibration 318
 - normal 318.B
 - parametrically sustained 318.B
 - self-excited 318.B
 - Viète, F. 444
 - Vietoris axiom 425.Q
 - Vietoris exact sequence, Mayer- (for a proper triple) 201.C
 - Villat integration formula App. A, Table 15.VI
 - Vinogradov mean value theorem 4.E
 - virtual arithmetic genus (of a divisor) 16.E
 - viscosity 205.B
 - coefficient of 205.C
 - coefficient of bulk 205.C
 - coefficient of shear 205.C
 - magnetic 259
 - visibility manifold 178.F
 - Vitali covering theorem 380.D
 - Vivanti theorem 339.A
 - Volterra integral equation 217.A
 - Volterra operator 68.J
 - Volterra theorem, Poincaré- 198.J
 - Volterra type
 - integral equation of 217.A
 - integrodifferential equation of 222.A
 - volume
 - (of an idele) 6.D
 - (of a lattice in a Euclidean space) 92.D
 - (of a polyhedron) 139.F
 - (of a simplex in an affine space) 7.E
 - inner 270.G
 - outer 270.G
 - volume element
 - (of an oriented C^∞ -manifold) 105.W
 - associated with a Riemannian metric 105.W
 - integral of a function with respect to a (on a C^∞ -manifold) 105.W
 - von Kármán transonic similarity 205.D
 - von Mises theorem 399.K
 - von Neumann, J. 445
 - von Neumann algebra 308.C
 - discrete 308.E
 - finite 308.E
 - induced 308.C
 - purely infinite 308.E
 - reduced 308.C
 - semifinite 308.E
 - structure theorem for, of type III 308.I
 - of type I 308.E
 - of type II 308.E
 - of type II_1 308.E
 - of type II_∞ 308.E
 - of type III 308.E
 - von Neumann condition 304.F
 - von Neumann density theorem 308.C
 - von Neumann-Halmos theorem 136.E
 - von Neumann inequality 251.M
 - von Neumann-Morgenstern solution 173.D
 - von Neumann reduction theory 308.G
 - von Neumann selection theorem 22.F
 - von Neumann theorem, Weyl- 390.I
 - von Neumann uniqueness theorem 351.C
 - vortex, vector field without 442.D
 - vortex line 205.B
 - vorticity 205.B
 - vorticity theorem, Helmholtz 205.B
 - Vossen theorem, Cohn- 111.I
 - VSVO algorithm 303.E
 - V.W.B. process 136.E
- W**
- $W_p^1(\Omega)$ (Sobolev space) 168.B
 - w-plane 74.D
 - w-point (of an entire function) 429.B
 - w-sphere 74.D
 - W-construction (of an Eilenberg-MacLane complex) 70.F
 - W-surface 111.I
 - W^* -algebra 308.C
 - Wagner function 39.F
 - waiting time 260.H
 - waiting time distribution 307.C
 - Wald theorem 399.H,M
 - Walker equation, Yule- 421.D
 - Walker metrics, Robertson- 359.E
 - wall
 - adiabatic 419.A
 - diathermal 419.A
 - Wall group 114.J

Wallis formula App. A, Table 10.VI
 Wallis test, Kruskal- 371.D
 Walsh system of orthogonal functions 317.C
 wandering 126.E
 weakly, under a group 136.E
 wandering point 126.E
 wandering set 136.C
 weakly 136.C
 Wang exact sequence (of a fiber space) 148.E
 Waring problem 4.E
 Warning second theorem 118.B
 Warning theorem 118.B
 wasan 230
 water wave(s) 205.F
 deep 205.F
 long 205.F
 shallow 205.F
 Watson formula 39.E, App. A, Table 19.IV
 Watson-Nicholson formula App. A, Table 19.III
 Watson process
 Galton- 44.B
 multi (k)-type Galton- 44.C
 Watson transform 160.C 220.B
 wave(s) 446
 Alfvén 259
 capillary 205.F
 dispersive 446
 electromagnetic 446
 electromagnetic, theory of 130.B
 fast 259
 gravity 205.F
 gravity, long 205.F
 longitudinal 446
 Mach 205.B
 partial 386.B
 partial, expansion 375.E 386.B
 plane 446
 polarized 446
 shock 205.B 446
 sine 446
 sinusoidal 446
 slow 259
 spherical 446
 stationary 446
 Stokes 205.F
 surface 446
 transverse 446
 water 205.F
 water, deep 205.F
 water, long 205.F
 water, shallow 205.F
 wave equation 325.A 446, App. A, Table 15.III
 wave expansion, partial 375.E 386.B
 wave front set 274.B 345.A
 analytic 274.D
 wave function 351.D
 spheroidal 133.E
 wave guide 130.B
 wavelength (of a sine wave) 446
 wave number (of a sine wave) 446
 wave operator 375.B, H
 generalized 375.B
 incoming 375.B
 modified 375.B
 outgoing 375.B
 wave propagation 446
 wave scattering amplitude, partial 375.E
 wave steepness 205.F
 W.B. process 136.E
 WC group 118.D

Subject Index

Weak topology

weak (boundary component) 77.E
 weak Bernoulli process 136.E
 very 136.E
 weak convergence
 (of operators) 251.C
 (of probability measures) 341.F
 (of a sequence of submodules) 200.J
 weak C^∞ topology 279.C
 weak derivative 125.E
 weak dimension (of a module) 200.K
 weaker
 (equivalence relation) 135.C
 (method of summation) 379.L
 (topology) 425.H
 (uniformity) 436.E
 weak extension (of a differential operator) 112.E, F
 weak form of the boundary value problem (of partial differential equations) 304.B
 weak global dimension (of a ring) 200.K
 weak homotopy equivalence 202.F
 weak lacuna 325.J
 weak law of large numbers 395.B
 weak Lefschetz theorem 16.U
 weakly, converge
 (in a normal linear space) 37.E
 (in a topological linear space) 424.H
 weakly almost complex manifold 114.H
 weakly compact (linear operator) 68.M
 weakly compact cardinal number 33.E
 weakly continuous (function with values in a Banach space) 37.K
 weakly continuous representation (of a topological group) 69.B
 weakly dominated (statistical structure) 396.F
 weakly equivalent (transformations) 136.F
 weakly G -stationary (system of random variables) 395.I
 weakly hyperbolic linear (differential operator) 325.H
 weakly inaccessible (ordinal number) 312.E
 weakly inaccessible (cardinal number) 33.E
 weakly integrable 443.F
 weakly isomorphic (automorphisms) 136.E
 weakly measurable 443.B, I
 weakly mixing (automorphism) 136.E
 weakly modular 351.L
 weakly nonlinear differential equations 290.D
 weakly 1-complete manifold 114.H
 weakly stationary process 395.A
 weakly stationary process of degree k 395.I
 weakly stationary random distribution 395.C
 weakly symmetric Riemannian space 412.J
 weakly wandering set 136.C
 weakly wandering under a group 136.F
 weak minimum 46.C
 weak Mordell-Weil theorem 118.E
 weak operator topology 251.C
 weak potential kernel 260.D
 weak solution 204.C 323.G 378.I
 weak solution, Hopf's 204.C
 weak* Dirichlet algebra 164.G
 weak star topology 37.E 424.H
 weak topology
 (in a cell complex) 70.D
 (on a class of measures) 338.E
 (on a direct product space) 425.K
 (on a direct sum) 425.M
 (on a locally convex space) 424.H
 (on a normed linear space) 37.E
 (relative to the pairing $\langle E, F \rangle$) 424.H

Weak type (p, q) , quasi-linear operator of

- hereditarily 425.M
- weak type (p, q) , quasi-linear operator of 224.E
- Weber differential equation 167.C, App. A, Table 20.III
- Weber formula App. A, Table 19.IV
- Weber function 167.C, App. A, Table 19.IV 20.III
- Weber-Hermite differential equation 167.C
- Weber-Sonine formula App. A, Table 19.III
- web group 234.B
- Wedderburn-Mal'tsev theorem (on algebras) 29.F
- Wedderburn theorem
 - (on commutativity of finite fields) 149.M
 - (on simple algebras on a field) 29.E
 - (on simple rings) 368.G
- wedge 125.V
 - infinitesimal 125.V
- wedge product (of derived functors) 200.K
- wedge theorem, edge of the 125.W
- Weierstrass, K. 447
 - analytic function in the sense of 198.I
- Weierstrass approximation theorem 336.A
- Weierstrass canonical form
 - (of elliptic curves) 9.D
 - (of the gamma function) 174.A
- Weierstrass canonical product 429.B
- Weierstrass criterion for uniform convergence 435.A
- Weierstrass elliptic functions 134.F, App. A, Table 16.IV
- Weierstrass-Enneper formula 275.A
- Weierstrass \wp -function 134.F, App. A, Table 16.IV
- Weierstrass point 11.D
- Weierstrass preparation theorem 21.E 370.B
- Weierstrass sigma function 134.F
- Weierstrass-Stone theorem 168.B
- Weierstrass theorem
 - (on compactness of subsets of \mathbf{R}) 355.D
 - (on continuous functions on a compact set) 84.C
 - (on essential singularities) 198.D
 - (on expansion of meromorphic functions) 272.A
 - (on transcendental entire functions) 124.B
- Bolzano- 140 273.F
- Casorati- (on essential singularities) 198.D
- of double series 379.H
- Lindemann- 430.D
- Weierstrass-type preparation theorem (for micro-differential operators) 274.F
- Weierstrass zeta function 134.F
- weight(s)
 - (of an automorphic form) 32.C
 - (in a barycenter) 7.C
 - (of a covariant) 226.D
 - (of a multiple covariant) 226.E
 - (of a representation of a complex semisimple Lie algebra) 248.W
 - (on a von Neumann algebra) 308.D
 - (of a weighted homogeneous analytic function) 418.D
- equal, principle of 402.E
- extremal length with 143.B
- highest (of a representation of a complex semisimple Lie algebra) 248.W
- Källén-Lehmann 150.D
- weighted homogeneous (analytic function) 418.D
- weighted moving average 397.N
- weight function
 - (for the mean of a function) 211.C
 - (in numerical integration) 299.A
 - (in orthogonality) 317.A
- weight group (of a pair (T, K)) 92.C
- weighting matrix 86.B
- weight k , automorphic form of 32.B
- weight k , Fuchsian form of 32.B
- weight k , Hilbert modular form of 32.B
- weight k , Siegel modular form of 32.F
- weight lattice (of a pair (T, K)) 92.C
- weight m , automorphic form of 32.A
- weight w , invariant of 226.D
- Weil-Châtelet group 118.D
- Weil cohomology 450.Q
- Weil conjecture 450.Q
 - Taniyama- 450.S
- Weil domain 21.G
- Weil group 6.E 450.H
- Weil L -function 450.H
- Weil measure 225.G
- Weil number 3.C
- Weil theorem
 - Borel- 437.Q
 - Mordell- 118.E
 - weak Mordell- 118.E
- Weinberg-Salam model, Glashow- 132.D
- Weingarten formula
 - (for an isometric immersion) 365.C
 - (in the theory of surface) 111.H, App. A, Table 4.I
- Weingarten surface 111.I
- Weirich formula App. A, Table 19.III
- Weisberger sum rule, Adler- 132.C
- Welch test 400.G
- well-behaved 36.K
- well-chained (metric space) 79.D
- well-measurable 407.B
- well-measurable σ -algebra 407.B
- well-ordered set 311.C
- well-ordering 311.C
- well-ordering theorem 34.B
- well-posed
 - (initial value problem) 321.E
 - (martingale problem) 115.C
 - (problems for partial differential equations) 322.A
- Wendroff scheme, Lax- 304.F
- Weyl, H. 448
- Weyl canonical basis 248.P
- Weyl chamber 13.J 248.R
 - positive 248.R
- Weyl character formula (on representation of compact Lie groups) 248.Z
- Weyl conformal curvature tensor 80.P
- Weyl criterion 182.H
- Weyl form 351.C
- Weyl form of the CCRs (canonical commutation relations) 337.A
- Weyl formula 323.M
- Weyl group
 - (of a BN pair) 13.R
 - (of a complex semisimple Lie algebra) 248.R
 - (of a connected algebraic group) 13.H
 - (of a Coxeter complex) 13.R
 - (of a root system) 13.J
 - (of a symmetric Riemannian space) 413.F
- affine (of a symmetric Riemannian space) 413.I
- k - 13.Q
- Weyl integral formula 225.I

Weyl lemma 112.D
 Weyl-Stone-Titchmarsh-Kodaira theory 112.O
 Weyl theorem
 (on Lie algebras) 248.F
 Cartan- 248.W
 Weyl theory
 Peter- (on compact groups) 69.B
 Peter- (on compact Lie groups) 249.U
 Weyl-von Neumann theorem 390.I
 white group 92.D
 Whitehead group (of a ring) 237.J
 Whitehead product 202.P
 Whitehead theorem 202.N
 generalized 202.N
 Whitehead torsion 65.C
 white noise 176.D
 Gaussian 407.C
 Whitney class, Stiefel- 56.B,F 147.M
 Whitney class, total Stiefel- 56.B
 Whitney class, universal Stiefel- 56.B
 Whitney condition (b) 418.G
 at a point 418.G
 Whitney extension theorem 168.B
 Whitney mapping (map) 201.J
 Whitney number, Stiefel- 56.F
 Whitney prestratification 418.G
 Whitney stratification 418.G
 Whitney sum (of vector bundles) 147.F
 Whitney theorem 105.K
 Whitney U -test, Mann- 371.C
 Whittaker differential equation 167.B, App. A,
 Tables 14.II 19.II
 Whittaker function 167.B, App. A, Table 19.II
 Whittaker model 450.O
 wider sense, Bayes solution in the 398.B
 Widom type, Parreau- 164.K
 width, curve of constant 111.E
 width of an oval 111.E
 Wiener, generalized Tauberian theorems of 192.D
 Wiener-Brelot solution, Perron- (of Dirichlet problem) 120.C
 Wiener compactification 207.C
 Wiener filter 86.E
 Wiener formula 160.B
 Wiener-Hopf integrodifferential equation 222.C
 Wiener-Ikehara-Landau theorem 123.B
 Wiener integral, multiple 176.I
 Wiener-Itô decomposition 176.I
 Wiener kernel 95
 Wiener-Lévy theorem 159.I
 Wiener martingales, $\{\mathcal{F}_t\}$ - 406.B
 Wiener measure 250.E
 Wiener measure with the initial distribution μ 45.B
 Wiener process 5.D 45.B 98.B
 Wiener test
 (for Brownian motion) 45.D
 (for Dirichlet problem) 338.G
 (for random walk) 260.E
 Wiener theorem, Paley- 125.O,BB
 Wightman axiom 150.D
 Wightman field 150.D
 Wightman function 150.D
 truncated 150.D
 Wigner coefficients 353.B
 Wigner rotation 258.C
 Wigner supermultiplet 351.J
 Wigner theorem 258.C 351.H
 Wilcoxon signed rank test 371.B
 Wilcoxon test 371.C

Subject Index

Yourden square design

Wilczynski, directrix of 110.B
 wild
 (knot) 235.A
 (space) 65.F
 Wilson-Hilferty approximations 374.F
 Wilson theorem 297.G
 Wiman theorem 429.B
 winding number 198.B
 window 421.C
 winning strategy 33.F
 Wirtinger inequality App. A, Table 8
 Wirtinger presentation (of a knot group) 235.B,D
 Wishart distribution 374.C
 p -dimensional noncentral 374.C
 witch of Agnesi 93.H
 within-group variance 397.L
 without source (vector field) 442.D
 without vortex (vector field) 442.D
 Witt decomposition (of a quadratic form) 348.E
 Witt group (of nondegenerate quadratic forms) 348.E
 Witt matrix, Hasse- 9.E
 Witt theorem
 (on ε -trace forms) 60.O
 (on quadratic forms) 348.E
 Poincaré-Birkhoff- (on Lie algebras) 248.J
 Witt vector 449
 of length n 449.B
 WKB method 25.B
 WKB method 25.B
 Wold decomposition 395.D
 Wolfowitz inequality 399.J
 word 31.B 190.M
 cord 63.A
 equivalent 31.B
 word problem (in a finitely presented group) 161.B
 worst-case complexity 71.A
 Wright differential equation, Cherwell- 291.F
 Wronskian (determinant) 208.E
 Wu class (of a topological manifold) 56.F
 Wu formula App. A, Table 6.V

X

x_+^4 (distribution) 125.EE
 \mathfrak{X} -minimal function 367.E
 x_i -axis (of a Euclidean space) 140

Y

Y -diffeomorphism 136.G
 Y -flow 136.G
 Yamabe problem 183 364.H
 Yang-Mills equation 80.Q
 Yang-Mills field 150.G
 Yang-Mills functional 80.Q
 Yang-Mills G -connection 80.Q
 Yates-Terry normal score test, Fisher- 371.C
 Yosida approximation 286.X
 Yosida theorem, Hille- 378.B
 Young diagram 362.H
 Young inequality 224.E, App. A, Table 8
 Hausdorff- 224.E
 Young modulus 271.G
 Young symmetrizer 362.H
 Young theorem, Hausdorff- 317.B
 Yourden square 102.K
 Yourden square design 102.K

Yukawa potential 338.M
Yule-Walker equation 421.D

Z

Z (integers) 294.A,C
Z (Zermelo set theory) 33.B
ZF (Zermelo-Fraenkel set theory) 33.B
z-distribution 374.B, App. A, Table 22
z-plane 74.D
z-sphere 74.D
z-transformation, Fisher 374.D
Z-distribution 341.D
Z-set 382.B
Z-action (continuous) 126.B
Z-action of class C^r 126.B
Z₁-extension 14.L
 basic 14.L
 cyclotomic 14.L
Zariski closed (set) 16.A
Zariski connectedness theorem 16.X
Zariski decomposition 15.D
Zariski dense (set) 16.A
Zariski main theorem 16.I
Zariski open (set) 16.A
Zariski ring 284.C
 complete 284.C
Zariski site 16.AA
Zariski topology
 (of a spectrum) 16.D
 (of a variety) 16.A
Zassenhaus group 151.H
Zassenhaus theorem
 Jordan- (on integral representation of a group) 362.K
 Schur- (on Hall subgroups) 151.E
Zeeman unknotting theorem 65.D
Zeller theorem, Brown-Shields- 43.C
zenith angle App. A, Table 3.V
Zeno, paradoxes of 319.C
Zermelo-Fraenkel set theory 33.A,B
Zermelo set theory 33.B
zero(s) 355.A
 (of a function on an algebraic curve) 9.C
 (of a function on an algebraic variety) 16.M
 homologous to 198.B
 homotopic to 202.B
 scattered, function with 208.C
zero algebra 29.A
zero cycles 16.R
zero-dimensional space, Baire 273.B
zero divisor
 (of a function on an algebraic variety) 16.M
 (with respect to M/P) 284.A
 (of a ring) 368.B
zero element
 (of an additive group) 2.E 190.A
 (of a field) 149.A
 (of a linear space) 256.A
 (of a ring) 368.A
zero homomorphism (between two A -modules) 277.H
zero matrix 269.B
zero object 52.N
zero-one law 342.G
 Blumenthal 261.B
 Hewitt-Savage 342.G
 Kolmogorov 342.G
zero point
 (of a holomorphic function) 198.C

 (of a polynomial) 337.B 369.C
 (of a subset of a polynomial ring) 369.C
 of the k th order (of a holomorphic function) 198.C
 of the $-k$ th order (of a complex function) 198.D
 order of (of a homomorphic function) 198.C
zero-point theorem
 Hilbert 369.D
 Rückert 23.B
zero representation (of an algebra) 362.C
zero ring 368.A
zero-section (of a block bundle) 147.Q
zero-sum (game) 173.A
zero-sum two-person game 108.B
zeroth law of thermodynamics 419.A
zero vector 442.A
zeta function(s) 450
 (associated with a prehomogeneous space) 450.V
 (of a homeomorphism of a compact metric space) 126.K
 of an algebraic function field 450.P
 of an algebraic variety over a finite field 450.Q
congruence 450.P
Dedekind 14.C 450.D
defined by Hecke operators 450.M
Epstein 450.K
Hasse 450.S
Hey 27.F
Hurwitz 450.B
Ihara 450.U
Riemann 450.B
of a scheme 450.R
Selberg 450.T
Siegel, of indefinite quadratic forms 450.K
of a simple algebra 27.F
Tamagawa 450.L
Weierstrass 134.F, App. A, Table 16.IV
ZFC 33.B
Zilber theorem, Eilenberg- 201.J
zonal harmonics 393.D
zonal polynomial 374.C
zonal spherical function (on a homogeneous space) 437.Y
Zorn lemma 34.C
Zygmund, smooth in the sense of 168.B
Zygmund class 159.E
Zygmund singular integral operator, Calderón- 251.O
Zygmund type, kernel of Calderón 217.J